INFLATING POLYHEDRAL SURFACES

IGOR PAK*

Abstract. We prove that all polyhedral surfaces in $\mathbb{R}^3$ have volume-increasing isometric deformations. This resolves the conjecture of Bleecker who proved it for convex simplicial surfaces [B1]. A version of this result is proved for all convex surfaces in $\mathbb{R}^d$. We also discuss limits on the volume of such deformations, present a number of conjectures and special cases.

Introduction

Metric geometry of convex surfaces goes back to Legendre and Cauchy and has blossomed in the 20th century with the works of Alexandrov and his school. Despite a large body of results, there remain glaring gaps in our understanding. About ten years ago, Bleecker showed that for every convex simplicial surface in $\mathbb{R}^3$ there exists a volume-increasing continuous isometric deformation [B1]. In this paper we prove Bleecker's conjecture by extending this result to all polyhedral surfaces in $\mathbb{R}^3$. Our approach heavily relies on the notion of a *submetric deformation*, defined as a deformation where the geodesic distances between the corresponding points are non-increasing[1]. We first prove that for every polyhedral surface in $\mathbb{R}^3$ there exists a volume-increasing piecewise-linear continuous submetric deformation. Moreover, we show that if the original surface was convex, the submetric deformation can also be made convex. This construction is then combined with Burago-Zalgaller's theorem [BZ3] to obtain the result. We further extend our approach to higher dimensions, for all convex polyhedral surfaces in $\mathbb{R}^d$.

Let us briefly outline the history of the subject. In the 1940's Alexandrov showed that the intrinsic geometry of the surface of a convex polyhedron in $\mathbb{R}^3$ uniquely determines the polyhedron, up to a rigid motion [A2]. This extends the Cauchy rigidity theorem as the faces are no longer required to be rigid. This result, called the *uniqueness theorem*, was partially extended by Olovianishnikoff and further extended by Pogorelov to all convex surfaces [Po2]. Alexandrov also proved the celebrated *existence theorem* which showed that under certain natural conditions a convex surface can be realized as a surface of a polyhedron.

Clearly, the uniqueness theorem cannot be extended to non-convex polyhedra, as polyhedra can be isometrically deformed (see Figure 1). Such continuous isomet-

Date: 20 August, 2006.

*Department of Mathematics, MIT, Cambridge, MA, 02138, USA; Email: pak@math.mit.edu.

[1]Such deformations (also immersions, embedding, etc.) are called *short* or *nowhere long* in the differential geometry context. We find the word *submetric* much clearer, and its derivations simplify some sentence constructions.

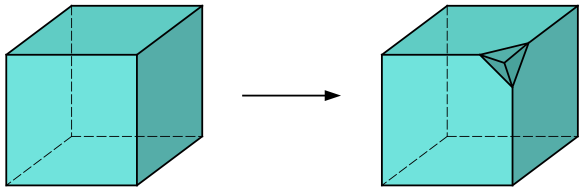

FIGURE 1. Bending of the surface of a cube.

ric deformations are called *bendings*. Of course, by the uniqueness theorem, in the bending of a convex polyhedron all other polyhedra must be non-convex.

Observe that the volume enclosed by the surface in Figure 1 is decreasing under the bending. About ten years ago Bleecker noticed that the volume can actually increase [B1]. This can be easily seen in an example shown in Figure 2, where a doubly covered equilateral triangle is deformed into a union of two simplices (the figure shows an unfolding and how the pieces are glued together).

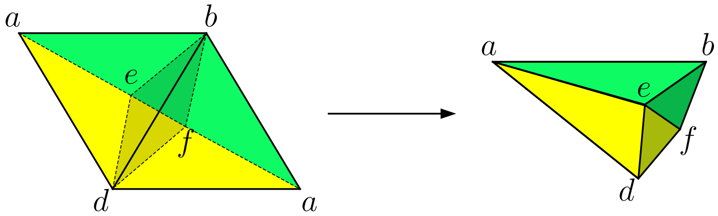

FIGURE 2. Volume-increasing bending of a doubly covered triangle.

Bleecker constructed some symmetric volume-increasing bendings of all simplicial polyhedra and gave symmetric construction for all regular polyhedra [B1]. Bleecker conjectured that this should be possible for all polyhedral surfaces in $\mathbb{R}^d$. Our first major result is the proof of his conjecture in $\mathbb{R}^3$ (see theorems 7 and 10).

Without volume considerations such bendings were further studied by Milka [Mi], and some of our constructions are based on his ideas (see also [P]). Bleecker obtained an about 37.7% increase over the volume of a regular tetrahedron and about 21.9% increase over the volume of a cube. He asked how much further this can be extended. This question was further reiterated in [Al3, S2] and by Connelly[2].

In a remarkable series of papers [BZ1, BZ3], Burago and Zalgaller first proved the analogue of the existence theorem for non-convex polyhedra. In the second paper they extended their approach to show that a non-convex polyhedron can be realized within an ε-neighborhood of a given submetric embedded polyhedral surface. Our proof of Bleecker's conjecture heavily relies on Burago-Zalgaller's theorem.

Our second main result (Theorem 8) is the existence of a continuous piecewise-linear volume-increasing deformation $\{S_t\}$ of the surface S of convex polyhedron in $\mathbb{R}^d$, such that all surfaces S_t are convex and submetric to S. When $d = 3$, this result is neither weaker nor stronger than Bleecker's theorem. On one hand, Bleecker

[2]Personal communication.

constructs an isometric deformation, while our deformations are only submetric. On the other hand, Bleecker works only with simplicial polyhedra and his deformations are necessarily non-convex, while ours are always convex.

The paper is structured as follows. We present the results in Section 1. The proofs are given in Section 2. In Section 3 we discuss the extend to which the volume can be extended and formulate a series of conjectures nd open problems. We elaborate on several interesting examples and special cases in Section 4. In a quick application of the mylar balloon, we obtain an upper bound on the maximum inflated volume in the case of doubly covered regular n-gons (Section 5). We conclude with final remarks in Section 6.

1. Main results

Throughout the paper we consider both convex and non-convex surfaces in $\mathbb{R}^3$, by which we mean embedded 2-dimensional orientable compact surfaces with no boundary. We also consider convex d-dimensional surfaces in $\mathbb{R}^{d+1}$ when $d \geq 3$. Unless explicitly stated otherwise, all surfaces are assumed to be polyhedral and compact.

1.1. **Convex surfaces.** Let $S = \partial P$ be the surface of a 3-dimensional convex polyhedron $P \subset \mathbb{R}^3$. We say that S is a *convex surface* in $\mathbb{R}^3$. We say that S is *simplicial* if P is simplicial, i.e. all its faces are triangles. The surface S' is called *isometric* to S, write $S \sim S'$, if there exist a piecewise-linear homeomorphism $\varphi : S \to S'$ which preserves geodesic distances: $|x, y|_S = |\varphi(x), \varphi(y)|_{S'}$ for all $x, y \in S$.

A continuous piecewise-linear isometric deformation $\{S_t \mid t \in [0, 1]\}$ is called a *bending* of S if all surfaces are isometric to $S = S_0$: $S_t \sim S$, for all $0 \leq t \leq 1$. We say that $\{S_t\}$ is *volume-increasing* if $\operatorname{vol}(S_t) < \operatorname{vol}(S_{t'})$ for all $0 \leq t < t' \leq 1$. The following is the main result in [B1] :

Theorem 1 (Bleecker). *For every convex simplicial surface S in $\mathbb{R}^3$, there exists a volume-increasing bending of S.*

Informally, this means that every convex simplicial polyhedron can be *inflated* without tearing and stretching. The proof in [B1] is based on an explicit technical construction. Let us emphasize that the surfaces in the theorem (and throughout this section) are always isometric embedding. For more on immersed surfaces see §1.3 below.

We say that a surface $S' \subset \mathbb{R}^d$ is *submetric* to S, write $S' \preccurlyeq S$, if there exist a homeomorphism $\varphi : S \to S'$ which does not increase the geodesic distances: $|x, y|_S \geq |\varphi(x), \varphi(y)|_{S'}$ for all $x, y \in S$. We refer to φ as *submetry map.* A *shrinking* of S is a continuous piecewise-linear deformation $\{S_t \mid t \in [0, 1]\}$, such that $S_0 = S$ and $S_t \preccurlyeq S_{t'}$, for all $0 \leq t' < t \leq 1$. We say that this shrinking is *from* S_0 *to* S_1 in this case.

We say that a deformation $\{S_t\}$ is *convex* if all surfaces S_t are convex. We are now ready to state our main result.

Theorem 2. *For every convex polyhedral surface S in $\mathbb{R}^3$, there exists a volume-increasing convex shrinking of S.*

Recall from the introduction that Theorem 2 is neither weaker nor stronger than Bleecker's theorem. Below we present several generalizations of both results, while all proofs are given in the next section. But first let us consider the following immediate corollaries of both theorems:

Corollary 3 (Bleecker)**.** *For every convex simplicial surface S in $\mathbb{R}^3$, there exists an isometric surface $S' \sim S$ of greater volume:* $\mathrm{vol}(S') > \mathrm{vol}(S)$.

Corollary 4. *For every convex surface S in $\mathbb{R}^3$, there exists a submetric convex surface $S' \preccurlyeq S$ of greater volume:* $\mathrm{vol}(S') > \mathrm{vol}(S)$.

Both corollaries are quite surprising. Corollary 3 says that for every convex polyhedron one can subdivide its surface into triangles which can be reassembled according to the same combinatorial rules to obtain a polyhedron of greater volume. Corollary 4 says that one can contract each triangle so that the (unique) convex polyhedron with these smaller triangles has greater volume. To see how that follows from the corollary, contract the whole surface S' by a factor $(1-\varepsilon)$ for $\varepsilon > 0$ small enough. By the definition of shrinking, all triangles will contract while the volume remains greater.

Let us show now how Corollary 3 follows from Corollary 4 using the following powerful result in [BZ3]. Let $S \sim S'$ be isometric surfaces in $\mathbb{R}^3$. We say that S' is in *ε-neighborhood* of S, if there exists an isometry $\varphi : S \to S'$ which satisfies $\|x, \varphi(x)\| \le \varepsilon$, for all $x \in S$ (here $\|x, y\|$ is the usual Euclidian distance in $\mathbb{R}^3$).

Theorem 5 (Burago–Zalgaller)**.** *Let S_1 be a surface submetric to surface S in $\mathbb{R}^3$, and let $\varepsilon > 0$ be any given constant. Then there exists a surface S_2 isometric to S, such that S_2 is in ε-neighborhood of S_1.*

Now, let S be a convex surface in $\mathbb{R}^3$. By Corollary 4, there exists a surface S_1 submetric to S with $\mathrm{vol}(S_1) > \mathrm{vol}(S)$. By Burago-Zalgaller's theorem, there is a surface $S_2 \sim S$ in ε-neighborhood of S_1. Taking ε small enough we can ensure that $\mathrm{vol}(S_2) > \mathrm{vol}(S_1)$. This extends Corollary 3 to *all* convex surfaces:

Corollary 6. *For every convex surface S in $\mathbb{R}^3$, there exists an isometric surface $S' \sim S$ of greater volume:* $\mathrm{vol}(S') > \mathrm{vol}(S)$.

Our next result is an extension of both Bleecker's theorem (Theorem 1) and Corollary 6:

Theorem 7. *For every convex polyhedral surface S in $\mathbb{R}^3$, there exists a volume-increasing bending of S.*

The result is obtained in the next section by a careful examination of the proof of Theorem 2. The proof is independent of the original proof Bleecker's theorem in [B1].

1.2. **Higher dimensional surfaces.** Let $S = \partial P$ be the surface of a $(d+1)$-dimensional convex polyhedron $P \subset \mathbb{R}^{d+1}$. We say that S is a *convex surface* in $\mathbb{R}^{d+1}$. As before, we say that a polyhedral surface $S' \subset \mathbb{R}^d$ is *submetric* to S, write $S' \preccurlyeq S$, if there exist a piecewise-linear (PL) homeomorphism $\varphi : S \to S'$ which does not increase the geodesic distances: $|x, y|_S \geq |\varphi(x), \varphi(y)|_{S'}$ for all $x, y \in S$. Definition of *shrinking* of S extends to higher dimensions without change.

Theorem 8. *For every d-dimensional convex polyhedral surface S in $\mathbb{R}^{d+1}$, there exists a volume-increasing convex shrinking of S.*

Corollary 9. *For every d-dimensional convex polyhedral surface S in $\mathbb{R}^{d+1}$, there exists a submetric convex surface $S' \preccurlyeq S$ of greater volume:* $\mathrm{vol}(S') > \mathrm{vol}(S)$.

Let us emphasize that the Theorem 8 is one of the very few results on metric geometry of polyhedra in higher dimensions (see §6.3). The proof is a technical iterative construction, which heavily uses convexity of P.

1.3. **Non-convex surfaces.** We present two natural generalizations of theorems 2 and 7: one for embedded and one for immersed surfaces. Let us start with the definitions.

Let S be an *abstract 2-dimensional polyhedral surface* defined as a collection of triangles $T_1, \ldots, T_m$ with combinatorial gluing rules. Here each triangle T_i is given by its edge lengths, and whenever two edges are glued, they have equal length. We always assume that S is a connected simplicial complex, and that it is *closed* (has no boundary) and *orientable*, i.e. homeomorphic to a sphere with $g \geq 0$ handles. Denote by V the set of vertices in S.

A (3-dimensional) *realization* of S is defined a map $f : V \to \mathbb{R}^3$ such that the Euclidean distance $\|v_1, v_2\|$ between vertices is equal to the edge length $|v_1, v_2|$ of any triangle T_i which contains v_1 and v_2.

An *immersion* is a realization where no two triangles have a 2-dimensional intersection. For example, a doubly covered triangle is a realization in $\mathbb{R}^3$ of a surface homeomorphic to a sphere, but not an immersion.

An *embedding* is a realization where two triangles intersect only by an edge or by a vertex they share. We always consider surfaces S up to *isometry*, so we speak of *isometric immersions* and *isometric embeddings*. An example in Figure 2 shows an isometric embedding of a doubly covered triangle.

Since S is orientable, for all immersions of S into $\mathbb{R}^3$ the $\mathrm{vol}(S)$ is well defined. When speaking about general isometric immersions or embeddings, it still makes sense to ask if there exist volume-increasing bendings (continuous piecewise-linear isometric deformations). The following results resolves Bleecker's conjecture in the strongest form.

