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Limit Shapes of Restricted Permutations

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Abstract of the Dissertation

Limit Shapes of Restricted Permutations

by

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Professor Igor Pak, Chair

Following the techniques initiated in [MP], we continue to study the limit shapes of random permutations avoiding a specific subset of patterns. We consider patterns in S_3 extensively, and also prove some results regarding pairs of permutations with one in S_3 and another in S_4. We analyze the limiting distribution of a pattern-avoiding permutation, and calculate the asymptotic behavior of distributions of positions of numbers in the permutations. The distributions vary significantly depending on which patterns are avoided. We also apply our results to obtain results on various permutation statistics.

The dissertation of Samuel Alexander Miner is approved.

Rafail Ostrovsky

Bruce Rothschild

Alexander Sherstov

Igor Pak, Committee Chair

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To my parents

–

for your boundless support and encouragement

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CHAPTER 1

Introduction

1.1 Introduction

The Catalan numbers are an extensively studied integer sequence which appears throughout combinatorics, counting numerous combinatorial objects. Euler first studied the sequence in the mid-1700's[Eul] while counting triangulations of polygons, though the history from that point on spans decades and mathematical objects. There are currently over 200 examples of objects enumerated by the Catalan numbers [S2]. Many statistics on such objects have been analyzed throughout the years as well.

The study of pattern-avoiding permutations dates back to MacMahon[Mac] in 1915. Knuth studied the topic as well[Knu] in the 1960s, while the notation and terminology we use today, including the term "pattern", was initiated by Simion and Schmidt[SS] in 1985. Since then, the field has expanded rapidly, with many enumerative results, generalizations, and extensions.

In this thesis, we view pattern-avoiding permutations as permutation matrices, and consider the expected distribution of the 1's within the matrices. For various permutation classes, we find and analyze the limiting distribution. Since permutations which avoid a given pattern of length three are enumerated by the Catalan numbers, these permutation matrices are a Catalan structure. By analyzing this limit shape in detail, we obtain results on statistics relating to this Catalan structure. We find results on statistics for the other permutation classes as well.

In the remainder of this introduction, we lay out some definitions which we will use throughout this thesis, describe the history of the relevant problems, and give an overview of our main results.

1.1.1 Asymptotics

Throughout this thesis we use $f(n) \sim g(n)$ to denote

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1\,.$$

We use $f(n) = O(g(n))$ to mean that there exists a constant M and an integer N such that

$$|f(n)| \leq\ M\,|g(n)| \quad \text{ for all } \ n > N.$$

Also, $f(n) = \Theta(g(n))$ denotes that $f(n) = O(g(n))$ and $g(n) = O(f(n))$. Similarly, $f(n) = o(g(n))$ is defined by

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0\,.$$

Recall Stirling's formula

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

We use C_n to denote the n-th Catalan number:

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \ \text{ and } \ C_n \sim \frac{4^n}{\sqrt{\pi}n^{\frac{3}{2}}}\,.$$

1.1.2 Pattern avoidance

We define a permutation σ of length n to be an ordering of the integers from 1 to n, inclusive. We denote the set of all permutations of length n by S_n. Let n and m be positive integers with $m \leq n$, and let

$$\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(n)) \in S_n,$$

and *pattern* $\tau = (\tau(1), \tau(2), \ldots, \tau(m)) \in S_m$. We say that σ *contains* τ if there exist indices $i_1 < i_2 < \ldots < i_m$ such that $(\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_m))$ is in the same relative order as $(\tau(1), \tau(2), \ldots, \tau(m))$. If σ does not contain τ then we say σ is τ-*avoiding*, or *avoiding pattern* τ. In this chapter we use only $\tau \in S_3$; to simplify the notation we use **123** and **132** to denote patterns $(1, 2, 3)$ or $(1, 3, 2)$, respectively. For example, $\sigma = (2, 4, 5, 1, 3)$ contains **132**, since the subsequence $(\sigma(1), \sigma(2), \sigma(5)) = (2, 4, 3)$ has the same relative order as $(1, 3, 2)$. However, $\sigma = (5, 3, 4, 1, 2)$ is **132**-avoiding.

We define two operations on patterns to help with notation. Let

$$\tau = (\tau_1, \ldots, \tau_n) \in S_n \,.$$

We say the *reverse* of τ is $\tau^R = (\tau_n, \ldots, \tau_1)$. We say the *complement* of τ is $\tau^C = (n + 1 - \tau_1, \ldots, n + 1 - \tau_n)$. Observe that $(\tau^R)^C = (\tau^C)^R = (n + 1 - \tau_n, \ldots, n + 1 - \tau_1)$. We define the *reverse-complement* of τ as $\tau^{RC} = (\tau^C)^R$. As an example, we see that $(1, 3, 4, 2)^R = (2, 4, 3, 1), (1, 3, 4, 2)^C = (4, 2, 1, 3)$, and $(1, 3, 4, 2)^{RC} = (3, 1, 2, 4)$.

If $S = \{\sigma, \tau, \pi, \ldots\} \subset S_n$, then $S^R = \{\sigma^R, \tau^R, \pi^R, \ldots\}$. The sets S^C and S^{RC} are defined similarly.

Given a permutation $\sigma \in S_n$, we say k is a fixed point of σ if $\sigma(k) = k$. We use $\mathrm{fp}_n(\sigma)$ to denote the number of fixed points of σ.

1.1.3 History

As mentioned above, the Catalan numbers were first studied by Euler, in the mid-1750's. Various researchers discovered them in connection with different mathematical objects, including Segner, Liouville, Catalan, and Cayley. The history of the subject is recounted and detailed by Pak in [P3]. In 1915, Percy MacMahon [Mac] enumerated permutations which avoid **123**, and realized that

$\mathcal{S}_n(\mathbf{123}) = C_n$. While Catalan numbers continued to be of interest, the concept of analyzing pattern-avoidance did not become widespread until the latter part of the 20th century. In 1965, Donald Knuth [Knu] showed that $\mathcal{S}_n(\mathbf{231}) = C_n$, in the context of stack-sortability in computer science. The field of pattern-avoidance did not gain traction until 1985, with Simion and Schmidt's paper [SS], in which they define "pattern", and enumerate $\mathcal{S}_n(A)$ for all $A \subset S_3$.

From this point forward, pattern-avoidance became a hot topic, with many extensions, generalizations, and new results. We mention the results most closely related to our research; the history and current state of the subject is very thoroughly explained by Kitaev [Kit]. While MacMahon and Knuth together showed that $\mathcal{S}_n(\tau) = C_n$ for every pattern $\tau \in S_3$, this was not the case for S_4. In fact, for $n \geq 7$, we have $|\mathcal{S}_n(\mathbf{1342})| < |\mathcal{S}_n(\mathbf{1234})| < |\mathcal{S}_n(\mathbf{1324})|$. Permutations avoiding **1342** were enumerated by Bóna [B4], and $\mathcal{S}_n(\mathbf{1234})$ by Gessel [Ges], while $|\mathcal{S}_n(1324)|$ is still open. Even the asymptotic growth rate of $|\mathcal{S}_n(1324)|$ is not known at this point; the best-known upper and lower bounds on the asymptotic growth rate of were proved by Bóna [B5] and Bevan [Bev], respectively. In a recent result, Conway and Guttmann [CG] estimate the asymptotic behavior with even more accuracy.

There are currently over 200 examples of combinatorial objects enumerated by the Catalan numbers [S2]. Another extensive area of study is to analyze statistics on these objects [B2, S1], including both their probabilistic and asymptotic behavior [Drm, FS].

1.1.4 Results

In Chapter 1, we consider $\mathcal{S}_n(\tau)$, for $\tau \in S_3$. We view permutations in these classes as 0-1 matrices, and we analyze the probability of a given permutation having a 1 in a given region of the matrix. We let $n \to \infty$, and analyze the limiting behavior of the probability of $\sigma(an) = bn$ for all values of $a, b \in [0,1]$. For $\tau = \mathbf{123}$ let $P_n(an, bn)$ be the number of permutations $\sigma \in \mathcal{S}_n(\tau)$ with $\sigma(j) = k$. For $\tau = \mathbf{132}$, let $Q_n(an, bn)$ denote the analogous quantity. On this first-order level of asymptotic behavior, we see that both

$$\frac{P_n(an, bn)}{C_n} < \varepsilon^n \quad \text{and} \quad \frac{Q_n(an, bn)}{C_n} < \varepsilon^n,$$

for $a + b < 1$. For $a + b > 1$, our two permutation classes exhibit different behavior. The differences in the two limit shapes help explain why there are numerous bijections between $\mathcal{S}_n(\mathbf{123})$ and $\mathcal{S}_n(\mathbf{132})$.

We also analyze the behavior of the fixed point statistic for permutations in $\mathcal{S}_n(\tau)$. Elizalde calculated the expected number of fixed points for each of these classes [E1]. We extend his results by calculating the probability of having fixed points in given positions in our matrices.

In Chapters 2 through 5, we consider $\mathcal{S}_n(A)$ for various sets A, and do the same sort of analysis. Chapter 2 involves sets $A \subset S_3$ with $|A| = 2$. The enumeration for these sets was completed by Simion and Schmidt [SS], and is either $|\mathcal{S}_n(A)| = 2^{n-1}$ or $\frac{n}{2} + 1$. The limit shapes for most classes have exponentially small probabilities away from one diagonal, as in chapter 1. One class, however, has limiting behavior which forms a V-shape, with the only non-exponential decay occurring at $a + 2b = 1$ and $2b - a = 1$.

In Chapter 3, we analyze sets $B \subset S_3$ with $|B| = 3$. Enumeration is either F_{n+1} or n depending on which set B we choose, where F_n is the n-th Fibonacci number. The limit shapes are not particularly exciting in this section, though we include it for thoroughness.

In Chapter 4 and 5, we analyze $\mathcal{S}_n(\tau, \rho)$ where $\tau \in S_3$ and $\rho \in S_4$. We calculate in detail the limiting shapes and expected fixed point behavior for two specific classes of permutations. In Chapter 4, with $\tau = \mathbf{123}, \rho = \mathbf{3412}$, we find that the limiting shape is exponentially low everywhere except where $a + 2b =$

$1, a+2b=2, 2a+b=1$, or $2a+b=2$. The shape is similar to that of $\mathcal{S}_n(\tau)$, as described in Chapter 1, though the distance from the diagonal where $a+b=1$ to the regions of highest probability remains constant as $n \to \infty$, whereas in $\mathcal{S}_n(\tau)$ this distance tends to 0 as n grows.

In Chapter 5, we consider $\tau = \mathbf{132}$ and $\rho = \mathbf{4231}$. Here, the limiting shape is only exponentially small in regions of measure less than 1, which matches the behavior of $\mathcal{S}_n(\mathbf{132})$. Here, however, the remainder of the square exhibits limiting behavior on the order of $\frac{c}{n}$, for $c = 1, 2$, or 3 depending on the values of a and b. It is not immediately clear why the relative heights of these regions should be so nicely distributed.

In Chapter 6, we consider the problem of calculating $L_k(n)$, the number of positive integer solutions to $x_1x_2 + x_2x_3 + \ldots + x_kx_{k+1} = n$, for fixed k. Somewhat surprisingly, formulas for this function are only known up to $k = 5$. Previously, the proofs of expressions for $L_4(n)$ and $L_5(n)$ involved manipulating generating functions. We give new combinatorial proofs of the expressions for $L_4(n)$ and $L_5(n)$.

In each of the following chapters, we include a shorter introduction which both presents the background and motivation of the current subject in more detail and also states our results.

CHAPTER 2

Avoiding a single pattern of length three

Introduction

The *Catalan numbers* is one of the most celebrated integer sequences, so much so that it is hard to overstate their importance and applicability. In the words of Manuel Kauers and Peter Paule, "It is not exaggerated to say that the Catalan numbers are the most prominent sequence in combinatorics" [KP] Richard Stanley called them "the most special" and his "favorite number sequence" [Kim]. To quote Martin Gardner, "they have the delightful propensity for popping up unexpectedly, particularly in combinatorial problems" [Gar]. In fact, Henry Gould's bibliography [Gou] lists over 450 papers on the subject, with many more in recent years.

Just as there are many combinatorial interpretations of Catalan numbers [S1, Exc. 6.19] (see also [P2, Slo, S2]), there are numerous results on statistics of various such interpretations (see e.g. [B1, S1]), as well as their probabilistic and asymptotic behavior (see [Drm, FS]). The latter results usually come in two flavors. First, one can study the probability distribution of statistics, such as the expectation, the standard deviation and higher moments. The approach we favor is to define the shape of a large random object, which can be then analyzed by analytic means (see e.g. [A2, Ver, VK]). Such objects then contain information about a number of statistics under one roof.

In this chapter we study the set $\mathcal{S}_n(\pi)$ of permutations $\sigma \in S_n$ *avoiding a pattern* π. This study was initiated by Percy MacMahon and Don Knuth, who showed that the size of $\mathcal{S}_n(\pi)$ is the Catalan number C_n, for all permutations $\pi \in S_3$ [Knu, Mac]. These results opened a way to a large area of study, with numerous connections to other fields and applications [Kit] (see also Subsection 2.9.2).

We concentrate on two classical patterns, the **123**- and **132**-avoiding permutations. Natural symmetries imply that other patterns in S_3 are equinumerous with these two patterns. We view permutations as 0-1 matrices, which we average, scale to fit a unit square, and study the asymptotic behavior of the resulting family of distributions. Perhaps surprisingly, behavior of these two patterns is similar on a small scale (linear in n), with random permutations approximating the reverse identity permutation $(n, n-1, \ldots, 1)$. However, on a larger scale (roughly, on the order n^α away from the diagonal), the asymptotics of shapes of random permutations in $\mathcal{S}_n(123)$ and $\mathcal{S}_n(132)$, are substantially different. This explains, perhaps, why there are at least nine different bijections between two sets, all with different properties, and none truly "ultimate" or "from the book" (see Subsection 2.9.3).

Our results are rather technical and contain detailed information about the random pattern avoiding permutations, on both the small and large scale. We exhibit several regimes (or "phases"), where the asymptotics are unchanged, and painstakingly compute the precise limits, both inside these regimes and at the phase transitions. Qualitatively, for **123**-avoiding permutations, our results are somewhat unsurprising, and can be explained by the limit shape results on the *brownian excursion* (see Subsection 2.9.7); still, our results go far beyond what was known. However, for the **132**-avoiding permutations, our results are extremely unusual, and have yet to be explained even on a qualitative level (see Subsection 2.9.8).

The rest of the chapter is structured as follows. In the next section we first present examples and calculations which then illustrate the "big picture" of our results. In Section 2.2 we give formal definitions of our matrix distributions and state basic observations on their behavior. We state the main results in Section 2.3, in a series of six theorems of increasing complexity, for the shape of random permutations in $\mathcal{S}_n(123)$ and $\mathcal{S}_n(132)$. Sections 2.4 and 2.5 contain proofs of the theorems. In the next three sections (sections 2.6, 2.7 and 2.8), we give a long series of corollaries, deriving the distributions for the positions of 1 and n, the number and location of fixed points, and the generalized rank. We conclude with final remarks and open problems (Section 2.9).

2.1 The big picture

In this section we attempt to give a casual description of our results, which basically makes this the second, technical part of the introduction.[1]

2.1.1 The setup

Let $P_n(j,k)$ and $Q_n(j,k)$ be the number of **123**- and **132**-avoiding permutations, respectively, of size n, that have j in the k-th position. These are the main quantities which we study in this chapter.

There are two ways to think of $P_n(\cdot,\cdot)$ and $Q_n(\cdot,\cdot)$. First, we can think of these as families of probability distributions

$$\frac{1}{C_n} P_n(j,\cdot)\,, \quad \frac{1}{C_n} P_n(\cdot,k)\,, \quad \frac{1}{C_n} Q_n(j,\cdot)\,, \quad \text{and} \quad \frac{1}{C_n} Q_n(\cdot,k)\,.$$

In this setting, we find the asymptotic behavior of these distributions, where they are concentrated and the tail asymptotics; we also find exactly how they depend

[1]Here and always when in doubt, we follow Gian-Carlo Rota's advice on how to write an Introduction [Rota].

on parameters j and k.

Alternatively, one can think of $P_n(\cdot,\cdot)$ and $Q_n(\cdot,\cdot)$ as single objects, which we can view as a bistochastic matrices:

$$\mathrm{P}_n = \frac{1}{C_n} \sum_{\sigma \in \mathcal{S}(123)} M(\sigma), \qquad \mathrm{Q}_n = \frac{1}{C_n} \sum_{\sigma \in \mathcal{S}(132)} M(\sigma),$$

where $M(\sigma)$ is a permutation matrix of $\sigma \in S_n$, defined so that

$$M(\sigma)_{jk} := \begin{cases} 1 & \sigma(j) = k \\ 0 & \sigma(j) \neq k. \end{cases}$$

This approach is equivalent to the first, but more conceptual and visually transparent, since both P_n and Q_n have nice geometric asymptotic behavior when $n \to \infty$. See Subsection 2.9.1 for more on this difference.

Let us present the "big picture" of our results. Roughly, we show that both matrices P_n and Q_n are very small for (j,k) sufficiently far away from the *anti-diagonal*

$$\Delta = \{(j,k) \mid j+k = n+1\},$$

and from the lower right corner (n,n) in the case of Q_n. However, already on the next level of detail there are large differences: P_n is exponentially small away from the anti-diagonal, while Q_n is exponentially small only above Δ, and decreases at a rate $\Theta(n^{-3/2})$ on squares below Δ.

At the next level of detail, we look inside the "phase transition", that is what happens when (j,k) are near Δ. It turns out, matrix P_n maximizes at distance $\Theta(\sqrt{n})$ away from Δ, where the values scale as $\Theta(n^{-1/2})$, i.e. much greater than the average $1/n$. On the other hand, on the anti-diagonal Δ, the values of P_n scale as $\Theta(n^{-3/2})$, i.e. below the average. A similar, but much more complicated phenomenon happens for Q_n. Here the "phase transition" splits into several phases, with different asymptotics for rate of decrease, depending on how the distance from (j,k) to Δ relates to $\Theta(\sqrt{n})$ and $\Theta(n^{3/8})$ (see Section 2.3).

At an even greater level of detail, we obtain exact asymptotic constants on the asymptotic behavior of P_n and Q_n, not just the rate of decrease. For example, we consider P_n at distance $\Theta(n^{1/2+\varepsilon})$ from Δ, and show that P_n is exponentially small for all $\varepsilon > 0$. We also show that below Δ, the constant term implied by the Θ notation in the rate $\Theta(n^{-3/2})$ of decrease of Q_n, is itself decreasing until "midpoint" distance $n/2$ from Δ, and is increasing beyond that, in a symmetric fashion.

Unfortunately, the level of technical detail of our theorems is a bit overwhelming to give a casual description; they are formally presented in Section 2.3, and proved in sections 2.4 and 2.5. The proofs rely on explicit formulas for $P_n(j,k)$ and $Q_n(j,k)$ which we give in Lemmas 2.4.2 and 2.5.3. These are proved by direct combinatorial arguments. From that point on, the proofs of the asymptotic behavior of P_n and Q_n are analytic and use no tools beyond Stirling's formula and the Analysis of Special Functions.

2.1.2 Numerical examples

First, in Figures 8.5 and 8.6 we compute the graphs of P_{250} and Q_{250} (see the Appendix). Informally, we name the diagonal mid-section of the graph of P_{250} the *canoe*; this is the section of the graph where the values are the largest. Similarly, we use the *wall* for the corresponding mid-section of Q_{250} minus the corner spike. The close-up views of the canoe and the wall are given in Figures 8.7 and 8.8, respectively. Note that both graphs here are quite smooth, since $n = 250$ is large enough to see the the limit shape, with $C_{250} \approx 4.65 \times 10^{146}$, and every pixelated value is computed exactly rather than approximated.

Observe that the canoe is symmetric across both the main and the anti-diagonal, and contains the high spikes in the corners of the canoe, both of which reach $1/4$. Similarly, the wall is symmetric with respect to the main diagonal, and has three spikes which reach 1/4. These results are straightforward and proved in the next section.

To see that the canoe is very thin, we compare graphs of the diagonal sections $P_n(k,k)/C_n$ for $n = 62, 125, 250$ and 500, as k varies from 90 to 160 (see Figures 8.1 to 8.4 in the Appendix). Observe that as n increases, the height of the canoes decreases, and so does the width and "bottom". As we mentioned earlier, these three scale as $\Theta(n^{-1/2})$, $\Theta(n^{-1/2})$, and $\Theta(n^{-3/2})$, respectively. Note also the sharp transition to a near flat part outside of the canoe; this is explained by an exponential decrease mentioned earlier. The exact statements of these results are given in Section 2.3.

Now, it is perhaps not clear from Figure 8.7 that the wall bends to the left. To see this clearly, we overlap two graphs in Figure 2.1. Note that the peak of $P_{250}(k,k)$ is roughly in the same place of $Q_{250}(k,k)$, i.e. well to the left of the midpoint at 125. The exact computations show that the maxima occur at 118 and at 119, respectively. Note also that $Q_{250}(k,k)$ has a sharp phase transition on the left, with an exponential decay, but only a polynomial decay on the right.

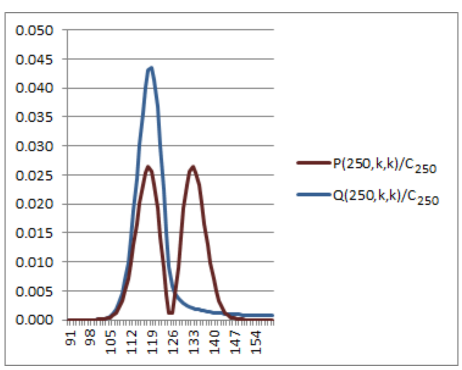


Figure 2.1: Comparison of $P_{250}(k,k)/C_{250}$ and $Q_{250}(k,k)/C_{250}$.

2.1.3 Applications

We mention only one statistic which was heavily studied in previous years, and which has a nice geometric meaning. Permutation σ is said to have a *fixed point* at k if $\sigma(k) = k$. Denote by $\mathrm{fp}_n(\sigma)$ the number of fixed points in σ.

For random permutations $\sigma \in S_n$, the distribution of fp is a classical problem; for example $\mathbf{E}[\mathrm{fp}] = 1$ for all n. In a fascinating paper [RSZ], the authors prove that the distribution of fp on $\mathcal{S}(321)$ and on $\mathcal{S}(132)$ coincide (see also [E2, EP]). Curiously, Elizalde used the generating function technique to prove that $\mathbf{E}[\mathrm{fp}] = 1$ in both cases, for all n, see [E1]. He also finds closed form g.f. formulas for the remaining two patterns (up to symmetry).

Now, the graphs of $P_n(k,k)/C_n$ and $Q_n(k,k)/C_n$ discussed above, give the expectations that k is a fixed point in a pattern avoiding permutation. In other words, fixed points of random permutations in $\mathcal{S}_n(123)$ and $\mathcal{S}_n(132)$ are concentrated under the canoe and under the wall, respectively. Indeed, our results immediately imply that w.h.p. they lie near $n/2$ in both cases. For random permutations in $\mathcal{S}_n(321)$ and $\mathcal{S}_n(231)$, the fixed points lie in the ends of the canoe and near the corners of the wall, respectively. In Section 2.7, we qualify all these statements and as a show of force obtain sharp asymptotics for $\mathbf{E}[\mathrm{fp}]$ in all cases, both known and new.

2.2 Basic observations

Recall that $\mathcal{S}_n(\pi)$ is the set of π-avoiding permutations in S_n. For the case of patterns of length 3, it is known that regardless of the pattern $\pi \in S_3$, we have $|\mathcal{S}_n(\pi)| = C_n$.

Theorem 2.2.1 (MacMahon, Knuth)**.** *For all $\pi \in S_3$, we have $|S_n(\pi)| = C_n$.*

While the equalities $|\mathcal{S}_n(132)| = |\mathcal{S}_n(231)| = |\mathcal{S}_n(213)| = |\mathcal{S}_n(312)|$ and $|\mathcal{S}_n(123)| = |\mathcal{S}_n(321)|$ are straightforward, the fact that $|\mathcal{S}_n(132)| = |\mathcal{S}_n(123)|$ is more involved.

2.2.1 Symmetries

Recall that $P_n(j,k)$ and $Q_n(j,k)$ denote the number of permutations in $\mathcal{S}_n(123)$ and $\mathcal{S}_n(132)$, respectively, of size n that have j in the k-th position. In this section we discuss the symmetries of such permutations.

Proposition 2.2.2. *For all n, j, k positive integers such that $1 \le j, k \le n$, we have $P_n(j,k) = P_n(k,j)$ and $Q_n(j,k) = Q_n(k,j)$. Also, $P_n(j,k) = P_n(n+1-k, n+1-j)$, for all k, j as above.*

The proposition implies that we can interpret $P_n(j,k)$ as either

$$|\{\sigma \in \mathcal{S}_n(123)\,\text{s.t.}\,\sigma(j) = k\}| \quad \text{or} \quad |\{\sigma \in \mathcal{S}_n(123)\,\text{s.t.}\,\sigma(k) = j\}|,$$

and we use both formulas throughout the chapter. The analogous statement holds with $Q_n(j,k)$ as well. Note, however, that $Q_n(j,k)$ is not necessarily equal to $Q_n(n+1-k, n+1-j)$; for example, $Q_3(1,2) = 2$ but $Q_3(2,3) = 1$. In other words, there is no natural analogue of the second part of the proposition for $Q_n(j,k)$, even asymptotically, as our results will show in the next section.

Proof. The best way to see this is to consider permutation matrices. Observe that $P_n(j,k)$ counts the number of permutation matrices $A = (a_{rs})$ which have $a_{kj} = 1$, but which have no row indices $i_1 < i_2 < i_3$ nor column indices $j_1 < j_2 < j_3$ such that $a_{i_1j_1} = a_{i_2j_2} = a_{i_3j_3} = 1$. If A is such a matrix, then $B = A^T$ is also a matrix which satisfies the conditions for $P_n(k,j)$, since $b_{jk} = 1$ and B has no indices which lead to the pattern **123**. This transpose map is clearly a bijection, so we have $P_n(j,k) = P_n(k,j)$.

Similarly, since any **132** pattern in a permutation matrix A will be preserved in $B = A^T$, we have $Q_n(j,k) = Q_n(k,j)$. Finally, observe that $P_n(j,k) = P_n(n+1-k, n+1-j)$, since σ is **123**-avoiding if and only if $\rho = (n+1-\sigma(n), n+1-\sigma(n-1), \ldots, n+1-\sigma(1))$ is **123**-avoiding. $\square$

2.2.2 Maxima and minima

Here we find all maxima and minima of matrices $P_n(\cdot,\cdot)$ and $Q_n(\cdot,\cdot)$. We separate the results into two propositions.

Proposition 2.2.3. *For all $n \geq 3$, the value of $P_n(j,k)$ is minimized when $(j,k) = (1,1)$ or (n,n). Similarly, $P_n(j,k)$ is maximized when*

$$(j,k) = (1,n), (1,n-1), (2,n) \quad \text{or } (n-1,1), (n,1), (n,2).$$

Proof. For any n, the only $\sigma \in \mathcal{S}_n(123)$ with $\sigma(1) = 1$ is $\sigma = (1, n, n-1, \ldots, 3, 2)$. This implies that $P_n(1,1) = 1$. Similarly, $P_n(n,n) = 1$, since the only such permutation is $(n-1, n-2, \ldots, 2, 1, n)$.

For every $j, k \leq n$, the maximum possible value of $P_n(j,k)$ is C_{n-1}, since the numbers from 1 to n excluding j must be **123**-avoiding themselves. Let us show that

$$P_n(1,n) = P_n(2,n) = P_n(1,n-1) = C_{n-1},$$

proving that this maximum is in fact achieved by the above values of j and k.

If $\sigma \in \mathcal{S}_n(123)$ has $\sigma(1) = n$, then n cannot be part of a **123**-pattern, since it is the highest number but must be the smallest number in the pattern. Therefore, any $\sigma \in \mathcal{S}_{n-1}(123)$ can be extended to a permutation $\tau \in \mathcal{S}_n(123)$ in the following way: let $\tau(1) = n$, and let $\tau(i) = \sigma(i-1)$ for $2 \leq i \leq n$. Since $|\mathcal{S}_{n-1}(123)| = C_{n-1}$, we have $P_n(1,n) = C_{n-1}$. Similarly, if $\sigma(2) = n$, then n cannot be part of a **123**-pattern, so $P_n(2,n) = C_{n-1}$. The same is true if $\sigma(1) = (n-1)$, so

$P_n(1, n-1) = C_{n-1}$. By Proposition 2.2.2, we also have

$$P_n(n-1, 1) = P_n(n, 1) = P_n(n, 2) = C_{n-1},$$

as desired. $\square$

Proposition 2.2.4. *For all $n \geq 4$, the value of $Q_n(j,k)$ is minimized when $(j,k) = (1,1)$. Similarly, $Q_n(j,k)$ is maximized when*

$$(j,k) = (1,n), (1,n-1), (n-1,1), (n,1), \quad \textit{or} \quad (n,n).$$

Proof. For $(j,k) = (1,1)$, the only **132**-avoiding permutation is

$$\sigma = (1, 2, 3, \ldots, n-1, n)\,.$$

Therefore, $Q_n(1,1) = 1$ for all n.

For the second part, we use the same reasoning as in Proposition 2.2.3, except for $(j,k) = (n,n)$. For $(j,k) = (n,n)$, we have $Q_n(n,n) = C_{n-1}$ as well, since n in the final position cannot be part of a 132-pattern. Observe that unlike $P_n(2,n)$, $Q_n(2,n) < C_{n-1}$, since $\sigma(2) = n$ requires $\sigma(1) = n-1$, in order to avoid a **132**-pattern. $\square$

2.3 Main results

In this section we present the main results of the chapter.

2.3.1 Shape of 123-avoiding permutations

Let $0 \leq a, b \leq 1, 0 \leq \alpha < 1$, and $c \in \mathbb{R}$ s.t. $c \neq 0$ for $\alpha \neq 0$ be fixed constants. Recall that $P_n(j,k)$ is the number of permutations $\sigma \in \mathcal{S}_n(123)$ with $\sigma(j) = k$.

Define

$$F(a,b,c,\alpha) = \sup\left\{ d \in \mathbb{R}_+ \Big| \lim_{n\to\infty} \frac{n^d P_n(an - cn^\alpha, bn - cn^\alpha)}{C_n} < \infty \right\}$$

$$\text{for } \alpha \neq 0 \text{ or } a + b \neq 1 \,,$$

and

$$F(a,b,c,\alpha) = \sup\left\{ d \in \mathbb{R}_+ \Big| \lim_{n\to\infty} \frac{n^d P_n(an - cn^\alpha + 1, bn - cn^\alpha)}{C_n} < \infty \right\},$$

$$\text{for } \alpha = 0 \text{ and } a + b = 1 \,.$$

Similarly, let

$$L(a,b,c,\alpha) = \lim_{n\to\infty} \frac{n^{F(a,b,c,\alpha)} P_n(an - cn^\alpha, bn - cn^\alpha)}{C_n} \text{ for } \alpha \neq 0 \text{ or } a + b \neq 1,$$

and

$$L(a,b,c,\alpha) = \lim_{n\to\infty} \frac{n^{F(a,b,c,\alpha)} P_n(an - cn^\alpha + 1, bn - cn^\alpha)}{C_n} \text{ for } \alpha = 0 \text{ and } a + b = 1,$$

defined for all a, b, c, α as above, for which $F(a,b,c,\alpha) < \infty$; let L be undefined otherwise.

