defined in Chapter 1, ... distance $\pm 1/2$ be an immersion at 6 points (the image points are the points at distance $\pm 1/2$ on each axis). There is a way of immersing $\mathbb{P}^2$ in $\mathbb{R}^3$, known as Boy's Surface. See Hilbert and Cohn-Vossen, *Geometry and the Imagination*, pp. 317–321.

•**30.** A continuous function $f\colon X \to Y$ is **proper** if $f^{-1}(C)$ is compact for every compact $C \subset Y$. The **limit set** $L(f)$ of f is the set of all $y \in Y$ such that $y = \lim f(x_n)$ for some sequence $x_1, x_2, x_3, \ldots \in X$ with no convergent subsequence.

(a) $L(f) = \emptyset$ if and only if f is proper.
(b) $f(X) \subset Y$ is closed if and only if $L(f) \subset f(X)$.
(c) There is a continuous $f\colon \mathbb{R} \to \mathbb{R}^2$ with $f(\mathbb{R})$ closed, but $L(f) \neq \emptyset$.
(d) A one-one continuous function $f\colon X \to Y$ is a homeomorphism (onto its image) if and only if $L(f) \cap f(X) = \emptyset$.
(e) A submanifold $M_1 \subset M$ is a closed submanifold if and only if the inclusion map $i\colon M_1 \to M$ is proper.
(f) If M is a connected manifold, there is a proper map $f\colon M \to \mathbb{R}$; the function f can be made C^∞ if M is a C^∞ manifold.
(g) The same is true if M has at most countably many components.

31. (a) Find a cover of $[0,1]$ which is not locally finite but which is "point-finite": every point of $[0,1]$ is in only finitely many members of the cover.
(b) Prove the Shrinking Lemma when the cover $\mathcal{O}$ is point-finite and countable (notice that local-finiteness is not really used).
(c) Prove the Shrinking Lemma when $\mathcal{O}$ is a (not necessarily countable) point-finite cover of any space. (You will need Zorn's Lemma; consider collections $\mathcal{C}$ of pairs (U, U') where $U \in \mathcal{O}$, $U' \subset U$, and the union of all U' for $(U, U') \in \mathcal{C}$, together with all other $U \in \mathcal{O}$ covers the space.)