Theorem 10. *For every immersed closed orientable polyhedral surface S in $\mathbb{R}^3$, there exists a volume-increasing bending $\{S_t\}$ of S. Moreover, if S is an embedding, the surfaces S_t can also be made embeddings.*

The proof of the theorem starts with a construction of a volume-increasing shrinking (no longer convex). We then use Burago-Zalgaller's theorem which we now restate emphasizing the difference between embedded and immersed versions:

Theorem 5′ (Burago–Zalgaller)**.** *Let S_1 be a polyhedral surface immersed into $\mathbb{R}^3$ submetric to an abstract polyhedral surface S. Then, for every constant $\varepsilon > 0$, there exists a isometric immersion S_2 of S into $\mathbb{R}^3$, such that S_2 is in ε-neighborhood of S_1. Moreover, if S_1 is an embedding, then S_2 can also be made an embedding.*

As of now, there is no analogue of the Burago-Zalgaller's theorem in higher dimensions for isometric immersions/embeddings into $\mathbb{R}^{d+1}$ when $d \geq 3$. Without such result, we are unable to generalize theorems 7 and 10 (see also §6.4).

2. Proof of theorems

We should start by saying that the proof of Theorem 2 is an important introduction to the constructive technique we employ, and is useful for proofs of theorems 7, 8 and 10. On the other hand, the reader interested in a clear path towards the proof of Theorem 10 should be able to skip the technical details in the proof below and proceed directly to §2.3 and then to 2.6.

2.1. **Proof of Theorem 1.** Let $P \subset \mathbb{R}^3$ be a convex polyhedron. For now we assume that P is simple. Later on we reduce the general case to the simple case.

Fix a parameter $\varepsilon > 0$. Think of ε as being very small. For every vertex v of P and every face F of P containing v, consider two edges e and e' such that $v \subset e, e' \subset F$. Denote by $x_{v,F}$ a vertex at distance ε from edges e and e'. Denote by $X_F \subset F$ the convex hull of points $x_{v,F}$, where $v \subset F$.

Let us subdivide the surface $S = \partial P$ into regions. For every vertex v, edge e containing it and faces F and F' containing e, connect points $x_{v,F}$ and $x_{v,F'}$ with a unique geodesic path crossing only edge e. Clearly, this geodesic path has length $\geq 2\varepsilon$. Also, connect every point $x_{v,F}$ to the vertex v (see Figure 3). Now each face F is subdivided into a polygon X_F, triangles (two per vertex), and trapezoids (one per edge).

Move every polygon X_F at distance $\alpha > 0$ away from P. The parameter $\alpha = \alpha(\varepsilon)$ will be determined later. Denote by $y_{v,F}$ the vertices of the obtained polygon, which we denote by Y_F. Let $Q = Q(\varepsilon)$ be the convex hull of points $\{y_{v,F}, v \subset F\}$. This is the desired polyhedron whose surface will be shown to be submetric to S. Of course, polygons X_F are congruent to Y_F, so we are left with triangles and trapezoids.

Denote by T_e the trapezoid obtained as a union of two trapezoids in S adjacent to edge e. Think of T_e as being unfolded on a plane. By T'_e denote the trapezoid shaped face of Q, parallel to e. Note that the trapezoids T_e and T'_e have parallel sides of equal length, but their heights can be different. Assume that the height h_e of T_e is greater or equal to the height h'_e of T'_e. Then there is a natural piecewise-linear map from T_e (folded as it lies on the surface S) to T'_e which sends the edges of X_F into the corresponding edges of Y_F and non-increases the distances.

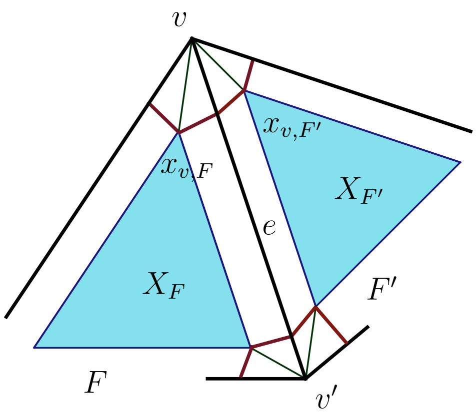

FIGURE 3. Subdivision of the surface $S = \partial P$ into regions around edge $e = (v, v')$.

Observe that when α is sufficiently small, our assumption $h_e \geq h'_e$ follows by continuity and from the triangle inequality, while when α is sufficiently large it fails. Define $\alpha = \alpha(\varepsilon)$ to be the largest possible so that $h_e \geq h'_e$ for every edge e in P. Having α determined, the polyhedron $Q = Q_\varepsilon$ is completely defined.

Recall that $|x, y|_S$ denotes the geodesic distance on the surface S, and $\|x, y\|$ denotes the usual Euclidean distance in $\mathbb{R}^3$. Note that for every vertex $v \subset e$, where edge $e = F \cap F'$, the condition

$$(*) \qquad \left|x_{v,F}, x_{v,F'}\right|_S \geq \left\|y_{v,F}, y_{v,F'}\right\|$$

is equivalent to $h_e \geq h'_e$. Furthermore, condition $(*)$ is linear in ε, i.e. depends only on the ratio α/ε. Therefore, parameter α is determined by the inequalities $\alpha \leq c_e\varepsilon$, for every edge e of P. We conclude that $\alpha = \alpha(\varepsilon)$ grows linearly with ε.

Recall our initial assumption that P is simple. Then, for every v in P, the convex hull of the points $\{y_{v,F}, v \subset F\}$ forms a triangular face U_v of Q. The union of triangles around each vertex v forms the surface of a triangular cone shape. We need to construct a map from this cone onto U_v. First, rearrange the triangular cone by stretching along edge of P and bending it along edges $(v, x_{v,F})$ (see Figure 4). In the new triangular cone the vertices will correspond to the faces F containing v, and in fact are the images of $x_{v,F}$. Similarly, the edges will correspond to the edges e containing v. Now shrink each side of the triangular cone so that it shrinks from $\left|x_{v,F}, x_{v,F'}\right|_S$ to $\left|y_{v,F}, y_{v,F'}\right|$ as in $(*)$. Finally, we obtain the surface of a triangular cone, which can be viewed as the top of a pyramid Δ_v over triangular base U_v.

We should warn the reader that when we shrink one or several sides of the triangle, we do not necessarily create a submetry, as can be seen in Figure 5.[3]

If vertex v of Δ_v projects in the interior of U_v, this maps non-increasing the distance and completes the construction. If v projects outside of U_v, one or two of the dihedral

[3]The only other argument we are using in this proof is a projection which always creates a submetry.

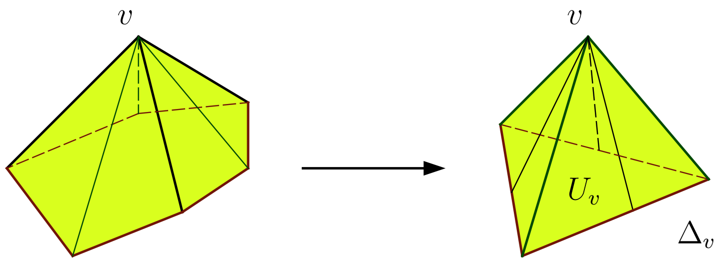

FIGURE 4. Transforming the cone.

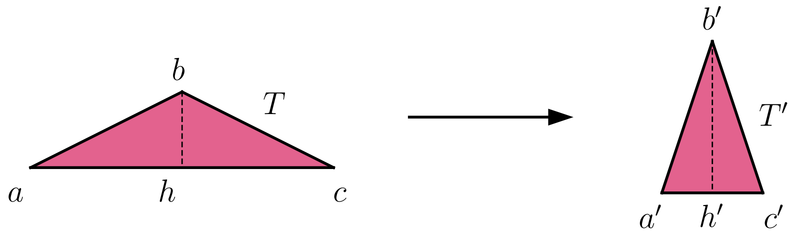

FIGURE 5. Two triangles $T = (a, b, c)$ and $T' = (a', b', c')$ with $|a, b| = |a', b'|$, $|b, c| = |b', c'|$, $|a, c| > |a', c'|$, but T' not submetric to T since $|b, h| < |b', h'|$.

angles of Δ_v, along an edge e in U_v, is obtuse. Suppose only one angle is obtuse, at edge e. Consider a plane H_e going through e, such that v projects onto interior of $H_e \cap \Delta_v$. Now project onto H_e the portions of the faces of Δ_v which lie on the same side of H_e as v (see Figure 6).

If there are two obtuse angles at edges e, e', we need to repeat this construction two more times. First, project v onto a hyperplane H_w going through a vertex $w = e \cap e'$ of U_v. Then, project the top surface of the resulting polyhedron onto a hyperplane H_e going though e' and $H_w \cap F_e$ (here F_e is the face of Δ_v containing e, other than U_v). It is easy to see that the resulting pyramid has only one obtuse angle (at e), so we can repeat the construction above.

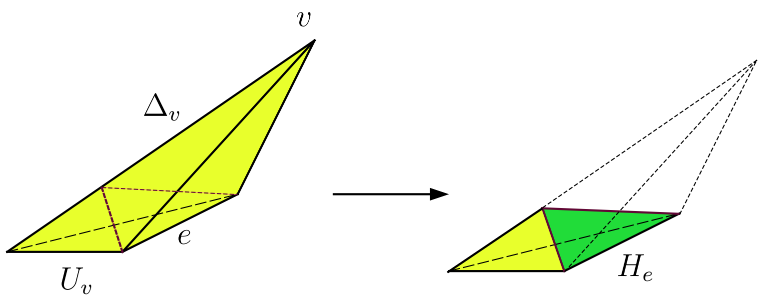

FIGURE 6. Cutting pyramid Δ_v with a plane H_e.

After at most three projections as above, we obtain a pyramid whose top vertex projects into the interior of U_v. Clearly, $Q_\varepsilon \to P$ as $\varepsilon \to 0$, by construction. This completes the construction of a PL-map $\varphi : S \to \partial Q_\varepsilon$.

Given the warning above, we do no claim that the obtained surface Q_ε is a submetry. However, for the purposes of this presentation, we assume that this is in fact a submetry and correct the problem later by a separate argument. Let us phrase this in the form of a claim which will be formalized and resolved in §2.3.

Claim. *The PL-map $\varphi : S \to \partial Q_\varepsilon$ constructed above can be "corrected" to become submetric, while leaving the volume argument unaffected.*

From here on, assume that $\{\partial Q_\varepsilon\}$ is a shrinking of S, for $\varepsilon \geq 0$ small enough. We need to prove that the deformation $\{\partial Q_\varepsilon\}$ is volume-increasing. Let us show that

$$(\star) \qquad \mathrm{vol}\big(Q_\varepsilon\big) \, = \, \mathrm{vol}(P) \, + \, \mathrm{area}(S) \cdot \alpha(\varepsilon) \, - \, O(\varepsilon^2), \quad \text{as } \ \varepsilon \to 0.$$

From above, $\alpha(\varepsilon) = c \cdot \varepsilon$ for some constant $c = c(P) > 0$. Therefore, from $(\star)$ we have:

$$\mathrm{vol}\big(Q_\varepsilon\big) \, = \, \mathrm{vol}(P) \, + \, \big(c \cdot \mathrm{area}(S)\big)\, \varepsilon \, - \, O(\varepsilon^2),$$

and $\mathrm{vol}(Q_\varepsilon)$ is increasing for ε small enough.

To obtain $(\star)$, observe that

$$\mathrm{area}(X_F) \, = \, \mathrm{area}(F) \, - \, O(\varepsilon)\,,$$

and

$$\mathrm{vol}(Q_\varepsilon) \, \geq \, \mathrm{vol}(P') \, + \sum_{F \subset S} \alpha(\varepsilon) \cdot \mathrm{area}(X_F) \, = \, \mathrm{vol}(P') \, + \, \alpha(\varepsilon) \cdot \big(\mathrm{area}(S) - O(\varepsilon)\big),$$

where P' is the convex hull of all points $x_{v,F}$.

Now, cut polyhedron P with hyperplanes spanned by triangles U_v and trapezoids $\big[x_{v,F}, x_{v',F}, x_{v,F'}, x_{v',F'}\big]$, corresponding to edges $e = (v, v') = F \cap F'$. They split P into triangular pyramids (one per vertex) and trapezoid shaped prisms (one per edge). The volume of each pyramid is $O(\varepsilon^3)$ and the volume of the prism corresponding to edge e is $O(\varepsilon^2) \cdot |e|$. Thus, $\mathrm{vol}(P')$ is equal to $\mathrm{vol}(P) - O(\varepsilon^2)$. This finishes the proof of $(\star)$ and completes the proof of the theorem in case when P is simple.

Before we consider the case of general (not-simple) polyhedra $P \subset \mathbb{R}^3$, we need to treat the case when P is flat, i.e. when S is a doubly covered convex polygon. Clearly, the above construction is inapplicable. In this case, for every vertex v of P, cut the unique (up to reflection) face F with a line $\ell \perp (v, x_v)$ at distance $|v, x_v|$ from x_v. Denote by x_v' and x_v'' two points of intersection of ℓ and X_F (see Figure 7). Connect the points x_v', x_v'' to vertex v, and drop perpendiculars to the sides of F. This makes a subdivision of F into one large region Z, rectangles (one per each side), and triangles around vertices (three per vertex).