Theorem 2.3.1. *For all* $0 \leq a, b \leq 1, c \in \mathbb{R}$ *and* $0 \leq \alpha < 1$, *we have*

$$F(a,b,c,\alpha) = \begin{cases} \infty & \text{if } a + b \neq 1\,, \\ \infty & \text{if } a + b = 1,\ c \neq 0,\ \alpha > \frac{1}{2}\,, \\ \frac{3}{2} & \text{if } a + b = 1,\ c = 0\,, \\ \frac{3}{2} - 2\alpha & \text{if } a + b = 1,\ c \neq 0,\ \alpha \leq \frac{1}{2}\,. \end{cases}$$

Here $F(a,b,c,\alpha) = \infty$ means that $P_n(an - cn^\alpha, bn - cn^\alpha) = o(C_n/n^d)$, for all $d > 0$. The following result proves the exponential decay of these probabilities.

Theorem 2.3.2. *Let* $0 \leq a, b \leq 1$ *s.t.* $a + b \neq 1, c \in \mathbb{R}$, *and* $0 < \alpha < 1$. *Then, for* n *large enough, we have*

$$\frac{P_n(an - cn^\alpha, bn - cn^\alpha)}{C_n} < \varepsilon^n,$$

where $\varepsilon = \varepsilon(a, b, c, \alpha)$ is independent of n, and $0 < \varepsilon < 1$. Similarly, let $0 \leq a \leq 1, c \neq 0$, and $\frac{1}{2} < \alpha < 1$. Then, for n large enough, we have

$$\frac{P_n(an - cn^\alpha, (1-a)n - cn^\alpha)}{C_n} < \varepsilon^{n^{2\alpha-1}},$$

where $\varepsilon = \varepsilon(a, c, \alpha)$ is independent of n and $0 < \varepsilon < 1$.

These theorems compare the growth of $P_n(an - cn^\alpha, bn - cn^\alpha)$ to the growth of C_n. Clearly,

$$\sum_{j=1}^{n} P_n(j,k) = |\mathcal{S}_n(123)| = C_n \quad \text{for all } 1 \leq k \leq n \ .$$

Therefore, if $P_n(j, k)$ were constant across all values of j, k between 1 and n, we would have $P_n(j, k) = C_n/n$ for all $1 \leq j, k \leq n$. Theorem 2.3.1 states that for $0 \leq a, b \leq 1$, $a + b \neq 1$, we have $P_n(an, bn) = o(C_n/n^d)$, for every $d \in \mathbb{R}$. For $a + b = 1$, we have $P_n(an, bn) = \Theta(C_n/n^{\frac{3}{2}})$. Theorem 2.3.1 is in fact stating slightly more. When we consider $P_n(an - cn^\alpha, bn - cn^\alpha)$ instead of $P_n(an, bn)$, we have

$$P_n(an - cn^\alpha, bn - cn^\alpha) = \Theta\left(C_n/n^{\frac{3}{2}-2\alpha}\right),$$

for all $\alpha \leq \frac{1}{2}$. This relationship can be seen in Figures 2.2 and 2.3.

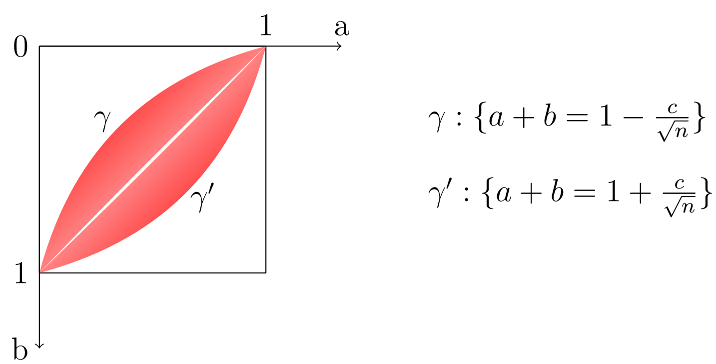
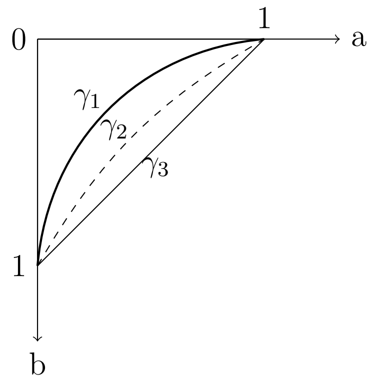


Figure 2.2: Region where $P_n(an, bn) \sim C_n/n^d$ for some d.

In Figure 2.3, on γ_1, we have

$$P_n\left(an - c\sqrt{n}, (1-a)n - c\sqrt{n}\right) = \Theta\left(\frac{C_n}{\sqrt{n}}\right).$$



$$\gamma_1 : \{a + b = 1 - \tfrac{2c}{\sqrt{n}}\}$$

$$\gamma_2 : \{a + b = 1 - \tfrac{2c}{n^{1-\alpha}} \text{ for some } 0 \le \alpha \le \tfrac{1}{2}\}$$

$$\gamma_3 : \{a + b = 1\}$$

Figure 2.3: Region where $P_n(an, bn) \sim C_n/n^d$ for some d.

On γ_2, where

$$a + b = 1 - \frac{2c}{n^{1-\alpha}}\,, \quad \text{for some} \ \ 0 \le \alpha \le \frac{1}{2},$$

we have

$$P_n(an - cn^\alpha, (1-a)n - cn^\alpha) = \Theta\left(\frac{C_n}{n^{\frac{3}{2}-2\alpha}}\right).$$

On γ_3, we have $P_n(an, (1-a)n) = \Theta(C_n/n^{\frac{3}{2}})$. Behavior is symmetric about the line $a + b = 1$.

The following result is a strengthening of Theorem 2.3.1 in a different direction. For a, b, c, and α as above, s.t. $F(a, b, c, \alpha) < \infty$, we calculate the value of $L(a, b, c, \alpha)$.

Theorem 2.3.3. *For all* $0 \le a \le 1, c \in \mathbb{R}$, *and* $0 \le \alpha \le 1/2$, *we have*

$$L(a, 1-a, c, \alpha) = \begin{cases} \xi(a,c) & \textit{if} \ \ c = 0 \ \ \textit{or} \ \ \alpha = 0\,, \\ \eta(a,c) & \textit{if} \ \ c \neq 0 \ \ \textit{and}\ 0 < \alpha < \frac{1}{2}\,, \\ \eta(a,c)\kappa(a,c) & \textit{if} \ \ c \neq 0 \ \ \textit{and}\ \alpha = \frac{1}{2}\,, \end{cases}$$

where

$$\xi(a,c) = \frac{(2c+1)^2}{4\sqrt{\pi}(a(1-a))^{\frac{3}{2}}},\ \eta(a,c) = \frac{c^2}{\sqrt{\pi}(a(1-a))^{\frac{3}{2}}}\ ,$$

$$\textit{and} \ \ \kappa(a,c) = \exp\left[\frac{-c^2}{a(1-a)}\right].$$

Let us note that for $\alpha = 0$ or $c = 0$ as in theorem, we actually evaluate

$$P_n(an - cn^\alpha, (1-a)n - cn^\alpha + 1) \quad \text{rather than} \quad P_n(an - cn^\alpha, (1-a)n - cn^\alpha).$$

We do this in order to ensure that we truly measure the distance away from the anti-diagonal where $j + k = n + 1$. This change only affects the asymptotic behavior of $P_n(\cdot,\cdot)$ when $\alpha = 0$ or $c = 0$.

2.3.2 Shape of 132-avoiding permutations

Recall that $Q_n(j,k)$ is the number of permutations $\sigma \in \mathcal{S}_n(132)$ with $\sigma(j) = k$. Let a, b, c and α be defined as in Theorem 2.3.1. Define

$$T(a,b,c,\alpha) = \sup\left\{ d \in \mathbb{R}_+ \,\Big|\, \lim_{n\to\infty} \frac{n^d Q_n(an - cn^\alpha, bn - cn^\alpha)}{C_n} < \infty \right\}.$$

Let

$$M(a,b,c,\alpha) = \lim_{n\to\infty} \frac{n^{T(a,b,c,\alpha)} Q_n(an - cn^\alpha, bn - cn^\alpha)}{C_n},$$

defined for all a, b, c, α as above for which $T(a,b,c,\alpha) < \infty$; M is undefined otherwise.

Theorem 2.3.4. *For $0 \le a, b \le 1$, $c \in \mathbb{Z}$ and $\alpha \ge 0$, we have* $T(a,b,c,\alpha) =$

$$= \begin{cases} \infty & \text{if } 0 \le a+b < 1, \\ \frac{3}{2} & \text{if } 1 < a+b < 2, \\ 0 & \text{if } a=b=1,\ \alpha = 0, \\ \frac{3}{4} & \text{if } a+b=1,\ c=0, \end{cases} \qquad \begin{cases} \frac{3}{2}\alpha & \text{if } a=b=1,\ 0<\alpha<1,\ c\neq 0, \\ \infty & \text{if } a+b=1,\ \frac{1}{2}<\alpha<1,\ c>0, \\ \frac{3}{4} & \text{if } a+b=1,\ 0\le\alpha\le\frac{3}{8},\ c\neq 0, \\ \frac{3}{4} & \text{if } a+b=1,\ \frac{3}{8}\le\alpha\le\frac{1}{2},\ c<0, \\ \frac{3}{2}\alpha & \text{if } a+b=1,\ \frac{1}{2}<\alpha<1,\ c<0, \\ \frac{3}{2}-2\alpha & \text{if } a+b=1,\ \frac{3}{8}\le\alpha\le\frac{1}{2},\ c>0. \end{cases}$$

As in Theorem 2.3.1, here $T(a,b,c,\alpha) = \infty$ means that $Q_n(an - cn^\alpha, bn - cn^\alpha) = o(C_n/n^d)$, for all $d > 0$. The following result proves exponential decay of these probabilities (cf. Theorem 2.3.2.)

Theorem 2.3.5. *Let $0 \leq a, b < 1$ such that $a + b < 1, c \neq 0$, and $0 < \alpha < 1$. Then, for n large enough, we have*

$$\frac{Q_n(an - cn^\alpha, bn - cn^\alpha)}{C_n} < \varepsilon^n,$$

where $\varepsilon = \varepsilon(a, b, c, \alpha)$ is independent of n, and $0 < \varepsilon < 1$. Similarly, let $0 \leq a \leq 1, 0 < c$, and $\frac{1}{2} < \alpha < 1$. Then, for n large enough, we have

$$\frac{Q_n(an - cn^\alpha, (1-a)n - cn^\alpha)}{C_n} < \varepsilon^{n^{2\alpha-1}},$$

where $\varepsilon = \varepsilon(a, c, \alpha)$ is independent of n, and $0 < \varepsilon < 1$.

The above theorems compare the relative growth rates of $Q_n(i, j)$ and C_n, as $n \to \infty$. Theorem 2.3.4 states that for $a + b < 1, Q_n(an, bn) = o(C_n/n^d)$ for all $d > 0$. For $1 < a + b < 2$, we have

$$Q_n(an, bn) = \Theta\left(\frac{C_n}{n^{\frac{3}{2}}}\right).$$

$Q_n(an, bn)$ is the largest when $a = b = 1$ or when $a + b = 1$. The true behavior of $Q_n(i, j)$ described in Theorems 2.3.4 and 2.3.6 takes the second order terms of i and j into account. In fact we have that

$$Q_n(n - cn^\alpha, n - cn^\alpha) = \Theta\left(\frac{C_n}{n^{\frac{3}{2}\alpha}}\right) \qquad \text{when } \alpha \leq \frac{1}{2}.$$

For $a + b = 1$, the asymptotic behavior of $Q_n(an - cn^\alpha, bn - cn^\alpha)$ varies through several regimes as α varies between 0 and 1, and c varies between positive and negative numbers. This relationship is illustrated in Figures 2.4 and 2.5.

In Figure 2.5, on the curve γ_1, we have

$$Q_n(an - c\sqrt{n}, (1-a)n - c\sqrt{n}) = \Theta\left(\frac{C_n}{n^{\frac{1}{2}}}\right).$$

Similarly, on γ_2, we have

$$Q_n(an - cn^\alpha, (1-a)n - cn^\alpha) = \Theta\left(\frac{C_n}{n^{\frac{3}{2}-2\alpha}}\right).$$

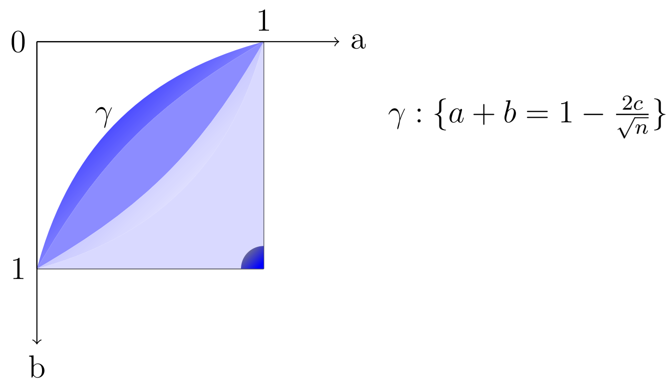


Figure 2.4: Region where $Q_n(an, bn) \sim C_n/n^d$ for some d.

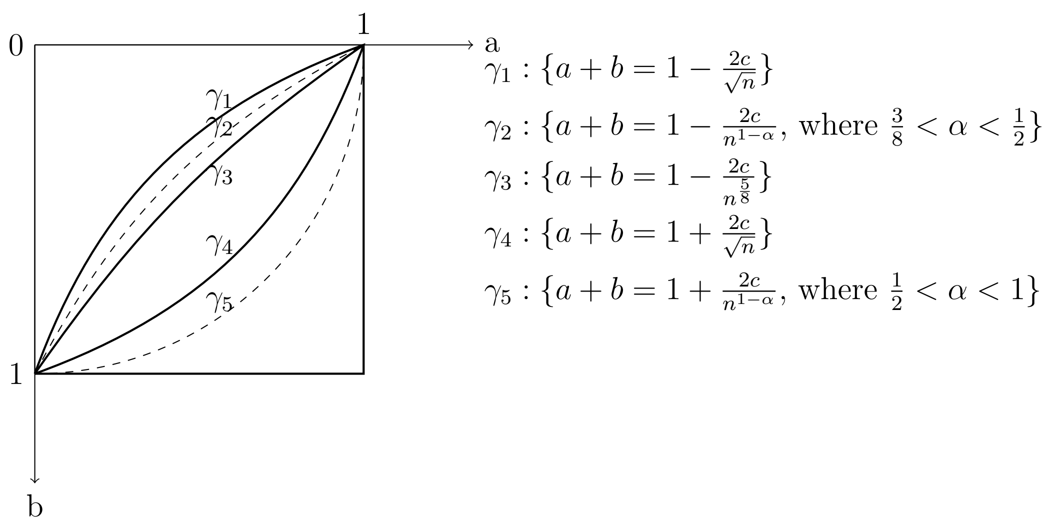


Figure 2.5: Region where $Q_n(an, bn) \sim C_n/n^d$ for some d.

In the space between γ_3 and γ_4, we have

$$Q_n(an + k, (1 - a)n + k) = \Theta\left(\frac{C_n}{n^{\frac{3}{4}}}\right),$$

where $-cn^{\frac{3}{4}} \leq k \leq cn^{\frac{1}{2}}$. Finally, on γ_5, we have

$$Q_n(an + cn^{\alpha}, (1 - a)n + cn^{\alpha}) = \Theta\left(\frac{C_n}{n^{\frac{3}{2}\alpha}}\right).$$

As in Theorem 2.3.3, the following result strengthens Theorem 2.3.4 in a different direction. For a, b, c, α where $T(a, b, c, \alpha) < \infty$, we calculate the value of $M(a, b, c, \alpha)$.

Theorem 2.3.6. *For a, b, c, α as above, we have*

$$M(a,b,c,\alpha) = \begin{cases} v(a,b) & \text{if } \ 1 < a+b < 2, \\ w(c) & \text{if } \ a=b=1,\ 0<\alpha<1, \\ u(c) & \text{if } \ a=b=1,\ c \geq 0,\ \alpha = 0, \\ w(c) & \text{if } \ a+b=1,\ c<0,\ \frac{1}{2} < \alpha, \\ x(a,c) & \text{if } \ a+b=1,\ c<0,\ \alpha=\frac{1}{2}, \end{cases}$$

and

$$M(a,b,c,\alpha) = \begin{cases} z(a) & \text{if } \ a+b=1,\ 0 \leq \alpha < \frac{3}{8}, \\ z(a) & \text{if } \ a+b=1,\ c<0,\ \frac{3}{8} \leq \alpha < \frac{1}{2}, \\ z(a)+y(a,c) & \text{if } \ a+b=1,\ c>0,\ \alpha=\frac{3}{8}, \\ y(a,c) & \text{if } \ a+b=1,\ c>0,\ \frac{3}{8} < \alpha < \frac{1}{2}, \\ y(a,c)\,\kappa(a,c) & \text{if } \ a+b=1,\ c>0,\ \alpha=\frac{1}{2}, \end{cases}$$

where

$$u(c) = \sum_{s=0}^{c} \left(\frac{s+1}{2c+1-s}\right)^2 \binom{2c+1-s}{c+1}^2 4^{s-2c-1}, \quad y(a,c) = \frac{2c^2}{\sqrt{\pi} a^{\frac{3}{2}} (1-a)^{\frac{3}{2}}},$$

$$w(c) = \frac{1}{2^{\frac{5}{2}}\, c^{\frac{3}{2}} \sqrt{\pi}}, \qquad x(a,c) = \frac{1}{4\pi a^{\frac{3}{2}}(1-a)^{\frac{3}{2}}} \int_0^\infty \frac{s^2}{(s+2c)^{\frac{3}{2}}} \exp\left[\frac{-s^2}{4a(1-a)}\right] ds,$$

$$v(a,b) = \frac{1}{2\sqrt{\pi}(2-a-b)^{\frac{3}{2}}(a+b-1)^{\frac{3}{2}}}, \qquad z(a) = \frac{\Gamma(\frac{3}{4})}{2^{\frac{3}{2}} \pi a^{\frac{3}{4}} (1-a)^{\frac{3}{4}}},$$

and $\kappa(a,c)$ is defined as in Theorem 2.3.3.

Observe that for $c = 0$ or $\alpha = 0$, values $Q_n(an - cn^\alpha, (1-a)n - cn^\alpha)$ behave the same asymptotically as $Q_n(an - cn^\alpha, (1-a)n - cn^\alpha + 1)$. We explain this in more detail in Lemma 2.5.8. This is in contrast with the behavior of $P_n(\cdot, \cdot)$, where we need to adjust when on the anti-diagonal. Note also that for $a = b = 1$, $\alpha = 0$ and $c = 0$, we have

$$u(0) = \frac{1}{4} = \lim_{n\to\infty} \frac{Q_n(n,n)}{C_n} = \lim_{n\to\infty} \frac{C_{n-1}}{C_n},$$

which holds since $Q_n(n,n) = C_{n-1}$ given in the proof of Proposition 2.2.4. We prove the theorem in Section 2.5.

2.4 Analysis of 123-avoiding permutations

2.4.1 Combinatorics of Dyck Paths

We say a Dyck path of length $2n$ is a path from $(0,0)$ to $(2n,0)$ in $\mathbb{Z}^2$ consisting of *upsteps* (1,1) and *downsteps* (1,-1) such that the path never goes below the x-axis. We denote by $\mathcal{D}_n$ the set of Dyck paths of length $2n$. We can express a Dyck path $\gamma \in \mathcal{D}_n$ as a word of length $2n$, where u represents an upstep and d represents a downstep.

Recall that $P_n(j,k)$ is the number of permutations $\sigma \in \mathcal{S}_n(123)$ with $\sigma(j) = k$. Let $f(n,k) = P_n(1,k)$ (or $P_n(k,1)$). Let $b(n,k)$ be the number of lattice paths consisting of upsteps and downsteps from $(0,0)$ to $(n+k-2, n-k)$ which stay above the x-axis. Here $b(n,k)$ are the *ballot numbers*, given by

$$b(n,k) = \frac{n-k+1}{n+k-1}\binom{n+k-1}{n}.$$

Lemma 2.4.1. *For all $1 \le k \le n$, we have $f(n,k) = b(n,k)$.*

Proof. We have that $f(n,k)$ counts the number of permutations $\sigma \in \mathcal{S}_n(123)$ such that $\sigma(1) = k$. By the RSK-correspondence (see e.g. [B1, S1]), we have $f(n,k)$ counts the number of Dyck paths $\gamma \in \mathcal{D}_n$ whose final upstep ends at the point $(n+k-1, n+1-k)$. Remove the last upstep from path γ, and all the steps after it. We get a path γ' from $(0,0)$ to $(n+k-2, n-k)$ which remains above the x-axis. These paths are counted by $b(n,k)$, and the map $\gamma \to \gamma'$ is clearly invertible, so $f(n,k) = b(n,k)$, as desired. $\square$

Lemma 2.4.2. *For all* $1 \le j, k \le n$*, we have*

$$P_n(j,k) = b(n-k+1, j)\, b(n-j+1, k), \quad \textit{where} \;\; j+k \le n+1.$$

Similarly, we have

$$P_n(j,k) = b(j, n-k+1)\, b(k, n-j+1), \quad \textit{where} \;\; j+k > n+1.$$

Proof. Let us show that the second case follows from the first case. Suppose $j+k > n+1$. By assuming the first case of the lemma, we have

$$\begin{aligned} P_n(j,k) &= P_n(n+1-j, n+1-k) \\ &= b(n-(n+1-k)+1, n+1-j)\, b(n-(n+1-j)+1, n+1-k) \\ &= b(k, n-j+1)\, b(j, n-k+1), \end{aligned}$$

by Proposition 2.2.2. Therefore, it suffices to prove the lemma for $j+k \le n+1$.

Let $j+k \le n+1$, and let σ be a **123**-avoiding permutation with $\sigma(j) = k$. We use decomposition $\sigma = \tau k \rho$, where

$$\tau = \{\sigma(1), \ldots, \sigma(j-1)\} \;\text{ and }\; \rho = \{\sigma(j+1), \ldots, \sigma(n)\}.$$

We now show that $\sigma(i) > k$, for all $1 \le i < j$. Suppose $\sigma(i) < k$ for some $i < j$. Then there are at most $(j-2)$ numbers $x < j$ with $\sigma(x) > k$. Since $\sigma(j) = k$, in total there are $(n-k)$ numbers y such that $\sigma(y) > k$. Since $j-2 < n-k$, there must be at least one number $z > j$ with $\sigma(z) > k$. However, this gives a **123** pattern consisting of (i, j, z), a contradiction. Therefore, $\sigma(i) > k$, for all $1 \le i < j$.

Consider the values of σ within τ. From above, the values within τ are all greater than k. Given a possible τ, the values within ρ which are greater than k must be in decreasing order, to avoid forming a **123**-pattern starting with k. Therefore, to count possible choices for τ, it suffices to count possible orderings within σ of the numbers x with $k \le x \le n$. The number of such orderings is

$b(n-k+1,j)$, since the smallest number is in the j-th position. Therefore, there are $b(n-k+1,j)$ possible choices for τ.

Now consider the values of σ within $\{k\} \cup \rho$. We have $(n-j+1)$ numbers to order, and $(k-1)$ of them are less than k. Our only restriction on ρ is that we have no **123**-patterns. There are $b(n-j+1,k)$ of these orderings, since the k-th smallest number is in the 1-st position. Therefore, we have $b(n-j+1,k)$ possible choices for ρ.

Once we have chosen τ and ρ, this completely determines the permutation σ. Therefore, there are $b(n-j+1,k)b(n-k+1,j)$ choices of such σ, as desired. $\square$

Example 2.4.3. *Let us compute $P_7(4,3)$, the number of permutations*

$$\sigma \in \mathcal{S}_7(123) \quad \textit{with} \quad \sigma(4)=3\,.$$

*The numbers 1 and 2 must come after 3 in any such permutation, since otherwise a **123** will be created with 3 in the middle. There are*

$$b(7-3+1,4) = \frac{2}{8}\binom{8}{5} = 14$$

ways to order the numbers between 3 and 7, shown here:

$$\begin{array}{ccccccc} (47635) & (54736) & (57436) & (57634) & (64735) & (65437) & (65734) \\ (67435) & (67534) & (74635) & (75436) & (75634) & (76435) & (76534). \end{array}$$

For each of these, there are

$$b(7-4+1,3) = \frac{2}{6}\binom{6}{4} = 5$$

ways to place the numbers between 1 and 3, shown here:

$$(***31*2) \quad (***321*) \quad (***32*1) \quad (***3*12) \quad (***3*21),$$

where the asterisks represent the positions of $4, 5, 6,$ and 7. In total, we have $P_7(4,3) = b(5,4)b(4,3) = (14)(5) = 70$.

2.4.2 Proof of theorems 2.3.1, 2.3.2, and 2.3.3

The proof follows from several lemmas: one technical lemma and one lemma for each case from Theorem 2.3.1.

Let $h : [0,1]^2 \to \mathbb{R}$ be defined as

$$h(a,b) = \frac{(1-a+b)^{(1-a+b)}(1-b+a)^{(1-b+a)}}{a^a(1-a)^{(1-a)}b^b(1-b)^{(1-b)}}.$$

Lemma 2.4.4 (Technical lemma)**.** *We have*

$$h(a,b) \leq 4, \quad \text{for all } \ 0 \leq a, b \leq 1.$$

Moreover, $h(a,b) = 4$ *if and only if* $b = 1-a$.

Proof. Observe that $h(a, 1-a) = 4$. Furthermore, we consider the partial derivatives of h with respect to a and b. We find that h has local maxima at each point where $b = 1-a$. In fact these are the only critical points within $[0,1]^2$. We omit the details.[2] □

Lemma 2.4.5 (First case)**.** *Let* $a, b \in [0,1], c \neq 0$, *and* $0 \leq \alpha < 1$, *such that* $a+b \neq 1$. *Then* $F(a,b,c,\alpha) = \infty$. *Moreover, for* n *sufficiently large, we have* $P_n(an-cn^\alpha, bn-cn^\alpha)/C_n < \varepsilon^n$, *where* ε *is independent of* n *and* $0 < \varepsilon < 1$.

Proof. By lemmas 2.4.1 and 2.4.2, we have

$$\begin{aligned} P_n(an-cn^\alpha, bn-cn^\alpha) &= b(n-(bn-cn^\alpha)+1, an-cn^\alpha)\ b(n-(an-cn^\alpha)+1, bn-cn^\alpha) \\ &= \frac{(n(1-a-b)+2cn^\alpha+2)^2}{n^2(1-b+a)(1-a+b)} \binom{n(1-b+a)}{n-(bn-cn^\alpha)+1} \binom{n(1-a+b)}{n-(an-cn^\alpha)+1}. \end{aligned}$$

Applying Stirling's formula gives

$$P_n(an-cn^\alpha, bn-cn^\alpha) \sim r(n,a,b) \cdot h(a,b)^n,$$

[2] The proof follows similar (and even somewhat simplified) steps as the proof of Lemma 2.5.5.

where $r(n,a,b) =$

$$= \frac{(n(1-a-b)+2cn^\alpha+2)^2(an-cn^\alpha)(bn-cn^\alpha)}{2\pi n^3(n(1-a)+cn^\alpha+1)(n(1-b)+cn^\alpha+1)(1-a+b)(1-b+a)} \times \left(\frac{(1-a+b)(1-b+a)}{ab(1-a)(1-b)}\right)^{\frac{1}{2}}.$$

Using

$$C_n \sim \frac{4^n}{\sqrt{\pi} n^{\frac{3}{2}}},$$

we obtain

$$\frac{n^d P_n(an-cn^\alpha, bn-cn^\alpha)}{C_n} \sim \sqrt{\pi}\; n^{d+\frac{3}{2}}\, r(n,a,b)\, h(a,b)^n\, 4^{-n}.$$

Clearly, for $h(a,b) < 4$, the r.h.s. $\to 0$ as $n \to \infty$, for all $d \in \mathbb{R}_+$. By Lemma 2.4.4, we have $h(a,b) < 4$ unless $b = 1-a$. Therefore, since $a+b \neq 1$, we have $F(a,b,c,\alpha) = \infty$. Also, when n is large enough,

$$\frac{P_n(an-cn^\alpha, bn-cn^\alpha)}{C_n} < \left(\frac{h(a,b)+4}{8}\right)^n,$$

as desired. □

Lemma 2.4.6 (Second case)**.** *For all $a \in (0,1), 0 < c$, and $\frac{1}{2} < \alpha < 1$, we have $F(a, 1-a, c, \alpha) = \infty$. Moreover, for n large enough, we have*

$$\frac{P_n(an-cn^\alpha, (1-a)n-cn^\alpha)}{C_n} < \varepsilon^{n^{2\alpha-1}},$$

where ε is independent of n and $0 < \varepsilon < 1$.

Proof. Let $k = cn^\alpha$. Evaluating $P_n(an-k, (1-a)n-k)$, we have

$$P_n(an-k,(1-a)n-k) = \frac{(2k+2)^2}{(2(1-a)n)(2an)}\binom{2(1-a)n}{(1-a)n+k+1}\binom{2an}{an+k+1}.$$

Using Stirling's formula and simplifying this expression gives

$$P_n(an-k,(1-a)n-k) \sim \frac{(k+1)^2}{\pi(a(1-a))^{\frac{3}{2}} n^3}\; 4^n \left(\frac{an}{an+k}\right)^{an+k} \left(\frac{an}{an-k}\right)^{an-k}$$

$$\times \left(\frac{(1-a)n}{(1-a)n+k}\right)^{(1-a)n+k} \left(\frac{(1-a)n}{(1-a)n-k}\right)^{(1-a)n-k}.$$

Clearly,

$$\ln\left[\left(\frac{an}{an+k}\right)^{an+k}\left(\frac{an}{an-k}\right)^{an-k}\right] \sim \frac{-k^2}{an} \quad \text{as} \;\; n \to \infty.$$

Therefore,

$$\frac{n^d P_n(an-k,(1-a)n-k)}{C_n} \sim \frac{n^d(k+1)^2}{\sqrt{\pi}(a(1-a)n)^{\frac{3}{2}}} \;\exp\left[\frac{-k^2}{a(1-a)n}\right].$$

Substituting $k \leftarrow cn^\alpha$, gives

$$\frac{n^d P_n(an-k,(1-a)n-k)}{C_n} \sim \frac{c^2 n^d}{\sqrt{\pi}(a(1-a))^{\frac{3}{2}} n^{\frac{3}{2}-2\alpha}} \;\exp\left[\frac{-c^2}{a(1-a)n^{1-2\alpha}}\right].$$

For $\alpha > \frac{1}{2}$, this expression $\to 0$ as $n \to \infty$, for all d. This implies that $F(a, 1-a, c, \alpha) = \infty$. In fact, we have also proved the second case of Theorem 2.3.6, as desired. □

Lemma 2.4.7 (Third case)**.** *For all $a \in (0,1), c > 0$, and $\alpha \in [0,1]$, we have*

$$F(a, 1-a, 0, \alpha) = F(a, 1-a, c, 0) = \frac{3}{2}.$$

Furthermore, we have $L(a, 1-a, c, 0) = \xi(a, c)$.

