$$\frac{\partial \psi^{ij}}{\partial x^{kl}}(A) = \begin{cases} a_{jl} & k = i \neq j \\ a_{il} & k = j \neq i \\ 2a_{il} & k = i = j \\ 0 & \text{otherwise.} \end{cases}$$

Show that the rank of this matrix is $n(n+1)/2$ at I (and hence at A for all $A \in O(n)$.) Conclude that $O(n)$ has dimension $n(n-1)/2$.
(f) Show that $\det A = \pm 1$ for all $A \in O(n)$. The group $O(n) \cap SL(n, \mathbb{R})$ is called the **special orthogonal group** $SO(n)$, or the **rotation group** $R(n)$.

* **34.** Let $M(m, n)$ denote the set of all $m \times n$ matrices, and $M(m, n; k)$ the set of all $m \times n$ matrices of rank k.

(a) For every $X_0 \in M(m, n; k)$ there are permutation matrices P and Q such that

$$PX_0Q = \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix}, \qquad \text{where } A_0 \text{ is } k \times k \text{ and non-singular.}$$

(b) There is some $\varepsilon > 0$ such that A is non-singular whenever all entries of $A - A_0$ are $< \varepsilon$.
(c) If

$$PXQ = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where the entries of $A - A_0$ are $< \varepsilon$, then X has rank k if and only if $D = CA^{-1}B$. *Hint*: If I_k denotes the $k \times k$ identity matrix, then

$$\begin{pmatrix} I_k & 0 \\ Y & I_{m-k} \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ YA + C & YB + D \end{pmatrix}.$$

(d) $M(m, n; k) \subset M(m, n)$ is a submanifold of dimension $k(m + n - k)$ for all $k \leq m, n$.