We first obtain an isometric embedding of positive volume. Bend the rectangles at right angle with respect to Z, and bend the triangles to form a pyramid whose apex v projects onto x_v. Two copies of such surface form the desired immersion. Finally, to obtain a convex submetry we project the pyramid constructed above onto rectangle

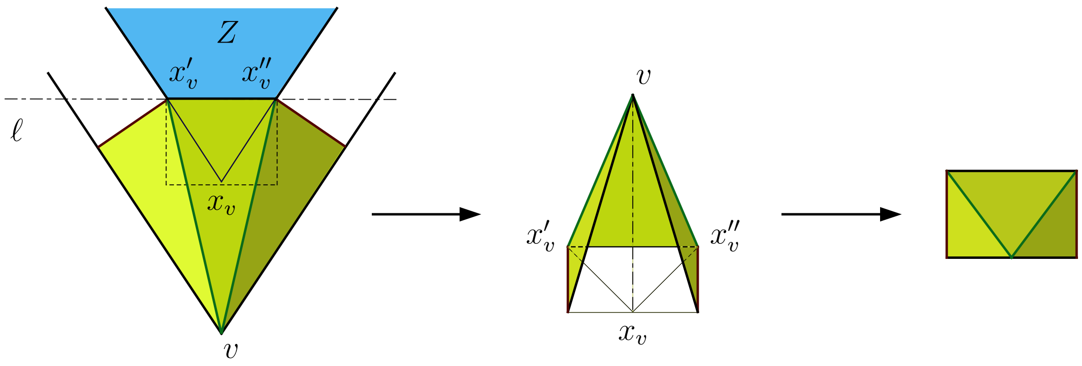

FIGURE 7. Subdivision of a polygon and its bending.

with vertices x'_v and x''_v. Of course, the volume of the resulting surface S_ε is positive. To check that it is increasing, the same argument as above for gives:

$$\mathrm{vol}(S_\varepsilon) \,=\, \mathrm{area}(F)\,\varepsilon \,+\, O(\varepsilon^2)\,, \quad \text{as} \quad \varepsilon \to 0\,.$$

Therefore, for $\varepsilon > 0$ small enough the volume is increasing, as desired. An extreme example of the construction, when S is a doubly covered square and $\varepsilon = 1/(2+2\sqrt{2})$ will be shown later in Figure 23.

For general (non-flat) P, consider cones C_v around each vertex v of P. Denote by L_v a ray which lies in the interior of the cone. For each v in P, cut C_v with a hyperplane H_v orthogonal to L_v and at distance δ from v. For $\delta > 0$ small enough, the resulting polyhedron P_δ is simple and we can apply the previous construction.

As we did earlier, project the top of the pyramid onto the face W_v of P_δ contained in H_v (see Figure 8). Here we are using the fact that v projects into the relative interior of W_v). This shrinks the surface of P to P_δ. It remains to set the dependence of ε on δ and check that the volume of the resulting Q_ε is still increasing.

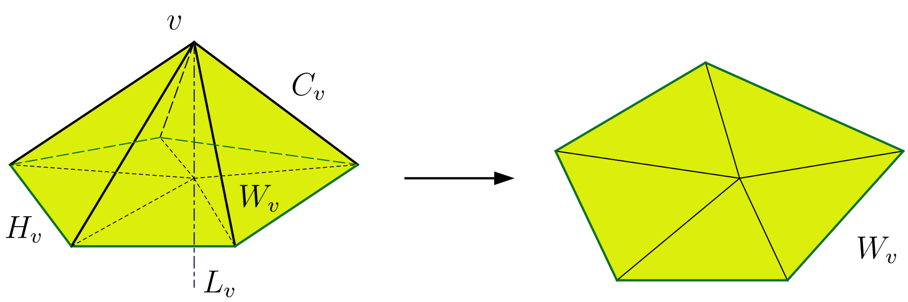

FIGURE 8. Cutting the cone C_v with hyperplane $H_v \perp L_v$.

By analogy with the earlier construction, set ε to be the largest possible so that Q_ε is well defined. Note that the only restriction on ε is that the sides of polygons X_F should not collapse. This condition is given by two linear inequalities $\varepsilon \le c\delta$ for every edge e of W_v (and both faces containing it). Thus, again one can take δ linear in ε. Since

$$\mathrm{vol}(P) = \mathrm{vol}(P_\delta) + O(\delta^3) = \mathrm{vol}(P_\delta) + O(\varepsilon^3),$$

we obtain the theorem in the general case. $\square$

2.2. **An example.** The case when P is a unit cube is especially symmetric (cf. [P]). Here the trapezoids T_e and T'_e are isometric to $2\varepsilon \times (1-2\varepsilon)$ rectangles, the triangles around vertices are right equilateral triangles, and parameter $\alpha = \varepsilon/\sqrt{2}$. Faces U_v are equilateral triangles with side lengths 2ε. The shrinking map is shown in Figure 9.

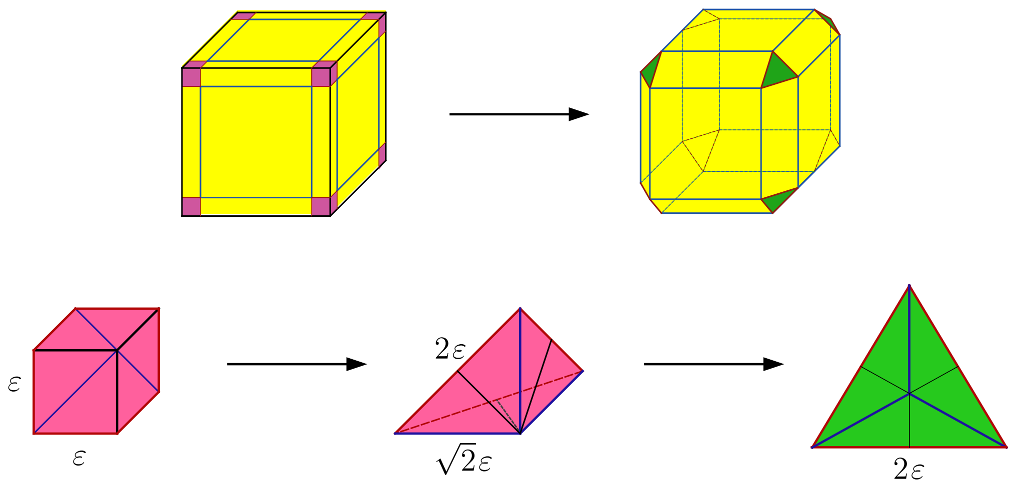

FIGURE 9. Volume-increasing shrinking of a cube.

Observe that if we attach pyramids to faces U_v rather project them, we obtain an isometric bending of a cube (see example 4.3 for further details).

When $\varepsilon = 1/4$, the polyhedron Q_ε is an Archimedean solid called *rhombicube-octahedron* with

$$\mathrm{vol}\big(Q_{\frac{1}{4}}\big) = \frac{1}{2} + \frac{5\sqrt{2}}{12} \approx 1.0892.$$

In general, polyhedra Q_ε are defined for all $\varepsilon \in [0, 1/2]$ and their volume is a cubic polynomial in ε. The maximum volume ≈ 1.1688 is achieved is achieved at $\varepsilon \approx 0.1425$. Let us note also that the polyhedron $Q_{\frac{1}{2}}$ is an octahedron with edge length 1 and volume $\sqrt{2}/3 \approx 0.4714$.

2.3. **Resolving the Claim: an addendum to the proof of Theorem 2.** As the following argument shows, there really is no reason to be careful in constructing of the PL-map $\varphi = \varphi_\varepsilon$ from the union $W_v \subset S$ of all triangles around v onto $U_v \subset Q_\varepsilon$, i.e. in constructing the map $\varphi : W_v \to U_v$. Indeed, consider *any* PL-homeomorphism

$\psi : W_v \to U_v$. By compactness, there is constant $C > 0$ such that the geodesic distances increase by a factor at most C:

$$(\mathcal{L}) \qquad |x,y|_S \le C \cdot |\psi(x), \psi(y)|_Q \,, \quad \text{for all } \; x,y \in W_v \,.$$

We call $(\mathcal{L})$ the *Lipschitz condition.*

We now shrink the whole resulting polyhedron Q_ε to Q'_ε in such a way that the volume of Q'_ε is close to that of Q_ε, and the composition map is really a submetry. The construction is based on a sequence of cuts with a plane and projections, similar to that in Figure 6 and 8.

Let us start with the cutting and projecting construction in case of 2-dimensional polygons. The following example is already nontrivial and is a good illustration of the general principle. Suppose we are given a polygon $P \subset \mathbb{R}^2$ and an edge $e = (x,y)$. Denote by β_1, β_2 the angles between e and edges $\ell_1 = (x', x)$ and $\ell_2 = (y, y')$ adjacent to e. If either of the angles is non-obtuse, say $\beta_1 \le \pi/2$, we can always cut P with a line L nearly orthogonal to e and project the side of P containing e onto L (see Figure 10). If the angle close enough to $\pi/2$, the length of the projection $e_1 \subset L$ of e can be made smaller than $1/C$.

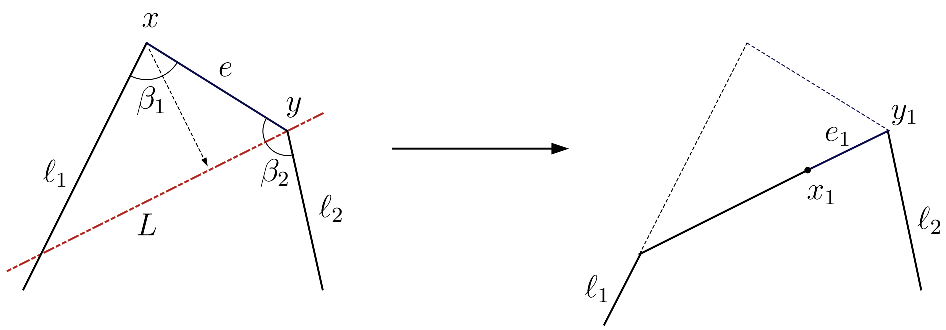

Figure 10. Cutting and projecting in case of an acute angle β_1.

In case when both angles β_1 and β_2 re obtuse, we need a more elaborate scheme. Let $\beta = \beta_1 + \beta_2 - \pi$ be the angle between ℓ_1 and ℓ_2. Fix an angle γ, which will determined later on.

We assume that the edges ℓ_1, ℓ_2 of P are long enough for the following construction to work. Cut P with a line L_1 going though x at angle γ with ℓ_2, and project the portion of ∂P on the same side as e onto L_1. Denote by $e_1 = (x_1, y_1) \subset L_1$ the projection of e. Then cut P with a line L_2 going though y_1 at angle γ with e_1, and project the portion of ∂P on the same side as e_1 onto L_2. Denote by $e_2 = (x_2, y_2) \subset L_2$ the projection of e_1. Repeat the procedure, alternating between sides e_1 and e_2 (see Figure 11).

Note that the angle between lines L_i and L_{i+1} is equal to $\sigma = \pi - \beta - 2\gamma$. Choose γ is such a way that $0 < \sigma < \pi/2$. Then, after m steps, the length of the projection $e_m = (x_m, y_m)$ is at most $\cos(\sigma)^m |e|$, which is $< 1/C$ for m large enough.

We are ready to present the construction. Let $\varphi : S \to Q_\varepsilon$ be a map which is a submetry map everywhere but triangles around vertices, and on them it satisfies the

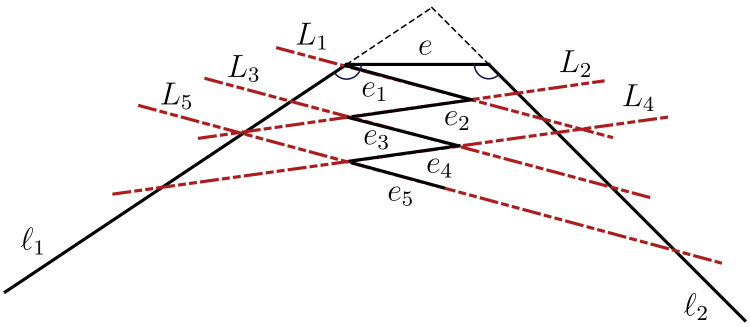

FIGURE 11. Iterated cutting and projecting.

Lipschitz condition ($\mathcal{L}$). For every vertex v of P, fix a plane H spanned by U_v and two pair of orthogonal lines E_1, E_2 in H. Use the cutting and projecting procedure described above (now with planes instead of lines) making all planes parallel to E_1. Here we keep the planes on one side of U_v, which will be repeatedly projected. Then repeat the same procedure now with planes parallel to E_2. Once finished, U_v will contract by a factor of at least C in every direction, i.e. we get a polyhedral surface submetric to S.

It remains to check that the above procedure is well defined and does not affect the volume. Indeed, if $\varepsilon > 0$ is small enough, all the cutting above affects Q_ε only locally, in the neighborhood of the vertices. The volume cut is thus $O(\varepsilon^3)$, and the rest of the argument follows as in the proof above. This suffice to establish the claim for simple polyhedra, and complete the prof of Theorem 2 in this case.

Finally, let us note that in construction of the PL-homeomorphism φ in §2.1, the constant C in the Lipschitz condition depends only on the face and dihedral angles of P, not on the edge lengths of P. Thus, for non-simple polyhedra, the requirement on the lengths of edges of P_δ is $\Omega(\varepsilon)$, and it still suffice to take $\delta = c \cdot \varepsilon$ for sufficiently large c. This implies Theorem 2. $\square$

2.4. **Proof of Theorem 8.** The proof starts as the the proof of Theorem 2 in §2.1 and continues as in §2.3. We refer to [Zi] for the introduction and basic definitions. When speaking about metric geometry in $\mathbb{R}^d$ we use the same notation and terminology as in [MP].

Let $P \subset \mathbb{R}^{d+1}$ be a convex polyhedron and $S = \partial P$ be its surface. The *facets* are d-dimensional faces of P, the *ridges* are faces of dimension $d-1$, and the *warp faces* are faces of dimension $\leq d-2$. For a face G of P, denote by $\dim(G)$ the dimension of G, i.e. of the subspace spanned by G. The points on warp faces are called *warp points*. The k-dimensional volume of a k-dimensional face G is denoted by $\mathrm{vol}_k(G)$.

Polyhedron P is called *simple* if every vertex v of P belongs to exactly d edges (1-dimensional faces of P). We first consider the case when P simple, and then make a reduction in the general case.

Fix a parameter $\varepsilon > 0$. For every facet F of P consider a subset X_F of points $x \in F$ at distance $\geq \varepsilon$ from the boundary ∂F. Note that for every ridge $A \subset F$ there is a parallel ridge $X_{A,F} \subset X_F$. Furthermore, X_F has the same combinatorial structure as F, and for every k-dimensional face $G \subset F$ there exists a k-dimensional face $X_{G,F} \subset X_F$.

Let Y_F be a translation of X_F at distance $\alpha = c_1\varepsilon$, where the constant $c_1 = c_1(P)$ will be determined later. Denote by Q_ε the convex hull of all Y_F over all facets F of P. It is easy to see that when $\varepsilon > 0$ is small enough, all vertices of Y_F are in convex position.