Proof. In this case, to ensure that cn^α measures the distance from the anti-diagonal, we need to analyze $P_n(an-cn^\alpha,(1-a)n-cn^\alpha+1)$. Evaluating as in Lemma 2.4.6, gives

$$\frac{n^d P_n(an-cn^\alpha,(1-a)n-cn^\alpha+1)}{C_n} \sim \frac{n^{d-\frac{3}{2}}(2k+1)^2}{4\sqrt{\pi}(a(1-a))^{\frac{3}{2}}} \exp\left[\frac{-k^2}{a(1-a)n}\right].$$

For $c = 0$, we get $F(a, 1-a, 0, \alpha) = 3/2$ and $L(a, 1-a, 0, \alpha) = \xi(a, 0)$. For $\alpha = 0$, we get $F(a, 1-a, c, 0) = 3/2$ and $L(a, 1-a, c, 0) = \xi(a, c)$, as desired. □

Lemma 2.4.8 (Fourth case). *For all $a \in (0,1), c > 0$ and $0 < \alpha \le \frac{1}{2}$, we have*

$$F(a, 1-a, c, \alpha) = \frac{3}{2} - 2\alpha.$$

Furthermore, for $0 < \alpha < \frac{1}{2}$, we have

$$L(a, 1-a, c, \alpha) = \eta(a,c) \quad \textit{and} \quad L\left(a, 1-a, c, \frac{1}{2}\right) = \eta(a,c)\kappa(a,c),$$

where $\eta(a,c)$ and $\kappa(a,c)$ are defined as in Theorem 2.3.3.

Proof. As in Lemma 2.4.6, we have

$$\frac{n^d P_n(an-k, (1-a)n-k)}{C_n} \sim \frac{c^2 n^d}{\sqrt{\pi}(a(1-a))^{\frac{3}{2}} n^{\frac{3}{2}-2\alpha}} \exp\left[\frac{-c^2}{a(1-a)n^{1-2\alpha}}\right].$$

We can rewrite this expression as

$$\frac{n^d P_n(an-k, (1-a)n-k)}{C_n} \sim \eta(a,c)\ n^{d-(\frac{3}{2}-2\alpha)} \exp\left[\frac{-c^2}{a(1-a)n^{1-2\alpha}}\right].$$

For $\alpha < \frac{1}{2}$, we clearly have

$$\exp\left[\frac{-c^2}{a(1-a)n^{1-2\alpha}}\right] \to 1 \quad \text{as} \quad n \to \infty,$$

so $F(a, 1-a, c, \alpha) = \frac{3}{2} - 2\alpha$ and $L(a, 1-a, c, \alpha) = \eta(a,c)$. For $\alpha = \frac{1}{2}$, by the definition of $\kappa(a,c)$, we have

$$\exp\left[\frac{-c^2}{a(1-a)n^{1-2\alpha}}\right] \to \kappa(a,c), \quad \text{as} \quad n \to \infty,$$

so $F(a, 1-a, c, 1/2) = 1/2$, and $L(a, 1-a, c, 1/2) = \eta(a,c)\kappa(a,c)$, as desired. $\square$

Let us emphasize that the results of the previous two lemmas hold for $c < 0$ as well as $c > 0$. This is true by the symmetry of $P_n(j,k)$ displayed in Lemma 2.2.2, and since c only appears in the formulas for $\eta(a,c)$ and $\kappa(a,c)$ as c^2. Therefore, we have proven all cases of Theorems 2.3.1 and 2.3.3.

2.5 Analysis of 132-avoiding permutations

2.5.1 Combinatorics of Dyck paths

We recall a bijection φ between $\mathcal{S}_n(132)$ and $\mathcal{D}_n$, which we then use to derive the exact formulas for $Q_n(j,k)$. This bijection is equivalent to that in [EP], itself a variation on a bijection in [Kra] (see also [B1, Kit] for other bijections between these combinatorial classes).

Given $\gamma \in \mathcal{D}_n$, for each downstep starting at point (x,y) record y, the *level* of (x,y). This defines $y_\gamma = (y_1, y_2, \ldots, y_n)$.

We create the **132**-avoiding permutation by starting with a string $\{n, n-1, \ldots, 2, 1\}$ and removing elements from the string one at a time each from the y_i-th spot in the string, creating a permutation $\varphi(\gamma)$. Suppose this permutation contains a **132**-pattern, consisting of elements a, b, and c with $a < b < c$. After a has been removed from the string, the level in the string must be beyond b and c. Since we can only decrease levels one at a time, we must remove b before removing c, a contradiction. Therefore, the map φ is well-defined, and clearly one-to-one. By Theorem 2.2.1, this proves that φ is the desired bijection.

Example 2.5.1. *Take the Dyck path $\gamma = (uuduuddudd)$. Then $z_\gamma = (2,3,2,2,1)$, as seen in Figure 2.6. We then create our **132**-avoiding permutation $\varphi(\gamma)$ by taking the string $\{5,4,3,2,1\}$ and removing elements one at a time. First we remove the 2-nd element (4), then we remove the 3-rd element from the remaining list $\{5,3,2,1\}$, which is 2, then the 2-nd from the remaining list $\{5,3,1\}$, which is 3, then the 2-nd from $\{5,1\}$, which is 1, then the last element (5), and we obtain $\varphi(\gamma) = (4,2,3,1,5)$.*

Let $g(n,k)$ be the number of permutations $\sigma \in \mathcal{S}_n(132)$ with $\sigma(1) = k$, so $g(n,k) = Q_n(1,k)$. Recall that since $Q_n(j,k) = Q_n(k,j)$, we can also think of $g(n,k)$ as the number of permutations $\sigma \in \mathcal{S}_n(132)$ with $\sigma(k) = 1$. Let $b(n,k)$

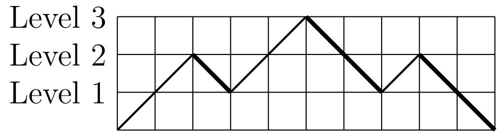


Figure 2.6: Dyck Path γ with downsteps at $y_\gamma = (2, 3, 2, 2, 1)$.

denote the ballot numbers as in Lemma 2.4.1.

Lemma 2.5.2. *For all $1 \le k \le n$, we have $g(n, k) = b(n, k)$.*

Proof. Let $\sigma \in \mathcal{S}_n(132)$ with $\sigma(1) = k$. Using bijection φ, we find that $\varphi^{-1}(\sigma)$ is a Dyck path with its final upstep from $(n+k-2, n-k)$ to $(n+k-1, n+1-k)$. The result now follows from the same logic as in the proof of Lemma 2.4.1. □

Lemma 2.5.3. *For all $1 \le j, k \le n$,*

$$Q_n(j,k) \;=\; \sum_r b(n-j+1, k-r)\; b(n-k+1, j-r)\, C_r,$$

where the summation is over values of r such that

$$\max\{0, j+k-n-1\} \le r \le \min\{j,k\} - 1.$$

Proof. Since our formula is symmetric in j and k except for the upper limit of summation, proving the lemma when $j \le k$ will suffice. When $j \le k$ the upper limit is $j-1$, rather than $k-1$ when $j > k$. $Q_n(j,k)$ represents the number of permutations $\sigma \in \mathcal{S}_n(132)$ with $\sigma(j) = k$. Let $q_n(j,k,r)$ be the number of **132**-avoiding permutations σ counted by $Q_n(j,k)$ such that there are exactly r values x with $x < j$ such that $\sigma(x) < k$. Below we show that

$$q_n(j,k,r) \;=\; b(n-j+1, k-r)\, b(n-k+1, j-r)\, C_r \quad \text{for all } \; 0 \le r \le j-1,$$

which implies the result.

Let $\sigma \in \mathcal{S}_n(132)$ such that $\sigma(j) = k$ and there are exactly r numbers x_i with $x_i < j$ and $\sigma(x_i) < \sigma(j) = k$. We use decomposition $\sigma = \tau\pi k\phi$, where

$$\tau = \{\sigma(1), \ldots, \sigma(j-r-1)\}, \;\; \pi = \{\sigma(j-r), \ldots, \sigma(j-1)\},$$

and

$$\phi = \{\sigma(j+1), \ldots, \sigma(n)\}.$$

Observe that either all elements of π are smaller than k, or there is some element of π greater than k, and some element of τ smaller than k. Suppose the second case is true, with a an element of τ smaller than k, and b an element of π larger than k. Then a, b, and k form a **132**-pattern, a contradiction. Therefore, all elements of π must be smaller than k.

Suppose some element x of π is smaller than $(k-r)$. Then some number y with $k-r \leq y < k$ is an element of ϕ. Then we have a **132**-pattern, formed by x, k, and y, which is a contradiction, so π consists of $\{k-r, k-r+1, \ldots, k-2, k-1\}$. There are C_r possible choices for π, since π must avoid the **132**-pattern.

Now consider the values of σ within τ. Observe that regardless of τ, the numbers s in ϕ with $s > k$ must be in decreasing order in ϕ, in order to avoid a **132**-pattern that starts with k. Therefore, the number of possible choices for τ is equal to the number of possible orderings of the numbers between k and n that avoid **132**, with k in the $(j-r)$-th position (since π only consists of numbers smaller than k). There are exactly $b(n-k+1, j-r)$ such possible choices.

Finally, consider values of σ within $\{k\} \cup \phi$. Here we need to order $n-j+1$ numbers so that they avoid the **132**-pattern, with the first number being the $(k-r)$-th smallest. There are exactly $b(n-j+1, k-r)$ ways to do this. Choosing π, τ, and ϕ completely determines σ. Therefore, there are $b(n-j+1, k-r)\, b(n-k+1, j-r)\, C_r$ possible choices for σ, as desired. $\square$

Example 2.5.4. *Let us compute $Q_7(4,3)$, the number of permutations $\sigma \in \mathcal{S}_7(132)$ with $\sigma(4) = 3$. We count the permutations separately depending on how many numbers smaller than 3 come ahead of 3. First suppose $r = 0$, so there are no numbers smaller than 3 ahead of 3 in the permutation. This means that 3 is* *in the 4-th position among those numbers greater than equal to 3. There are*

$$b(7-3+1, 4-0) = \frac{5-4+1}{5+4-1}\binom{5+4-1}{5} = \frac{2}{8}\binom{8}{5} = 14$$

ways to order the numbers between 3 and 7, displayed here:

$$(45637) \quad (54637) \quad (56437) \quad (56734) \quad (64537) \quad (65437) \quad (65734)$$
$$(67435) \quad (67534) \quad (74536) \quad (75436) \quad (75634) \quad (76435) \quad (76534).$$

For each of these there are

$$b(7-4+1, 3-0) = \frac{2}{6}\binom{6}{4} = 5$$

ways to place the numbers between 1 and 3, shown here:

$$(***312*) \quad (***321*) \quad (***32*1) \quad (***3*12) \quad (***3*21),$$

where the stars represent the positions of 4,5,6 and 7. In total we find that $q_7(4, 3, 0) = b(5, 4)b(4, 3) = (14)(5) = 70.$

*Similarly, for $r = 1$ there is 1 number smaller than 3 ahead of 3 in the permutation. This number has to be 2, since otherwise a **132** pattern would be formed with 1,2, and 3. Also this number must be directly in front of 3 in the permutation, since otherwise a **132** would be formed with 2 as the 1 and 3 as the 2. This means that 3 is now in the 3-rd position among those numbers greater than equal to 3. There are*

$$b(7-3+1, 4-1) = \frac{3}{7}\binom{7}{5} = 9$$

ways to order the numbers between 3 and 7, displayed here:

$$(45367) \quad (54367) \quad (56347) \quad (64357) \quad (65347)\,,$$

$$(67345) \quad (74356) \quad (75346) \quad (76345)\,.$$

For each of those orderings there are

$$b(7-4+1, 3-1)C_1 = \frac{3}{5}\binom{5}{4} = 3$$

ways to place the numbers between 1 and 3, displayed here:

$$(**231**)\quad(**23*1*)\quad(**23**1).$$

Therefore, overall we have $q_7(4,3,1) = b(5,3)b(4,2)C_1 = (9)(3) = 27$*. In other words, there are 27 distinct* ***132****-avoiding permutations of length 7 with 3 in the 4-th position, and one number smaller than 3 and ahead of 3.*

The final case is when $r = 2$*, in which case both 1 and 2 come ahead of 3 in the permutation. This means that 3 is in the 2-nd position among those numbers between 3 and 7. Therefore there are*

$$b(7-3+1, 4-2) = \frac{4}{6}\binom{6}{5} = 4$$

ways to order the numbers between 3 and 7, displayed here:

$$(43567)\quad(53467)\quad(63457)\quad(73456).$$

For each of these we have

$$b(7-4+1, 3-2)C_2 = \binom{4}{4}(2) = 2$$

ways to place the numbers between 1 and 3, displayed here:

$$(*123***)\quad(*213***).$$

Together we have that $q_7(4,3,2) = b(5,2)b(4,1)C_2 = (4)(1)(2) = 8$*, so there are 8 different* ***132****-avoiding permutations of length 7 with 3 in the 4-th position, and two numbers smaller than 3 and ahead of 3. We have shown that*

$$Q_7(4,3) = q_7(4,3,0) + q_7(4,3,1) + q_7(4,3,2) = 70 + 27 + 8 = 105.$$

2.5.2 Proof of theorems 2.3.4, 2.3.5, and 2.3.6

The proof again involves one technical lemma and several cases corresponding to the statements of Theorems 2.3.4 and 2.3.6.

Let $h : [0,1]^3 \to \mathbb{R}$ be defined so that

$$h(a,s,t) = \frac{4^{ast}(1-at+a-ast)^{1-at+a-ast}(1-a+at-ast)^{1-at+a-ast}}{(1-at)^{(1-at)}(a-ast)^{(a-ast)}(1-a)^{(1-a)}(at-ast)^{(at-ast)}}.$$

Lemma 2.5.5. *For all $(a,s,t) \in [0,1]^3$, we have $h(a,s,t) \le 4$. Moreover,*

$$h(a,s,t) = 4 \quad \textit{if and only if} \quad s = \frac{at+a-1}{at}.$$

Proof. Take the logarithmic derivative of h to obtain

$$\begin{aligned}
\frac{d(\ln h)}{ds} &= at\ln 4 + (-at(1+\ln(1-at+a-ast))) \\
&\qquad - at(1+\ln(1-a+at-ast)) \\
&\qquad - [-at(1+\ln(a-ast)) - at(1+\ln(at-ast))] \\
&= at\ln[4(a-ast)(at-ast)] \\
&\qquad - at\ln[(1-at+a-ast)(1-a+at-ast)].
\end{aligned}$$

Set this derivative equal to 0 to get

$$4(a-ast)(at-ast) - (1-at+a-ast)(1-a+at-ast) = 0,$$

or

$$(ast-(at+a-1))(3ast-(at+a+1)) = 0,$$

giving

$$s = \frac{at+a-1}{at} \quad \text{or} \quad s = \frac{at+a+1}{3at}.$$

Since

$$\frac{at+a+1}{3at} = \frac{1}{3} + \frac{1}{3t} + \frac{1}{3at} \ge \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1,$$

this value of s is greater than 1, and is only equal to 1 if $a = s = t = 1$. Similarly, the ratio $(at + a - 1)/at$ is between 0 and 1 if $at + a > 1$. It is easy to see that the second derivative

$$\frac{d^2(\ln h)}{(ds)^2} < 0 \quad \text{at} \quad s = \frac{at + a - 1}{at},$$

which implies that this value of s does indeed maximize $h(a, s, t)$. We can also verify that

$$h\Big(a, \frac{at + a - 1}{at}, t\Big) = \frac{4^{(at+a-1)}(2 - 2at)^{(2-2at)}(2 - 2a)^{(2-2a)}}{(1 - at)^{(1-at)}(1 - at)^{(1-at)}(1 - a)^{(1-a)}(1 - a)^{(1-a)}} = 4.$$

Observe that $h(a, s, t) < 4$ on the boundary of $[0, 1]^3$ except for where $a = s = t = 1$, completing the proof. □

Lemma 2.5.6. *Let $a, b \in [0, 1]^2, c \neq 0$ and $0 \leq \alpha < 1$, such that $a + b < 1$. Then $T(a, b, c, \alpha) = \infty$. Moreover, for n sufficiently large, we have*

$$\frac{Q_n(an - cn^\alpha, bn - cn^\alpha)}{C_n} < \varepsilon^n,$$

where ε is independent of n, and $0 < \varepsilon < 1$.

Proof. By Lemma 2.5.3, we have

$$\begin{aligned} Q_n(atn, an) &= \sum_r b(n - atn + 1, an - r)\, b(n - an + 1, atn - r)\, C_r \\ &= \sum_r \left[\frac{n - atn + 1 - (an - r) + 1}{n - atn + 1 + (an - r) - 1}\binom{n - atn + an - r}{n - atn + 1}\right] \\ &\qquad \times \left[\frac{n - an + 1 - atn + r + 1}{n - an + 1 + atn - r - 1}\binom{n - an + 1 + atn - r - 1}{n - an + 1}\right] \\ &\qquad \times \left[\frac{1}{r + 1}\binom{2r}{r}\right], \end{aligned}$$

where the summation is over values of r such that

$$\max\{0, j + k - n - 1\} \leq r \leq \min\{j, k\} - 1.$$

Let $r = atsn$, so s varies from 0 to $\left(1 - \frac{1}{atn}\right)$ by increments of $\frac{1}{atn}$. Applying Stirling's formula, we get

$$Q_n(atn, an) \sim \sum_r \chi(n, atn, an, astn)\ h(a, s, t)^n,$$

where

$$\begin{aligned}\chi(n, atn, an, astn) = &\sqrt{\frac{(1 - at + a - ats)(1 - a + at - ats)}{(1 - at)(a - ats)(1 - a)(at - ats)(ats)}}\\ &\times \frac{a^2t(1 - s)(1 - st)(n(1 - a - at + ats) + 2)^2}{2(\pi n)^{3/2}(n(1 - at) + 1)(n(1 - a) + 1)}\\ &\times \frac{1}{(1 + a - at - ats)(1 - a + at - ats)(atsn + 1)}.\end{aligned}$$

We now have

$$\frac{n^d Q_n(an - cn^\alpha, bn - cn^\alpha)}{C_n} \sim \sqrt{\pi} n^{d+\frac{3}{2}} \sum_{r=0}^{an-cn^\alpha-1} \nu_r(n),$$

where

$$\nu_r(n) = \chi(n, an - cn^\alpha, bn - cn^\alpha, r)\, h(b, r/an, a/b)^n\, 4^{-n}.$$

From Lemma 2.5.5, we have that $h(b, r/an, a/b) < 4$ for $r \neq (a + b - 1)n$. For values of r where $h(b, r/an, a/b) < 4$, $\nu_r(n)$ decreases exponentially as $n \to \infty$ for fixed d. Therefore, for these values of r,

$$\sqrt{\pi} n^{d+\frac{3}{2}} \nu_r(n) \to 0 \text{ as } n \to \infty \text{ for all } d.$$

The only values of r which could potentially have $\lim_{n\to\infty} \nu_r(n) \neq 0$, are when $r \sim (a + b - 1)n$, as $n \to \infty$. Observe that since $a + b - 1 < 0$, there are no such possible values of r. In this case,

$$\lim_{n\to\infty} \frac{n^d Q_n(an - cn^\alpha, bn - cn^\alpha)}{C_n} = 0 \quad \text{for all} \quad d > 0.$$

This implies $T(a, b, c, \alpha) = \infty$ when $a + b < 1$. Also, for n large enough, we have

$$\frac{Q_n(an - cn^\alpha, bn - cn^\alpha)}{C_n} < \left(\frac{1 + h(b, 0, a/b)}{2}\right)^n,$$

as desired. □

Lemma 2.5.7. *Let $a \in [0,1], c > 0,$ and $\frac{1}{2} < \alpha < 1$. Then $T(a, 1-a, c, \alpha) = \infty$. Moreover, for n large enough, we have*

$$\frac{Q_n(an - cn^\alpha, (1-a)n - cn^\alpha)}{C_n} < \varepsilon^{n^{2\alpha-1}},$$

where $\varepsilon = \varepsilon(a, c, \alpha)$ is independent of n, and $0 < \varepsilon < 1$.

Proof. Let $k = cn^\alpha$. Then

$$Q_n(an - k, (1-a)n - k) = q_n(an - k, (1-a)n - k, 0)\ S_{a,n,k}\ ,$$

where

$$S_{a,n,k} = \left(1 + \sum_{r=1}^{an-k-1} g_r(n)\right),$$

and

$$g_r(n) = \left(\frac{q_n(an - k, (1-a)n - k, r)}{q_n(an - k, (1-a)n - k, 0)}\right).$$

Observe that

$$q_n(an - k, (1-a)n - k, 0) = P_n(an - k, (1-a)n - k).$$

Applying Stirling's formula and using the Taylor expansion for $\ln(1 + x)$ gives

$$g_r(n) \sim \frac{(2k + r + 2)^2 C_r}{(2k + 2)^2 4^r}\ \exp\left[\frac{-(4kr + r^2)}{4a(1-a)n}\right].$$

Therefore,

$$\frac{n^d Q_n(an - cn^\alpha, bn - cn^\alpha)}{C_n} \sim \left(\frac{n^d P_n(an - cn^\alpha, bn - cn^\alpha)}{C_n}\right)$$
$$\times \left(1 + \sum_{r=1}^{an-k-1} \frac{(2k + r + 2)^2 C_r}{(2k + 2)^2 4^r}\ \exp\left[\frac{-(4kr + r^2)}{4a(1-a)n}\right]\right).$$

For $\alpha > \frac{1}{2}$, by Theorem 2.3.1, we have

$$\lim_{n\to\infty} \frac{n^d P_n(an - cn^\alpha, (1-a)n - cn^\alpha)}{C_n} = 0 \text{ for all } d.$$

Therefore, $T(a, 1-a, c, \alpha) = \infty$ for $\alpha > \frac{1}{2}$. Also, we have proven the second case of Theorem 2.3.5, as desired. □

For the next three cases, we denote $r = hn^p$, and let h and p be fixed as $n \to \infty$. For $p > \frac{1}{2}$, we have

$$\frac{n^d g_r(n)}{C_n} \to 0 \quad \text{as} \quad n \to \infty, \quad \text{for all} \quad d,$$

since $g_r(n)$ decreases exponentially for fixed d.

For $\alpha < p < \frac{1}{2}$, we have

$$g_r \sim \frac{r^{\frac{1}{2}}}{4\sqrt{\pi}k^2} \sim \left(\frac{\sqrt{h}}{4\sqrt{\pi}}\right) n^{\frac{p}{2}-2\alpha}.$$

For $p \leq \alpha < \frac{1}{2}$ or $p < \alpha = \frac{1}{2}$, we have

$$g_r = \Theta\left(r^{-\frac{3}{2}}\right) = \Theta\left(n^{-\frac{3p}{2}}\right).$$

Similarly, for $p = \frac{1}{2}$, we obtain

$$g_r = \Theta\left(n^{\frac{1}{4}-2\alpha}\right).$$

Lemma 2.5.8. *Let $a \in [0,1], c \in \mathbb{R}$ and $0 \leq \alpha < \frac{3}{8}$. Then*

$$T(a, 1-a, c, \alpha) = \frac{3}{4} \quad \textit{and} \quad M(a, 1-a, c, \alpha) = z(a),$$

where

$$z(a) = \frac{\Gamma(\frac{3}{4})}{2^{\frac{9}{4}}\pi\,[a(1-a)]^{\frac{3}{4}}} \quad \textit{as in Theorem 2.3.6.}$$

Proof. As in Lemma 2.5.7, we write

$$Q_n(an - cn^\alpha, (1-a)n - cn^\alpha) = P_n(an - cn^\alpha, (1-a)n - cn^\alpha)\, S_{a,n,k}\,,$$

where

$$S_{a,n,k} = 1 + \sum_{r=0}^{an-k-1} g_r(n).$$

Again, as $n \to \infty$, we have

$$g_r(n) \sim \frac{(2k+r+2)^2 C_r}{(2k+2)^2 4^r} \ \exp\left[\frac{-(4kr + r^2)}{4a(1-a)n}\right].$$

Fix $s > 0$, and observe that for any $0 \leq \alpha < \frac{3}{8}$, we have

$$g_{n^\delta}(n) = o(g_{s\sqrt{n}}(n)), \quad \text{for every} \quad \delta \neq \frac{1}{2}.$$

Therefore, as $n \to \infty, t \to 0$, and $u \to \infty$, we have

$$S_{a,n,k} \sim \sum_{r=t\sqrt{n}}^{u\sqrt{n}} g_r(n).$$

Interpreting this sum as a Riemann sum, we have

$$S_{a,n,k} \sim \sqrt{n} \int_t^u g_{v\sqrt{n}}(n)dv$$

$$\sim \sqrt{n} \int_t^u \frac{(2k + v\sqrt{n} + 2)^2}{(2k+2)^2\sqrt{\pi}(v\sqrt{n})^{\frac{3}{2}}} \left(\exp\left[\frac{-(4kv\sqrt{n} + (v\sqrt{n})^2)}{4a(1-a)n} \right] \right) dv.$$

Therefore, we have

$$S_{a,n,k} \sim \sqrt{n} \int_t^u \frac{v^{\frac{1}{2}} n^{\frac{1}{4}}}{4k^2\sqrt{\pi}} \left(\exp\left[\frac{-v^2}{4a(1-a)} \right] \right) dv.$$

A direct calculation gives

$$S_{a,n,k} = \frac{n^{\frac{3}{4}-2\alpha}}{c^2}\; z(a) \left(\sqrt{\pi}\, [a(1-a)]^{\frac{3}{2}} \right).$$

Now we see that

$$\frac{n^d Q_n(an - cn^\alpha, (1-a)n - cn^\alpha)}{C_n} \sim \frac{n^d P_n(an - cn^\alpha, (1-a)n - cn^\alpha)}{C_n} S_{a,n,k}$$

$$\sim z(a)\, n^{d-\frac{3}{4}},$$

by the proof of Theorem 2.3.1 and the analysis of $S_{a,n,k}$. Therefore, $T(a, 1-a, c, \alpha) = \frac{3}{4}$. For $\alpha < \frac{3}{8}$ this also gives $M(a, 1-a, c, \alpha) = z(a)$, as desired. $\square$

This case displays why we do not need to adjust our analysis to be on the anti-diagonal. Since the behavior of Q depends on values of r on the order of $\sqrt{n}$, adding 1 to the second coordinate is a lower-order term and does not affect G or M at all. In fact, the whole value of cn^α has no effect on G or M for this case.

Lemma 2.5.9. *Let $a \in [0,1], c > 0$, and $\frac{3}{8} < \alpha \leq \frac{1}{2}$. Then*

$$T(a, 1-a, c, \alpha) = \frac{3}{2} - 2\alpha\,.$$

Moreover,

$$M(a, 1-a, c, \alpha) = \begin{cases} y(a,c) & \text{if } \ \frac{3}{8} < \alpha < \frac{1}{2}\,, \\ y(a,c)\,\kappa(a,c) & \text{if } \ \alpha = \frac{1}{2}\,, \end{cases}$$

where $y(a,c)$ and $\kappa(a,c)$ are defined as in Theorem 2.3.3.

Proof. As in the previous lemma, we have $Q_n(an - cn^\alpha, (1-a)n - cn^\alpha) = P_n(an - cn^\alpha, (1-a)n - cn^\alpha)\, S_{a,n,k}$. We analyze $S_{a,n,k}$ to see which values of r contribute the most. In this case, for $p = \frac{1}{2}$, we have $g_r = \Theta(n^{\frac{1}{4} - 2\alpha})$, which is on the order of n^d with d strictly less than $-\frac{1}{2}$. Therefore, even if we sum over all values of r where $p = \frac{1}{2}$, we will end up with an expression on the order of $n^{d + \frac{1}{2}}$ which is lower order than a constant. For $p \leq \alpha < \frac{1}{2}$ or $p < \alpha = \frac{1}{2}$, since $g_r = \Theta(n^{-\frac{3p}{2}})$, the terms with the highest order will come when $p = 0$. Therefore, the values of r which contribute the most to $S_{a,n,k}$ will be constants in this case. From this, we have

$$S_{a,n,k} \sim 1 + \sum_{r=1}^{s} g_r \sim 1 + \sum_{r=1}^{s} \frac{C_r}{4^r} \sim 2, \quad \text{as } \ n \to \infty.$$

Therefore,

$$\frac{n^d Q_n(an-k, (1-a)n-k)}{C_n} \sim \frac{2n^d P_n(an-k, (1-a)n-k)}{C_n}.$$

Referring back to Theorems 2.3.1 and 2.3.3 gives us the desired results. □

Lemma 2.5.10. *Let $a \in [0,1]$ and $c > 0$. Then $T(a, 1-a, c, 3/8) = \frac{3}{4}$ and $M(a, 1-a, c, 3/8) = z(a) + y(a,c)$.*

Proof. Here we are essentially on the intersection of the last two cases, which provides some intuition to the reason that $M(a, 1-a, c, 3/8) = z(a) + y(a,c)$.

Again we let $r = hn^p$. When $\alpha = \frac{3}{8}$, values of g_r which contribute the highest order to $S_{a,n,k}$ are when $p = 0$ and when $p = \frac{1}{2}$. We get $z(a)$ from the terms where $p = \frac{1}{2}$, and $y(a, c)$ when $p = 0$.

$$S_{a,n,k} \sim \frac{z(a)}{c^2} \left(\sqrt{\pi}\,[a(1-a)]^{\frac{3}{2}}\right) + 2,$$

so

$$\frac{n^d Q_n(an-k,(1-a)n-k)}{C_n} \sim \frac{2c^2}{\sqrt{\pi}(a(1-a))^{\frac{3}{2}}} n^{d-\frac{3}{4}} + z(a)n^{d-\frac{3}{4}},$$

so $T(a, 1-a, c, \alpha) = \frac{3}{4}$ and $M(a, 1-a, c, \alpha) = z(a) + y(a, c)$, as desired. $\square$

Lemma 2.5.11. *Let $c > 0$ and $0 \leq \alpha < 1$. Then*

$$T(1, 1, c, \alpha) = \frac{3}{2}\alpha.$$

Moreover, for $\alpha > 0$, we have $M(1, 1, c, \alpha) = w(c)$, and $M(1, 1, c, 0) = u(c)$, where $w(c)$ and $u(c)$ are defined as in Theorem 2.3.6.

Proof. We first consider $Q_n(n-k, n-k)$ with $k = cn^\alpha$, and $\alpha > 0$. We have

$$Q_n(n-k, n-k) = \sum_{i=n-2k-1}^{n-k-1} b(k+1, n-k-i)^2 C_i \ ,$$

which is equivalent to

$$Q_n(n-k, n-k) = C_k C_k C_{n-2k-1} \left(1 + \sum_{r=1}^{k} h_r(n)\right),$$

where

$$h_r(n) = \frac{(r+1)^2 \binom{2k-r}{k}^2}{\binom{2k}{k}^2} \left(\frac{C_{n-2k-1+r}}{C_{n-2k-1}}\right).$$

Observe that $h_r(n) \sim r^2 \exp\left[\frac{-r^2}{2k}\right]$, so $h_{k^\delta}(n) = o(h_{k^{\frac{1}{2}}}(n))$ for all $\delta \neq \frac{1}{2}$. From here we have

$$\frac{n^d Q_n(n-k, n-k)}{C_n} \sim \frac{n^d}{4\pi k^3} \int_0^\infty n^{\frac{3}{2}\alpha} s^2 \exp\left[\frac{-s^2}{2c}\right] ds$$

$$\sim \frac{n^{d-\frac{3}{2}\alpha}}{4\pi c^3}\int_0^\infty s^2 \exp\left[\frac{-s^2}{2c}\right]ds.$$

Therefore $T(1,1,c,\alpha) = \frac{3}{2}\alpha$ and

$$M(1,1,c,\alpha) = \frac{1}{2^{\frac{5}{2}}\pi^{\frac{1}{2}}c^{\frac{3}{2}}} = w(c),$$

as desired.