Denote by $e_1, \ldots, e_{d+1}$ the edges of P containing vertex v. Similarly, by $F_1, \ldots, F_{d+1}$ denote the facets of P containing v, and such that $e_i \notin F$, for all i. To simplify notation, let $x_i = x_{F_i}$. Consider the set of points $w_1, \ldots, w_{d+1}$ on the edges: $w_i \in e_i$, such that the vectors (v, w_i) and (v, x_{F_i}) satisfy the following system of equations:

$$(v, x_i) \, = \, (v, w_1) + \ldots + (v, w_{i-1}) + (v, w_{i+1}) + \ldots + (v, w_{d+1}) \, , \quad \text{for all} \;\; i = 1 \ldots d+1.$$

Clearly, the system has a unique solution which determines points w_i, and the distances $|v, w_i|$ are linear in ε. Consider also all points

$$w_I \, = \, w_{i_1} + \ldots + w_{i_r}$$

for every subset $I \subset \{1, 2, \ldots, d+1\}$, such that $|I| \leq d$. From above, we have $w_\varnothing = v$ and $w_I = x_i$, where $I = 1, \ldots, \widehat{\imath}, \ldots, d+1$, i is omitted. Inside each facet F_i, consider a parallelepiped W_F spanned by the vertices $w_I \in F_i$. To make a global notation, denote by $W_{v,F}$ the parallelepiped containing v and lies inside F.

We can now construct a subdivision of the surface $S = \partial P$, generalizing the subdivision in the proof of Theorem 2 (see Figure 3). First, do this separately on each facet F and then take a union of some of the resulting pieces. For a k-dimensional face $G \subset F$, denote by $W_{v,F,G}$ the unique *interior* $(d-k)$-dimensional face of $W_{v,F}$ (i.e such that $W_{v,F,G} \not\subseteq \partial F$), which is *transversal* to G (i.e. such that $G \cap W_{v,F,G}$ is a point). Denote by $T_{G,F} \subset F$ the convex hull of all $W_{v,F,G}$, over all $v \in G$.

By construction, all polyhedral regions $W_{v,F,G}$ lie in the intersection of $(d-k)$ pairs of parallel hyperplanes, all at distance ε from each other. For example, for $d = 3$ the resulting 3-dimensional polyhedral regions are either nearly flat polygonal prism-like polyhedra (case $k = 2$), or square pencil-like polyhedra (case $k = 1$), or parallelepipeds (case $k = 0$), as in Figure 12. Similarly, $T_{F,F} = X_F$, but we want to keep a separate notation in this case. Finally, let $T_G = \cup_F T_{G,F}$. This completes the subdivision of S.

The surface ∂Q_ε has a natural subdivision into facets. For each k-dimensional face G of P, denote by T'_G the the facet of Q_ε which is a convex hull of faces $X_{G,F}$, over all $F \supset G$. Around v (which is not a vertex of Q_ε but is a limit of them when $\varepsilon \to \mathbf{0}$), the structure of T'_G is that of a $(k+1)$-simplex times the cone of G. For example, when $d = 3$, polyhedra T'_G look the similar to T_G (case $k = 2$), like triangular pencils (case $k = 1$), and like tetrahedra (case $k = 0$).

We say that a facet U of Q_ε has *order* k if $\mathrm{vol}_d(U) = \theta(\varepsilon^k)$ as $\varepsilon \to 0$. The order of a region W in the above subdivision of S can be defined in a similar way. From above, it is easy to see that facets T'_G and regions T_G have the same order $d - \dim(G)$.

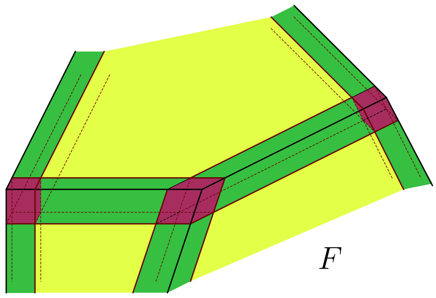

FIGURE 12. Subdivision of a 3-dimensional facet F of $P \subset \mathbb{R}^4$.

Looking at the subspace spanned by G and its orthogonal complement, we can speak of *long* and *short directions*, respectively. By construction, both T_G and T'_G have $\dim(G)$ short and $d - \dim(G)$ long directions.

It is easy to see now that both T_G and T'_G have the same adjacency rule as the polyhedron P. This means that for every two faces $G \subset G'$ such that $\dim(G) = \dim(G') - 1$, the facet T_G is adjacent to a facet $T_{G'}$ (by a ridge). The same holds for regions T'_G.

Let us now construct a PL-homeomorphism $\psi : P \to Q_\varepsilon$ which maps X_F to Y_F congruently, and maps $T_G \to T'_G$ so that there is neither stretching nor shrinking in the long directions. We use the adjacency rules to construct a PL-homeomorphism ψ as follows. Start with the identity map $\psi : X_F \to Y_F$ and take at the adjacent regions T_F and faces T'_F, respectively. Once T_F is unfolded into $\mathbb{R}^d$, T_F and T'_F are both prism-like polyhedra which differ by height. There is a natural extension of ψ to them. On the next level we have more freedom of choice as we get again prism/pencil-like polyhedra but with a 2-dimensional cross-section in the short directions. Any PL-map on the cross-section respecting the boundary will suffice (the boundary contains one-dimensional stretches coming from $\psi : T_F \to T'_F$). Repeat this procedure until the rest of ψ is obtained. At the end we get to facets T'_v of order d, which correspond to vertices of P. Here polyhedra $T_v \subset S$ are parallelepipeds, and T'_v are d-simplices in Q_ε. The map ψ is already fixed on the boundary $\psi : \partial T_v \to \partial T'_v$, and any PL-homeomorphism of the interior will suffice for our purposes.

By construction of ψ, we have an isometry on large portion of the facets, but may have some stretching along small portions of the surface S. We have an identity map in the long directions and some more involved map in the short directions of every T'_G. By compactness and the isoperimetric inequality in $\mathbb{R}^{d+1}$ [BZ2], the stretching is bounded from above by a constant C, as in the Lipschitz condition $(\mathcal{L})$.

Now we start the cutting and projecting procedure, as described in §2.3. Start with the d-dimensional polyhedra T'_A, which as we recall have one short and $(d-1)$ long directions. In the 2-dimensional plane orthogonal to $A \subset \mathbb{R}^{d+1}$, use the cutting and projecting procedure, with the lines now substituted with hyperplanes extended in the $(d-1)$ long dimensions. After contracting these facets of Q_ε by a factor of C, we may have changed geometry and even combinatorics of the facets of lower order, but the general structure (of facets parallel to G) is still the same and the resulting PL-map is still identity in the long directions.

We proceed to cutting and projecting for facets T_G or order 2, which will have two short and $(d-2)$ long directions. Take a 3-subspace $H_G \subset \mathbb{R}^{d+1}$ orthogonal to G. The cross-section of Q_ε in H_G becomes a 3-dimensional cone. Cut and project this cone on a 2-plane L_G, and choose two orthogonal lines E_1 and E_2. Then cut and project the result on hyperplanes parallel to G and E_1. Do the same for hyperplanes parallel to G and E_2. This gives the desired contraction of facets U_G. Again, the combinatorics and geometry of facets may change, but no facets of lower order is removed or created and the map remains identity on long directions. We repeat this for all facets of order 2, then for all facets of order 3, etc. We finish with the cutting and projecting procedure for the (usual) cones C_v corresponding to every vertex v of P. The details are straightforward.

Denote by Q'_ε the resulting polyhedra. By construction, they are always submetric to S. It remains to compute the volume $\mathrm{vol}(Q'_\varepsilon)$ and show that $\{Q'_\varepsilon\}$ is indeed volume-increasing. As before, let $\alpha = c_1\varepsilon$, where c_1 is the largest possible so that the regions X_F do not collapse. By the same argument as in §2.1, we have:

$$\mathrm{vol}_{d+1}(Q'_\varepsilon) \,=\, \mathrm{vol}_{d+1}(Q_\varepsilon) \,+\, O(\varepsilon^2) \,=\, \mathrm{vol}_{d+1}(P) \,+\, \mathrm{vol}_d(\partial P)\,\varepsilon \,+\, O(\varepsilon^2)\,,$$

since every time we cut and project we do this along faces of dimension $\le d-1$. This finishes the proof in the case when P is simple.

For general $P \subset \mathbb{R}^d$, we construct a simple polyhedron by the following cutting procedure. We will use the cut and project construction for all warp faces of the dimension $d-2$, in any order. Start with a $(d-2)$-dimensional face $B \subset P$. At a generic point $b \in B$ the neighborhood of $S = \partial P$ is a product of a $(d-2)$-dimensional subspace V and a cone C_B. Let H be a hyperplane orthogonal to the interior ray R_B in C_B, parallel to V, and at distance δ from B. Now cut and project P onto H. Repeat this for all $(d-2)$-dimensional faces. If we choose rays R_B generically, we obtain a simple polyhedron P_δ, since every warp point is in the closure of a $(d-2)$-dimensional warp face.[4]

Note that we no longer have a bound on the size of edges of P_δ. However, we never used in our construction the fact that the regions X_F were constructed with the same uniform ε. Instead, let us set $\delta = \varepsilon$ and modify the above construction as follows.

When $\delta \to 0$ is small enough, the combinatorics of faces will stabilize. Define the sets X_F the same way as before, but only for facets which don't degenerate when $\delta \to 0$. This is still well defined, and at the limit $\varepsilon \to 0$ we get the same rate of increase of $\mathrm{vol}(Q_\varepsilon)$. Now, the only obstacle is that these facets F and the corresponding to them regions T_F will disappear when the cut and project procedure is done. In effect, we simply make deeper cuts into P. But again, this does not affect the volume calculation nor does this change the properties of the constructed

[4]Formally, an extra argument is required to show that this can be done with *the same* δ. To avoid this technicality one can allow δ to vary slightly for different B. Alternatively, even if the rays are chosen non-generically, one can continue cutting and projecting for warp faces of smaller dimension, including all vertices at the final stage.

PL-maps, for example, that they are identity in the long directions. We omit the details.

Finally, we still need to treat separately the case of a "flat" P, i.e. when P is a doubly covered convex d-dimensional polyhedron F in $\mathbb{R}^{d+1}$. Note that to satisfy the theorem it suffice to contract F to the standard unfolding of the "top half" of a $(d+1)$-dimensional cube with side $\varepsilon > 0$. This can be easily done by making two central conical triangulations of both, overlapping them, and triangulating them again. Then we contract separately the simplices in each cone (see Figure 13). This is a contraction when $\varepsilon > 0$ is small enough. Of course, the volume of a d-cube is equal to ε^d, and thus increasing. This completes the proof. $\square$

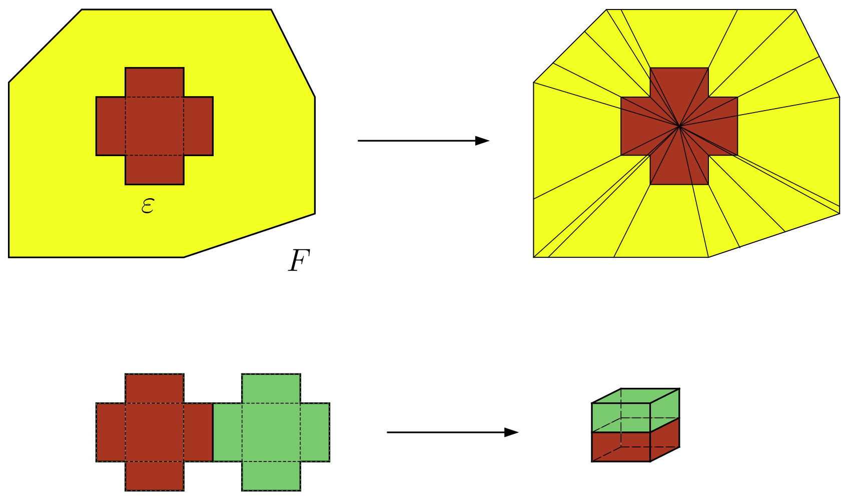

Figure 13. Contracting a face of a flat polyhedron onto top half of a cube. Putting two halves together.

2.5. **Proof of Theorem 7.** For flat surfaces the result is already established in the proof of Theorem 2. Let us now present an isometry construction when $\mathrm{vol}(S) > 0$.

Fix a volume-increasing shrinking $\{S_\varepsilon, \varepsilon \in [0,\rho]\}$ as constructed above, for some $\rho > 0$. Since $S_\rho \preccurlyeq S$, by the Burago-Zalgaller's theorem (Theorem 5), there exists an isometric embedding $S' \sim S$ is the ϵ-neighborhood of S_ρ, where $\epsilon > 0$ can be made as small as desired. Before we set ϵ, we need to briefly recall the proof of Theorem 5 given in [BZ3].

The surface S' is constructed by first using a very refined triangulation of S and the corresponding triangulation of S_ρ (this part is based on [BZ1]), and then by making certain embeddings of the triangles. The original Burago-Zalgaller's construction is quite robust and allows a much flexibility in choosing the triangulation as long as the triangles are small and the angles are acute.[5] In our case, one can take a regular triangulation on the faces of F, slightly modified around the edges, as prescribed

[5]In fact, there is another technical condition on the centers of circumscribed circles. As shown in [BZ3], this is easy to get once an acute triangulation is found (see also 6.4).

in [BZ1]. Moreover, one can then embed these triangles in a trivial way when they lie in polygons Y_F and in translation-invariant way along trapezoids T'_e, making the cross-sections and the trapezoid as shown Figure 14. The crimps here are small and clustered closer to the middle to avoid intersections with those of other edges.

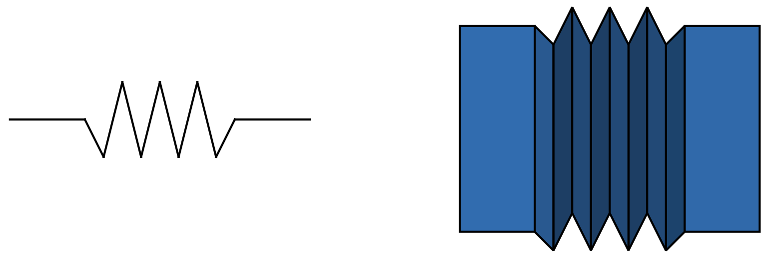

Figure 14. Immersed trapezoid and its cross-section.

Now recall that in construction of the shrinking, the map $\varphi_\varepsilon : S \to S_\varepsilon$ is linear in ε around each vertex. Fix a map $\psi = \psi_\rho : S \to S_\rho$ and consider a family of embeddings corresponding to the maps $\psi_\varepsilon : S \to S'_\varepsilon$ for $0 \le \varepsilon \le \rho$, such that $S'_\rho := S'$, $S_0 = S$, and where ψ_ε is defined to be linear around each vertex with faces and trapezoids mapped as above. It is easy to see that this is well defined. Note also that the construction easily extends to non-simple polyhedra verbatim.

It remains to check that $\Xi_\epsilon = \{S'_\varepsilon\}$ is volume-increasing. This is straightforward if the parameter $\epsilon = \epsilon(\rho) > 0$ is chosen small enough to begin with. Indeed, by the isoperimetric inequality applied to regions on surfaces around the edges and vertices of P, the volume difference $|\mathrm{vol}(S'_\varepsilon) - \mathrm{vol}(S_\varepsilon)| = O(\varepsilon^2)$, where the constant implied by the $O(\cdot)$ notation depends ϵ and is independent of ε. Therefore, taking $\rho > 0$ small enough we can ensure that $\mathrm{vol}(S'_\varepsilon) - \mathrm{vol}(S)$ be greater than this difference for $\varepsilon \in [0, \rho]$, and in fact grows with ε. $\square$

2.6. **Proof of Theorem 10.** We start with the second part of the theorem which is easier to visualize. Assume that a polyhedral surface S is embedded into $\mathbb{R}^3$, and let P be a (possibly non-convex) polyhedron in $\mathbb{R}^3$, such that $S = \partial P$.

Fix a plane H in general position, such that $H \cap P = \varnothing$. Start moving H towards P until it hits a vertex v. Continue moving H in this direction at distance $\varepsilon > 0$ and then at distance $\delta > \varepsilon$. Denote by H_ε and H_δ the corresponding hyperplanes parallel to H. If $\varepsilon, \delta > 0$ are small enough, the hyperplanes H_ε and H_δ intersect only the cone C_v starting at vertex v. We fix δ and present a volume-increasing shrinking $\{S_\varepsilon\}$, for all $\varepsilon > 0$ small enough compared to δ.

Denote by $R \subset S$ the region on the surface S which lies on the other side of H_δ from v. The surfaces S_ε we consider will agree with S on R. Denote by $F_1, \ldots, F_m$ the triangular faces of the cone C_v cut with H_δ, labeled in the cyclic order. For convenience, we keep all indices (mod m) throughout the proof, so that $F_0 = F_m$, etc.

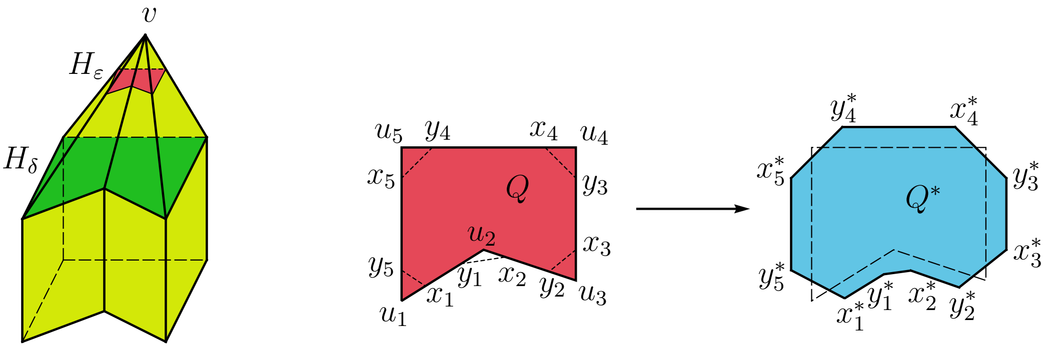

FIGURE 15. Polyhedron P cut with hyperplanes H_ε and H_δ. Polygons Q and Q^*.

For each F_i, denote by $e_i = F_i \cap H_\delta$ and $h_i = F_i \cap H_\varepsilon$ the lines of intersection of cone faces with the hyperplanes. Let $u_i = h_i \cap h_{i-1}$ and $w_i = e_i \cap e_{i-1}$ be the vertices of polygons $Q = S \cap H_\varepsilon$ and $G = S \cap H_\delta$, respectively.

Fix parameters $\beta > 0$ and $\gamma > 0$. Let x_i, y_i be the points on h_i such that $|x_i, u_i| = |y_i, u_{i+1}| = \varepsilon\beta$. Now, move each interval (x_i, y_i) *away* from the interior of Q at distance $\varepsilon\gamma$. Denote by (x_i^*, y_i^*) the resulting intervals, and by Q^* the polygon $[x_1^*, y_1^*, x_2^*, y_2^*, \ldots, x_m^*, y_m^*]$. Note that when ε decreases, polygons Q and Q^* contract at the same rate. Thus, when $\beta > 0$ and $\gamma > 0$ are sufficiently small, one can ensure that Q^* is not self-intersecting, and this restriction is independent of ε. However, for the construction we need a stronger condition.

Denote by $T_i \subset F_m$ a trapezoid formed by (x_i, y_i) and edge e_i. Rotate each trapezoid T_i around its edge e_i *away* from P, so that the resulting trapezoid T_i' contains the interval (x_i^*, y_i^*). For all i, denote by (x_i', y_i') the rotation of (x_i, y_i), and by Δ_i the triangle $[y_{i-1}', w_i, x_i']$.

The trapezoids T_i and T_{i+1} are called *adjacent*. We claim that there exist constants $\beta > 0$ and $\gamma > 0$ independent of ε, such that trapezoids T_i' and triangles Δ_j do not intersect between themselves and each other, except if they share an edge or are adjacent trapezoids (i.e. unless they intersect for combinatorial reasons).

First, set $\beta > 0$ small enough, so that when $\gamma = 0$ the polygon Q^* in not self-intersecting. Since G is also not self-intersecting, neither are the triangles and trapezoids in between. Now start increasing γ. Observe that the first time the intersection can occur is at a vertex x_i' or y_i', for some i. It is easy to show that $\|x_i, x_i'\| \le 2\varepsilon\gamma$, $\|y_i, y_i'\| \le 2\varepsilon\gamma$, and the same is true for their projections $\pi(x_i'), \pi(y_i')$ on H_ε. This is easily satisfied when $\gamma > 0$ is sufficiently small.

Consider also triangles $\Upsilon_i = [x_i', y_i', v]$ and $\Lambda_i = [y_{i-1}', x_i', v]$. We need these triangles not to intersect each other, trapezoids T_i and triangles Δ_i. Again, just as in the previous case, this is satisfied when $\gamma > 0$ is sufficiently small, independently of ε. We skip the details.

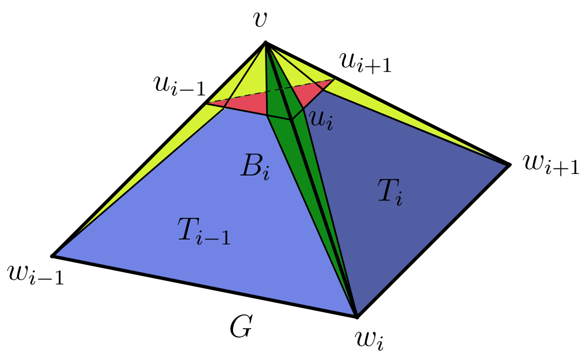

FIGURE 16. Trapezoids T_{i-1}, T_i and a 4-gon hinge B_i in between.

We need one more condition. We want to make sure that the "short" edges of triangles Δ_i are sufficiently small:

$$\|y'_{i-1}, x'_i\| \le |y_{i-1}, x_i|_S\,, \quad \text{for all } \; i\,.$$

Observe that both sides grow linearly with ε and that the inequality is obviously satisfied when γ is small enough. We can now fix parameters $\beta, \gamma > 0$ and proceed to the construction.

We construct S_ε in two stages as follows. First, attach to R all trapezoids T_i and triangles Δ_i constructed above (recall that they are completely determined by the parameters β, γ). Then, add triangles Υ_i and Λ_i constructed above. Denote by A_ε the resulting embedded surface. Note that we are not claiming that A_ε is submetric to S. Instead, we construct a PL-homeomorphism $\psi : S \to A_\varepsilon$ which will be "corrected" later.

Let ψ be identity on R and a rotation on trapezoids: $\psi(T_i) = T'_i$ for all $1 \le i \le m$. Let $\psi : [x_i, y_i, v] \to \Upsilon_i$ be a linear map. Take a surface quadrilateral $B_i = [y_{i-1}, w_i, x_i, v] \subset S$, defined as a union of two triangles in F_i and F_{i-1}. Think of regions B_i as hinges. Turn them, increasing or decreasing the angle between two triangles to fit on Δ_i. Then project them on a plane spanned by Δ_i, for all $1 \le i \le m$. Project this unhinged B_i onto Δ_i and let ψ be this projection, wherever defined. Finally, set any PL-map from the remaining portion of B_i onto Λ_i. This completes the construction of a PL-homeomorphism ψ.

By construction, map ψ is an isometry on T'_i and a submetry on the preimage of Δ_i. Let us show how to modify the cut and project construction as in §2.3, so we can change ψ into a submetry. The idea is clear from Figure 17 where a 2-dimensional case is shown. Instead of projecting onto a line, decrease the heights of all points to make them nearly flat, while avoiding the overlaps. The lengths are then contracted by about the same factor. In fact, we can make the heights between the layers smaller and smaller, to avoid the explosion after repeated projections and to ensure that ever that at every iteration we contract by the same constant factor.

In our situation we need to do this cutting and near-projecting procedure in two directions parallel to plane H. Formally, we need to choose two orthogonal lines

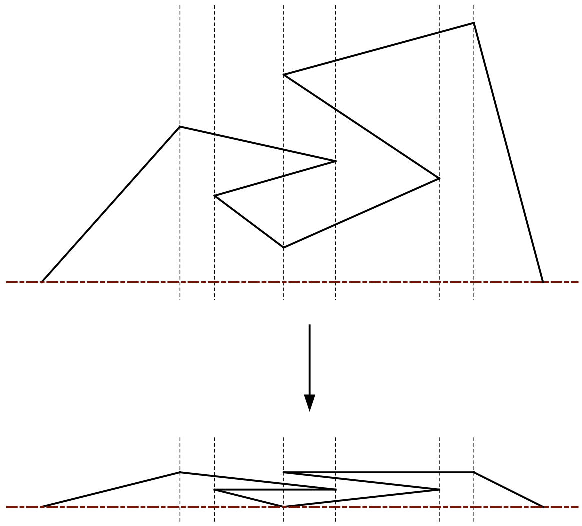

FIGURE 17. Cutting and near projecting.

$E_1 \perp E_2 \subset H$. Then perform the cutting and near-projecting first onto planes parallel to E_1, and then onto planes parallel to E_2. At the end, a portion of trapezoids T_i and triangles Δ_i will be changed. By construction, the edges cut from them will have $O(\varepsilon)$ length. When $\varepsilon > 0$ is small these cuts will not reach R and will change the area of T_i' by at most $O(\varepsilon^2)$. Let S_ε be the resulting surface.

Let us make a volume calculation to show that $\mathrm{vol}(S_\varepsilon)$ is increasing. Since $\beta, \gamma > 0$ are fixed constants, we have

$$\mathrm{area}(T_i) \;=\; \mathrm{area}(F_i) \;+\; O(\varepsilon)\,,$$

and the height of (x_i', y_i') over the plane containing F_i is $\theta(\varepsilon)$ when ε is sufficiently smaller than fixed parameter δ. Therefore, the asymptotic behavior for the volume of polyhedra $Y_i = [x_i, y_i, x_i', y_i', w_i, w_{i+1}]$ is based on the volume of a pyramid formula:

$$\mathrm{vol}(Y_i) \;=\; \frac{1}{3}\,\mathrm{area}(F_i)\cdot O(\varepsilon) \;+\; O(\varepsilon^2)\,.$$

Here the volume change along the hinges is $O(\varepsilon^2)$. Therefore, we have:

$$\mathrm{vol}(S_\varepsilon) \;=\; \mathrm{vol}(S) + \sum_{i=1}^{m} \mathrm{vol}(Y_i) + O(\varepsilon^2) \;=\; \mathrm{vol}(S) + \frac{1}{3}\sum_{i=1}^{m} \mathrm{area}(F_i)\cdot O(\varepsilon) + O(\varepsilon^2)\,.$$

This implies that $\mathrm{vol}(S_\varepsilon)$ is increasing for ε sufficiently small, and proves that $\{S_\varepsilon\}$ is a volume-increasing shrinking.

To apply the Burago-Zalgaller's theorem (Theorem $5'$) to S_ε, we need to use essentially the same argument as in the proof of Theorem 7. We need to fix a trivial flat triangulation on trapezoids and map the hinges as a cone starting at w_i over the same cross-section as in Figure 14. The remaining part is linear in ε and any triangulation will work there. The details are straightforward.

For the case when S is *immersed*, the proof is actually easier, as we have certain extra flexibilities in the construction. We can now ignore the intersections (some of which can be forced) and need only to avoid the overlaps of the triangles and trapezoids. Both the constructive proof above and the application of Burago-Zalgaller's theorem can be repeated verbatim. This completes the proof of the theorem. $\square$

3. The volume of inflated polyhedra

3.1. **Convex shrinking.** Given a convex polyhedron $P \subset \mathbb{R}^3$, Corollary 4 says that one can construct another convex polyhedron Q of bigger volume and with a submetric surface. Since the proof of Theorem 2 is constructive, we actually know how to do that. One can apply the same construction to Q, obtaining yet another polyhedron of bigger volume, etc. Each time we repeat the construction the volume increases while the surface shrinks. Of course, the convex body we get in the limit is no longer polyhedral, but it still compact and its volume is well defined. In fact, by the isometric inequality the volume of a body obtained by a shrinking of ∂P is always bounded by a constant which depends on the area of ∂P. An interesting question is how large this volume can be.

Problem 1. *For a convex polyhedral surface S in $\mathbb{R}^3$, compute:*

$$\zeta(S) := \sup_{S' \preccurlyeq S} \operatorname{vol}(S'),$$

where the supremum is over all convex *polyhedral surfaces S' submetric to S.*

Clearly, $\zeta(S) > 0$ by Corollary 4. It would be interesting if there was a unique convex piecewise-smooth surface $Z = Z(S)$ submetric to S, and such that $\operatorname{vol}(Z) = \zeta(S)$. However, we will not try to state this as a conjecture.[6] Note that in the open problem above we consider all surfaces submetric to S, not just those obtained by a volume-increasing shrinking. In fact, we believe in the following conjecture of independent interest, which, if true, removes this difference.