For $\alpha = 0$,

$$Q_n(n-k,n-k) = Q_n(n-c,n-c) = \sum_{i=n-2c-1}^{n-c-1} b(c+1,n-c-i)^2 C_i,$$

$$= \sum_{s=0}^{c}\left(\frac{s+1}{2c+1-s}\binom{2c+1-s}{c+1}\right)^2 C_{n-2c-1+s}.$$

Therefore, we have

$$\frac{Q_n(n-k,n-k)}{C_n} \sim \frac{1}{4^{2c+1}}\sum_{s=0}^{c}\left(\frac{s+1}{2c+1-s}\binom{2c+1-s}{c+1}\right)^2 4^s \quad \text{as } n\to\infty,$$

completing the proof. □

Lemma 2.5.12. *Let $a\in[0,1], c<0$ and $0<\alpha<1$. Then*

$$T(a,1-a,c,\alpha) = \begin{cases} \frac{3}{4} & \text{if } 0<\alpha\le\frac{1}{2}\,, \\ \frac{3}{2}\alpha & \text{if } \frac{1}{2}<\alpha<1\,, \end{cases}$$

and

$$M(a,1-a,c,\alpha) = \begin{cases} z(a) & \text{if } 0<\alpha<\frac{1}{2}\,, \\ x(a,c) & \text{if } \alpha=\frac{1}{2}\,, \\ w(c) & \text{if } \frac{1}{2}<\alpha<1\,, \end{cases}$$

where $z(a), x(a,c)$, and $w(c)$ are defined as in Theorem 2.3.6.

Proof. We now analyze $Q_n(an+k,(1-a)n+k)$, where $k=cn^\alpha$. We have

$$Q_n(an+k,(1-a)n+k)$$
$$= \sum_{r=2k-1}^{an+k-1} b((1-a)n-k+1,(1-a)n+k-r)$$
$$b(an-k+1,an+k-r)\,C_r\,.$$

We can rewrite this as

$$\begin{aligned}
&Q_n(an+k, an+k)\\
&\quad = \sum_{d=0}^{an-k} b((1-a)n-k+1, (1-a)n-k-d+1)\\
&\qquad\qquad \times b(an-k+1, an-k-d+1)\, C_{d+2k-1}\,.
\end{aligned}$$

Denote by g_d the d-th term of this summation. Then we can express $Q_n(an+k, an+k)$ as

$$Q_n(an+k, an+k) = g_0\left(1 + \sum_{d=1}^{an-k} \frac{g_d}{g_0}\right).$$

Denote h_d as $h_d = \frac{g_d}{g_0}$. We can express h_d as

$$h_d = \frac{(d+1)^2\binom{2(an-k)-d}{an-k}\binom{2((1-a)n-k)-d}{(1-a)n-k}}{\binom{2(an-k)}{an-k}\binom{2((1-a)n-k)}{(1-a)n-k}}\left(\frac{C_{d+2k-1}}{C_{2k-1}}\right).$$

We now have

$$h_d \sim d^2 \exp\left[\frac{-d^2}{4an}\right] \exp\left[\frac{-d^2}{4(1-a)n}\right]\left(\frac{C_{d+2k-1}}{4^d C_{2k-1}}\right).$$

Fix $s > 0$ and $0 \leq \beta < 1$. We set $d = sn^\beta$, and consider the behavior of h_d as $n \to \infty$. Observe that for any α,

$$h_{n^\delta} = o(h_{sn^{\frac{1}{2}}})$$

when $\delta \neq \frac{1}{2}$. For $\frac{1}{2} = \beta < \alpha$, we have

$$h_d \sim d^2 \exp\left[\frac{-s^2}{4a(1-a)}\right].$$

For $\frac{1}{2} = \beta = \alpha$, we have

$$h_d \sim d^2 \exp\left[\frac{-s^2}{4a(1-a)}\right]\left(\frac{2c}{s+2c}\right)^{\frac{3}{2}}.$$

Similarly, for $\alpha < \beta = \frac{1}{2}$,

$$h_d \sim d^{\frac{1}{2}}(2k)^{\frac{3}{2}} \exp\left[\frac{-s^2}{4a(1-a)}\right].$$

Therefore, as $n \to \infty$, $t \to 0$, and $u \to \infty$,

$$Q_n(an+k, an+k) \sim g_0 \left(1 + \sum_{d=tn^{\frac{1}{2}}}^{un^{\frac{1}{2}}} h_d \right).$$

Recall that

$$g_0 \sim \frac{4^{n-1}}{(2\pi a(1-a)k)^{\frac{3}{2}} n^3}.$$

We are now ready to analyze $T(a, 1-a, c, \alpha)$. For $\alpha < \frac{1}{2}$ we have

$$\begin{aligned}\frac{n^d Q_n(an+cn^\alpha, (1-a)n+cn^\alpha)}{C_n} &\sim \frac{n^d}{4\pi(2a(1-a)c)^{\frac{3}{2}} n^{\frac{3}{2}(1+\alpha)}} \int_0^\infty n^{\frac{3}{4}} (2k)^{\frac{3}{2}} s^{\frac{1}{2}} \\ &\qquad \times \exp\left[\frac{-s^2}{4a(1-a)}\right] ds \\ &\sim \frac{n^{d-\frac{3}{4}}}{4\pi(a(1-a))^{\frac{3}{2}}} \left(\frac{\Gamma(\frac{3}{4})(4a(1-a))^{\frac{3}{4}}}{2} \right),\end{aligned}$$

so $T(a, 1-a, c, \alpha) = \frac{3}{4}$ and $M(a, 1-a, c, \alpha) = z(a)$, as desired. For $\alpha > \frac{1}{2}$ we have

$$\begin{aligned}\frac{n^d Q_n(an+cn^\alpha, (1-a)n+cn^\alpha)}{C_n} &\sim \frac{n^d}{4\pi(2a(1-a)c)^{\frac{3}{2}} n^{\frac{3}{2}(1+\alpha)}} \\ &\qquad \times \int_0^\infty n^{\frac{3}{2}} s^2 \exp\left[\frac{-s^2}{4a(1-a)}\right] ds \\ &\sim \frac{n^{d-\frac{3}{2}\alpha}}{4\pi(2a(1-a)c)^{\frac{3}{2}}} \left(\frac{\pi^{\frac{1}{2}}(4a(1-a))^{\frac{3}{2}}}{4} \right),\end{aligned}$$

so $T(a, 1-a, c, \alpha) = \frac{3}{2}\alpha$ and $M(a, 1-a, c, \alpha) = w(c)$.

For $\alpha = \frac{1}{2}$ we have

$$\begin{aligned}\frac{n^d Q_n(an+cn^\alpha, (1-a)n+cn^\alpha)}{C_n} &\sim \frac{n^d}{4\pi(2a(1-a)c)^{\frac{3}{2}} n^{\frac{3}{2}(1+\alpha)}} \\ &\qquad \times \int_0^\infty n^{\frac{3}{2}} s^2 \exp\left[\frac{-s^2}{4a(1-a)}\right] \left(\frac{2c}{s+2c}\right)^{\frac{3}{2}} ds \\ &\sim \frac{n^{d-\frac{3}{4}}}{4\pi(a(1-a))^{\frac{3}{2}}} \int_0^\infty \frac{s^2}{(s+2c)^{\frac{3}{2}}} \exp\left[\frac{-s^2}{4a(1-a)}\right] ds \sim n^{d-\frac{3}{4}} x(a,c),\end{aligned}$$

proving that $T(a, 1-a, c, \frac{1}{2}) = \frac{3}{4}$ and $M(a, 1-a, c, \frac{1}{2}) = x(a, c)$. This completes the proof. $\square$

Our final case is where $1 < a + b < 2$.

Lemma 2.5.13. *Let $a, b \in [0, 1]$, such that $1 < a + b < 2$, and let $c \in \mathbb{R}$, $0 < \alpha < 1$. Then*

$$T(a, b, c, \alpha) = \frac{3}{2} \quad \textit{and} \quad M(a, b, c, \alpha) = v(a, b),$$

where $v(a, b)$ is defined as in Theorem 2.3.6.

Proof. By considering $Q_n(an + k, bn + k)$, we obtain

$$Q_n(an + k, bn + k) = C_{(1-a)n+k} C_{(1-b)n+k} C_{(a+b-1)n-2k-1} \left(1 + \sum_{d=1}^{(1-b)n} g_d \right),$$

where

$$g_d = \frac{(d+1)^2 \binom{2((1-a)n+k)-d}{(1-a)n+k} \binom{2((1-b)n+k)-d}{(1-b)n+k}}{\binom{2((1-a)n+k)}{(1-a)n+k} \binom{2((1-b)n+k)}{(1-b)n+k}} \left(\frac{C_{(a+b-1)n-1+d}}{C_{(a+b-1)n-1}} \right).$$

We find that

$$g_d \sim d^2 \exp \left[\frac{-d^2(2-a-b)}{4(1-a)(1-b)n} \right],$$

so $g_{n^\beta} = o(g_{n^{\frac{1}{2}}})$ if $\beta \neq \frac{1}{2}$.

Therefore, letting $k = cn^\alpha$, we get

$$\frac{n^d Q_n(an + cn^\alpha, bn + cn^\alpha)}{C_n} \sim \frac{n^{d-3}}{4\pi((1-a)(1-b)(a+b-1))^{\frac{3}{2}}} \times \int_0^\infty n^{\frac{3}{2}} s^2 \exp \left[\frac{-s^2(2-a-b)}{4(1-a)(1-b)} \right] ds\,.$$

We can now see that $T(a, b, c, \alpha) = \frac{3}{2}$ and $M(a, b, c, \alpha) = v(a, b)$, as desired. We have now proved all cases of Theorems 2.3.4 and 2.3.6. $\square$

2.6 Expectation of basic permutation statistics

2.6.1 Results

The following result describes the behavior of the first and last elements of σ and τ.

Theorem 2.6.1. *Let $\sigma \in \mathcal{S}_n(123)$ and $\tau \in \mathcal{S}_n(132)$ be permutations chosen uniformly at random from the corresponding sets. Then*

$$\mathbf{E}[\sigma(1)] = \mathbf{E}[\sigma^{-1}(1)] = \mathbf{E}[\tau(1)] = \mathbf{E}[\tau^{-1}(1)] \to n-2 \quad \textit{as } n \to \infty, \tag{1}$$

$$\mathbf{E}[\sigma(n)] = \mathbf{E}[\sigma^{-1}(n)] \to 3, \quad \textit{as } n \to \infty, \tag{2}$$

and

$$\mathbf{E}[\tau(n)] = \mathbf{E}[\tau^{-1}(n)] = \frac{(n+1)}{2} \quad \textit{for all } n. \tag{3}$$

We remark here that the above theorem can be proved by using the exact formulas for $P_n(j,k)$ and $Q_n(j,k)$ shown in Lemmas 2.4.2 and 2.5.3. We will instead prove the theorem by analyzing bijections between $S_n(123)$ and $S_n(132)$. Before proving Theorem 2.6.1 we need some notation and definitions.

2.6.2 Definitions

Let A, B be two finite sets. We say A and B are *equinumerous* if $|A| = |B|$. Let $\alpha : A \to \mathbb{Z}$ and $\beta : B \to \mathbb{Z}$. We say α and β are *statistics* on A and B, respectively. We say that two statistics α and β on equinumerous sets are *equidistributed* if $|\alpha^{-1}(k)| = |\beta^{-1}(k)|$ for all $k \in \mathbb{Z}$; in this case we write $\alpha \sim \beta$. Note that being equidistributed is an equivalence relation.

For example, let $A = \mathcal{S}_n(132)$, let $\alpha : A \to \mathbb{Z}$, such that $\alpha(\sigma) = \sigma(1)$, and let $\beta : A \to \mathbb{Z}$ such that $\beta(\sigma) = \sigma^{-1}(1)$. Then $\alpha \sim \beta$, by Proposition 2.2.2. Given

$\pi \in S_n$, let

$$\text{rmax}(\pi) = \#\{i \text{ s.t. } \pi(i) > \pi(j) \text{ for all } j > i, \text{ where } 1 \le i \le n\}.$$

We call rmax(π) the number of *right-to-left maxima* in π. Let

$$\text{ldr}(\pi) = \max\{i \text{ s.t. } \pi(1) > \pi(2) > \ldots > \pi(i)\}.$$

We call ldr(π) the *leftmost decreasing run* of π.

2.6.3 Position of first and last elements

To prove Theorem 2.6.1, we first need two propositions relating statistics on different sets. Let $\mathcal{T}_n$ be the set of rooted plane trees on n vertices. For all $T \in \mathcal{T}_n$, denote by $\delta_r(T)$ the degree of the root vertex in T. Recall that $\mathcal{D}_n$ is the set of Dyck paths of length $2n$. For all $\gamma \in \mathcal{D}_n$, denote by $\alpha(\gamma)$ the number of points on the line $y = x$ in γ. Recall that

$$|\mathcal{D}_n| = |\mathcal{S}_n(123)| = |\mathcal{S}_n(132)| = C_n.$$

Recall that $|\mathcal{T}_{n+1}| = C_n$ (see e.g. [B1, S1]), so $\{\mathcal{T}_n, \mathcal{D}_n, \mathcal{S}_n(123), \mathcal{S}_n(132)\}$ are all equinumerous.

Proposition 2.6.2. *Define the following statistics:*

$$\begin{array}{lll}
\delta_n : \mathcal{T}_{n+1} \to \mathbb{Z} & \textit{such that} & \delta_n(T) = \delta_r(T), \\
\alpha_n : \mathcal{D}_n \to \mathbb{Z} & \textit{such that} & \alpha_n(\gamma) = \alpha(\gamma), \\
\mathit{rmax}_n : \mathcal{S}_n(132) \to \mathbb{Z} & \textit{such that} & \mathit{rmax}_n(\sigma) = \mathit{rmax}(\sigma), \\
\mathit{ldr}_n : \mathcal{S}_n(123) \to \mathbb{Z} & \textit{such that} & \mathit{ldr}_n(\sigma) = \mathit{ldr}(\sigma), \\
\mathit{ldr}'_n : \mathcal{S}_n(132) \to \mathbb{Z} & \textit{such that} & \mathit{ldr}'_n(\sigma) = \mathit{ldr}(\sigma), \quad \textit{and} \\
\mathit{first}_n : \mathcal{S}_n(123) \to \mathbb{Z} & \textit{such that} & \mathit{first}_n(\sigma) = \sigma(1).
\end{array}$$

Then, we have

$$\delta_n \sim \alpha_n \sim \mathit{rmax}_n \sim \mathit{ldr}_n \sim \mathit{ldr}'_n \sim (n + 1 - \mathit{first}_n) \quad \textit{for all } n.$$

Proof of Proposition 2.6.2. Throughout the proof we refer to specific bijections described and analyzed in [CK]; more details and explanation of equidistribution are available there. We now prove equidistribution of the statistics one at a time.

• $\underline{\delta_n \sim \alpha_n}$ Recall the standard bijection $\phi : \mathcal{T}_{n+1} \to \mathcal{D}_n$. Observe that

$$\phi : \delta_n \to \alpha_n \,.$$

Therefore, $\delta_n \sim \alpha_n$, as desired.

• $\underline{\alpha_n \sim \mathrm{rmax}_n}$ Let $\psi : \mathcal{D}_n \to \mathcal{S}_n(132)$ be the bijection φ presented in Section 2.5.1. Observe that $\psi : \alpha_n \to \mathrm{rmax}_n$. Therefore, $\alpha_n \sim \mathrm{rmax}_n$, as desired.

• $\underline{\mathrm{rmax}_n \sim \mathrm{ldr}_n}$ Let Φ be the West bijection between $\mathcal{S}_n(132)$ and $\mathcal{S}_n(123)$. Observe that $\Phi : \mathrm{rmax}_n \to \mathrm{ldr}_n$. Therefore, $\mathrm{rmax}_n \sim \mathrm{ldr}_n$, as desired.

• $\underline{\mathrm{ldr}_n \sim \mathrm{ldr}'_n}$ Let Ψ be the Knuth-Richards bijection between

$$\mathcal{S}_n(123) \;\text{ and }\; \mathcal{S}_n(132)\,.$$

Observe that if $\Psi(\sigma) = \tau$ for some $\sigma \in \mathcal{S}_n(123)$, then

$$\mathrm{ldr}_n(\sigma) = \mathrm{ldr}'_n(\tau^{-1})\,.$$

Since $\mathcal{S}_n(132)$ is closed under inverses, we get $\mathrm{ldr}_n(\mathcal{S}_n(123)) \sim \mathrm{ldr}'_n(\mathcal{S}_n(132))$, as desired.

• $\underline{\mathrm{ldr}'_n \sim (n+1-\mathrm{first}_n)}$ Let Δ be the Knuth-Rotem bijection between $\mathcal{S}_n(132)$ and $\mathcal{S}_n(321)$, and let $\Gamma : \mathcal{S}_n(321) \to \mathcal{S}_n(123)$ such that $\Gamma(\sigma) = (\sigma(n), \ldots, \sigma(1))$. Observe that if $\Gamma \circ \Delta(\sigma^{-1}) = \tau$ for some $\sigma \in \mathcal{S}_n(132)$, then $\mathrm{ldr}'_n(\sigma^{-1}) = n+1-\mathrm{first}_n(\tau)$. Since σ^{-1} ranges over all $\mathcal{S}_n(132)$, we get $\mathrm{ldr}'_n \sim (n+1-\mathrm{first}_n)$, as desired. □

The second proposition explains the behavior of $\tau(n)$, for $\tau \in \mathcal{S}_n(132)$.

Proposition 2.6.3. *Let $\tau \in \mathcal{S}_n(132)$ be chosen uniformly at random. Let $last_n : \mathcal{S}_n(132) \to \mathbb{Z}$ be defined as $last_n(\tau) = \tau(n)$. Then we have*

$$last_n \sim n + 1 - last_n.$$

Proof. By applying Lemma 2.5.3 for $k = n$, we obtain

$$Q_n(j, n) = C_{j-1}C_{n-j},$$

implying that

$$Q_n(j, n) = Q_n(n + 1 - j, n) \text{ for all integers } j \text{ and } n \text{ where } j \leq n.$$

Since for all $1 \leq j \leq n$, we have

$$|\text{last}_n^{-1}(j)| = Q_n(j, n) \text{ and } |(n + 1 - \text{last}_n)^{-1}(j)| = Q_n(n + 1 - j, n)\ ,$$

we have $\text{last}_n \sim (n + 1 - \text{last}_n)$, by the definition of equidistribution. $\square$

2.6.4 Proof of Theorem 2.6.1

Let $\sigma \in \mathcal{S}_n(123)$ and $\tau \in \mathcal{S}_n(132)$ be chosen uniformly at random. It is known that

$$\mathbf{E}[\delta_n] \to 3 \text{ as } n \to \infty,$$

where δ_n is defined as in Proposition 2.6.2 (see e.g. [FS, Example III.8]). Since δ_n and $(n + 1 - \text{first}_n)$ are equidistributed by Proposition 2.6.2, we have

$$\mathbf{E}[\delta_n] = n + 1 - \mathbf{E}[\sigma(1)].$$

Therefore, we have $\mathbf{E}[\sigma(1)] \to (n - 2)$ as $n \to \infty$, as desired.

Due to the symmetries explained in the proof of Proposition 2.2.2, we have

$$\mathbf{E}[\sigma^{-1}(1)] = \mathbf{E}[\sigma(1)] \to n - 2\,,$$

and

$$\mathbf{E}[\sigma(n)] = \mathbf{E}[\sigma^{-1}(n)] = n + 1 - \mathbf{E}[\sigma(1)] \to 3\,,$$

as $n \to \infty$, completing the proof of (1).

Since $P_n(1,k) = b(n,k) = Q_n(1,k)$ for all n and k by lemmas 2.4.1 and 2.5.2, we have

$$\mathbf{E}[\tau(1)] = \mathbf{E}[\sigma(1)] \to n - 2 \quad \text{as} \quad n \to \infty.$$

Also, by Proposition 2.2.2, we have

$$\mathbf{E}[\tau^{-1}(1)] = \mathbf{E}[\tau(1)] \to n - 2 \quad \text{as} \quad n \to \infty,$$

completing the proof of (2).

In order to complete the proof of Theorem 2.6.1, it suffices to prove (3). By Proposition 2.6.3, we have $\mathrm{last}_n \sim (n + 1 - \mathrm{last}_n)$, so

$$\mathbf{E}[\mathrm{last}_n] = \mathbf{E}[n + 1 - \mathrm{last}_n].$$

By the linearity of expectation, we have $\mathbf{E}[\mathrm{last}_n] = (n+1)/2$, so

$$\mathbf{E}[\tau(n)] = \frac{n+1}{2},$$

as desired. By the symmetry in Proposition 2.2.2, we have

$$\mathbf{E}[\tau^{-1}(n)] = \mathbf{E}[\tau(n)] = \frac{n+1}{2},$$

completing the proof.

2.7 Fixed points in random permutations

2.7.1 Results

Let $\sigma \in S_n$. The number of fixed points of σ is defined as

$$\mathrm{fp}_n(\sigma) = \#\{i \ \text{ s.t. } \ \sigma(i) = i, 1 \le i \le n\}.$$

In [E1, §5.6], Elizalde uses bijections and generating functions to obtain results on the expected number of fixed points of permutations in $\mathcal{S}_n(321), \mathcal{S}_n(132)$, and $\mathcal{S}_n(123)$. In the following three theorems, the first part is due to Elizalde, while the second parts are new results.

We use $I_{n,\varepsilon}(a) = [(a-\varepsilon)n, (a+\varepsilon)n]$ to denote the intervals of elements in $\{1, \ldots, n\}$.

Theorem 2.7.1. *Let $\varepsilon > 0$, and let $\sigma \in \mathcal{S}_n(321)$ be chosen uniformly at random. Then*

$$\mathbf{E}[fp_n(\sigma)] = 1 \quad \textit{for all} \quad n.$$

Moreover, for $a \in (\varepsilon, 1-\varepsilon)$,

$$\mathbf{P}\left(\sigma(i) = i \quad \textit{for some} \quad i \in I_{n,\varepsilon}(a)\right) \to 0 \quad \textit{as} \quad n \to \infty.$$

Theorem 2.7.2. *Let $\varepsilon > 0$, and let $\sigma \in \mathcal{S}_n(132)$ be chosen uniformly at random. Then*

$$\mathbf{E}[fp_n(\sigma)] = 1, \quad \textit{as} \quad n \to \infty.$$

Moreover, for $a \in (0, 1/2-\varepsilon) \cup (1/2+\varepsilon, 1-\varepsilon)$,

$$\mathbf{P}\left(\sigma(i) = i \quad \textit{for some} \quad i \in I_{n,\varepsilon}(a)\right) \to 0 \quad \textit{as } n \to \infty.$$

Theorem 2.7.3. *Let $\varepsilon > 0$, and let $\sigma \in \mathcal{S}_n(123)$ be chosen uniformly at random. Then*

$$\mathbf{E}[fp_n(\sigma)] \to \frac{1}{2}, \quad \textit{as} \quad n \to \infty.$$

Moreover, for $a \in (0, 1/2-\varepsilon) \cup (1/2+\varepsilon, 1)$,

$$\mathbf{P}\left(\sigma(i) = i \quad \textit{for some} \quad i \in I_{n,\varepsilon}(a)\right) \to 0 \quad \textit{as } n \to \infty.$$

In fact, Elizalde obtains exact formulas for $\mathbf{E}[\mathrm{fp}_n(\sigma)]$ in the last case as well [E1, Prop. 5.3]. We use an asymptotic approach to give independent proofs of all three theorems.

The final case to consider, of fixed points in **231**-avoiding permutations, is more involved. In the language of Section 2.1, the expectation is equal to the sum of entries of Q_n along anti-diagonal Δ, parallel to the wall. It is larger than in the case of the canoe since the decay from the wall towards Δ is not as sharp as in the case of the canoe.

In [E1], Elizalde calculated the (algebraic) generating function for the expected number of fixed points in $\mathcal{S}_n(231)$, but only concluded that $\mathbf{E}[\mathrm{fp}_n(\sigma)] > 1$ for $n \geq 3$. Our methods allow us to calculate the asymptotic behavior of this expectation, but not the location of fixed points.

Theorem 2.7.4. *Let $\sigma \in \mathcal{S}_n(231)$ be chosen uniformly at random. Then*

$$\mathbf{E}[fp_n(\sigma)] \sim \frac{2\Gamma(\frac{1}{4})}{\sqrt{\pi}}\, n^{\frac{1}{4}}, \quad as \;\; n \to \infty.$$

Recall that if $\sigma \in S_n$ is chosen uniformly at random, then $\mathbf{E}[\mathrm{fp}_n(\sigma)] = 1$, so the number of fixed points statistic does not distinguish between random permutation in $\mathcal{S}_n(132)$, $\mathcal{S}_n(321)$ and random permutations in S_n. On the other hand, permutations in $\mathcal{S}_n(123)$ are less likely to have fixed points, in part because they can have at most 2 of them, and permutations in $\mathcal{S}_n(231)$ are much more likely to have fixed points than the typical permutation in S_n.

2.7.2 Proof of Theorem 2.7.1

Let $\sigma \in \mathcal{S}_n(321)$ be chosen uniformly at random, and $\sigma' = (\sigma(n), \ldots, \sigma(1))$. Since $\sigma \in \mathcal{S}_n(321)$, we have $\sigma' \in \mathcal{S}_n(123)$. For $\tau \in S_n$, define $\mathrm{afp}(\tau) = \#\{i \text{ s.t. } \tau(i) = n+1-i, 1 \leq i \leq n\}$. Let $\mathrm{afp}_n : \mathcal{S}_n(123) \to \mathbb{Z}$ such that $\mathrm{afp}_n(\tau) = \mathrm{afp}(\tau)$. Let $\mathrm{fp}_n : \mathcal{S}_n(321) \to \mathbb{Z}$ such that $\mathrm{fp}_n(\sigma) = \mathrm{fp}(\sigma)$. By these symmetries, we have $\mathrm{fp}_n \sim \mathrm{afp}_n$.

Clearly,

$$\mathbf{E}[\mathrm{afp}_n] = \frac{1}{C_n}\sum_{k=1}^{n} P_n(k, n+1-k).$$

Observe that

$$P_n(k, n+1-k) = C_{k-1}C_{n-k}$$

by Lemma 2.4.2. By the recurrence relation for the Catalan numbers, we have

$$\sum_{k=1}^{n} P_n(k, n+1-k) = C_n.$$

Therefore,

$$\mathbf{E}[\mathrm{afp}_n] = 1, \quad \text{so} \quad \mathbf{E}[\mathrm{fp}_n] = 1.$$

For the proof of the second part, let $\varepsilon > 0$ and $a \in (\varepsilon, 1-\varepsilon)$, and define $\delta = \min\{\varepsilon, a-\varepsilon\}$. Let

$$\mathrm{P} = \mathbf{P}\left(\sigma(i) = i \ \text{ for some } i \in [\delta n, (1-\delta)n+1]\right).$$

By the definition of P, we have

$$\mathbf{P}(\sigma(i) = i \ \text{ for some } i \in I_{n,\varepsilon}(a)) \leq \mathrm{P} \ \text{ for all } n,$$

so it suffices to show that $\mathrm{P} \to 0$ as $n \to \infty$. Observe that for all n, we have

$$\mathrm{P} = \frac{1}{C_n}\sum_{k=\delta n}^{(1-\delta)n+1} P_n(k, n+1-k).$$

By Lemma 2.4.2, we have

$$\frac{P_n(\delta n, (1-\delta)n+1)}{C_n} = \frac{C_{\delta n-1}C_{(1-\delta)n}}{C_n} \sim \frac{1}{4\sqrt{\pi}(\delta(1-\delta)n)^{\frac{3}{2}}}, \quad \text{as} \quad n \to \infty.$$

Observe that for $d \in (\delta, 1-\delta)$ and for n sufficiently large, we have $P_n(dn, (1-d)n+1) < P_n(\delta n, (1-\delta)n+1)$. Therefore, for n sufficiently large, we have

$$\begin{aligned}
\mathrm{P} &= \frac{1}{C_n}\sum_{k=\delta n}^{(1-\delta)n+1} P_n(k, n+1-k) \\
&\leq \frac{(1-2\delta)n+2}{C_n}P_n(\delta n, (1-\delta)n+1) \\
&\sim \frac{1-2\delta}{4\sqrt{\pi}(\delta(1-\delta))^{\frac{3}{2}}\sqrt{n}}.
\end{aligned}$$

Therefore, $\mathrm{P} \to 0$ as $n \to \infty$, as desired.

2.7.3 Proof of theorems 2.7.2 and 2.7.3

We omit the proofs of the probabilities tending to 0 since they are very similar to the proof of Theorem 2.7.1, and instead prove the following proposition.

Proposition 2.7.5. *Let $\sigma \in \mathcal{S}_n(123)$ and $\tau \in \mathcal{S}_n(132)$ be chosen uniformly at random. Then*

$$\mathbf{E}[fp_n(\tau)] = 1 \quad \textit{for all } n, \quad \textit{and} \quad \mathbf{E}[fp_n(\sigma)] \to \frac{1}{2}, \quad \textit{as} \quad n \to \infty.$$

Proof. Let $\sigma \in \mathcal{S}_n(123)$ and $\tau \in \mathcal{S}_n(132)$ be chosen uniformly at random. By a bijection between **132**-avoiding and **321**-avoiding permutations in [EP] (see also [CK, Rob, RSZ]), fixed points are equidistributed between $\mathcal{S}_n(132)$ and $\mathcal{S}_n(321)$. Therefore, by the proof of Theorem 2.7.1, we have $\mathbf{E}[\mathrm{fp}_n(\tau)] = 1$.