Conjecture 1. *Let S_0 be a convex polyhedral surface in $\mathbb{R}^3$ and let S_1 be a convex polyhedral surface submetric to S_0 of greater volume:* $\operatorname{vol}(S_1) > \operatorname{vol}(S_0)$. *Then there exist a volume-increasing shrinking from S_0 to S_1. Similarly, if* $\operatorname{vol}(S_1) = \operatorname{vol}(S_0)$, *there exist a volume-preserving*[7] *shrinking from S_0 to S_1.*

The second part means that you never have to first increase the volume and then decease it in order to shrink S_0 to S_1 of the same volume. In the shrinking of a cube in Example 1, there exist a value $\varepsilon \approx 0.3026$ such that $\operatorname{vol}(Q_\varepsilon) = \operatorname{vol}(Q_0) = 1$. In this case, an explicit volume-preserving shrinking (as in the conjecture) can be obtained in a way similar to the construction in the proof of Theorem 2. There we need to

[6] This is a pure speculation at this point as not a single example of this has been established. We would be just as happy to have a C^2-smooth surface. Note also that the geodesic distance can be defined on general convex surfaces (see e.g. [AZ, Po2]), so it makes sense to say that a general convex surface S' is submetric to S.

[7] Usually, the term *volume-preserving* means something different, but in this context we mean "*volume-unchanging*". Neither notion will be used anywhere else in the paper.

have parameter $\delta = \delta(\varepsilon)$ to grow much faster than prescribed in the proof. We omit the details.

3.2. **Non-convex shrinking.** Let us now switch to the volume of surfaces isometric to S, and thus no longer convex.

Problem 2. *For a convex polyhedral surface S in $\mathbb{R}^3$, compute:*

$$\eta(S) := \sup_{S' \sim S} \mathrm{vol}(S'),$$

where the supremum is over all immersed polyhedral surfaces S' isometric to S.

From Burago-Zalgaller's theorem (Theorem 5), we know that in the definition of $\eta(S)$ one can use *submetric* in place of *isometric* surfaces. From here we immediately have

$$\zeta(S) \le \eta(S)$$

for all convex S. We believe that the equality is never achieved.

Conjecture 2. *For every convex polyhedral surface S in $\mathbb{R}^3$ we have:*

$$\zeta(S) < \eta(S).$$

In other words, conjecture 2 says that for every convex S there exist a parameter $\alpha > 0$ and an immersed surface S_1 submetric to S, such that every convex surface S_2 we have $\mathrm{vol}(S_2) < \mathrm{vol}(S_1) - \alpha$. The conjecture can be explained by the following stronger claim.

Observe that one can speak of a general surface S' *submetric* to S if the geodesic distance $|x, y|_{S'}$ is defined for all $x, y \in S'$. This can be done for all piecewise-smooth surfaces, and in fact a lower smoothness class will suffice.

Conjecture 3. *For every convex polyhedral surface S in $\mathbb{R}^3$ there exist a unique (up to rigid motions) piecewise-smooth surface $K = K(S)$ which is embedded into $\mathbb{R}^3$, submetric to S and such that $\mathrm{vol}(K) = \eta(S)$. Moreover, the surface K is strictly non-convex and smooth everywhere except at the vertices.*

Of course, all polyhedral surfaces are piecewise-smooth, so the supremum of $\mathrm{vol}(S')$ over all piecewise-smooth surfaces S' submetric to S is at least $\eta(S)$. On the other hand, one can approximate any piecewise-smooth surface S' submetric to S with polyhedral surfaces submetric to S. Therefore, the point of the first part of the conjecture is the fact that $K(S)$ is piecewise-smooth and unique up to rigid motions.

Finally, if $K(S)$ is non-convex, it cannot be approximated with convex polyhedral surfaces $S' \preccurlyeq S$, which suggests (but does not formally imply) that $\zeta(S) < \eta(S)$.

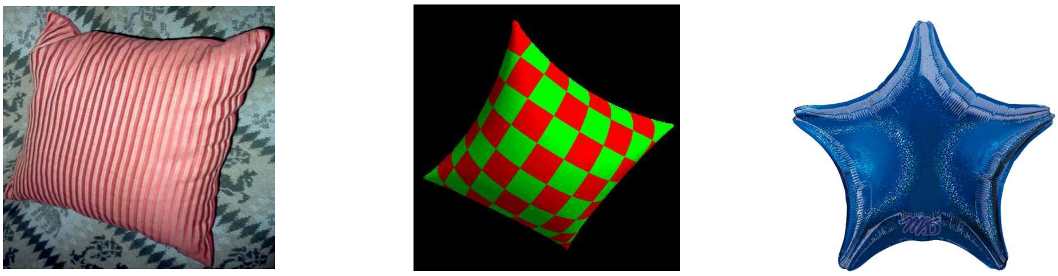

Figure 18. Real cushion pillow, the ideal pillow shape and a star shaped party balloon.

3.3. **Further speculations.** To see the reasoning behind Conjecture 3, consider the following physical experiment. Think of the surface S made out of bendable but non-stretchable material, much like cellophane or polyester. Blow the air into S until it no longer possible. The resulting shape is the desired surface $K(S)$, and the physical intuition is suggests that this surface is indeed unique.

When S is a doubly covered square the experiments produces a cushion pillow shape, so one can this of the surface $K(S)$ as an ideal pillow shape (see Figure 18). This is the only example for which an investigation of $K(S)$ has been attempted (see §4.4 below). One can see that the shape is strictly non-convex as predicted by Conjecture 3. In a way, it also suggests that the volume-increasing deformation shown in Figure 2 is far from optimal: the optimal would be concave along all three sides of the triangle. This can be illustrated with a party balloon in Figure 18. Here we essentially have an ideal inflated surface $K(S)$, where S is a doubly covered regular pentagon.

Now, we are so confident in our Conjecture 3 and the physical experiments, we are willing to strengthen it and venture further guesses on the shape of $K(S)$. First, it is natural to assume that the first part of Conjecture 3 holds for all convex piecewise-smooth surfaces S, not just polyhedral surfaces. As we show below, the second part is no longer true in this generality.

Second, note the crimps on the sides of the cushion pillow in Figure 2 or (much smaller) crimps on the surface of the party balloon in Figure 18. We believe this a universal phenomenon, and the crimps always appear when the ideal inflated surface K is approached.

Conjecture 4. *For every convex polyhedral surface S in $\mathbb{R}^3$, the* ideal inflated surface $K = K(S)$ *has smaller area:*

$$\operatorname{area}(S) \, < \, \operatorname{area}(K).$$

Moreover, the submetry $K \preccurlyeq S$ is strict *almost everywhere:*

$$|x,y|_S \, < \, |x',y'|_K \quad a.s. \quad for \;\, x,y \in S\,,$$

where $x' = \varphi(x)$, $y' = \varphi(y)$, and φ is the submetry map $\varphi : S \to K(S)$.

The second part of the conjecture suggests that in fact the cushion pillow in Figure 2 is actually quite far from the ideal surface. In a better experiment, the surface should have small crimps almost everywhere (the crimps should become smaller closer to the center of each side of the pillow).

For a convex piecewise-smooth surface S, define the *crimping ratio* $\operatorname{cr}(S)$ as follows:

$$\operatorname{cr}(S) \,=\, \frac{\operatorname{area}\bigl(K(S)\bigr)}{\operatorname{area}(S)}\,.$$

Assuming the first part of Conjecture 3 extends to convex piecewise-smooth surfaces S, the crimping ratio is well defined, and $\operatorname{cr}(S) \leq 1$. Our final conjecture has an isoperimetric flavor:

Conjecture 5. *Among all convex piecewise-smooth surfaces S, the crimping ratio minimizes on a doubly covered disc.*

The crimping ratio of a doubly covered disc is given in §4.5 below. Let us note here that this is not the first time this shape appears in the context of isoperimetric problem. An old Alexandrov's conjecture claims that the doubly covered disc maximizes the ratio $\operatorname{area}(S)/\operatorname{diam}^2(S)$, where $\operatorname{diam}(S)$ is the geodesic diameter of the surface S [A1] (see also [Gh] and references therein).

3.4. **Non-convex surfaces.** One can ask if Conjecture 3 extends to non-convex surfaces S. Perhaps so, but not in the obvious way, even if S embedded into $\mathbb{R}^3$ and homeomorphic to a sphere. Take for example a doubly covered ring with a small gap as in Figure 19. As can be seen in a physical experiment in the beginning of the inflation process we get an embedding, but very soon two ends collapse into a swim ring shape. Thus, one has to either allow realized (not even immersed) surfaces or accept the non-uniqueness of the limit shape.

FIGURE 19. Non-convex inflated balloons and a doubly covered shape which collapses when inflated.

3.5. **Smooth surfaces.** One can ask to what extend the conjectures above apply to smooth convex surfaces. It seems that the first part of Conjecture 3 extends to this case, but as the obvious example of a sphere shows, we can have $S = K(S)$. Obviously, $K(S)$ is convex and has no crimping, i.e. $\operatorname{cr}(S) = 1$. Thus Conjecture 2, the second part of Conjecture 3 and Conjecture 4 are inapplicable to the smooth case.

We call the surface S *non-inflatable*, if $K(S) = S$. The following problem is of interest from the differential geometry point of view.

Problem 3. *Characterize all smooth convex non-inflatable surfaces.*

Clearly, $K(K(S)) = K(S)$, so when $K(S)$ is a smooth surface we get an example of a non-inflatable surface. Beside the case of a sphere, the mylar balloon is another example of a non-inflatable surface. It is well studied in the literature and is a motivation of our Conjecture 5. We describe the mylar balloon and its properties in §4.5. In the other direction, it is known that certain surfaces are inflatable i.e. have continuous volume-increasing isometric deformation. Among other things, Bleecker showed that oblate ellipsoids with semi-axis $(1, 1, r)$, such that $r < \sqrt{3/8}$, are always inflatable [B2]. Similarly, Bleecker showed that a smooth non-inflatable surface cannot have flat regions [B2].

4. Examples and special cases

4.1. **Doubly covered triangle.** Let S be the surface of a doubly covered triangle with side length equal to 1. Already in this simplest case neither $\eta(S)$ not $\zeta(S)$ are not known. The maximal volume attained by the isometry given in Figure 2 is about 0.0309. Bleecker found an isometry construction based on a symmetric version of the same idea, with all three sides bent inside reaching the volume 0.0430. Interestingly, the optimal Bleecker's isometry coincides with the optimal case of our isometry (see the proof of Theorem 2).

We should mention the isoperimetric inequality (see e.g. [BZ2])

$$(\circledast) \qquad \operatorname{vol}(S) \le \frac{1}{6\sqrt{\pi}} \operatorname{area}(S)^{3/2},$$

which gives an upper bound of 0.0758 for the volume. Of course, if conjectures 3 and 4 hold in this case, then the optimal surface $K(S)$ is non-convex, i.e. far away from a sphere, and has surface area smaller than $\operatorname{area}(S)$.

4.2. **Tetrahedron.** Consider the surface S of a regular tetrahedron with side length equal to 1. Bleecker's isometric bending of a tetrahedron is given in Figure 20 (only one quarter of a tetrahedron is shown). The volume maximizes at 0.1628 compared with 0.1179, the volume of a regular tetrahedron. This is still quite far from the isoperimetric inequality bound of 0.2143, which is not expected to be reached.

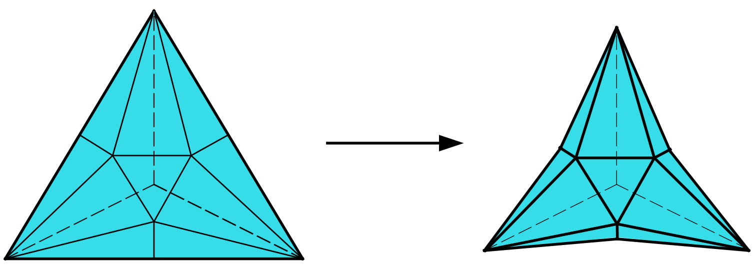

Figure 20. Bleecker's isometric bending of a tetrahedron.

4.3. **Cube.** As we mentioned in the introduction, in the unit cube case the Bleecker's construction gives the volume 1.2187. The construction in Example 2.2 maximizes the volume at 1.1820, while the isoperimetric inequality gives the upper bound of 1.3820.

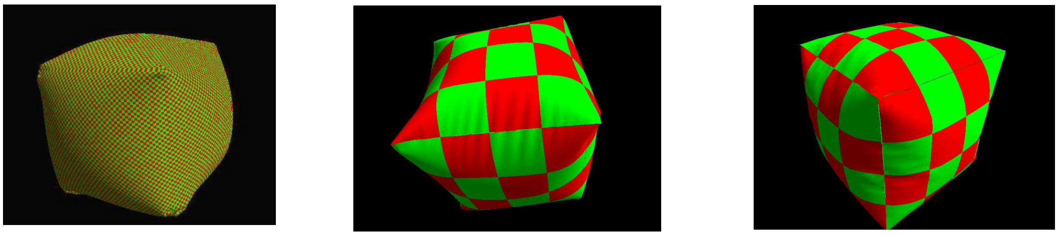

FIGURE 21. Andreas Gammel's computer simulation of the ideal inflated surface of a cube.

4.4. **Doubly covered square.** Let S be a doubly covered unit square. Informally, we refer to the surface $K = K(S)$ as the *ideal pillow shape* (see Figure 18). The problem of computing the vol(K) has attracted much attention in the recreational literature under different names such as *the tea bag problem* [W], *the paper bag problem* [R], etc. Here we give a quick survey of the known results, most of which are largely unavailable in the mainstream mathematical literature.

FIGURE 22. Andreas Gammel's computer simulation of the ideal pillow shape: an intermediate step and the final shape (view from the side).

In a recent paper [R], Robin reports physical experiments and an empirical formula for the volume of the ideal rectangular pillow. His formula gives an estimate 0.1910 for the vol(K). Andreas Gammel did a number of computer experiments simulating an ideal pillow shape K (see Figure 22). He estimated the volume vol(K) as about 0.208 [Ga]. This should be compared with the lower bound of 0.1129 given by our general construction (see Figure 23), which in this case is similar to Bleecker's construction of bending of a cube [B1]. In the opposite direction, the isoperimetric inequality ($\circledast$) gives the upper bound of 0.2660.

An interesting study was undertaken by Andrew Kepert [K], who obtained a succession of explicit constructions of isometric embeddings of S. His first construction, shown in Figure 24, has volume 0.1902. It can be viewed as a simple modification of our general construction in this case[8]

[8]Of course, Andrew Kepert's construction was made much earlier, in 1997 [K].

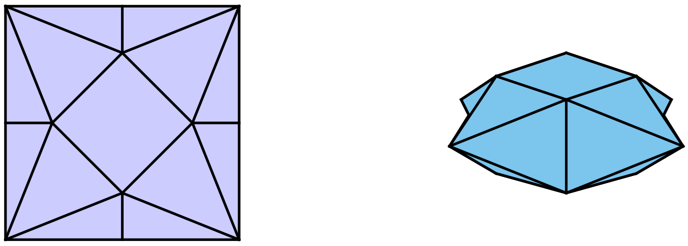

Figure 23. Isometric embedding of S as constructed in the proof of Theorem 2.

Kepert introduced a number of pleats and utilized the symmetry. The idea can be understood from Figure 24, where only one octant of the construction is shown. The volume in this case is about 0.2055. Note the circular boundary in the octant construction which is close but not exactly optimal, as can be seen in the ideal shape simulation 18. Kepert also found a heuristic upper bound 0.2183 based on several unproven assumptions [K] (see §6.5).

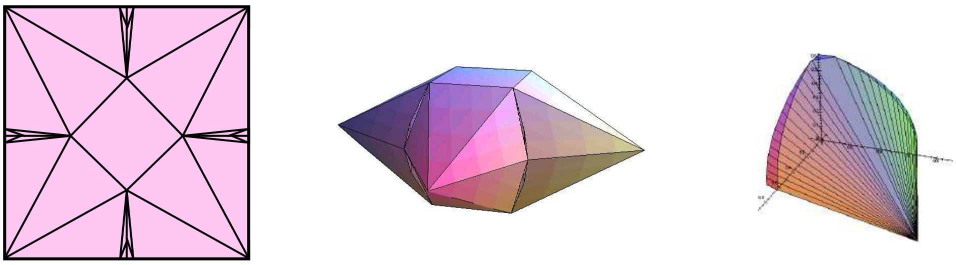

Figure 24. Kepert's two constructions.

4.5. **Mylar balloon.** Let S be the surface of doubly covered unit circle. The ideal inflated surface $K = K(S)$ is commonly known as *mylar balloon* [Pa, MO], named after inelastic polyester material it is commonly made of (see Figure 25).

By the symmetry, the mylar balloon K is a surface of revolution. To describe it, we first write its inflated radius:

$$r = \frac{4\sqrt{2\pi}}{\Gamma\left(\frac{1}{4}\right)^2} \approx 0.7627\,.$$

The *elliptic integrals* of the first and second kind are defined as follows:

$$F(z,k) = \int_0^z \frac{dt}{\sqrt{1-t^2}\,\sqrt{1-k^2t^2}} \quad \text{and} \quad E(z,k) = \int_0^z \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}\,dt\,.$$

The *Jacobi sine function* $\operatorname{sn}(u,k)$ is defined as the inverse to $F(z,k)$, for all $k \in \mathbb{R}$. The *Jacobi cosine function* $\operatorname{cn}(u,k)$ is defined by

$$\operatorname{sn}(u,k)^2 + \operatorname{cn}(u,k)^2 = 1\,.$$

It was shown in [Pa, MO] that the mylar balloon surface

$$K = \big\{\big(x(u,v), y(u,v), z(u,v)\big),\ u \in [-A, A], v \in [0, 2\pi]\big\}$$

is defined by the following equations:

$$x(u,v) = r\,\mathrm{cn}(u,a)\cos(v), \quad y(u,v) = r\,\mathrm{cn}(u,a)\sin(v),$$

$$z(u,v) = \frac{r}{a}\left[E\big(\mathrm{sn}(u,a),a\big) - \frac{1}{2}F\big(\mathrm{sn}(u,a),a\big)\right],$$

where $a = \frac{1}{\sqrt{2}}$ and $A = F(1,a)$.

The submetry map $\varphi : S \to K$ is determined via geodesics from North and South poles to the equator: each such geodesics has length 1. Note that K is smooth, convex and flat only at the North and South poles. Thus $\zeta(S) = \eta(S)$ in this case and Conjecture 4 holds for the surface S.

FIGURE 25. Mylar party balloon and NASA's Ultra Long Duration Balloon.

The area and the volume of K are given by

$$\mathrm{area}(K) = \pi^2 r^2 = \frac{32\pi^3}{\Gamma\big(\frac{1}{4}\big)^4} \approx 5.7422,$$

$$\mathrm{vol}(K) = \frac{2\pi r^2}{3} = \frac{64\pi^2}{3\,\Gamma\big(\frac{1}{4}\big)^4} \approx 1.2185.$$

Note that according to Conjecture 5, the crimping ratio minimizes in this case and is equal to

$$\mathrm{cr}(S) = \frac{16\pi^2}{\Gamma\big(\frac{1}{4}\big)^4} \approx 0.9139.$$

Interestingly, the crimping ratio $\mathrm{cr}(S)$ is equal to the distance between the North and South poles in this case. Thus, from Bleecker's result mentioned in § 3.5, the oblate ellipsoids with the same axis as mylar balloon is inflatable. We refer to [MO, Pa] for complete calculations and further details.

5. Applications of the mylar balloon

5.1. Denote by P_n the regular polygon inscribed into a unit circle, and let S_n be a doubly covered polygon P_n, a convex surface in $\mathbb{R}^3$. The following result is a rare general result on the function $\eta(\cdot)$.

Proposition 1. *Denote by $S_\circ$ the doubly covered unit circle and let $\daleth = \eta(S_\circ)$. Then, for all $n \geq 3$ we have: $\eta(S_n) < \daleth$. Moreover, if $K = K(S_\circ)$ is well defined and is equal to the mylar balloon, then we have:*

$$\lim_{n\to\infty} \eta(S_n) = \daleth.$$

Recall from the previous section that the volume of the mylar balloon $\daleth = \mathrm{vol}(K) \approx 1.2185$. Thus, under assumptions as in the proposition, we can compare what bound do we get for different n. When $n = 6$, we have $\mathrm{area}(S_6) = 3\sqrt{3}$ and the isoperimetric inequality ($\circledast$) gives $\eta(S_6) < 1.1138$, which is better than the inequality in the theorem. On the other hand, when $n = 12$, we have $\mathrm{area}(S_{12}) = 6$ and the isoperimetric inequality gives $\eta(S_{12}) < 1.3820$, which is weaker than the inequality in the theorem.

One would assume that $\eta(S_n)$ is increasing with n, but with the tools we have at the moment we cannot establish even that. Similarly, it is natural to conjecture that $\zeta(S_n) \to \daleth$ as $n \to \infty$. However, a careful examination of the proof below shows that the submetric embeddings constructed there are not necessarily convex.

5.2. **Proof of Proposition 1.** For the first part, take the doubly covered unit circle $S_\circ$ and fold onto S_n as shown in Figure 26. Every submetry $S_n' \preccurlyeq S_n$ now corresponds to a submetry $S' \preccurlyeq S_\circ$ of the same volume. By definition, $\eta(S_\circ)$ is the supremum of the volume of the submetric embeddings of $S_\circ$. Therefore, $\eta(S_\circ) \geq \mathrm{vol}(S') = \mathrm{vol}(S_n')$. Taking S_n' such that $\mathrm{vol}(S_n')$ is as close to $\eta(S_n)$ as desired, we obtain the result.

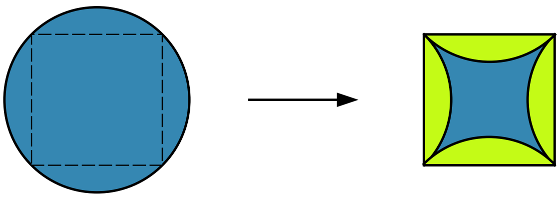

Figure 26. Folding the circle onto a square.

For the second part, let us approximate the mylar balloon with submetric embeddings of S_n. Consider a triangulation τ_n of the n-gon P_n into triangles such that the length of each side of each triangle is at most the side-length of P_n, and there are no extra vertices added on the sides of P_n. Denote by K the surface of the mylar balloon. The submetric map $\varphi : S \to K$ maps the vertices of τ_n onto K so that the geodesic distances are non-increasing (in fact, strictly decreasing in this case). Note also that triangles in τ_n attached to the boundary are mapped into triangular regions on K.

The boundary edge is mapped onto a geodesic, which is smaller than the circular arc of K. The latter is smaller than the circular arc of the unit circle by a factor of $r < 0.77$ (see Figure 27).

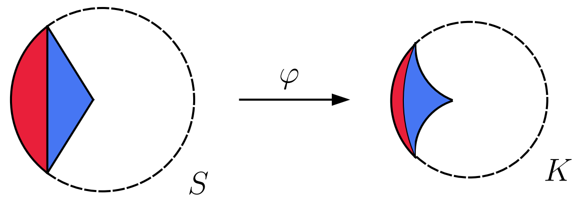

FIGURE 27. Triangle on a circle and mylar balloon.

Now replace all triangles on the surface of K with straight triangles on vertices of $\varphi(\tau_n)$. Contract the resulting surface X_n by a factor $(1-\varepsilon)$, for some fixed $\varepsilon > 0$ independent on n. Let us show the submetry on all triangles in the triangulation τ_n. First, the edge lengths of all triangles is $O(1/n)$ by construction. Because of smoothness of K, the geodesic and straight distances on K are in fact within factor $1 + O(1/n^2)$ of each other. Roughly, for all but the boundary triangles this implies a near-isometry, and, after contracting by a factor $(1-\varepsilon)$, all these triangle become submetric. For the boundary triangles, we have a similar argument for the interior edges and the constant shrinking factor for the exterior edges.

Finally, for the volume the convergence argument above gives:

$$\mathrm{vol}(X_n) \, = \, \mathrm{vol}(K) \, + \, \mathrm{area}(K) \cdot \left(1 - O\left(\frac{1}{n^2}\right)\right).$$

After we do the contraction, the volume decreases by a factor of $\alpha = (1-\varepsilon)^3$. Since ε is independent of n, the volume converges to $\alpha \, \mathrm{vol}(K)$, as $n \to \infty$. Therefore, by choosing $\varepsilon > 0$ small enough, we can get submetric embedding with the volume as close to K as desired. $\square$

6. Final remarks

6.1. One can view the results in this paper as the first step in characterization of non-inflatable surfaces: we show that such surfaces cannot be polyhedral. It seems that the fundamental idea in this paper can be employed to obtain more delicate regularity results.

Suppose, for example, that S is a piecewise-smooth non-inflatable surface with a *crease* along a smooth curve $f \subset S$. Fix a point $v \in f$ and consider two geodesics $g, h \subset S$ orthogonal to f in v. Choose points $x \in g$, $y \in h$ on the geodesics at equal geodesic distance ε from v. Similarly, choose points $w_1, w_2 \in f$ so that the distance from them to v along f is also ε. Connect x, y with w_1, w_2 by geodesics and denote by U the resulting region $[x, w_1, y, w_2] \subset S$ (see Figure 28). As $\varepsilon \to 0$, the region U converges to a square bent along a diagonal.

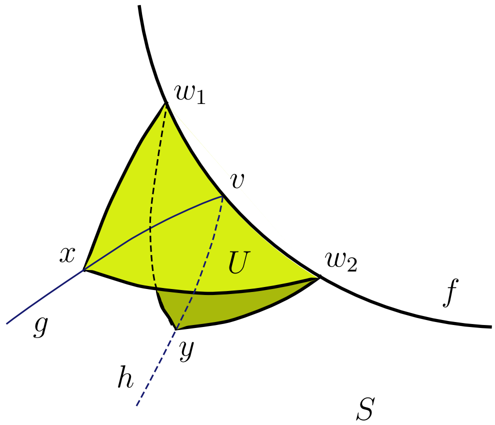

FIGURE 28. Inflating a portion of the surface.

One would assume that it is possible to "inflate" the curved tetrahedron spanned by the region, while keeping intact the geodesics on the boundary ∂U. This is impossible to do when f is a straight line on the interval (w_1, w_2), but if f is concave as in Figure 28, there is clearly a room to push v "inside" to make the acute angle between g and h a bit larger. Making rigorous an argument of this kind is an interesting challenge which lies outside the scope of this paper.

6.2. The proof of Theorem 8 is based on a classical idea of truncating convex polyhedra, an approach which often appears in a different context. For example, all Archimedean solids can be obtained from Euclidean solids by truncating the vertices. Similarly, the permutohedron in $\mathbb{R}^d$ can be obtained from a simplex by first (symmetrically) cutting along vertices, then along edges, etc., and eventually along all ridges (see [Zi]). The same approach was recently used in [DP] as a way to realize generalized associahedra.

6.3. With some notable exceptions, very little is known about metric geometry of polyhedra in dimensions higher than three [Po2] (see also [MP]). In a certain sense which can be made precise, the higher the dimension d, the "more rigid" are convex polyhedra. A number of 3-dimensional results are simply false in higher dimensions. For example, it is easy to see that Alexandrov's existence theorem cannot be extended to d-dimensional convex surfaces in $\mathbb{R}^{d+1}$ when $d \geq 3$, since there is not enough degrees of freedom. The same is true for the Burago-Zalgaller existence theorem in [BZ1], since there exist topological obstacles for immersions of d-dimensional surfaces into $\mathbb{R}^{d+1}$, such as the classical Hopf theorem.

In the same way, while in $\mathbb{R}^3$ a number of non-rigid immersed polyhedra is known, such as the Bricard octahedra, not a single example is known for $d \geq 5$. In fact, it is conjectured that higher dimensional cross-polyhedra cannot be rigid. We refer to [C2] for a comprehensive survey on rigidity, to [Gr] for a general setting of the Burago-Zalgaller type results (including the foundational results by Nash and Kuiper), and to [Y] for a general overview of the modern geometry, which puts Bleecker's results into context.

6.4. It is possible that Burago-Zalgaller's theorem (Theorem 5′) holds in higher dimensions, but their approach is specific to three dimensions and non-generalizable. In fact, their starting point is a construction of an acute triangulation of the surface (Lemma 1 in [BZ1]). This result, largely unknown in the West, has been recently rediscovered, but only a weaker versions have been obtained (see [Ma, Za]). As of now, very little is known about acute triangulations is higher dimensions [Za].