Now we prove that

$$\mathbf{E}[\mathrm{fp}_n(\sigma)] \to \frac{1}{2}, \quad \text{as} \quad n \to \infty.$$

Observe that for all n, we have

$$\mathbf{E}[\mathrm{fp}_n(\sigma)] = \sum_{k=1}^{n} \frac{P_n(k,k)}{C_n}.$$

Let c_1 and c_2 be constants such that $0 < c_1 < c_2$. By Theorems 2.3.1 and 2.3.3, we have

$$\mathbf{E}[\mathrm{fp}_n(\sigma)] \sim \sum_{k=n/2-c_2\sqrt{n}}^{n/2-c_1\sqrt{n}} \frac{P_n(k,k)}{C_n} + \sum_{k=n/2+c_1\sqrt{n}}^{n/2+c_2\sqrt{n}} \frac{P_n(k,k)}{C_n} + e(c_1,c_2,n),$$

where the error term $e(c_1,c_2,n) \to 0$ as $c_1 \to 0, c_2 \to \infty$, and $n \to \infty$. By the symmetry in Proposition 2.2.2, and by Lemma 2.4.8, we have

$$\begin{aligned}\mathbf{E}[\mathrm{fp}_n(\sigma)] &\sim 2 \sum_{k=n/2-c_2\sqrt{n}}^{n/2-c_1\sqrt{n}} \frac{P_n(k,k)}{C_n} + e(c_1,c_2,n) \\ &\sim 2 \sum_{k=c_1\sqrt{n}}^{c_2\sqrt{n}} \frac{8k^2 n^{-\frac{3}{2}}}{\sqrt{\pi}} e^{-\frac{4k^2}{n}} + e(c_1,c_2,n)\,.\end{aligned}$$

As $c_1 \to 0$ and $c_2 \to \infty$, we have

$$\mathbf{E}[\mathrm{fp}_n(\sigma)] \sim 2\sqrt{n}\int_0^\infty \frac{8c^2}{\sqrt{\pi}\sqrt{n}}e^{-4c^2}dc = \frac{16}{\sqrt{\pi}}\int_0^\infty c^2e^{-4c^2}.$$

Since

$$\int_0^\infty r^2e^{-4r^2}dr = \frac{\sqrt{\pi}}{32},$$

we get $\mathbf{E}[\mathrm{fp}_n(\sigma)] \to 1/2$, as desired. $\square$

2.7.4 Proof of Theorem 2.7.4

Proof. Let $\sigma \in \mathcal{S}_n(231)$ be chosen uniformly at random. Observe that $\tau = (\sigma(n), \sigma(n-1), \ldots, \sigma(1)) \in \mathcal{S}_n(132)$ and is distributed uniformly at random. Therefore,

$$\mathbf{E}[\mathrm{fp}_n(\sigma)] = \mathbf{E}[\mathrm{afp}_n(\tau)] = \sum_{k=1}^{n} \frac{Q_n(k, n+1-k)}{C_n}.$$

Let $A_k = Q_n(k, n-k+1)/C_n$, so $\mathbf{E}[\mathrm{fp}_n(\sigma)] = \sum_{k=1}^n A_k$. By Lemma 2.5.3 and Stirling's formula, we have

$$\mathbf{E}[\mathrm{fp}_n(\sigma)] \sim 2\sum_{k=1}^{n/2} \frac{n^{\frac{3}{2}}}{\sqrt{\pi}k^{\frac{3}{2}}(n-k)^{\frac{3}{2}}}\left(1 + \sum_{r=1}^{k-1} \frac{\sqrt{r}}{\sqrt{\pi}}\exp\left[-\frac{r^2n}{4k(n-k)}\right]\right).$$

Let $B_K = \sum_{k=1}^{K-1} A_k, C_K = \sum_{k=K}^{n} A_k$, and let $K = \sqrt{n}$. We show that $B_K = o(C_K)$. For $k < K$, we have

$$\left(\sum_{r=1}^{k-1} \frac{\sqrt{r}}{\sqrt{\pi}}\exp\left[-\frac{r^2n}{4k(n-k)}\right]\right) = O(k^{\frac{3}{4}}), \quad \text{since} \quad \sum_{r=1}^{C\sqrt{k}} \frac{\sqrt{r}}{\sqrt{\pi}} \sim \frac{2Ck^{\frac{3}{4}}}{3\sqrt{\pi}}.$$

Therefore, $A_k = O(k^{-\frac{3}{4}})$, and

$$B_K = O(\sqrt{n}(\sqrt{n})^{-\frac{3}{4}}) = O(n^{\frac{1}{8}}).$$

For $k \to \infty$ as $n \to \infty$, we have

$$\begin{aligned}\left(1 + \sum_{r=1}^{k-1} \frac{\sqrt{r}}{\sqrt{\pi}}\exp\left[-\frac{r^2n}{4k(n-k)}\right]\right) &\sim \frac{k^{\frac{3}{4}}}{\sqrt{\pi}}\int_0^\infty \sqrt{x}\exp\left[-\frac{x^2n}{4k(n-k)}\right]dx \\ &= \frac{\sqrt{2}\,\Gamma\left(\frac{3}{4}\right)}{\sqrt{\pi}}\left(\frac{k(n-k)}{n}\right)^{\frac{3}{4}}.\end{aligned}$$

Therefore, for these values of k, we have

$$A_k \sim \frac{\sqrt{2}\,\Gamma\left(\frac{3}{4}\right)}{\pi} \cdot \frac{n^{\frac{3}{4}}}{k^{\frac{3}{4}}(n-k)^{\frac{3}{4}}}.$$

Consequently, we have

$$C_K = \sum_{k=\sqrt{n}}^{n} A_k \sim \frac{2\sqrt{2}\,\Gamma\left(\frac{3}{4}\right)}{\pi}\, n^{\frac{1}{4}} \int_0^{\frac{1}{2}} (x-x^2)^{-\frac{3}{4}} dx$$

$$\sim \frac{2\sqrt{2}\,\Gamma\left(\frac{3}{4}\right)}{\pi}\, n^{\frac{1}{4}} \left(\frac{2\sqrt{2\pi}\,\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}\right) = \frac{2\,\Gamma\left(\frac{1}{4}\right)}{\sqrt{\pi}}\, n^{\frac{1}{4}}.$$

As a result, we have

$$\mathbf{E}[\mathrm{fp}_n(\sigma)] = B_K + C_K \sim \frac{2\,\Gamma\left(\frac{1}{4}\right)}{\sqrt{\pi}} n^{\frac{1}{4}},$$

as desired. □

2.8 Generalized rank and the longest increasing subsequence

2.8.1 Results

For $\lambda > 0$, define the *rank*$_\lambda$ of a permutation $\sigma \in S_n$ as the largest integer r such that $\sigma(i) > \lambda r$ for all $1 \leq i \leq r$. Observe that for $\lambda = 1$, we have $\mathrm{rank}_\lambda = \mathrm{rank}$ as defined in [EP] (see also [CK, Kit]).

Theorem 2.8.1. *Let $c_1, c_2, \lambda > 0$ and $0 < \varepsilon < \frac{1}{2}$. Also, let $\sigma \in \mathcal{S}_n(123)$ and $\tau \in \mathcal{S}_n(132)$ be chosen uniformly at random. Let $R_{\lambda,n} = \mathrm{rank}_\lambda(\sigma)$ and $S_{\lambda,n} = \mathrm{rank}_\lambda(\tau)$. Then $\mathbf{E}[R_{1,n}] = \mathbf{E}[S_{1,n}]$ for all n. Furthermore, for n sufficiently large, we have*

$$\frac{n}{\lambda+1} - c_1 n^{\frac{1}{2}+\varepsilon} \leq \mathbf{E}[R_{\lambda,n}] \leq \frac{n}{\lambda+1} - c_2\sqrt{n}.$$

The following corollary rephrases the result in a different form.

Corollary 2.8.2. *Let λ and $R_{\lambda,n}$ be as in Theorem 2.8.1. Then we have*

$$\lim_{n\to\infty} \frac{\log\left(\frac{n}{\lambda+1} - \mathbf{E}[R_n]\right)}{\log n} = \frac{1}{2}.$$

For $\lambda = 1$, the corollary is known and follows from the following theorem of Deutsch, Hildebrand and Wilf. Define $\mathrm{lis}(\sigma)$, the length of the longest increasing subsequence in σ, to be the largest integer k such that there exist indices $i_1 < i_2 < \ldots < i_k$ which satisfy $\sigma(i_1) < \sigma(i_2) < \ldots < \sigma(i_k)$. Let $\mathrm{lis}_n : \mathcal{S}_n(321) \to \mathbb{Z}$ such that $\mathrm{lis}_n(\sigma) = \mathrm{lis}(\sigma)$.

Theorem 2.8.3 ([DHW]). *Let $\sigma \in \mathcal{S}_n(321)$. Define*

$$X_n(\sigma) = \frac{lis_n(\sigma) - \frac{n}{2}}{\sqrt{n}}.$$

Then we have

$$\lim_{n\to\infty} \mathbf{P}(X_n(\sigma) \le \theta) = \frac{\Gamma(\frac{3}{2}, 4\theta^2)}{\Gamma(\frac{3}{2})},$$

where $\Gamma(x,y)$ *is the* incomplete Gamma function

$$\Gamma(x,y) = \int_0^y u^{x-1} e^{-u} du.$$

Theorem 2.8.3 states that $\mathbf{E}[\mathrm{lis}_n\sigma] \to \frac{n}{2} + c\sqrt{n}$ as $n \to \infty$, for some constant $c > 0$. By [EP], we have $S_{1,n} \sim (n - \mathrm{lis}_n)$. By Knuth-Richards' bijection between $\mathcal{S}_n(123)$ and $\mathcal{S}_n(132)$, we have $S_{1,n} \sim R_{1,n}$, so $\mathbf{E}[R_{1,n}] = \mathbf{E}[n - \mathrm{lis}_n]$. Therefore,

$$\mathbf{E}[R_{1,n}] \to n - (\frac{n}{2} + c\sqrt{n}) = \frac{n}{2} - c\sqrt{n}, \quad \text{as} \quad n \to \infty.$$

2.8.2 Another technical lemma

By Knuth-Richards' bijection (also Simion-Schmidt's bijection) between $\mathcal{S}_n(123)$ and $\mathcal{S}_n(132)$, the rank statistic is equidistributed in these two classes of permutations, so $\mathbf{E}[R_{1,n}] = \mathbf{E}[S_{1,n}]$ for all n (see [CK, Kit]). Therefore, it suffices to prove the inequalities for $R_{\lambda,n}$.

We prove the lower bound first, followed by the upper bound. To prove the lower bound, we first need a lemma regarding this sum.

Lemma 2.8.4. *Let $\varepsilon > 0, c_1 > 0, \lambda > 0, n$ a positive integer, and i, j be integers such that*

$$1 \le i \le r, 1 \le j \le \lambda r, \quad \textit{where} \;\; r = \left\lfloor \frac{n}{\lambda+1} - c_1 n^{\frac{1}{2}+\varepsilon} \right\rfloor.$$

Then the function $P_n(i,j)$ is maximized for $(i,j) = (r, \lambda r)$, as $n \to \infty$.

Proof of Lemma 2.8.4. By reasoning similar to the proof of Lemma 2.4.6,

$$\frac{P_n(r,\lambda r)}{C_n} \sim \frac{(\lambda+1)^5 c_1^2 n^{2\varepsilon-\frac{1}{2}}}{4\lambda^{\frac{3}{2}}\sqrt{\pi}} \exp\left[-\frac{(\lambda+1)^4 c_1^2 n^{2\varepsilon}}{4\lambda}\right], \quad \text{as} \;\; n \to \infty.$$

Let $1 \le i \le r$ and $1 \le j \le \lambda r$ such that $i + j \sim sn$ for some $0 \le s < 1$. Then by Lemma 2.4.5, there exists some $0 < \delta < 1$ such that

$$\frac{P_n(i,j)}{C_n} < \delta^n \;\; \text{for } n \text{ sufficiently large.}$$

For n sufficiently large, we have

$$\delta^n < \frac{(\lambda+1)^5 c_1^2 n^{2\varepsilon-\frac{1}{2}}}{4\lambda^{\frac{3}{2}}\sqrt{\pi}} \exp\left[-\frac{(\lambda+1)^4 c_1^2 n^{2\varepsilon}}{4\lambda}\right],$$

so $P_n(i,j) < P_n(r,\lambda r)$ as $n \to \infty$.

It remains to consider $1 \le i \le r, 1 \le j \le \lambda r$ such that $i + j \sim n$. Since $r \sim n/(\lambda+1)$, we need

$$i \sim \frac{n}{\lambda+1} \;\; \text{and} \;\; j \sim \frac{\lambda n}{\lambda+1}$$

as well. Let $i = n/(\lambda+1) - c$ and $j = \lambda n/(\lambda+1) - d$, where

$$c = a n^{\frac{1}{2}+\varepsilon+\alpha} \;\; \text{and} \;\; d = b n^{\frac{1}{2}+\varepsilon+\beta},$$

$0 \le \alpha, \beta < 1/2 - \varepsilon$, and if $\alpha = 0$ (or $\beta = 0$), then $a \ge c_1$ (or $b \ge \lambda c_1$, respectively).

We have

$$\frac{P_n(i,j)}{C_n} \sim \frac{(\lambda+1)^3(c+d+2)^2}{4\lambda^{\frac{3}{2}}\sqrt{\pi}n^{\frac{3}{2}}} \exp\left[-\frac{(\lambda+1)^2(c+d)^2}{4\lambda n}\right],$$

by similar logic to that used in the proof of Lemma 2.4.6. Plugging in for c and d in the exponent gives

$$\frac{P_n(i,j)}{C_n} \sim \frac{(\lambda+1)^3(c+d+2)^2}{4\lambda^{\frac{3}{2}}\sqrt{\pi}n^{\frac{3}{2}}} \exp\left[-\frac{(\lambda+1)^2(an^\alpha+bn^\beta)^2 n^{2\varepsilon}}{4\lambda}\right].$$

Clearly if $\alpha > 0$ or $\beta > 0$ we have $P_n(i,j) < P_n(r, \lambda r)$ as $n \to \infty$. Similarly, if $\alpha = \beta = 0$ but $a + b > (\lambda+1)c_1$, then we again have $P_n(i,j) < P_n(r, \lambda r)$ as $n \to \infty$. Therefore, the function $P_n(i,j)$ is indeed maximized at $(i,j) = (r, \lambda r)$, as desired. □

2.8.3 Proof of the lower bound in Theorem 2.8.1

Let $\varepsilon > 0, \sigma \in \mathcal{S}_n(123)$, and let $0 < c_1$. Consider

$$\mathbf{P}(R_{\lambda,n} \leq r), \quad \text{where } r = \frac{n}{\lambda+1} - c_1 n^{\frac{1}{2}+\varepsilon}.$$

By the union bound,

$$\mathbf{P}(R_{\lambda,n} \leq r) \leq \sum_{i=1}^{r}\sum_{j=1}^{\lambda r} \frac{P_n(i,j)}{C_n}.$$

By Lemma 2.8.4, we have

$$\mathbf{P}(R_{\lambda,n} \leq r) \leq \lambda r^2 \, \frac{P_n(r,\lambda r)}{C_n},$$

and by Lemma 2.4.6, there exists $\delta > 0$ so that for n sufficiently large, we have

$$\lambda r^2 \, \frac{P_n(r,\lambda r)}{C_n} < \lambda r^2 \delta^{n^{2\varepsilon}} \to 0.$$

Therefore,

$$\mathbf{P}\left(R_{\lambda,n} \leq \frac{n}{\lambda+1} - c_1 n^{\frac{1}{2}+\varepsilon}\right) \to 0 \quad \text{as } n \to \infty,$$

and

$$\mathbf{E}[R_{\lambda,n}] \geq \frac{n}{\lambda+1} - c_1 n^{\frac{1}{2}+\varepsilon},$$

as desired.

2.8.4 Proof of the upper bound in Theorem 2.8.1

We can express $\mathbf{E}[R_{\lambda,n}]$ as

$$\mathbf{E}[R_{\lambda,n}] = \sum_{k=0}^{n/(\lambda+1)} k\mathbf{P}(R_{\lambda,n} = k) = \frac{n}{\lambda+1} - \sum_{k'=0}^{n/(\lambda+1)} k'\mathbf{P}\left(R_{\lambda,n} = \frac{n}{\lambda+1} - k'\right),$$

if we let $k' = n/(\lambda+1) - k$. From here, for every $0 < a < b$, we have

$$\begin{aligned}\mathbf{E}[R_{\lambda,n}] &\le \frac{n}{\lambda+1} - \sum_{k'=a\sqrt{n}}^{b\sqrt{n}} k'\mathbf{P}\left(R_{\lambda,n} = \frac{n}{\lambda+1} - k'\right) \\ &\le \frac{n}{\lambda+1} - a\sqrt{n}\sum_{k'=a\sqrt{n}}^{b\sqrt{n}} \mathbf{P}\left(R_{\lambda,n} = \frac{n}{\lambda+1} - k'\right).\end{aligned}$$

Therefore, it suffices to show that for some choice of $0 < a < b$, we have

$$\mathbf{P}\left(\frac{n}{\lambda+1} - b\sqrt{n} \le R_{\lambda,n} \le \frac{n}{\lambda+1} + a\sqrt{n}\right) = A > 0,$$

for some constant $A = A(a, b, \lambda)$.

Let

$$F = \left\lfloor \frac{\lambda-1}{\lambda+1} n \right\rfloor.$$

Let $\sigma \in \mathcal{S}_n(123)$, and suppose we have

$$i, j < \frac{n}{\lambda+1}, \sigma(i) = i + F, \quad \text{and} \quad \sigma(j) = j + F.$$

Then for any $r > j$, we have $\sigma(r) < j + F$, since otherwise a **123**-pattern would exist with (i, j, r). However, this is a contradiction, since $\sigma : \mathbb{Z} \cap [j+1, n] \to \mathbb{Z} \cap [1, j+F-1]$ must be injective, but $n - j > j + F - 1$. Consequently, σ can have at most one value of $i < n/(\lambda+1)$ with $\sigma(i) = i + F$.

Let $k' = n/(\lambda+1) - d\sqrt{n}$ for some constant d. Then we have

$$\mathbf{P}(R_{\lambda,n} \le k') \ge \sum_{i=1}^{k'} \frac{P_n(i, i+F)}{C_n},$$

since $\sum_{i=1}^{k'} P_n(i, i+F)$ counts the number of values $i \le k'$ such that $\sigma(i) = i + F$ for some $\sigma \in \mathcal{S}_n(123)$, and each σ is counted by at most one i. In the notation of Theorem 2.3.3, we obtain

$$\mathbf{P}(R_{\lambda,n} \le k') \ge \int_d^\infty \eta\left(\frac{1}{\lambda+1}, t\right) \kappa\left(\frac{1}{\lambda+1}, t\right) dt \quad \text{as} \quad n \to \infty.$$

For any $d > 0$, this integral is a positive constant which is maximized at $d = a$ for $d \in [a, b]$. Therefore,

$$\mathbf{P}\left(\frac{n}{\lambda+1} - b\sqrt{n} \le R_{\lambda,n} \le \frac{n}{\lambda+1} + a\sqrt{n}\right) \to \int_a^b \eta\left(\frac{1}{\lambda+1}, t\right) \kappa\left(\frac{1}{\lambda+1}, t\right) dt\,,$$

as $n \to \infty$. Denote

$$A(a, b, \lambda) = \int_a^b \eta\left(\frac{1}{\lambda+1}, t\right) \kappa\left(\frac{1}{\lambda+1}, t\right) dt.$$

For any λ we can choose $0 < a(\lambda) < b(\lambda)$ so that $A(a, b, \lambda)$ is bounded away from 0. Plugging back into our upper bound gives

$$\mathbf{E}[R_{\lambda,n}] \le \frac{n}{\lambda+1} - a(\lambda)A(a, b, \lambda)\sqrt{n},$$

completing the proof of the upper bound and of Theorem 2.8.1.

2.9 Final remarks and open problems

2.9.1

The history of asymptotic results on Catalan numbers goes back to Euler who noticed in 1758, that $C_{n+1}/C_n \to 4$ as $n \to \infty$, see [Eul]. In the second half of the 20th century, the study of various statistics on Catalan objects, became of interest first in Combinatorics and then in Analysis of Algorithms. Notably, binary and plane trees proved to be especially fertile ground for both analysis and applications, and the number of early results concentrate on these. We refer to

[A3, BPS, Dev, DFHNS, DG, FO, GW, GP, Ort, Tak] for an assortment of both recent and classical results on the distributions of various statistics on Catalan objects, and to [FS] for a compendium of information on asymptotic methods in combinatorics.

The approach of looking for a limiting object whose properties can be analyzed, is standard in the context of probability theory. We refer to [A1, A2] for the case of limit shapes of random trees (see also [Drm]), and to [Ver, VK] for the early results on limit shapes of random partitions and random Young tableaux. Curiously, one of the oldest bijective approach to pattern avoidance involves infinite "generating trees" [West].

2.9.2

The study of pattern avoiding permutations is very rich, and the results we obtain here can be extended in a number of directions. First, most naturally, one can ask what happens to patterns of size 4, especially to classes of equinumerous permutations not mapped into each other by natural symmetries (see [B2]). Of course, multiple patterns with nice combinatorial interpretations, and other generalizations are also of interest (see e.g. [B1, Kit]). We return to this problem in later sections.

Second, there are a number of combinatorial statistics on $\mathcal{S}_n(\mathbf{123})$ and $\mathcal{S}_n(\mathbf{132})$, which have been studied in the literature, and which can be used to create a bias in the distribution. In other words, for every such statistic $\alpha : \mathcal{S}_n(\pi) \to \mathbb{Z}$ one can study the limit shapes of the weighted average of matrices

$$\sum_{\sigma \in \mathcal{S}_n(\pi)} q^{\alpha(\sigma)} M(\sigma)\,, \quad \text{where } \ q \geq 0 \ \text{ is fixed}$$

(cf. Subsection 2.1.1). Let us single out statistic α which counts the number of times pattern ω occurs in a permutation σ. When π is empty, that is when the summation above is over the whole S_n, these averages interpolate between S_n for $q = 1$, and $\mathcal{S}_n(\omega)$ for $q \to 0$. We refer to [B3, Hom] for closely related results (see also [MV1, MV2])

Finally, there are natural extensions of pattern avoidance to 0-1 matrices, see [KMV, Spi], which, by the virtue of their construction, seem destined to be studied probabilistically. We plan to make experiments with the simple patterns, to see if they have interesting limit shapes.

2.9.3

There are at least nine different bijections between $\mathcal{S}_n(123)$ and $\mathcal{S}_n(132)$, not counting symmetries which have been classified in the literature [CK] (see also [Kit, §4]). Heuristically, this suggests that none of these is the most "natural" or "canonical". From the point of view of [P1], the reason is that such a natural bijection would map one limit shape into the other. But this is unlikely, given that these limit shapes seem incompatible.

2.9.4

The integral which appears in the expression for $x(a, c)$ in Theorem 2.3.5 is not easily evaluated by elementary methods. After a substitution, it is equivalent to

$$\int_0^\infty \frac{z^2}{(z+c)^{\frac{3}{2}}}\, e^{-z^2} dz,$$

which can be then computed in terms of hypergeometric and Bessel functions[3]; we refer to [AS] for definitions. Similarly, it would be nice to find an asymptotic formula for $u(c)$ in Theorem 2.3.6.

[3]For more discussion of this integral, see http://tinyurl.com/akpu5tk

2.9.5

Let us mention that the results in Section 2.3 imply few other observations which are not immediately transparent from the figures. First, as we mentioned in Subsection 2.1.1, our results imply that the curve $Q_n(k,k)$ is symmetric for $(1/2+\varepsilon)n < k < (1-\varepsilon)n$, reaching the minimum at $k = 3n/4$, for large n. Second, our results imply that the ratio

$$\frac{Q_n(n/2-\sqrt{n}, n/2-\sqrt{n})}{P_n(n/2-\sqrt{n}, n/2-\sqrt{n})} \to 2 \qquad \text{as} \;\; n \to \infty,$$

which is larger than the apparent ratios of peak heights visible in Figure 2.1. Along the main diagonal, the location of the local maxima of $P_n(k,k)$ and $Q_n(k,k)$ seem to roughly coincide and have a constant ratio, as $n \to \infty$. Our results are not strong enough to imply this, as extra multiplicative terms can appear. It would be interesting to see if this is indeed the case.

2.9.6

The generalized rank statistics rank_λ we introduce in Section 2.8 seem to be new. Our numerical experiments suggest that for all $\lambda > 0$ and for all n, rank_λ is equidistributed between $\mathcal{S}_n(132)$ and $\mathcal{S}_n(123)$. This is known for $\lambda = 1$ (see §2.8.2). We conjecture that this is indeed the case and wonder if this follows from a known bijection. If true, this implies that the "wall" and the left side of the "canoe" are located at the same place indeed, as suggested in the previous subsection.

Note that $\text{rank}(\sigma) \le n/2$ for every $\sigma \in S_n$, since otherwise $M(\sigma)$ is singular. Using the same reasoning, we obtain $\text{rank}_\lambda(\sigma) \le n/(1+\lambda)$ for $\lambda \le 1$. It would be interesting to see if there is any connection of generalized ranks with the longest increasing subsequences, and if these inequalities make sense from the point of view of the Erdős-Szekeres inequality [ES]. We refer to [AD, BDJ] for more on the distribution of the length of the longest increasing subsequences in random permutations.

2.9.7

From Lemma 2.4.2, it is easy to see that $P_n(j,k)$ for $j+k \leq n+1$, coincided with the probability that a random Dyck path of length $2n$ passes through point $(n-j+k-1, n+j-k-1)$. This translates the problem of computing the limit shape of the "canoe" to the shape of Brownian excursion, which is extremely well understood (see [Pit] and references therein). As mentioned in the introduction, this explains all qualitative phenomena in this case. For example, the expected maximum distance from the anti-diagonal is known to be $\sqrt{\pi n}(1+o(1))$ (see [Chu, DI]). Similarly, the exponential decay of $P_n(k-t, n-k-t)$ for $t = n^{1/2+\varepsilon}$, follows from the setting, and seems to correspond to tail estimates for the expected maximal distance. However, because of the emphasis on the maxima and occupation time of Brownian excursions, it seems there are no known probabilistic analogues for results such as our Theorem 2.3.3 despite similarities of some formulas. For example, it is curious that for $c \neq 0$ and $\alpha = 1/2$, the expression

$$\eta(a,c)\kappa(a,c) \; = \; \frac{c^2}{\sqrt{\pi}\, a^{\frac{3}{2}}(1-a)^{\frac{3}{2}}} \exp\left[\frac{-c^2}{a(1-a)}\right]$$

is exactly the density function of a Maxwell-distributed random variable, which appears in the contour process of the Brownian excursion (cf. [GP]).

2.9.8

Since the paper containing the results of this chapter was published, a probabilistic model with the same limit shape as $\mathcal{S}_n(132)$ was found, by Christopher Hoffman, Doug Rizzolo, and Erik Slivken [HRS].

2.9.9

After this paper was written and posted on the arXiv, we learned of two closely related papers. In [ML], the authors set up a related random pattern avoiding permutation model and make a number of Monte Carlo simulations and conjectures, including suggesting an empiric "canoe style" shape. Rather curiously, the authors prove the exponential decay of the probability $P\big(\tau(1) > 0.71n\big)$, for random $\tau \in \mathcal{S}_n(4231)$.

In [AM], the authors prove similar "small scale" results for patterns of size 3, i.e. exponential decay above anti-diagonal and polynomial decay below anti-diagonal for random $\sigma \in \mathcal{S}_n(132)$. They also study a statistic similar but not equal to rank.

CHAPTER 3

Avoiding two patterns of length three simultaneously

3.1 Introduction

In this chapter, we again analyze permutations which avoid patterns of length three. The difference is that we now require permutations to avoid multiple patterns simultaneously. For a given set A of patterns of length three, with $|A| \geq 2$, the number of permutations of length n avoiding all patterns in A is less than the n-th Catalan number, since the set is more restrictive. Depending on the size of A, and the specific patterns in A, we find a range of limit shapes and asymptotic behavior.

3.2 Definitions and Basic Observations

Let $A = \{\tau_1, \ldots, \tau_r\}$, where $\tau_i \in S_3$ for all i. Denote by $\mathcal{S}_n(A)$ the set of permutations in S_n which avoid each τ_i simultaneously. As seen in Chapter 2.9.9, if $|A| = 1$, then $|\mathcal{S}_n(A)| = C_n$, regardless of which pattern of length 3 is in A.

For $A \subset S_3$ with $|A| > 1$, initially we see there are $2^6 - 6 - 1 = 57$ such distinct sets. We use symmetry to simplify the analysis.

3.2.1 Symmetries

For A of size 2, up to symmetry there are five classes to consider. Let $\mathcal{A} \subset 2^{S_3}$, such that $A \in \mathcal{A}$ if and only if $|A| = 2$. To simplify the argument, we define an equivalence relation on $\mathcal{A}$. Let $S, T \in \mathcal{A}$, with $S = \{a, b\}$ and $T = \{c, d\}$. Then we say $S \equiv T$ if

$$T \in \{S, S^C, S^R, S^{RC}\}.$$

Since the reverse and complement operations act on sets in $\mathcal{A}$, and form a Klein four-group, this equivalence relation is well-defined. This definition is natural for us, since sets which are equivalent under this definition will have limit shapes which are equal under some rotation or reflection.

We now give a proposition describing elements of $\mathcal{A}$.

Proposition 3.2.1. *Let $\mathcal{A} \subset 2^{S_3}$, such that $A \in \mathcal{A}$ if and only if $|A| = 2$. Then the equivalence classes under $\equiv$ are given by $A^{(1)}, A^{(2)}, A^{(3)}, A^{(4)}$, and $A^{(5)}$, where*

$$A^{(1)} = \{\boldsymbol{132},\ \boldsymbol{213}\} \sim \{\boldsymbol{231},\ \boldsymbol{312}\},$$

$$A^{(2)} = \{\boldsymbol{132},\ \boldsymbol{231}\} \sim \{\boldsymbol{132},\ \boldsymbol{312}\} \sim \{\boldsymbol{213},\ \boldsymbol{231}\} \sim \{\boldsymbol{213},\ \boldsymbol{312}\},$$

$$A^{(3)} = \{\boldsymbol{123},\ \boldsymbol{132}\} \sim \{\boldsymbol{123},\ \boldsymbol{213}\} \sim \{\boldsymbol{231},\ \boldsymbol{321}\} \sim \{\boldsymbol{312},\ \boldsymbol{321}\},$$

$$A^{(4)} = \{\boldsymbol{123},\ \boldsymbol{231}\} \sim \{\boldsymbol{123},\ \boldsymbol{312}\} \sim \{\boldsymbol{132},\ \boldsymbol{321}\} \sim \{\boldsymbol{213},\ \boldsymbol{321}\},$$

$$A^{(5)} = \{\boldsymbol{123},\ \boldsymbol{321}\}.$$

Proof. We prove the result for $A^{(3)}$, since the other cases are similar. Let A be defined as $A = \{\mathbf{123}, \mathbf{132}\}$. Then $A^R = \{\mathbf{123}^R, \mathbf{132}^R\} = \{\mathbf{321}, \mathbf{231}\}$. Similarly, $A^C = \{\mathbf{321}, \mathbf{312}\}$, and $A^{RC} = \{\mathbf{123}, \mathbf{213}\}$. Therefore, $A^{(3)} = \{A, A^R, A^C, A^{RC}\}$. By the definition of $\equiv$, we see that $A^{(3)}$ consists precisely of the sets equivalent to A, as desired. $\square$

Because of the forementioned symmetries, when calculating the limiting distribution of $A^{(i)}$ it suffices to analyze the first pair of patterns listed, and then rotate the distribution as needed to calculate for the other pairs of patterns.

For sets A of length two, $\mathcal{S}_n(A)$ was studied and enumerated by Simion and Schmidt [SS].

Theorem 3.2.2 ([SS]). *Let $A^{(1)}, A^{(2)}, A^{(3)}, A^{(4)}$, and $A^{(5)}$ be as above. Then*

$$|\mathcal{S}_n(A^{(1)})| = |\mathcal{S}_n(A^{(2)})| = |\mathcal{S}_n(A^{(3)})| = 2^{n-1}\,,$$

$$\text{while } |\mathcal{S}_n(A^{(4)})| = \binom{n}{2} + 1 \text{ and } |\mathcal{S}_n(A^{(5)})| = 0\,, \text{ for } n \geq 5\,.$$

Unlike $\mathcal{S}_n(\tau)$ with $\tau \in S_3$, the size of $\mathcal{S}_n(A)$ depends on which two patterns A contains. The fact that $\mathcal{S}_n(A^{(5)}) = \emptyset$ for $n \geq 5$ follows from Erdős and Szekeres's result on longest increasing and decreasing subsequences [ES].

In the following section, we analyze the limit shapes of $\mathcal{S}_n(A)$ for each $A \in \mathcal{A}$.

3.3 Main results

In this section we present the main results of the paper.

3.3.1 Shape of A-avoiding permutations

Let $0 \leq a, b \leq 1, c, d \in \mathbb{R}, \alpha, \beta \in [0, 1)$ be fixed constants. Recall that $A_n^{(i)}(j, k)$ measures the probability that a permutation σ chosen uniformly at random from $\mathcal{S}_n(A^{(i)})$ has $\sigma(j) = k$. Define

$$F^i(a, b, c, d, \alpha, \beta) = \sup\left\{ r \in \mathbb{R}_+ \,\middle|\, \lim_{n\to\infty} n^r A_n^{(i)}(an + cn^\alpha, bn + dn^\beta) < \infty \right\}\,.$$

Similarly, let

$$L^i(a, b, c, d, \alpha, \beta) = \lim_{n\to\infty} n^{F^i(a,b,c,d,\alpha,\beta)} A_n^{(i)}(an + cn^\alpha, bn + dn^\beta)\,,$$

defined for all a, b as above, for which $F^i(a, b, c, d, \alpha, \beta) < \infty$; let L be undefined otherwise.

Note that unlike in Chapter 2.9.9, we allow deviations cn^{α} and bn^{β} to potentially differ - this is because we do not always retain the same symmetries which were present in $\mathcal{S}_n(\mathbf{123})$ and $\mathcal{S}_n(\mathbf{132})$.

Theorem 3.3.1. *For all* $0 \leq a, b \leq 1, c, d \in \mathbb{R}, \alpha, \beta \in [0, 1)$*, we have*

$$F^{(1)}(a, b, c, d, \alpha, \beta) = \begin{cases} 0 & \text{if } a + b = 1, \alpha = \beta = 0 \\ \infty & \text{otherwise.} \end{cases}$$

Theorem 3.3.2. *Let* $0 \leq a, b \leq 1, c, d \in \mathbb{R}, \alpha, \beta \in [0, 1)$*. Then we have*

$$F^{(3)}(a, b, c, d, \alpha, \beta) = \begin{cases} 0 & \text{if } a + b = 1, \alpha = \beta = 0, c + d \geq 0\,, \\ \infty & \text{otherwise.} \end{cases}$$

The limiting distribution for $A^{(2)}$ is noticeably different from $A^{(1)}$ and $A^{(3)}$.

Theorem 3.3.3. *For all* $a, b \in [0, 1], c, d \in \mathbb{R}, \alpha, \beta \in [0, 1)$*, we have*

$$F^{(2)}(a, b, c, d, \alpha, \beta) = \begin{cases} \infty & b \notin \{1 - 2a, 2a - 1\}\,, \\ \infty & b \in \{1 - 2a, 2a - 1\}, b \neq 1, \max\{\alpha, \beta\} > \frac{1}{2}\,, \\ \frac{1}{2} & b \in \{1 - 2a, 2a - 1\}, b \neq 1, \alpha, \beta \leq \frac{1}{2}\,, \\ \infty & b = 1, a \in \{0, 1\}, \alpha \neq \beta \\ \infty & b = 1, a \in \{0, 1\}, \alpha = \beta > 0, 2c \notin \{d, -d\} \\ \frac{\alpha}{2} & b = 1, a = 0, \alpha = \beta > 0, 2c = -d\,, \\ \frac{\alpha}{2} & b = 1, a = 1, \alpha = \beta > 0, 2c = d\,, \\ 0 & b = 1, a = 0, \alpha = \beta = 0, -d \geq c - 1 \geq 0\,, \\ 0 & b = 1, a = 1, \alpha = \beta = 0, -d \geq -c \geq 0\,, \\ \infty & b = 1, a = 0, \alpha = \beta = 0, -d < c - 1 \\ \infty & b = 1, a = 1, \alpha = \beta = 0, -d < -c\,. \end{cases}$$

In both of the results above, $F^i(a,b)=\infty$ means that $A_n^{(i)}(an,bn)=o(1/n^r)$, for all $r>0$.

In fact, we have results on exponential decay for each of these permutation classes as well.

Theorem 3.3.4. *Let* $0\le a,b\le 1, c,d\in\mathbb{R}, \alpha,\beta\in[0,1)$. *Then there exists* $\varepsilon(a,b,c,d,\alpha,\beta)\in(0,1)$ *such that*

$$A_n^{(1)}(an+cn^\alpha, bn+dn^\beta) < \begin{cases} \varepsilon^n & \text{for } a+b\neq 1\,,\\ \varepsilon^{n^\alpha} & \text{for } a+b=1, \alpha>0, \alpha\ge\beta\,,\\ \varepsilon^{n^\beta} & \text{for } a+b=1, \beta>\alpha\,,\end{cases}$$

for n large enough.

Theorem 3.3.5. *Let* $0\le a,b\le 1, c,d\in\mathbb{R}, \alpha,\beta\in[0,1)$. *Then there exists* $\varepsilon(a,b,c,d,\alpha,\beta)\in(0,1)$ *such that*

$$A_n^{(3)}(an+cn^\alpha, bn+dn^\beta) \le \begin{cases} \varepsilon^n & \text{for } a+b>1\,,\\ \varepsilon^{n^\alpha} & \text{for } a+b=1, \alpha>\beta, c>0\,,\\ \varepsilon^{n^\beta} & \text{for } a+b=1, \beta>\alpha, d>0\,,\\ \varepsilon^{n^\alpha} & \text{for } a+b=1, \alpha=\beta>0, c+d\ge 0\\ 0 & \text{for } a+b<1\,,\\ 0 & \text{for } a+b=1, \alpha>\beta, c<0\,,\\ 0 & \text{for } a+b=1, \beta>\alpha, d<0\,,\\ 0 & \text{for } a+b=1, \alpha=\beta, c+d<0\,,\end{cases}$$

for n large enough.

We have a related result for $\mathcal{S}_n(A^{(2)})$.

Theorem 3.3.6. *Let* $0 \leq a, b \leq 1, c, d \in \mathbb{R}, \alpha, \beta \in [0, 1)$. *Let* $\gamma = \max\{\alpha, \beta\}$. *Then there exists* $\varepsilon(a, b, c, d, \alpha, \beta) \in (0, 1)$ *such that*

$$A_n^{(2)}(an + cn^\alpha, bn + dn^\beta) \leq \begin{cases} \varepsilon^n & \text{for } b \notin \{2a-1, 1-2a\}\,, \\ \varepsilon^{n^{2\gamma-1}} & \text{for } b \in \{1-2a, 2a-1\}, b \neq 1, \gamma > \frac{1}{2}\,, \\ \varepsilon^{n^\gamma} & \text{for } b = 1, a \in \{0,1\}, \alpha \neq \beta\,, \\ \varepsilon^{n^\alpha} & \text{for } b = 1, a \in \{0,1\}, \\ & \quad \text{and } \alpha = \beta > 0, 2c \notin \{d, -d\} \\ 0 & \text{for } b = 1, a = 0, \alpha = \beta = 0, -d < c - 1\,, \\ 0 & \text{for } b = 1, a = 1, \alpha = \beta = 0, -d < -c\,, \end{cases}$$

for n *large enough.*

We also have the following result for $A^{(4)}$.

Theorem 3.3.7. *For all* $0 \leq a, b \leq 1$, *we have*

$$F^{(4)}(a, b, c, d, \alpha, \beta) = \begin{cases} 0 & \text{if } a + b = 1, a \in [0, 1)\,, \\ 2 & \text{if } a = 1, b = 0 \\ 2 & \text{if } a = 0 \text{ or } b = 1 \\ 1 & \text{otherwise.} \end{cases}$$

As well as analyzing the rate of decay of $A_n^{(i)}(an + cn^\alpha, bn + dn^\beta)$, we also analyze the constant on the leading term of $A_n^{(i)}(an + cn^\alpha, bn + dn^\beta)$, denoted by $L^i(a, b, c, d, \alpha, \beta)$.

Theorem 3.3.8. *Let* $a \in [0, 1]$. *Then*

$$L^{(1)}(a, 1-a, c, d, 0, 0) = \begin{cases} \frac{1}{3}\left(2^{-|c+d-1|} + 2^{-|c-d|}\right) & \text{if } a \in \{0, 1\} \\ \frac{1}{3}\left(2^{-|c+d-1|}\right) & \text{otherwise.} \end{cases}$$

Theorem 3.3.9. *Let $a, b \in [0,1], c, d \in \mathbb{R}, \alpha, \beta \in [0,1)$. Then for $b = 0$, we have*

$$L^{(2)}(a,b,c,d,\alpha,\beta) = \begin{cases} \sqrt{\frac{2}{\pi}} & \text{if } a = \frac{1}{2}, \alpha, \beta < \frac{1}{2}\,, \\ \sqrt{\frac{2}{\pi}} \exp\left[-2c^2\right] & \text{if } a = \frac{1}{2}, \alpha = \frac{1}{2} > \beta \\ \sqrt{\frac{2}{\pi}} \exp\left[-\frac{d^2}{2}\right] & \text{if } a = \frac{1}{2}, \beta = \frac{1}{2} > \alpha \\ \sqrt{\frac{2}{\pi}} \exp\left[-\frac{(d-2c)^2}{2}\right] & \text{if } a = \frac{1}{2}, \alpha = \beta = \frac{1}{2}\,. \end{cases}$$

For $b = 2a - 1$, we have

$$L^{(2)}(a,b,c,d,\alpha,\beta) = \begin{cases} \sqrt{\frac{1}{4(1-a)\pi}} & \text{if } b \neq 1, \alpha, \beta < \frac{1}{2}\,, \\ \sqrt{\frac{1}{4(1-a)\pi}} \exp\left[-\frac{c^2}{1-a}\right] & \text{if } b \neq 1, \alpha = \frac{1}{2} > \beta \\ \sqrt{\frac{1}{4(1-a)\pi}} \exp\left[-\frac{d^2}{4(1-a)}\right] & \text{if } b \neq 1, \beta = \frac{1}{2} > \alpha \\ \sqrt{\frac{1}{4(1-a)\pi}} \exp\left[-\frac{(d-2c)^2}{4(1-a)}\right] & \text{if } b \neq 1, \alpha = \beta = \frac{1}{2}\,. \end{cases}$$

For $b = 1 - 2a < 1$, we have

$$L^{(2)}(a,b,c,d,\alpha,\beta) = \begin{cases} \sqrt{\frac{1}{4a\pi}} & \text{if } \alpha, \beta < \frac{1}{2}\,, \\ \sqrt{\frac{1}{4a\pi}} \exp\left[-\frac{c^2}{a}\right] & \text{if } \alpha = \frac{1}{2} > \beta \\ \sqrt{\frac{1}{4a\pi}} \exp\left[-\frac{d^2}{4a}\right] & \text{if } \beta = \frac{1}{2} > \alpha \\ \sqrt{\frac{1}{4a\pi}} \exp\left[-\frac{(d+2c)^2}{4a}\right] & \text{if } \alpha = \beta = \frac{1}{2}\,. \end{cases}$$

Finally, we have

$$L^{(2)}(a,b,c,d,\alpha,\beta) = \begin{cases} \sqrt{\frac{1}{4c\pi}} & \text{if } b = 1, a = 0, \alpha = \beta > 0, 2c = -d\,, \\ \sqrt{\frac{1}{-4c\pi}} & \text{if } b = 1, a = 1, \alpha = \beta > 0, 2c = d\,, \\ 2^{d-1}\binom{-d}{c-1} & \text{if } b = 1, a = 0, \alpha = \beta = 0, -d \geq c - 1 \geq 0\,, \\ 2^{d-1}\binom{-d}{-c} & \text{if } b = 1, a = 1, \alpha = \beta = 0, -d \geq -c \geq 0\,. \end{cases}$$

Theorem 3.3.10. *Let $a \in [0,1]$. Then*

$$L^{(3)}(a,1-a,c,d,0,0) = \begin{cases} 2^{-c-d-1} & \text{if } a \notin \{0,1\}, c+d \geq 0\,, \\ 2^{-c-d-1} & \text{if } a = 1, c \neq 0, c+d \geq 0\,, \\ 2^{-c-d-1} & \text{if } a = 0, d \neq 0, c+d \geq 0\,, \\ 2^{-c} & \text{if } a = 0, d = 0\,, \\ 2^{-d} & \text{if } a = 1, c = 0\,. \end{cases}$$

Theorem 3.3.11. *Let $a, b \in [0,1]$. Then*

$$L^{(4)}(a,b,c,d,\alpha,\beta) = \begin{cases} 2 & \text{if } \ a = 0, b < 1 \ \text{ or } a > 0, b = 1\,, \\ 2 & \text{if } \ a = 1, b = 0\,, \\ 2a & \text{if } \ a + b < 1\,, \\ b^2 & \text{if } \ a + b = 1\,, \\ 2(1-b) & \text{if } \ a + b > 1\,. \end{cases}$$

3.4 Analysis of $A^{(1)}$-avoiding permutations

3.4.1 Symmetry

The permutations $\sigma \in \mathcal{S}_n(A^{(1)})$ obey some simple symmetry.

Lemma 3.4.1. *For all $1 \leq j, k \leq n$, we have $A_n^{(1)}(j,k) = A_n^{(1)}(k,j) = A_n^{(1)}(n+1-k, n+1-j)$.*

Proof. Since $\mathbf{132}^{-1} = \mathbf{132}$, and $\mathbf{213}^{-1} = \mathbf{213}$, for every $\sigma \in \mathcal{S}_n(A^{(1)})$, we have $\sigma^{-1} \in \mathcal{S}_n(A^{(1)})$ as well. Therefore, $A_n^{(1)}(j,k) = A_n^{(1)}(k,j)$. Let

$$\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(n)) \in \mathcal{S}_n(A^{(1)})\,,$$

with $\sigma(j) = k$. Then the permutation $\tau = \sigma^r \in \mathcal{S}_n(A^{(1)})$, with $\tau(n+1-k) = n+1-j$.

Here we give a lemma which is the main step towards proving Theorem 3.3.1. Recall that $A_n^{(1)}(j,k)$ is the probability that a random $\sigma \in \mathcal{S}_n(A^{(1)})$ has $\sigma(j) = k$.

Lemma 3.4.2. *For all* $1 \le j,k \le n$, *we have*

$$A_n^{(1)}(j,k) = \begin{cases} \frac{2^{k+j-2}+2^{k-j-1}}{3\cdot 2^{n-1}} & j<k, j+k \le n+1 \ , \\ \frac{2^{2n-k-j}+2^{k-j-1}}{3\cdot 2^{n-1}} & j<k, j+k > n+1 \ , \\ \frac{2^{2k-2}+2}{3\cdot 2^{n-1}} & j=k, j+k \le n+1 \ , \\ \frac{2^{2n-2k}+2}{3\cdot 2^{n-1}} & j=k, j+k > n+1 \ , \\ \frac{2^{k+j-2}+2^{j-k-1}}{3\cdot 2^{n-1}} & j>k, j+k \le n+1 \ , \\ \frac{2^{2n-k-j}+2^{j-k-1}}{3\cdot 2^{n-1}} & j>k, j+k > n+1 \ . \end{cases}$$

Proof. Due to the symmetry explained in Lemma 3.4.1, it suffices to prove the first and third cases. First, suppose that $j < k$ and $j+k \le n+1$. Let $\sigma \in \mathcal{S}_n(A^{(1)})$ with $\sigma(j) = k$. Let $A_\sigma = \{i < j : \sigma(i) > k\}$, let $|A_\sigma| = r$, and let $B_\sigma = \{i > j : \sigma(i) > k\}$. Since σ avoids **132**, $A_\sigma = [r]$, otherwise a **132** would be formed with k as the 2. Also, $B_\sigma = \{j+1, \ldots, j+(n-k-r)\}$, since otherwise σ contains a **213** with k as the 2.

Now for $i \in \mathbb{N} \cap [r+1, j+n-k-r]$, we have $\sigma(i) = i+k-j$, since otherwise either a **132** or a **213** would be formed. Therefore, to determine σ, we have to choose the values of $\sigma(i)$ for $i \in A_\sigma \cup (\{j+n-k-r+1, n\} \cap \mathbb{N})$.

For $r = 0$, we must have $\sigma(i) = i+k-j$ for $1 \le i \le j+n-k$, while there are

$$|\mathcal{S}_{k-j}(A^{(1)})| = 2^{k-j-1}$$

ways to complete σ.

For $1 \le r \le j-1$, there are $2^{r-1}2^{k-j+r-1}$ ways to complete σ. Therefore, the probability that $\sigma \in \mathcal{S}_n(A^{(1)})$ has $\sigma(j) = k$ is

$$\begin{aligned}
A_n^{(1)}(j,k) &= \frac{2^{k-j-1} + \sum_{r=1}^{j-1} 2^{r-1}2^{k-j+r-1}}{2^{n-1}} \\
&= 2^{k-j-n}\left(1 + \sum_{r=1}^{j-1} 2^{2r-1}\right) \\
&= 2^{k-j-n}\left(1 + 2\frac{4^{j-1}-1}{4-1}\right) \\
&= \frac{2^{k-j-n}\left(2 \cdot 4^{j-1} + 1\right)}{3} \\
&= \frac{2^{k+j-1-n} + 2^{k-j-n}}{3}, \text{ as desired.}
\end{aligned}$$

For the third and fourth cases, for $j = k$, the only difference is that for $r = 0$, there is one such permutation σ, instead of $2^{k-j-1} = \frac{1}{2}$. Therefore,

$$\begin{aligned}
A_n^{(1)}(j,k) &= \left(2^{1-n}\right)\left(1 + \sum_{r=1}^{k-1} 2^{r-1}2^{k-j+r-1}\right) \\
&= \left(2^{1-n}\right)\left(1 + \sum_{r=1}^{k-1} 4^{r-1}\right) \\
&= \left(2^{1-n}\right)\left(1 + \frac{4^{k-1}-1}{4-1}\right) \\
&= \frac{2^{2k-1-n} + 2^{2-n}}{3}, \text{ as desired.}
\end{aligned}$$

□

In Figure 3.1, we display a random permutation matrix $\sigma \in \mathcal{S}_{19}(A^{(1)})$. The typical permutation has almost all of its nonzero entries along the diagonal where $j + k = n + 1$. In Figure 3.2, we display the limit shape of $A_n^{(1)}$ for $n = 250$.

3.4.2 Proof of Theorems 3.3.1, 3.3.4, and 3.3.8

The proof follows from two lemmas: one for each case from Theorem 3.3.1.

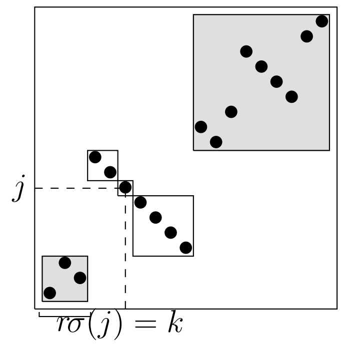


Figure 3.1: Random $\sigma \in \mathcal{S}_n(A^{(1)})$

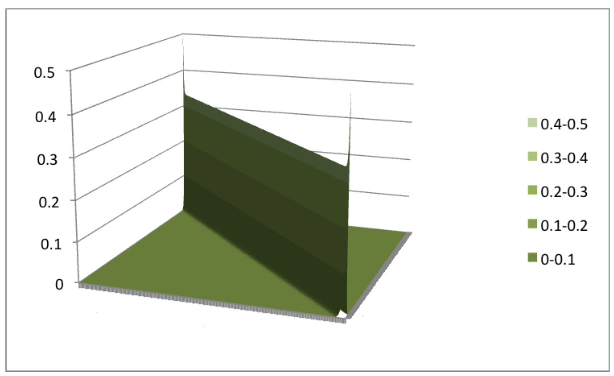


Figure 3.2: Limit shape of $A^{(1)}_{250}$.

Lemma 3.4.3 (First case)**.** *Let* $a, b \in [0,1], c, d \in \mathbb{R}, \alpha, \beta \in [0,1)$ *such that* $a + b \neq 1$. *Then* $F^{(1)}(a, b, c, d, \alpha, \beta) = \infty$. *Moreover, for* n *sufficiently large, we have*

$$A_n^{(1)}(an + cn^{\alpha}, bn + dn^{\beta}) < \varepsilon^n,$$

where ε *is independent of* n *and* $0 < \varepsilon < 1$.

Observe that this lemma proves Theorem 3.3.4, and one case of Theorem 3.3.1.

Proof. Suppose $a + b < 1$, and $a < b$. (We omit the proofs of the other cases, since they are very similar.) Then by Lemma 3.4.2, we have

$$\begin{aligned} A_n^{(1)}(an + cn^\alpha, bn + dn^\beta) &= \frac{2^{an+cn^\alpha+bn+dn^\beta-2} + 2^{bn+dn^\beta-an-cn^\alpha-1}}{3 \cdot 2^{n-1}} \\ &= \frac{1}{6}\left(2^{a+b-1+cn^{\alpha-1}+dn^{\beta-1}}\right)^n + \frac{1}{3}\left(2^{b-a-1+dn^{\beta-1}-cn^{\alpha-1}}\right)^n \\ &\sim \frac{1}{6}\left(2^{a+b-1}\right)^n + \frac{1}{3}\left(2^{b-a-1}\right)^n , \end{aligned}$$

as $n \to \infty$.

Since $a + b < 1$, we see that

$$\varepsilon = \frac{2^{a+b-1} + 1}{2} < 1 .$$

Then for n large enough, we have

$$A_n^{(1)}(an + cn^\alpha, bn + dn^\beta) < \varepsilon^n .$$

We also get $F(1, a, b) = \infty$, as desired.

Proofs of the cases where $a + b > 1$ or $a > b$ are very similar, so we omit them to avoid repetition. □

Lemma 3.4.4 (Second case)**.** *Let $a \in (0, 1), c, d \in \mathbb{R}, \alpha, \beta \in [0, 1)$ with α, β not both equal to 0. Let $\gamma = \max\{\alpha, \beta\}$, so $\gamma > 0$. Then we have*

$$F^{(1)}(a, 1 - a, c, d, \alpha, \beta) = \infty .$$

Furthermore, there exists $\varepsilon \in (0, 1)$ such that

$$A_n^{(1)}(an + cn^\alpha, bn + dn^\beta) < \varepsilon^{n^\gamma} ,$$

for n large enough.

Observe that this lemma proves the remaining cases of Theorem 3.3.1 for which

$$F^{(1)}(a, b, c, d, \alpha, \beta) = \infty .$$

Proof. Suppose that $\alpha < \beta$, and $d < 0$. We omit proofs of the other possible relationships between α and β, and between c, d and 0, since the proofs are very similar. As in the proof of Lemma 3.4.3, we use Lemma 3.4.2 to see that

$$\begin{aligned} A_n^{(1)}(an + cn^\alpha, (1-a)n + dn^\beta) &= \frac{2^{an+cn^\alpha+(1-a)n+dn^\beta-2} + 2^{(1-a)n+dn^\beta-an-cn^\alpha-1}}{3 \cdot 2^{n-1}} \\ &= \frac{1}{6}\left(2^{cn^\alpha+dn^\beta}\right) + \frac{1}{3}\left(2^{dn^\beta-cn^\alpha}\right) \\ &\sim \frac{1}{2}2^{dn^\beta}\,, \end{aligned}$$

as $n \to \infty$.

Letting

$$\varepsilon = \frac{2^d+1}{2}\,,$$

we see that

$$A_n^{(1)}(an + cn^\alpha, (1-a)n + dn^\beta) < \varepsilon^{n^\beta}\,,$$

for n large enough. Also, we get $F^{(1)}(a, 1-a, c, d, \alpha, \beta) = \infty$, as desired. □

Finally, we examine cases where $F^1 \neq \infty$.

Lemma 3.4.5. *Let $a \in [0,1], c, d \in \mathbb{R}$. Then we have*

$$F^{(1)}(a, 1-a, c, d, 0, 0) = 0\,.$$

Furthermore, for $a \in (0,1)$ we have

$$L^{(1)}(a, 1-a, c, d, 0, 0) = \frac{1}{3}2^{-|c+d-1|}\,,$$

and for $a \in \{0,1\}$, we have

$$L^{(1)}(a, 1-a, c, d, 0, 0) = \frac{1}{3}\left(2^{-|c+d-1|} + 2^{-|c-d|}\right)\,.$$

Proving this lemma will show that entries in permutations which avoid $A^{(1)}$ only have nonzero probabilities if they are a constant distance away from the diagonal with $a + b = 1$.

Proof. We prove two cases of this lemma, and omit the remaining proofs since they are very similar. First, suppose $a \in (0, \frac{1}{2})$, and $c + d \leq 1$. Then by Lemma 3.4.2, we have

$$\begin{aligned} A^{(1)}(an + cn^{\alpha}, bn + dn^{\beta}) &= \frac{2^{an+c+(1-a)n+d-2} + 2^{(1-a)n+d-an-c-1}}{3 \cdot 2^{n-1}} \\ &= \frac{1}{3} 2^{c+d-1} + \frac{1}{3} 2^{-2an+d-c} \\ &\sim \frac{1}{3} 2^{-|c+d-1|}\,, \end{aligned}$$

as desired.

Second, suppose $a = 0$, and $c + d \leq 1$. Again, by Lemma 3.4.2, we have

$$\begin{aligned} A^{(1)}(an + cn^{\alpha}, bn + dn^{\beta}) &= \frac{2^{c+n+d-2} + 2^{n+d-c-1}}{3 \cdot 2^{n-1}} \\ &= \frac{1}{3} 2^{c+d-1} + \frac{1}{3} 2^{d-c}\,. \end{aligned}$$

Since $k \leq n$, and $k = n + d$, we need $d \leq 0$. Also, since $j \geq 1$, and $j = 0n + c = c$, we have $c \geq 1$. Therefore, $d - c \leq 0$. Similarly, $c + d - 1 \leq 0$, so we see that

$$A^{(1)}(an + cn^{\alpha}, bn + dn^{\beta}) = \frac{1}{3} \left(2^{-|c+d-1|} + 2^{-|d-c|} \right) ,$$

matching the claim in the lemma.

The remaining cases are very similar calculations, and in each possibility we see that

$$F^{(1)}(a, 1 - a, c, d, 0, 0) = 0\,.$$

This completes all cases of Theorems 3.3.1, 3.3.4, and 3.3.8, as desired.

□

3.5 Analysis of $A^{(2)}$-avoiding permutations

Here we analyze the limit shape of $\mathcal{S}_n(A^{(2)})$, where $A^{(2)} = \{\mathbf{132},\ \mathbf{231}\}$. We first observe a result on symmetry of $A_n^{(2)}(j, k)$.

3.5.1 Symmetry and explicit formulas for $A^{(2)}$-avoiding permutations

Lemma 3.5.1. *For all $n \in \mathbb{N}, 1 \leq j, k \leq n$, we have*

$$A_n^{(2)}(j,k) = A_n^{(2)}(n+1-j,k)\,.$$

Proof. Since $\mathbf{132} = \mathbf{231}^R$, we observe that $\sigma \in \mathcal{S}_n(A^{(2)})$ with $\sigma(j) = k$ implies that $\sigma^R \in \mathcal{S}_n(A^{(2)})$, with $\sigma^R(n+1-j) = k$. Therefore, $A_n^{(2)}(j,k) = A_n^{(2)}(n+1-j,k)$, as desired.

$\square$

We can observe more about the typical $\sigma \in A_n^{(2)}(j,k)$ as well; any such $\sigma = (\sigma(1), \ldots, \sigma(n))$ must have

$$\sigma(1) > \sigma(2) > \ldots \sigma(i-1) > 1 = \sigma(i) < \sigma(i+1) < \ldots < \sigma(n),$$

for some i.

In the next lemma, we calculate the explicit value of $A_n^{(2)}(j,k)$ for any j, k.

Lemma 3.5.2. *For all $n \in \mathbb{N}, 1 \leq j, k \leq n$, we have*

$$A_n^{(2)}(j,k) = \frac{\binom{n-k}{n-j} + \binom{n-k}{j-1}}{2^{n-k+1}}\ ,$$

where $\binom{a}{b} = 0$ for $b < 0$ or $b > a$.

Proof. We prove Lemma 3.5.2 by a counting argument. Let $\sigma \in \mathcal{S}_n(A^{(2)})$ with $\sigma(j) = k$. Let $c, d \in [k-1]$. Since σ avoids $\mathbf{132}$ and $\mathbf{231}$, we must have $\sigma^{-1}(c) < j$ if and only if $\sigma^{-1}(d) < j$. In other words, the values smaller than k must appear on the same side of k in σ. So either $\sigma^{-1}(1) < j$ or $\sigma^{-1}(1) > j$.

If $\sigma^{-1}(1) < j$, then there are 2^{k-2} ways to arrange the elements in $[k-1]$. When arranging the elements d greater than k, we only need to choose which $n-j$ of them have $\sigma^{-1}(d) > j$. This case gives us $2^{k-2}\binom{n-k}{n-j}$ such possible permutations σ.

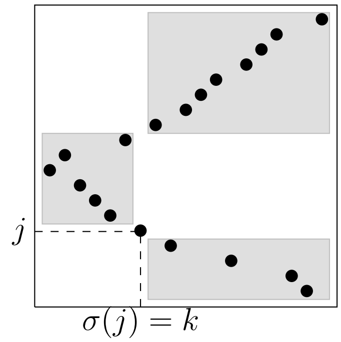


Figure 3.3: Random $\sigma \in \mathcal{S}_n(A^{(2)})$

The other possibility is $\sigma^{-1}(1) > j$. Again, there are 2^{k-2} ways to arrange the elements in $[k-1]$. When arranging the elements d greater than k, we need to choose which $j-1$ of them have $\sigma^{-1}(d) < j$. Overall, there are $2^{k-2}\binom{n-k}{j-1}$ possibilities for σ.

Therefore, in all, we have

$$A_n^{(2)}(j,k) = \frac{2^{k-2}\left(\binom{n-k}{n-j} + \binom{n-k}{j-1}\right)}{2^{n-1}} = \frac{\binom{n-k}{n-j} + \binom{n-k}{j-1}}{2^{n-k+1}},$$

as desired. $\square$

This argument can be visualized with the help of Figure 3.3. Also, in Figure 3.4, we display the limit shape of $A_n^{(2)}$ for $n = 250$.

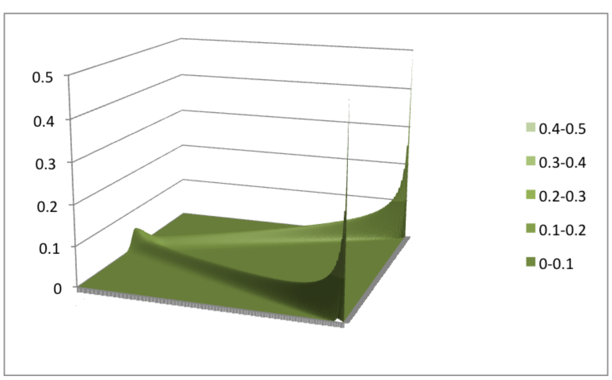


Figure 3.4: Limit shape of $A_{250}^{(2)}$.

3.5.2 Proof of Theorems 3.3.3, 3.3.6, and 3.3.9

The proof of these theorems follows from several lemmas: a technical lemma, Lemma 3.5.2, and a lemma for each case of Theorem 2.3.3.