It would be nice to further generalize Bleecker's theorem and our theorems 7 and 10 to $\mathbb{R}^d$, but we see no way of proving them directly without Burago-Zalgaller's theorem even when $d = 3$. There is one exception, however. If one considers only isometric immersions rather than embeddings, the first part of Theorem 10 can be obtained nearly verbatim, with Theorem 5′ replaced by an appropriate strengthening of Tasmuratov's theorem [T1] (see also [T2]). While Tasmuratov's proof is significantly simpler than the proof of Burago and Zalgaller, it remains unclear whether his approach can be extended to higher dimensions. We plan to return to this problem in the future.

Let us also mention here a completely elementary treatment of a very special case of Burago-Zalgaller's theorem in [Z].

6.5. Kepert's heuristic upper bound mentioned in 4.4 is based on a subdivision of the surface S of the doubly covered unit square into four cones C with side length $1/2$ and the remaining region. Then the idea is to consider all surfaces S' which contain four cones isometric to C and have the same area: $\text{area}(S') = \text{area}(S)$. It is then assumed that among such surfaces the largest volume is attained by a sphere with four cones attached to it. If true, this gives an upper bound on $\eta(S)$.

While the above assumption vaguely resembles the *double bubble problem* (see e.g. [HMRR, Mo]), there are several important differences. With four cones the problem loses symmetry, so the resulting shape is no longer the surface of revolution, an important first step in many isoperimetric problems.

6.6. The crimping of non-inflatable surfaces was first noticed by Paulsen in the context of the mylar balloon [Pa] (see also [B1, §5.2]). However, the physical nature of the crimping during inflation process has hardly been understood. For example, one would assume that the pleated surface has crimps which go along the surface, but the experiments seem to suggest otherwise. Open mathematical problems remain as well. For example, it takes an argument to prove that there really does exist a volume-increasing continuous shrinking from the doubly covered circle S to the final state. This might not be difficult, but has not been done.

Another interesting open problem is the question of uniqueness, as in Conjecture 3, i.e. whether the mylar balloon is a unique shape up to rigid motions, which attains the largest possible volume while remains submetric to S. There is, of course, little doubt that this is true. In fact, we even used this result explicitly in the assumptions of Proposition 1. In view of the symmetry of the mylar balloon, it is natural to assume that the volume maximizes when the shape is a surface of revolution, and Paulsen's analysis completes the proof. Unfortunately, it is unclear whether a complete proof of the assumption is attainable with the available techniques.

6.7. There is a fairly large literature on isometric deformations of convex surfaces under *buckling*, especially for surfaces of revolution (see [Po1, Po3]). In this case, the volume actually decreases, so in a way this process is an inverse of inflation. To get an idea of these constructions, a buckling of a cube from [Sh1] is shown in Figure 29 (see also [Sh2]).

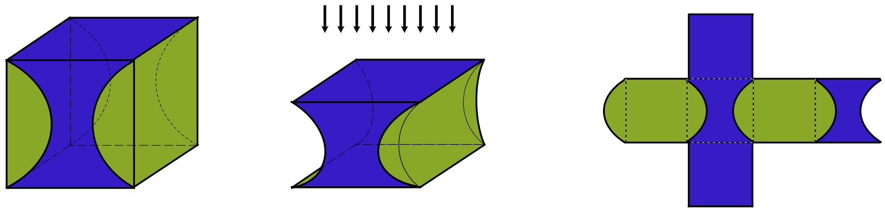

FIGURE 29. Buckling of a cube and its unfolding (scaled down).

In a different direction, when the surface of a polyhedron is *elastic*, there has been work on simulation of the inflation (see e.g. [SHM] and references therein). Of course, the volume is no longer bounded. As it increases, the inflated polyhedron starts to approximate a sphere (see Figure 30).

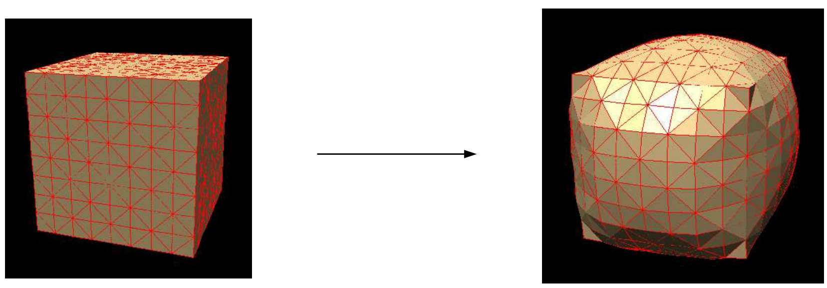

FIGURE 30. Smirnov's simulation of an elastic cube inflation [SHM].

6.8. Observe that in all our constructions of isometric deformations, the geometry of faces changes, even if combinatorics remains the same. In view of the flexible surfaces and Connelly's flexor examples (see [C1, C2]), one can ask whether there exist volume-increasing flexible surfaces. The answer turns out to be no: the volume is invariant under flexing. This is the statement of the *bellows conjecture* which was proved recently by Sabitov for polyhedral spheres [S1] and then extended to orientable surfaces of every genus in [CSW]. On the other hand, there exists a convex (even strictly convex) polyhedron P and another (non-convex) polyhedron Q with congruent faces and the same combinatorial structure, and such that $\mathrm{vol}(P) < \mathrm{vol}(Q)$. In the non-strictly convex case the answer follows immediately from Corollary 3. For more about realizations of convex polyhedra with given faces see [FP] and references therein.

Let us note here that constructing of a volume-increasing bendings may prove easier for polyhedra in spaces of constant curvature. This is another problem suggested by Bleecker in [B1]. As was shown by V. Alexandrov in [Al2], the bellows conjecture fails in this case. Similarly, the infinitesimal isometric deformations in higher dimensions may be easier to construct than the (usual) isometric deformation (cf. [B1]). The analogue of the bellows conjecture in $\mathbb{R}^3$ was also refuted by V. Alexandrov [Al1].

Note on figures. While most figures in the paper are made by the author, we also use several pictures made by others. For the latter, we either have a written permission to use them or they are publicly available.

Mylar balloon pictures (figures 18 and 25) were made by `BalloonManiacs.com`. The cube and pillow simulations are made by Andreas Gammel (figures 22 and 21). Pictures of isometric embeddings of a cube (Figure 24) are made by Andrew Kepert. The inflated cube in Figure 30 is made by Andrei Smirnov. Artistic vision of NASA's ULDB (Figure 25) is available from the `NASA.gov` web site. Finally, two pillow shapes in Figure 18 are available from [W], and were donated to public by their creators.

Acknowledgements. I am grateful to David Bleecker, Tobias Colding, Joel Haas, Michael Hutchings, David Jerison and Ezra Miller for the interesting discussions and help with the references. Special thanks to Andreas Gammel and Andrew Kepert who kindly allowed me to use their pictures and refer to their calculations on the tea bag problem (§4.4), and to Andrei Smirnov for his his picture of the inflated elastic cube (see §6.7). I am also indebted to Shelly Jenson from `BalloonManiacs.com` for the permission to use their high quality pictures of balloons. The author was partially supported by the NSF.

References

[A1] A. D. Alexandrov, *Intrinsic geometry of convex surfaces* (in Russian), M.–L.: Gostekhizdat, 1948; English translation in *Selected Works. Part 2*, CRC Press, Boca Raton, FL, 2005.

[A2] A. D. Alexandrov, *Convex polyhedra* (in Russian), M.: Gostekhizdat, 1950; English translation in Springer Monographs in Mathematics, Springer, Berlin, 2005.

[AZ] A. D. Aleksandrov and V. A. Zalgaller, *Intrinsic Geometry of Surfaces*, AMS, Providence, RI, 1967; available at `http://www.ams.org/online_bks/mmono15/`

[Al1] V. A. Alexandrov, Remarks on Sabitov's conjecture on volume rigidity with infinitesimal bending of a surface, *Sib. Math. J.* **30** (1989), 678–684.

[Al2] V. A. Alexandrov, An example of a flexible polyhedron with nonconstant volume in the spherical space, *Beiträge Algebra Geom.* **38** (1997), 11–18; `arXiv: math.MG/0111003`.

[Al3] V. A. Alexandrov, How one can crush a milk carton in such a way as to enlarge its volume (in Russian), *Soros Educational J.* **51** (2000), 121–127.

[B1] D. D. Bleecker, Volume increasing isometric deformations of convex polyhedra, *J. Diff. Geom.* **43** (1996), 505–526.

[B2] D. Bleecker, Isometric deformations of compact hypersurfaces, *Geom. Dedicata* **64** (1997), 193–297.

[BZ1] Ju. D. Burago and V. A. Zalgaller, Polyhedral embedding of a net (in Russian), *Vestnik Leningrad. Univ.* **15** (1960), no. 7, 66–80.

[BZ2] Yu. D. Burago and V. A. Zalgaller, *Geometric inequalities*, Springer, Berlin, 1988.

[BZ3] Yu. D. Burago and V. A. Zalgaller, Isometric piecewise-linear embeddings of two-dimensional manifolds with a polyhedral metric into $\mathbf{R}^3$, *St. Petersburg Math. J.* **7** (1996), 369–385.

[C1] R. Connelly, Conjectures and open questions in rigidity, in *Proc. ICM Helsinki (1978)*, 407–414, Acad. Sci. Fennica, Helsinki, 1980.

[C2] R. Connelly, Rigidity, in *Handbook of Convex Geometry*, vol. A, 223–271, North-Holland, Amsterdam, 1993.

[CSW] R. Connelly, I. Sabitov and A. Walz, The bellows conjecture, *Beiträge Algebra Geom.* **38** (1997), 1–10.

[DP] S. L. Devadoss and I. Pak, Realization of graph-associahedra and statistics on trees, in preparation.

[FP] M. Fedorchuk and I. Pak, Rigidity and polynomial invariants of convex polytopes, *Duke Math. J.* **129** (2005), 371-404.

[Ga] A. Gammel, *The teabag constant*, web page `http://www.dse.nl/~andreas/teabag.html`

[Gh] M. Ghomi, Classical open problems in differential geometry, preprint (2006); available at `www.math.gatech.edu/~ghomi/Papers/op.pdf`

[Gr] M. Gromov, *Partial Differential Relations*, Springer, Berlin, 1986.

[HMRR] M. Hutchings, F. Morgan, M. Ritoré and A. Ros, Proof of the double bubble conjecture, *Ann. of Math.* **155** (2002), 459–489.

[K] A. Kepert, *Teabag problem*, web page `http://frey.newcastle.edu.au/~andrew/teabag` (defunct; cache on different dates is available at `http://web.archive.org`).

[Ma] H. Maehara, Acute triangulations of polygons, *Europ. J. Combin.* **23** (2002), 45–55.

[Mi] A. D. Milka, Linear bendings of regular convex polyhedra (in Russian), *Mat. Fiz. Anal. Geom.* **1** (1994), 116–130.

[MP] E. Miller and I. Pak, Metric combinatorics of convex polyhedra: cut loci and nonoverlapping unfoldings, to appear in *Discrete & Comput. Geom.*; arXiv:math.MG/0312253.

[MO] I. Mladenov and J. Oprea, The mylar balloon revisited, *Amer. Math. Monthly* **110** (2003), 761–784.

[Mo] F. Morgan, Proof of the double bubble conjecture, *Amer. Math. Monthly* **108** (2001), 193–205.

[P] I. Pak, Inflating the cube without stretching, arXiv:math.MG/0607754.

[Pa] W. Paulsen, What is the shape of a mylar balloon?, *Amer. Math. Monthly* **101** (1994), 953–958.

[Po1] A. V. Pogorelov, *Geometric methods in the nonlinear theory of elastic shells* (in Russian), Nauka, Moscow, 1967, 280 pp.

[Po2] A. V. Pogorelov, *Extrinsic geometry of convex surfaces*, AMS, Providence, RI, 1973.

[Po3] A. V. Pogorelov, *Bendings of surfaces and stability of shells* (in Russian), second edition, Naukova Dumka, Kiev, 1998, 200 pp. See also English translation of the first edition: Translations of Mathematical Monographs **72**, AMS, Providence, RI, 1988, 77 pp.

[R] A. C. Robin, Paper bag problem, *Mathematics today*, Bulletin of the Institute of Mathematics and its Applications **40** (June 2004), 104–107.

[S1] I. Kh. Sabitov, The volume as a metric invariant of polyhedra, *Discrete & Comput. Geom.* **20** (1998), 405–425.

[S2] I. Kh. Sabitov, On some recent results in the metric theory of polyhedra, *Rend. Circ. Mat. Palermo (2)*, Suppl. No. 65, part II, (2000), 167–177.

[Sh1] M. I. Shtogrin, Piecewise smooth embedding of a cube, *Russian Math. Surveys* **59** (2004), 979–981.

[Sh2] M. I. Shtogrin, Special isometric transformations of the surfaces of the Platonic solids, *Russian Math. Surveys* **60** (2005), 799–801.

[SHM] A. V. Smirnov, W. Huebsch and C. Menchini, A flow solver with flexible boundaries, in *Proc. IASTED Int. Conf.: Modeling and Simulation*, Vol. 174, IASTED, Palm-Springs, FL, 258–263; available at `http://mulphys.com/doc/flowsolver.pdf`

[T1] S. S. Tasmuratov, Deformation of a polygon into a polyhedron with a given boundary, *Sib. Math. J.* **15** (1974), 947–953.

[T2] S. S. Tasmuratov, Bending of a finitely connected polygon into a polyhedron with a given boundary (in Russian), *Geometry* **6** (1977), 102–107; a periodic publication of Leningrad. Gos. Ped. Inst., Leningrad, USSR.

[W] Paper bag problem, in *Wikipedia*, `http://en.wikipedia.org/wiki/Paper_bag_problem`

[Y] S.-T. Yau, Review of geometry and analysis. Kodaira's issue, *Asian J. Math.* **4** (2000), 235–278.

[Z] V. A. Zalgaller, Some bendings of a long cylinder, *J. Math. Sci.* (New York) **100** (2000), 2228–2238.

[Za] T. Zamfirescu, Acute triangulations: a short survey, in *Proc. Sixth National Conference of S.S.M.R.*, Sibiu, Romania, 2002, 9–17.

[Zi] G. M. Ziegler, *Lectures on Polytopes*, Springer, New York, 1995.