Lemma 3.5.3. *Let $f : [0,1]^2 \to \mathbb{R}$ be defined so that*

$$f(x,y) = \frac{(x+y)^{(x+y)}}{(2x)^x(2y)^y}.$$

Then $f(x,y)$ achieves a maximum of 1 at $x = y$, for $x \in [0,1]$.

Proof. First we consider the boundaries of the unit square.

For $x = 0$, we have

$$f(0,y) = \frac{y^y}{(2y)^y} = \left(\frac{1}{2}\right)^y,$$

which is maximized at $y = 0$, with $f(0,0) = 1$.

For $x = 1$, we have

$$f(1,y) = \frac{(1+y)^{(1+y)}}{2(2y)^y} = \frac{1+y}{2}\left(\frac{1+y}{2y}\right)^y.$$

Differentiating with respect to y gives

$$f'(1,y) = \left(\frac{1+y}{2y}\right)^y \left(\frac{1}{2} + \frac{1+y}{2}\left(\ln\frac{1+y}{2y} + \frac{y^2}{1+y}\right)\right),$$

which is always positive since each term is positive.

Therefore, $f(1,y)$ is maximized at $y = 1$, with

$$f(1,1) = \frac{2^2}{2(2)} = 1.$$

The analysis for $y = 0$ and $y = 1$ yields the same results, since the function is symmetrical in x and y.

Finally, we consider $f(x,y)$ with $x, y \in (0,1)$. Here we take partial derivatives with respect to x and y. We have

$$\frac{\partial f}{\partial x} = \frac{(2x)^x(2y)^y(x+y)^{x+y}\left(\ln(x+y) - \ln 2x\right)}{(2x)^{2x}(2y)^{2y}},$$

which is 0 when $x = y$.

Due to the symmetry of $f(x, y)$, the other partial derivative has the same behavior. Since

$$f(x, x) = \frac{(2x)^{2x}}{(2x)^x (2x)^x} = 1,$$

we have completed the proof.

$\square$

Lemma 3.5.4. *Let $a, b \in [0, 1], c, d \in \mathbb{R}, \alpha, \beta \in [0, 1)$, with $b \notin \{1 - 2a, 2a - 1\}$. Then*

$$F^{(2)}(a, b, c, d, \alpha, \beta) = \infty\,,$$

and there exists $\varepsilon \in (0, 1)$ such that $A_n^{(2)}(a, b, c, d, \alpha, \beta) < \varepsilon^n$ for n large enough.

Proof. By applying Lemma 3.5.2 and Stirling's formula, and using the Taylor expansion for $\ln(1 + x)$, we get

$$\begin{aligned}
A_n^{(2)}(an + cn^\alpha, bn + dn^\beta) &\sim \frac{\binom{(1-b)n - dn^\beta}{(1-a)n - cn^\alpha} + \binom{(1-b)n - dn^\beta}{an + cn^\alpha - 1}}{2^{(1-b)n - dn^\beta + 1}} \\
&\sim \sqrt{\frac{1 - b}{8\pi(1 - a)(a - b)n}}\, (f(1 - a, a - b))^n \\
&\quad \times \left(\frac{(1 - b)^{-dn^\beta}}{(2 - 2a)^{-cn^\alpha} (2a - 2b)^{cn^\alpha - dn^\beta}} \right) \\
&\quad \times \exp\left[-\frac{(dn^\beta - 2cn^\alpha)^2}{4(1 - a)n} \right] \\
&\quad + \sqrt{\frac{1 - b}{8a\pi(1 - a - b)n}}\, (f(a, 1 - a - b))^n \\
&\quad \times \left(\frac{(1 - b)^{-dn^\beta}}{(2a)^{-cn^\alpha} (2 - 2a - 2b)^{cn^\alpha - dn^\beta}} \right) \\
&\quad \times \exp\left[-\frac{(dn^\beta + 2cn^\alpha)^2}{4an} \right],
\end{aligned}$$

as $n \to \infty$.

Since $b \notin \{1 - 2a, 2a - 1\}$, we see that $1 - a \neq a - b$ and $a \neq 1 - a - b$. Therefore, by Lemma 3.5.3, we see that each term decays exponentially with n.

Letting

$$\varepsilon = \frac{\max f(1-a, a-b), f(a, 1-a-b)+1}{2},$$

we see that for n large enough we have $A_n^{(2)}(an+cn^\alpha, bn+dn^\beta) < \varepsilon^n$. Multiplying by n^r will not change this fact for any positive r, so we have $F^{(2)}(a,b,c,d,\alpha,\beta) = \infty$ as well, as desired.

$\square$

Lemma 3.5.5. *Let $a, b \in [0,1], c, d \in \mathbb{R}, \alpha, \beta \in [0,1)$ with $b \neq 1, b \in \{1-2a, 2a-1\}$. Let $\gamma = \max\{\alpha, \beta\}$, and suppose $\gamma > \frac{1}{2}$. Then we have*

$$F^{(2)}(a,b,c,d,\alpha,\beta) = \infty\,.$$

Furthermore, there exists $\varepsilon \in (0,1)$ such that we have

$$A^{(2)}(a,b,c,d,\alpha,\beta) < \varepsilon^{n^{2\gamma-1}},$$

for n large enough.

Proof. We prove the lemma for $b = 1-2a$; the proof for $b = 2a-1$ is almost identical. Since $b = 1-2a$, we have either $f(1-a, a-b) = 1$, by Lemma 3.5.3. By the same analysis as in the proof of Lemma 3.5.4, we see that

$$\begin{aligned} A_n^{(2)}(an+cn^\alpha, bn+dn^\beta) = & \sqrt{\frac{1-b}{8\pi(1-a)(a-b)n}} \exp\left[-\frac{(dn^\beta - 2cn^\alpha)^2}{4(1-a)n}\right], \\ & + \sqrt{\frac{1-b}{8a\pi(1-a-b)n}} \left(f(a, 1-a-b)\right)^n \\ & \times \left(\frac{(1-b)^{-dn^\beta}}{(2a)^{-cn^\alpha}(2-2a-2b)^{cn^\alpha - dn^\beta}}\right) \\ & \times \exp\left[-\frac{(dn^\beta + 2cn^\alpha)^2}{4an}\right] \end{aligned}$$

Since $\gamma > \frac{1}{2}$, the first term grows asymptotically like $\exp -Cn^{2\gamma-1}$, for some constant C. Even if the second term has $f(a,1,a,b) \neq 1$, we still get

$$A_n^{(2)}(a,b,c,d,\alpha,\beta) < \varepsilon^{n^{2\gamma-1}},$$

for n large enough. Also, as in the previous lemma, multiplying by any positive power of n does not change this fact, so $F^{(2)}(a,b,c,d,\alpha,\beta)=\infty$, as desired.

As mentioned before, the argument is very similar if $b=1-2a$, rather than $b=2a-1$, so we omit it.

$\square$

Lemma 3.5.6. *Let $a,b\in[0,1], c,d\in\mathbb{R}, \alpha,\beta\in[0,1)$. For $b\in\{1-2a,2a-1\}, b\neq 1, \alpha,\beta\leq\frac{1}{2}$, we have*

$$F^{(2)}(a,b,c,d,\alpha,\beta)=\frac{1}{2}\,.$$

Furthermore, we have

$$L^{(2)}(a,b,c,d,\alpha,\beta)=\begin{cases}\sqrt{\frac{2}{\pi}} & b=0,\alpha,\beta<\frac{1}{2}\,,\\ \sqrt{\frac{2}{\pi}}\exp\left[-2c^2\right] & b=0,\alpha=\frac{1}{2}>\beta\,,\\ \sqrt{\frac{2}{\pi}}\exp\left[-\frac{d^2}{2}\right] & b=0,\beta=\frac{1}{2}>\alpha\,,\\ \sqrt{\frac{2}{\pi}}(\exp\left[-\frac{(d-2c)^2}{2}\right] & \\ \qquad\qquad +\exp\left[-\frac{(d+2c)^2}{2}\right]) & b=0,\alpha=\beta=\frac{1}{2}\,,\end{cases}$$

and

$$L^{(2)}(a,b,c,d,\alpha,\beta)=\begin{cases}\sqrt{\frac{1}{4(1-a)\pi}} & b=2a-1>0,\alpha,\beta<\frac{1}{2}\,,\\ \sqrt{\frac{1}{4(1-a)\pi}}\exp\left[-\frac{c^2}{1-a}\right] & b=2a-1>0,\alpha=\frac{1}{2}>\beta\,,\\ \sqrt{\frac{1}{4(1-a)\pi}}\exp\left[-\frac{d^2}{4(1-a)}\right] & b=2a-1>0,\beta=\frac{1}{2}>\alpha\,,\\ \sqrt{\frac{1}{4(1-a)\pi}}\exp\left[-\frac{(d-2c)^2}{4(1-a)}\right] & b=2a-1>0,\alpha=\beta=\frac{1}{2}\,,\\ \sqrt{\frac{1}{4a\pi}} & b=1-2a>0,\alpha,\beta<\frac{1}{2}\,,\\ \sqrt{\frac{1}{4a\pi}}\exp\left[-\frac{c^2}{a}\right] & b=1-2a>0,\alpha=\frac{1}{2}>\beta\,,\\ \sqrt{\frac{1}{4a\pi}}\exp\left[-\frac{d^2}{4a}\right] & b=1-2a>0,\beta=\frac{1}{2}>\alpha\,,\\ \sqrt{\frac{1}{4a\pi}}\exp\left[-\frac{(d+2c)^2}{4a}\right] & b=1-2a>0,\alpha=\beta=\frac{1}{2}\,,\end{cases}$$

Proof. We consider three separate cases of b, α, and β, and omit since the rest since the proofs are very similar. First, suppose $b = 2a - 1 > 0$, with $\alpha, \beta < \frac{1}{2}$.

By the same argument as in the proof of Lemma 3.5.5, we have

$$\begin{aligned} A_n^{(2)}(an + cn^\alpha, bn + dn^\beta) \sim & \sqrt{\frac{1-b}{8\pi(1-a)(a-b)n}} \exp\left[-\frac{(dn^\beta - 2cn^\alpha)^2}{4(1-a)n}\right] \\ & + \sqrt{\frac{1-b}{8a\pi(1-a-b)n}} \left(f(a, 1-a-b)\right)^n \\ & \times \left(\frac{(1-b)^{-dn^\beta}}{(2a)^{-cn^\alpha}(2-2a-2b)^{cn^\alpha - dn^\beta}}\right) \\ & \times \exp\left[-\frac{(dn^\beta + 2cn^\alpha)^2}{4an}\right]. \end{aligned}$$

as $n \to \infty$. Since $b > 0$, we have $a \neq 1 - a - b$, so $f(a, 1-a-b) < 1$. Also, since $\alpha, \beta < \frac{1}{2}$, we have

$$\frac{-(dn^\beta - 2cn^\alpha)^2}{4(1-a)n} \to 0$$

as $n \to \infty$.

Plugging these in, we see that

$$n^r A_n^{(2)}(an + cn^\alpha, bn + dn^\beta) \sim n^{r-\frac{1}{2}} \sqrt{\frac{1}{4\pi(1-a)}},$$

so $F^{(2)}(a, b, c, d, \alpha, \beta) = \frac{1}{2}$ and

$$L^{(2)}(a, b, c, d, \alpha, \beta) = \sqrt{\frac{1}{4\pi(1-a)}},$$

as desired.

Now, we consider another case with $b = 1 - 2a > 0, \alpha = \frac{1}{2} > \beta$. As in the proof of Lemma 3.5.4, we have

$$
\begin{aligned}
A_n^{(2)}(an+cn^\alpha, bn+dn^\beta) &\sim \frac{\binom{(1-b)n-dn^\beta}{(1-a)n-cn^\alpha}+\binom{(1-b)n-dn^\beta}{an+cn^\alpha-1}}{2^{(1-b)n-dn^\beta+1}} \\
&\sim \sqrt{\frac{1-b}{8\pi(1-a)(a-b)n}}\,(f(1-a,a-b))^n \\
&\quad\times\left(\frac{(1-b)^{-dn^\beta}}{(2-2a)^{-cn^\alpha}(2a-2b)^{cn^\alpha-dn^\beta}}\right) \\
&\quad\times\exp\left[-\frac{(dn^\beta-2cn^\alpha)^2}{4(1-a)n}\right] \\
&\quad+\sqrt{\frac{1-b}{8a\pi(1-a-b)n}}\,(f(a,1-a-b))^n \\
&\quad\times\left(\frac{(1-b)^{-dn^\beta}}{(2a)^{-cn^\alpha}(2-2a-2b)^{cn^\alpha-dn^\beta}}\right) \\
&\quad\times\exp\left[-\frac{(dn^\beta+2cn^\alpha)^2}{4an}\right] \\
&\sim \varepsilon^n+\sqrt{\frac{1-b}{8a\pi(1-a-b)n}}\exp\left[-\frac{(dn^\beta+2cn^\alpha)^2}{4an}\right] \\
&\sim \sqrt{\frac{1}{4a\pi n}}\exp\left[-\frac{c^2}{a}\right],
\end{aligned}
$$

as $n \to \infty$.

We see that $F^{(2)}(a,b,c,d,\alpha,\beta)=\frac{1}{2}$, and

$$
L^{(2)}(a,b,c,d,\alpha,\beta)=\sqrt{\frac{1}{4a\pi}}\exp\left[-\frac{c^2}{a}\right],
$$

as desired.

Finally, we consider one more case, with $b=0, a=\frac{1}{2}$, and $\alpha=\beta=\frac{1}{2}$. As in the previous cases, we see that

$$
\begin{aligned}
A_n^{(2)}(an + cn^\alpha, bn + dn^\beta) &\sim \frac{\binom{(1-b)n-dn^\beta}{(1-a)n-cn^\alpha} + \binom{(1-b)n-dn^\beta}{an+cn^\alpha-1}}{2^{(1-b)n-dn^\beta+1}} \\
&\sim \sqrt{\frac{1-b}{8\pi(1-a)(a-b)n}}\,(f(1-a, a-b))^n \\
&\quad \times \left(\frac{(1-b)^{-dn^\beta}}{(2-2a)^{-cn^\alpha}(2a-2b)^{cn^\alpha-dn^\beta}}\right) \\
&\quad \times \exp\left[-\frac{(dn^\beta - 2cn^\alpha)^2}{4(1-a)n}\right] \\
&\quad + \sqrt{\frac{1-b}{8a\pi(1-a-b)n}}\,(f(a, 1-a-b))^n \\
&\quad \times \left(\frac{(1-b)^{-dn^\beta}}{(2a)^{-cn^\alpha}(2-2a-2b)^{cn^\alpha-dn^\beta}}\right) \\
&\quad \times \exp\left[-\frac{(dn^\beta + 2cn^\alpha)^2}{4an}\right] \\
&\sim \sqrt{\frac{1}{2\pi n}}\exp\left[-\frac{(d-2c)^2}{2}\right] \\
&\quad + \sqrt{\frac{1}{2\pi n}}\exp\left[-\frac{(d+2c)^2}{2}\right],
\end{aligned}
$$

as $n \to \infty$.

We see that $F^{(2)}(a, b, c, d, \alpha, \beta) = \frac{1}{2}$, and

$$
L^{(2)}(a, b, c, d, \alpha, \beta) = \sqrt{\frac{1}{2\pi}}\left(\exp\left[-\frac{(d-2c)^2}{2}\right] + \exp\left[-\frac{(d+2c)^2}{2}\right]\right),
$$

as desired. The remaining cases are very similar, so we omit their proofs for brevity.

$\square$

The remaining cases of Theorems 3.3.3, 3.3.6, and 3.3.9 all involve $b = 1$.

Lemma 3.5.7. *Let $a \in \{0, 1\}, c, d \in \mathbb{R}, \alpha, \beta \in [0, 1)$. Suppose $\alpha \neq \beta$, and let $\gamma = \max\{\gamma\}$. Then*

$$
F^{(2)}(a, 1, c, d, \alpha, \beta) = \infty,
$$

and there exists $\varepsilon \in (0,1)$ *such that*

$$A_n^{(2)}(an + cn^\alpha, bn + dn^\beta) < \varepsilon^{n^\gamma},$$

for n *large enough.*

Proof. Let $a = 0$; the proof for $a = 1$ is very similar, so we omit it. By Lemma 3.5.2, we have

$$\begin{aligned} A_n^{(2)}(an + cn^\alpha, bn + dn^\beta) &= \frac{\binom{-dn^\beta}{n-cn^\alpha} + \binom{-dn^\beta}{cn^\alpha - 1}}{2^{-dn^\beta+1}}, \\ &\sim \frac{\binom{-dn^\beta}{cn^\alpha-1}}{2^{-dn^\beta+1}}. \end{aligned}$$

This function is 0 for $\alpha > \beta$, so we must have $\alpha < \beta$. Using Stirling's formula, and the Taylor expansion for $\ln(1+x)$, we have

$$A_n^{(2)}(an + cn^\alpha, bn + dn^\beta) \sim \frac{1}{2^{-dn^\beta+1}}\sqrt{\frac{1}{2\pi cn^\alpha}}\left(\frac{-ed}{c}n^{\beta-\alpha}\right)^{cn^\alpha}.$$

Letting $\varepsilon = \frac{\frac{1}{2-d}+1}{2}$, we see that

$$A_n^{(2)}(an + cn^\alpha, bn + dn^\beta) < \varepsilon^{n^\beta},$$

for n large enough, as desired. Furthermore, multiplying by any positive power of n will not change this fact, so we have

$$F^{(2)}(a,b,c,d,\alpha,\beta) = \infty,$$

as desired. □

Lemma 3.5.8. *Let* $a \in \{0,1\}, c,d \in \mathbb{R}, \alpha \in (0,1)$, *and suppose* $2c \notin \{d,-d\}$. *Then*

$$F^{(2)}(a,1,c,d,\alpha,\alpha) = \infty,$$

and there exists $\varepsilon \in (0,1)$ *such that*

$$A_n^{(2)}(an + cn^\alpha, bn + dn^\beta) < \varepsilon^{n^\alpha},$$

as $n \to \infty$.

Proof. Let $a = 1$; again, the proof for $a = 0$ is very similar so we omit it for brevity. As in the proof of Lemma 3.5.7, we have

$$A_n^{(2)}(an + cn^\alpha, bn + dn^\beta) \sim \frac{\binom{-dn^\alpha}{-cn^\alpha} + \binom{-dn^\alpha}{n+cn^\alpha-1}}{2^{-dn^\alpha+1}} .$$

For $-c > -d$, we have $A_n^{(2)}(an + cn^\alpha, bn + dn^\beta) = 0$. For $-c \le -d$, we have

$$A_n^{(2)}(an + cn^\alpha, bn + dn^\beta) \sim \sqrt{\frac{-d}{2\pi(-c)(-d+c)}} \left(f(\frac{c}{d}, 1 - \frac{c}{d})\right)^{n^\alpha} ,$$

using notation from Lemma 3.5.3. Since $2c \neq d$, we have $f(\frac{c}{d}, 1 - \frac{c}{d}) < 1$, and letting

$$\varepsilon = \frac{f(\frac{c}{d}, 1 - \frac{c}{d}) + 1}{2} ,$$

we see that

$$A_n^{(2)}(an + cn^\alpha, bn + dn^\beta) < \varepsilon^{n^\alpha} ,$$

for n large enough, as desired. □

Lemma 3.5.9. *Let $a \in \{0, 1\}, c \in \mathbb{R}, \alpha \in (0, 1)$. Then we have*

$$F^{(2)}(a, 1, c, 4c\left(a - \frac{1}{2}\right), \alpha, \alpha) = \frac{\alpha}{2} ,$$

and

$$L^{(2)}(a, 1, c, d, \alpha, \alpha) = \sqrt{\frac{1}{4\pi|c|}} .$$

Proof. Let $a = 0$, so $d = -2c$, and $c > 0$, since $an + cn^\alpha \ge 1$. As in the proof of Lemma 3.5.7, and by Stirling's formula, we have

$$\begin{aligned} A_n^{(2)}(an + cn^\alpha, bn + dn^\beta) &\sim \frac{\binom{2cn^\alpha}{n-cn^\alpha} + \binom{2cn^\alpha}{cn^\alpha-1}}{2^{2cn^\alpha+1}} , \\ &\sim \frac{1}{2}\sqrt{\frac{2cn^\alpha}{2\pi c^2 n 2\alpha}} = \sqrt{\frac{1}{4cn^\alpha}} , \end{aligned}$$

as $n \to \infty$. Clearly, we have

$$F^{(2)}(a, 1, c, 4c\left(a - \frac{1}{2}\right), \alpha, \alpha) = \frac{\alpha}{2} ,$$

and

$$L^{(2)}(a, 1, c, 4c\left(a - \frac{1}{2}\right), \alpha, \alpha) = \sqrt{\frac{1}{4\pi|c|}}\,,$$

as desired. □

Lemma 3.5.10. *Let $c, d \in \mathbb{R}$. Then we have*

$$F^{(2)}(0, 1, c, d, 0, 0) = \begin{cases} 0 & -d \geq c - 1 \\ \infty & -d < c - 1 \end{cases}.$$

Also, for $-d \geq c - 1$ we have

$$L^{(2)}(0, 1, c, d, 0, 0) = 2^{d-1}\binom{-d}{c-1}\,,$$

and for $-d < c - 1$ there exists $\varepsilon \in (0, 1)$ such that

$$A_n^{(2)}(c, n + d) = 0\,.$$

for all n.

Proof. By Lemma 3.5.2, we have

$$\begin{aligned} A_n^{(2)}(c, n + d) &= \frac{\binom{-d}{n-c} + \binom{-d}{c-1}}{2^{-d+1}} \\ &= \begin{cases} 0 & -d < c - 1\,, \\ \frac{\binom{-d}{c-1}}{2^{-d+1}} & -d \geq c - 1\,, \end{cases} \end{aligned}$$

as desired. □

Lemma 3.5.11. *Let $c, d \in \mathbb{R}$. Then we have*

$$F^{(2)}(1, 1, c, d, 0, 0) = \begin{cases} 0 & -d \geq -c \\ \infty & -d < -c \end{cases}.$$

Also, for $-d \geq -c$ we have

$$L^{(2)}(1, 1, c, d, 0, 0) = 2^{d-1}\binom{-d}{-c}\,,$$

and for $-d < -c$ there exists $\varepsilon \in (0,1)$ such that

$$A_n^{(2)}(n+c, n+d) = 0\,.$$

for all n.

Proof. By Lemma 3.5.2, we have

$$\begin{aligned} A_n^{(2)}(n+c, n+d) &= \frac{\binom{-d}{-c} + \binom{-d}{n+c-1}}{2^{-d+1}} \\ &= \begin{cases} 0 & -d < -c\,, \\ \frac{\binom{-d}{-c}}{2^{-d+1}} & -d \geq -c\,, \end{cases} \end{aligned}$$

completing the proof of Lemma 3.5.11, and of Theorems 3.3.3, 3.3.6, and 3.3.9. □

3.6 Analysis of $A^{(3)}$-avoiding permutations

3.6.1 Explicit formulas for $A^{(3)}$-avoiding permutations

In this section we consider the limit shape of $\sigma \in \mathcal{S}_n(A^{(3)})$, where

$$A^{(3)} = \{\mathbf{123,132}\}\,.$$

We first prove a lemma calculating the explicit value of $A_n^{(3)}(j,k)$.

Lemma 3.6.1. *For all $n \in \mathbb{N}, 1 \leq j,k \leq n$, we have*

$$A_n^{(3)}(j,k) = \begin{cases} 0 & j+k < n \\ 2^{1-n} & j=n, k=n \\ 2^{-j} & k=n, j \neq n \\ 2^{-k} & j=n, k\neq n \\ 2^{n-j-k-1} & \textit{otherwise}\,. \end{cases}$$

Proof. Recall that $A_n^{(3)}(j,k)$ represents the probability of a permutation $\sigma \in \mathcal{S}_n(A^{(3)})$ chosen uniformly at random has $\sigma(j) = k$.

We calculate this probability by creating σ uniformly at random by defining the elements $\sigma(1), \sigma(2), \ldots, \sigma(n)$ sequentially . We will show that for each $i \leq n-1$, there are two choices for i, no matter what $\sigma(1), \ldots, \sigma(i-1)$ are. This matches the fact that $|\mathcal{S}_n(A^{(3)})| = 2^{n-1}$, and also gives us $A_n^{(3)}(j,k)$ in the meantime.

The procedure is defined as follows: At step 0, let $i = 0$ and let $r_i = n$. For $1 \leq i \leq n-1$, at step i, with probability $\frac{1}{2}$, choose $\sigma(i) = n-i$ and choose $r_i = r_{i-1}$. With probability $\frac{1}{2}$, choose $\sigma(i) = r_{i-1}$ and choose $r_i = n-i$. After $n-1$ steps, let $\sigma(n) = r_{n-1}$.

Observe that this procedure leads to 2^{n-1} possible permutations. Also, each permutation avoids **123** and **132**, since at each step i, $|\{a > i : \sigma(a) > \sigma(i)\}| \in \{0, 1\}$.

Clearly, $\sigma(j) \geq n-j$, so $A_n^{(3)}(j,k) = 0$ for $k < n-j$. Also, $A_n^{(3)}(j, n-j) = \frac{1}{2} = 2^{n-j-(n-j)-1}$, as desired. For $j = k = n$, the procedure must select $n-i$ at each step, so $A_n^{(3)}(n,n) = 2^{1-n}$. For $j = n$ and $k \neq n$, we must have $r_{n-1} = k$, which means that the procedure must select $\sigma(n-k) = r_{n-k-1}$, and then $\sigma(i) = n-i$ for steps $i = n-k+1$ through $i = n-1$. Therefore, $A_n^{(3)}(n,k) = 2^{-(1+k-1)} = 2^{-k}$. Similarly, for $j \neq n$ and $k = n$, we must have $\sigma(i) = n-i$ for steps $i = 1$ through $i = j-1$, and then $\sigma(j) = k$. Therefore, $A_n^{(3)}(j,n) = 2^{-j}$. Finally, for $j+k \geq n+1$, with $j \neq n$ and $k \neq n$, the procedure must select $\sigma(n-k) = r_{n-k-1}$, $\sigma(i) = n-i$ for steps $i = n-k+1$ through $i = j-1$, and then $\sigma(j) = r_{j-1} = k$. The probability of this happening is $2^{-(j-1-(n-k+1)+1+2)} = 2^{n-k-j-1}$, as desired.

$\square$

In Figure 3.5, we display a random $\sigma \in \mathcal{S}_n(A^{(3)})$ to illustrate the limit shape.

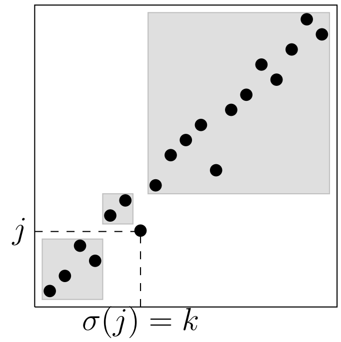


Figure 3.5: Random $\sigma \in \mathcal{S}_n(A^{(3)})$

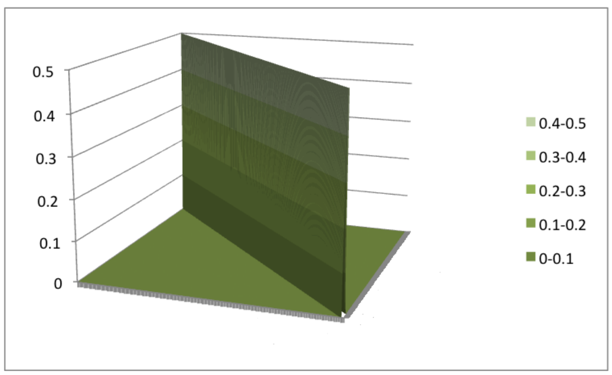


Figure 3.6: Limit shape of $A_{250}^{(3)}$.

3.6.2 Proof of Theorems 3.3.2, 3.3.5, and 3.3.10

We prove these theorems in several lemmas, which exhaust all the cases of the theorem. First, we consider values of a, b with $a + b \neq 1$.

Lemma 3.6.2. *Let* $a, b \in [0, 1], c, d \in \mathbb{R}, \alpha, \beta \in [0, 1)$, *such that* $a + b \neq 1$. *Then*

$$F^{(3)}(a, b, c, d, \alpha, \beta) = \infty\,.$$

Moreover, there exists $\varepsilon \in (0, 1)$ *such that*

$$A_n^{(3)}(an + cn^{\alpha}, bn + dn^{\beta}) < \varepsilon^n\,.$$

Proof. First, let $a + b < 1$. By Lemma 3.6.1, we see that

$$A_n^{(3)}(an + cn^{\alpha}, bn + dn^{\beta}) = 0\,,$$

so any $\varepsilon \in (0,1)$ would work. Also, multiplying by n^r is still zero for all positive r, so $F^{(3)}(a,b,c,d,\alpha,\beta) = \infty$, as desired.

Now, consider $a+b>1$. Again, by Lemma 3.6.1, we have

$$A_n^{(3)}(an+cn^\alpha, bn+dn^\beta) \leq 2^{n-(an+cn^\alpha)-(bn+dn^\beta)+1}\,,$$

since each nonzero case of Lemma 3.6.1 is no bigger than $2^{n-j-k+1}$. Since $a+b>1$, setting ε equal to

$$\varepsilon = \frac{1+2^{1-a-b}}{2}$$

gives us $\varepsilon \in (0,1)$, and also gives

$$A_n^{(3)}(an+cn^\alpha, bn+dn^\beta) < \varepsilon^n$$

for n large enough, as desired. □

Now, we have several lemmas corresponding to cases with $a+b=1$.

Lemma 3.6.3. *Let $a \in [0,1], c,d \in \mathbb{R}, \alpha,\beta \in [0,1)$. Let $\gamma = \max\{\alpha,\beta\}$, and suppose $\gamma > 0$. Then*

$$F^{(3)}(a,1-a,c,d,\alpha,\beta) = \infty\,.$$

Furthermore, there exists $\varepsilon \in (0,1)$ such that

$$A_n^{(3)}(an+cn^\alpha, bn+dn^\beta) < \varepsilon^{n^\gamma}\,,$$

for n large enough.

Proof. First, for $cn^\alpha + dn^\beta < 0$, by Lemma 3.6.1, we have $A_n^{(3)}(an+cn^\alpha,(1-a)n+dn^\beta) = 0$, which satisfies the lemma.

Now, consider $cn^\alpha + dn^\beta \geq 0$. Again, by Lemma 3.6.1, we have

$$\begin{aligned} A_n^{(3)}(an+cn^\alpha, bn+dn^\beta) &\leq 2^{n-(an+cn^\alpha)-(bn+dn^\beta)+1}\,, \\ &= 2^{-cn^\alpha-dn^\beta+1}\,. \end{aligned}$$

Now, suppose $\alpha > \beta$. With

$$\varepsilon = \frac{1 + 2^{-c}}{2}\,,$$

we have

$$A_n^{(3)}(an + cn^\alpha, bn + dn^\beta) < \varepsilon^{n^\alpha}\,,$$

for n large enough, as desired.

The proofs for $\alpha = \beta$ and $\alpha < \beta$ are very similar, so we omit them here for brevity.

□

Lemma 3.6.4. *Let $a \in [0,1], c, d \in \mathbb{R}$. For $c + d < 0$, we have*

$$F^{(3)}(a, 1-a, c, d, 0, 0) = \infty\,,$$

with $A_n^{(3)}(an + c, (1-a)n + d) = 0$. On the other hand, for $c + d \geq 0$, we have

$$F^{(3)}(a, 1-a, c, d, 0, 0) = 0\,,$$

and

$$L^{(3)}(a, 1-a, c, d, 0, 0) = \begin{cases} 2^{-c} & \text{for } a = 0, d = 0\,, \\ 2^{-d} & \text{for } a = 1, c = 0\,, \\ 2^{-c-d-1} & \text{for } a \notin \{0,1\} \text{ or } a = 1, c \neq 0\,, \\ & \quad \text{or } a = 0, d \neq 0\,. \end{cases}$$

Proof. By Lemma 3.6.1, for $c + d < 0$ we have $A_n^{(3)}(an + c, (1-a)n + d) = 0$ for all n. The other cases all follow from Lemma 3.6.1 as well - we exhibit one clearly here and omit the others for brevity.

Suppose $a = 1, c \neq 0, c + d \geq 0$. We are analyzing

$$A_n^{(3)}(an + cn^\alpha, (1-a)n + dn^\beta) = A_n^{(3)}(n + c, d)\,.$$

Since $c \neq 0$, and since $c + d \geq 0$, Lemma 3.6.1 gives us

$$A_n^{(3)}(an + cn^\alpha, (1-a)n + dn^\beta) = 2^{-c-d-1}\,.$$

Since this is a constant, we also get $F^{(3)}(a, 1-a, c, d, 0, 0) = 0$, as desired. This completes the proof of Lemma 3.6.4 and of Theorems 3.3.2, 3.3.5, and 3.3.10.

$\square$

3.7 Analysis of $A^{(4)}$-avoiding permutations

3.7.1 Explicit formulas for $A^{(4)}$-avoiding permutations

Here we analyze permutations which avoid $A^{(4)}$. Unlike the previous three cases, we have

$$|\mathcal{S}_n(A^{(4)})| = \binom{n}{2} + 1\,.$$

We will pay particular attention to $\mathcal{S}_n(\mathbf{123{,}231})$ as we analyze the asymptotics. Note that unlike when analyzing $\mathcal{S}_n(A^{(1)}), \mathcal{S}_n(A^{(2)})$, and $\mathcal{S}_n(A^{(3)})$, the number of permutations avoiding $A^{(4)}$ only grows quadratically with n, rather than exponentially, so there is no possibility of exponentially small probabilities for specific values of $A_n^{(4)}(j,k)$.

The following lemma gives the explicit value of $A_n^{(4)}(j,k)$.

Lemma 3.7.1. *For all $n \in \mathbb{N}, 1 \leq j, k \leq n$, we have*

$$A_n^{(4)}(j,k) = \begin{cases} \frac{j}{\binom{n}{2}+1} & j + k \leq n\,, \\ \frac{n+1-k}{\binom{n}{2}+1} & j + k > n+1\,, \\ \frac{\binom{k-1}{2}+1}{\binom{n}{2}+1} & j + k = n+1\,, \end{cases}$$

where $\binom{k-1}{2} = 0$ for $k = 1, 2$.

Before proving the lemma, we first consider what a typical permutation $\sigma \in \mathcal{S}_n(A^{(4)})$ looks like. Since σ avoids **231**, all elements ahead of 1 must be in decreasing order. Similarly, since σ avoids **123**, all elements after 1 must be in decreasing order as well. Therefore, σ consists of two concatenated decreasing sequences, with the first one ending in 1. As long as σ has this form, it will avoid **123**. Since we also need to avoid **231**, the second decreasing sequence must consist of consecutive elements, since if there is ever a gap between numbers in the second decreasing sequence, they would form the 3 and the 1 in a **231**. One way to count the permutations is to choose the element one bigger than the starting element of the second decreasing sequence, and the ending element of the second decreasing sequence. These numbers are between $n+1$ and 2, and once these two elements are chosen the permutation is uniquely determined. Therefore, there are $\binom{n}{2}$ choices for the starting and ending point, plus one permutation which comes when the second decreasing sequence is empty.

Now we are ready to prove the lemma.

Proof. We prove the lemma in cases.

First, let $j+k \leq n$, and let $\sigma \in \mathcal{S}_n(A^{(4)})$ with $\sigma(j) = k$. Suppose for all $r > j$, we have $\sigma(r) < k$. There are $n-j$ such r, and only $k-1$ numbers smaller than k. This gives a contradiction, since $k-1 < k \leq n-j$, so not every such r can have a unique $\sigma(r)$. Therefore, there must be some $r > j$ with $\sigma(r) > k$.

Because of this, k must be in the first decreasing sequence, and in fact, for all $1 \leq i \leq k-1$, we must have $\sigma(j+i) = k-i$. Therefore, the length of the second decreasing sequence is fixed, at $n-(j+k-1) = n+1-j-k$, which is at least 1 since $j+k \leq n$. The value which then determines the permutation is that of $\sigma(j+k)$, which must be at least $k+(n+1-j-k) = n+1-j$, and can be at most n. There are j such values, and each yields exactly one $\sigma \in \mathcal{S}_n(A^{(4)})$ with $\sigma(j) = k$. This completes the first case of the lemma.

Second, suppose $j + k > n + 1$. Since $j - 1 > n - k$, there must be some $r < j$ such that $\sigma(r) < j$. Therefore, $\sigma(j) = k$ is part of the second decreasing sequence. Because of this, and the fact that the second decreasing sequence is consecutive, σ must have $\sigma(j + i) = k - i$ for all $1 \leq i \leq n - j$.

In particular, $\sigma(n) = k - (n - j) = k + j - n$, so the final element of the second decreasing sequence (and of σ) is fixed. The value which then determines the permutation is the element which starts the second decreasing sequence, which can be any number between k and n. There are $n-k+1$ options, and each yields exactly one $\sigma \in \mathcal{S}_n(A^{(4)})$ with $\sigma(j) = k$, completing the second case of the lemma.

Finally, suppose $j + k = n + 1$. Suppose $\sigma^{-1}(1) < j$, so $\sigma(j)$ is part of the second decreasing sequence. Then for all $j + 1 \leq r \leq n$, we must have $\sigma(r) < k$. There are $n - j = k - 1$ such r, and exactly $k - 1$ possible such values for $\sigma(r)$. Therefore $\sigma^{-1}(1) > j$, and we have a contradiction.

Therefore, $\sigma(j)$ must be part of the first decreasing sequence. For each $1 \leq s \leq j - 1$, we must have $\sigma(s) > \sigma(j) = k$. There are exactly $n - k = j - 1$ possible values for $\sigma(s)$, so we must have $\sigma(s) = n + 1 - s$ for all such s. Therefore $\sigma = (n, n-1, \ldots, k+1, k, \tau)$, where $\tau \in \mathcal{S}_{k-1}(A^{(4)})$. There are $\binom{k-1}{2} + 1$ possible permutations τ, so $\binom{k-1}{2} + 1$ possible permutations σ, completing the lemma.

$\square$

This argument can be visualized with the help of Figure 3.7.

3.7.2 Proof of Theorem 2.3.6

We prove the theorem in cases.

First, suppose $a \in [0, 1)$, and consider $A_n^{(4)}(an, (1-a)n)$. From Lemma 3.7.1, we know that

$$A_n^{(4)}(an, (1-a)n) = \frac{\binom{(1-a)n-1}{2} + 1}{\binom{n}{2} + 1},$$

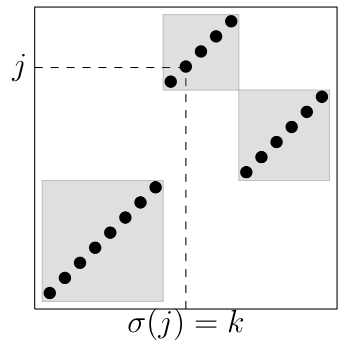


Figure 3.7: Random $\sigma \in \mathcal{S}_n(A^{(4)})$

since we are in the case of the lemma where $j + k = n + 1$.

Letting $n \to \infty$, we see that

$$A_n^{(4)}(an, (1-a)n) \to (1-a)^2 \quad \text{as} \quad n \to \infty\,.$$

Therefore, for any $r > 0$, we have $n^r A_n^{(4)}(an, (1-a)n) \sim n^r(1-a)^2$. Since $0 \leq a < 1$, this quantity is unbounded as $n \to \infty$, so $F^{(4)}(a, 1-a) = 0$ for $a \in [0, 1)$, completing the first case of the theorem.

Second, suppose $a = 1, b = 0$. Again, from Lemma 3.7.1, we know that

$$n^r A_n^{(4)}(n, 1) = n^r \frac{\binom{1-1}{2} + 1}{\binom{n}{2} + 1} \sim 2n^{r-2}\,,$$

as $n \to \infty$. Clearly, $F^{(4)}(1, 0) = 2$, as desired.

Third, suppose $a + b < 1$. From Lemma 3.7.1, we have

$$A_n^{(4)}(an, bn) = \frac{an}{\binom{n}{2} + 1}\,,$$

since $j + k \leq n$.

Multiplying by n^r, we get

$$n^r A_n^{(4)}(an, bn) = n^r \frac{an}{\binom{n}{2} + 1}\,.$$

Letting $n \to \infty$, for $a = 0$, we have

$$n^r A_n^{(4)}(an, bn) \sim 2n^{r-2}\,,$$

so $F^{(4)}(0,b) = 2$.

For $a > 0$, we instead have

$$n^r A_n^{(4)}(an, bn) \sim 2an^{r-1}\,,$$

so $F^{(4)}(a,b) = 1$.

Finally, suppose $a + b > 1$. Again, from Lemma 3.7.1, we have

$$A_n^{(4)}(an, bn) = \frac{\binom{n+1-bn}{\binom{n}{2}+1}}{},$$

since we have $j + k > n + 1$ when n is large.

For $b = 1$, multiplying by n^r and letting $n \to \infty$, we get

$$n^r A_n^{(4)}(an, n) = n^r \frac{\binom{1}{\binom{n}{2}+1}}{\sim} 2n^{r-2}\,.$$

We see that $F^{(4)}(a, 1) = 2$.

Now consider $b < 1$. Multiplying by n^r and letting $n \to \infty$, we get

$$n^r A_n^{(4)}(an, bn) = n^r \frac{(1-b)n + 1}{\binom{n}{2} + 1} \sim 2(1-b)n^{r-1}\,.$$

Here, we see that $F^{(4)}(a,b) = 1$, completing the proof of Theorem 2.3.6. $\square$

3.8 Expected number of fixed points

As in Chapter 2.9.9, the results in Lemmas 3.4.2, 3.5.2, 3.6.1, 3.7.1 help us prove results on the expected number of fixed points in each of these permutation classes.

Theorem 3.8.1. *Let σ be chosen uniformly at random in $\mathcal{S}_{2n}(A^{(1)})$, and τ be chosen uniformly at random in $\mathcal{S}_{2n+1}(A^{(1)})$. Then as $n \to \infty$, we have*

$$\mathbf{E}[fp_{2n}(\sigma)] \to \frac{4}{9}\,,$$

and

$$\mathbf{E}[fp_{2n+1}(\tau)] \to \frac{5}{9}\,.$$

Similarly, we have a result for permutations in $\mathcal{S}_n(A^{(1)^R})$.

Theorem 3.8.2. *Let σ be chosen uniformly at random in $\mathcal{S}_n(\mathbf{231}, \mathbf{312})$. Then as $n \to \infty$, we have*

$$\mathbf{E}[fp_n(\sigma)] \sim \frac{n}{3} + \frac{4}{9}\,.$$

Avoiding **132** and **213** simultaneously makes fixed points less likely to occur than in an arbitrary permutation, or even an arbitrary permutation avoiding either **132** or **213** separately. On the other hand, avoiding **231** and **312** simultaneously makes the expected number of fixed points linear - in fact each position has a probability tending to $\frac{1}{3}$ of being a fixed point.

For $\mathcal{S}_n(A^{(2)})$, our results are similar, though since $A^{(2)} = A^{(2)^R}$, the expected number of fixed points equal the expected number of anti-fixed points.

Theorem 3.8.3. *Let σ be chosen uniformly at random in $\mathcal{S}_n(A^{(2)})$. Then*

$$\mathbf{E}[fp_n(\sigma)] \to \frac{4}{3} \quad \textit{and } \mathbf{E}[afp_n(\sigma)] \to \frac{4}{3}\,,$$

as $n \to \infty$.

For $\mathcal{S}_n(A^{(3)})$, we see that the expected number of fixed points depends on the parity of n, while the expected number of anti-fixed points grows linearly with n.

Theorem 3.8.4. *Let σ be chosen uniformly at random in $\mathcal{S}_{2n}(A^{(3)})$, and τ be chosen uniformly at random in $\mathcal{S}_{2n+1}(A^{(3)})$. Then*

$$\mathbf{E}[fp_{2n}(\sigma)] \to \frac{2}{3} \quad \textit{and } \mathbf{E}[fp_{2n+1}(\tau)] \to \frac{1}{3}\,,$$

as $n \to \infty$.

Theorem 3.8.5. *Let σ be chosen uniformly at random in $\mathcal{S}_n(\boldsymbol{321}, \boldsymbol{231})$. Then*

$$\mathbf{E}[fp_n(\sigma)] = \frac{n}{4} + \frac{1}{2}\,,$$

for all $n \in \mathbb{N}$.

Finally, for $\mathcal{S}_n(A^{(4)})$, we see that the expected number of fixed points again depends on the parity of n, while the expected number of anti-fixed points grows linearly with n.

Theorem 3.8.6. *Let σ be chosen uniformly at random in $\mathcal{S}_{2n}(A^{(4)})$, and τ be chosen uniformly at random in $\mathcal{S}_{2n+1}(A^{(4)})$. Then*

$$\mathbf{E}[fp_{2n}(\sigma)] \to \frac{1}{2} \qquad \text{and } \mathbf{E}[fp_{2n+1}(\tau)] \to \frac{3}{4}\,,$$

as $n \to \infty$.

Theorem 3.8.7. *Let σ be chosen uniformly at random in $\mathcal{S}_n(\boldsymbol{321}, \boldsymbol{231})$. Then*

$$\mathbf{E}[fp_n(\sigma)] = \frac{n}{3} - \frac{2}{3}\,,$$

for all $n \in \mathbb{N}$.

3.8.1 Proof of Theorem 3.8.1

First, suppose σ is uniform in $\mathcal{S}_{2n}(A^{(1)})$. Then

$$\mathbf{E}[\mathrm{fp}_{2n}(\sigma)] = \sum_{j=1}^{2n} A_{2n}^{(1)}(j,j)\,,$$

by linearity of expectation. We evaluate this sum with the help of Lemma 3.4.2. By symmetry, we see that

$$\begin{aligned}
\mathbf{E}[\mathrm{fp}_{2n}(\sigma)] &= 2\sum_{j=1}^{n} \frac{2^{2j-2}+2}{3\cdot 2^{2n-1}} \\
&= \frac{2}{3\cdot 2^{2n-1}}\left(\sum_{j=1}^{n} 2^{2j-2} + \sum_{j=1}^{n} 2\right) \\
&= \frac{2}{3\cdot 2^{2n-1}}\left(\frac{2^{2n}-1}{3} + 2n\right) \\
&= \frac{4}{9} + \frac{4n}{3\cdot 2^{2n-1}} \\
&\sim \frac{4}{9}\,,
\end{aligned}$$

as $n \to \infty$, as desired.

Now, suppose τ is uniform in $\mathcal{S}_{2n+1}(A^{(1)})$. Then

$$\mathbf{E}[\mathrm{fp}_{2n+1}(\tau)] = \sum_{j=1}^{2n+1} A^{(1)}_{2n+1}(j,j)$$

again by linearity of expectation. By Lemma 3.4.2, we have

$$\begin{aligned}
\mathbf{E}[\mathrm{fp}_{2n+1}(\tau)] &= A^{(1)}_{2n+1}(n+1,n+1) + 2\sum_{j=1}^{n} \frac{2^{2j-2}+2}{3\cdot 2^{2n}} \\
&= \frac{2^{2n}+2}{3\cdot 2^{2n}} + \frac{2}{3\cdot 2^{2n}}\left(\sum_{j=1}^{n} 2^{2j-2} + \sum_{j=1}^{n} 2\right) \\
&= \frac{1}{3} + \frac{2}{3\cdot 2^{2n}} + \frac{2}{3\cdot 2^{2n}}\left(\frac{2^{2n}-1}{3} + 2n\right) \\
&= \frac{5}{9} + \frac{2+4n}{3\cdot 2^{2n}} \\
&\sim \frac{5}{9}\,,
\end{aligned}$$

as $n \to \infty$, as desired.

$\square$

3.8.2 Proof of Theorem 3.8.2

Let $\tau \in \mathcal{S}_n(A^{(1)})$ be chosen uniformly at random, and let $\sigma = \tau^R$. Then

$$\mathbf{E}[\mathrm{fp}_n(\sigma)] = \mathbf{E}[\mathrm{afp}_n(\tau)] = \sum_{j=1}^{n} A_n^{(1)}(j, n+1-j)\,.$$

We calculate the cases where n is even and odd separately. First, suppose n is even, so $n = 2r$. By Lemma 3.4.2, we have

$$\begin{aligned}
\mathbf{E}[\mathrm{fp}_n(\sigma)] &= 2\sum_{j=1}^{r} \frac{2^{n-1} + 2^{n-2j}}{3 \cdot 2^{n-1}} \\
&= \frac{2}{3 \cdot 2^{n-1}} \left(\sum_{j=1}^{r} 2^{n-1} + \sum_{j=1}^{r} 2^{n-2j} \right) \\
&= \frac{2}{3 \cdot 2^{n-1}} \left(r2^{n-1} + \frac{2^n - 1}{3} \right) \\
&= \frac{2r}{3} + \frac{4}{9} - \frac{2}{9 \cdot 2^{n-1}} \\
&\sim \frac{n}{3} + \frac{4}{9}\,,
\end{aligned}$$

as $n \to \infty$.

Second, suppose n is odd, so $n = 2r + 1$, for some integer r. Again, by Lemma 3.4.2, we have

$$\begin{aligned}
\mathbf{E}[\mathrm{fp}_n(\sigma)] &= A_{2r+1}^{(1)}(r+1, r+1) + 2\sum_{j=1}^{r} \frac{2^{n-1} + 2^{n-2j}}{3 \cdot 2^{n-1}} \\
&= \frac{2^{2r} + 2}{3 \cdot 2^{n-1}} + \frac{2}{3 \cdot 2^{n-1}} \left(\sum_{j=1}^{r} 2^{n-1} + \sum_{j=1}^{r} 2^{n-2j} \right) \\
&= \frac{1}{3} + \frac{2}{3 \cdot 2^{n-1}} + \frac{2}{3 \cdot 2^{n-1}} \left(r2^{n-1} + \frac{2^n - 1}{3} \right) \\
&= \frac{2r+1}{3} + \frac{4}{9} + \frac{4}{9 \cdot 2^{n-1}} \\
&\sim \frac{n}{3} + \frac{4}{9}\,,
\end{aligned}$$

as $n \to \infty$, as desired. Since this holds for n even and n odd, the proof is complete. □

3.8.3 Proof of Theorem 3.8.3

First, observe that since $A^{(2)} = \{\mathbf{132}, \mathbf{231}\}$ and $A^{(2)^R} = \{\mathbf{231}, \mathbf{132}\} = A^{(2)}$, we have $\mathbf{E}[\mathrm{fp}_n(\sigma)] = \mathbf{E}[\mathrm{afp}_n(\sigma)]$. Therefore, it suffices to prove that $\mathbf{E}[\mathrm{fp}_n(\sigma)] \to \frac{4}{3}$ as $n \to \infty$.

Let $\sigma \in \mathcal{S}_n(A^{(2)})$ be chosen uniformly at random. By Lemma 3.5.2, we have

$$\begin{aligned}
\mathbf{E}[\mathrm{fp}_n(\sigma)] &= \sum_{j=1}^{n} A_n^{(2)}(j,j) \\
&= \sum_{j=1}^{n} \frac{1}{2^{n-j+1} + \sum_{j=1}^{n} \frac{\binom{n-j}{j-1}}{2}^{n-j+1}} \\
&= 1 - \frac{1}{2^n} + \frac{1}{2^{n-1}} \sum_{j=1}^{n} 2^{j-2} \binom{n-j}{j-1}.
\end{aligned}$$

As $n \to \infty$, by Stirling's formula, we see that there is some $\varepsilon < 1$ such that expression $2^{j-n-1}\binom{n-j}{j-1} < \varepsilon^n$ unless $j = \frac{n}{3} + cn^{\alpha}$, with $\alpha \leq \frac{1}{2}$.

When $j = \frac{n}{3} + cn^{\alpha}$ we get

$$2^{j-n-1}\binom{n-j}{j-1} \sim 2^{j-n-1}2^{n-j}\sqrt{\frac{3}{\pi n}} \exp\left[\frac{-27c^2}{4}\right].$$

Summing over these values of j from $c = -\infty$ to $c = \infty$, and interpreting this as a Riemann sum, we obtain

$$\sum_{j=1}^{n} 2^{j-n-1}\binom{n-j}{j-1} \sim \frac{1}{2}\sqrt{\frac{3}{\pi n}}\sqrt{n}\sqrt{\pi}\frac{2}{3\sqrt{3}} = \frac{1}{3}.$$

Therefore, we see that

$$\mathbf{E}[\mathrm{fp}_n(\sigma)] \sim 1 + \frac{1}{3} = \frac{4}{3}, \qquad \text{as } n \to \infty, \qquad \text{as desired.}$$

□

3.8.4 Proof of Theorem 3.8.4

Let $\sigma \in \mathcal{S}_{2n}(A^{(3)})$. By Lemma 3.6.1, we have

$$\begin{aligned}
\mathbf{E}[\mathrm{fp}_{2n}(\sigma)] &= \sum_{j=1}^{2n} A_{2n}^{(3)}(j,j) \\
&= \sum_{j=n}^{2n-1} 2^{2n-2j-1} + A_{2n}^{(3)}(2n,2n) \\
&= \frac{2}{3} - \frac{1}{6\cdot 2^{2n}} + \frac{1}{2^{2n-1}} \\
&\sim \frac{2}{3}\,, \qquad \text{as } n \to \infty\,, \qquad \text{as desired.}
\end{aligned}$$

Now, let $\tau \in \mathcal{S}_{2n+1}(A^{(3)})$. Again, by Lemma 3.6.1, we have

$$\begin{aligned}
\mathbf{E}[\mathrm{fp}_{2n+1}(\tau)] &= \sum_{j=1}^{2n+1} A_{2n+1}^{(3)}(j,j) \\
&= \sum_{j=n+1}^{2n} 2^{2n+1-2j-1} + A_{2n+1}^{(3)}(2n+1,2n+1) \\
&= \frac{1}{3} - \frac{1}{3\cdot 2^{2n}} + \frac{1}{2^{2n}} \\
&\sim \frac{1}{3}\,, \qquad \text{as } n \to \infty\,, \qquad \text{as desired.}
\end{aligned}$$

This completes the proof. $\square$

3.8.5 Proof of Theorem 3.8.5

Let τ be chosen uniformly at random in $\mathcal{S}_n(A^{(3)})$, and let $\sigma = \tau^R$. Then σ is uniform in $\mathcal{S}_n(\mathbf{321}, \mathbf{231})$, and $\mathbf{E}[\mathrm{fp}_n(\sigma)] = \mathbf{E}[\mathrm{afp}_n(\tau)]$.

Therefore, by Lemma 3.6.1, we have

$$\begin{aligned}
\mathbf{E}[\mathrm{fp}_n(\sigma) &= \sum_{j=1}^{n} A_n^{(3)}(j, n+1-j) \\
&= A_n^{(3)}(1,n) + A_n^{(3)}(n,1) + \sum_{j=2}^{n-1} 2^{n-j-(n+1-j)-1} \\
&= \frac{1}{2} + \frac{1}{2} + \sum_{j=2}^{n-1} \frac{1}{4} \\
&= 1 + \frac{n-2}{4} = \frac{n}{4} + \frac{1}{2}\,,
\end{aligned}$$

as desired.

□

3.8.6 Proof of Theorem 3.8.6

Let $\sigma \in \mathcal{S}_{2n}(A^{(4)})$. By Lemma 3.7.1, we have

$$\begin{aligned}
\mathbf{E}[\mathrm{fp}_{2n}(\sigma)] &= \sum_{j=1}^{2n} A_{2n}^{(4)}(j,j) \\
&= \sum_{j=1}^{n} \frac{j}{\binom{2n}{2}+1} + \sum_{j=n+1}^{2n} \frac{2n+1-j}{\binom{2n}{2}+1} \\
&= \frac{\binom{n+1}{2}}{\binom{2n}{2}+1} + \frac{\binom{n+1}{2}}{\binom{2n}{2}+1} \\
&\sim 2\left(\frac{1}{4}\right) = \frac{1}{2}\,, \qquad \text{as } n \to \infty\,, \qquad \text{as desired.}
\end{aligned}$$

Now, let $\tau \in \mathcal{S}_{2n+1}(A^{(4)})$. Again, by Lemma 3.7.1, we have

$$\begin{aligned}
\mathbf{E}[\mathrm{fp}_{2n+1}(\tau)] &= \sum_{j=1}^{2n+1} A_{2n+1}^{(3)}(j,j) \\
&= \sum_{j=1}^{n} \frac{j}{\binom{2(n+1)}{2}+1} + A_{2n+1}^{(3)}(n+1,n+1) + \sum_{j=n+2}^{2n+1} \frac{2n+1-j}{\binom{2n+1}{2}+1} \\
&= \frac{\binom{n+1}{2}}{\binom{2n+1}{2}+1} + \frac{\binom{n}{2}+1}{\binom{2n+1}{2}+1} + \frac{\binom{n+1}{2}}{\binom{2n+1}{2}+1} \\
&\sim 3\left(\frac{1}{4}\right) = \frac{3}{4}\,, \qquad \text{as } n \to \infty\,, \qquad \text{as desired.}
\end{aligned}$$

This completes the proof. □

3.8.7 Proof of Theorem 3.8.7

Let τ be chosen uniformly at random in $\mathcal{S}_n(A^{(4)})$, and let $\sigma = \tau^R$. Then σ is uniform in $\mathcal{S}_n(\mathbf{321}, \mathbf{132})$, and $\mathbf{E}[\mathrm{fp}_n(\sigma)] = \mathbf{E}[\mathrm{afp}_n(\tau)]$.

Therefore, by Lemma 3.6.1, we have

$$\begin{aligned}
\mathbf{E}[\mathrm{fp}_n(\sigma) &= \sum_{j=1}^{n} A_n^{(4)}(j, n+1-j) \\
&= \sum_{j=1}^{n} \frac{\binom{n+1-j-1}{2} + 1}{\binom{n}{2} + 1} \\
&= \frac{\binom{n}{3} + n}{\binom{n}{2} + 1} \\
&\sim \frac{n-2}{3},
\end{aligned}$$

as $n \to \infty$, as desired.

□

CHAPTER 4

Avoiding three patterns of length three simultaneously

4.1 Introduction

We now consider permutations which simultaneously avoid three patterns of length three.

4.1.1 Symmetries

Let $\mathcal{B} \subset 2^{S_n}$ such that $B \in \mathcal{B}$ if and only if $|B| = 3$. As in the previous chapter, the sets in B can be partitioned into equivalence classes based on symmetries of the square. For $S, T \in \mathcal{B}$, let $S \sim T$ if

$$S \in \{S, S^C, S^R, S^{RC}, S^{-1}, (S^C)^{-1}, (S^R)^{-1}, (S^{RC})^{-1}\}.$$

Here these actions on sets in $\mathcal{B}$ generate all eight symmetries of the square, and form the dihedral group on the square. Because the complement, reverse, and inverse of a permutation all maintain the same matrix up to rotation and reflection, matrices within the same equivalence classes will have the same limit shape. We summarize this argument in the following lemma.

Proposition 4.1.1. *Let $\mathcal{B} \subset 2^{S_3}$, such that $B \in \mathcal{B}$ if and only if $|B| = 3$. Then* *the equivalence classes under* $\sim$ *are given by* $B^{(1)}, B^{(2)}, B^{(3)}, B^{(4)}$*, and* $B^{(5)}$*, where*

$$\begin{aligned}
B^{(1)} &= \{\boldsymbol{123},\ \boldsymbol{132},\ \boldsymbol{213}\} \sim \{\boldsymbol{231},\ \boldsymbol{312},\ \boldsymbol{321}\}\,, \\
B^{(2)} &= \{\boldsymbol{123},\ \boldsymbol{132},\ \boldsymbol{231}\} \sim \{\boldsymbol{123},\ \boldsymbol{132},\ \boldsymbol{312}\} \sim \{\boldsymbol{123},\ \boldsymbol{213},\ \boldsymbol{231}\} \\
&\quad \sim \{\boldsymbol{123},\ \boldsymbol{213},\ \boldsymbol{312}\} \sim \{\boldsymbol{132},\ \boldsymbol{231},\ \boldsymbol{321}\} \sim \{\boldsymbol{132},\ \boldsymbol{312},\ \boldsymbol{321}\} \\
&\quad \sim \{\boldsymbol{213},\ \boldsymbol{231},\ \boldsymbol{321}\} \sim \{\boldsymbol{213},\ \boldsymbol{312},\ \boldsymbol{321}\}\,, \\
B^{(3)} &= \{\boldsymbol{132},\ \boldsymbol{213},\ \boldsymbol{231}\} \sim \{\boldsymbol{132},\ \boldsymbol{213},\ \boldsymbol{312}\} \\
&\quad \sim \{\boldsymbol{132},\ \boldsymbol{231},\ \boldsymbol{312}\} \sim \{\boldsymbol{213},\ \boldsymbol{231},\ \boldsymbol{312}\}\,, \\
B^{(4)} &= \{\boldsymbol{123},\ \boldsymbol{231},\ \boldsymbol{312}\} \sim \{\boldsymbol{132},\ \boldsymbol{213},\ \boldsymbol{321}\}\,, \\
B^{(5)} &= \{\boldsymbol{123},\ \boldsymbol{132},\ \boldsymbol{321}\} \sim \{\boldsymbol{123},\ \boldsymbol{213},\ \boldsymbol{321}\} \\
&\quad \sim \{\boldsymbol{123},\ \boldsymbol{231},\ \boldsymbol{321}\} \sim \{\boldsymbol{123},\ \boldsymbol{312},\ \boldsymbol{321}\}\,.
\end{aligned}$$

Proof. We prove the result for $B^{(2)}$, the other cases are similar. Let $B = \{\mathbf{123},\mathbf{132},\ \mathbf{231}\}$. Then $B^R = \{\mathbf{123}^R, \mathbf{132}^R, \mathbf{231}^R\} = \{\mathbf{321},\mathbf{231},\mathbf{132}\}$. Similarly, $B^C = \{\mathbf{321},\mathbf{312},\mathbf{213}\}$, and $B^{RC} = \{\mathbf{123},\mathbf{213},\mathbf{312}\}$. Also,

$$\begin{aligned}
B^{-1} &= \{\mathbf{123},\mathbf{132},\mathbf{312}\}\,, \\
(B^R)^{-1} &= \{\mathbf{321},\mathbf{312},\mathbf{132}\}\,, \\
(B^C)^{-1} &= \{\mathbf{321},\mathbf{231},\mathbf{213}\}\,, \\
\text{and} \quad (B^{RC})^{-1} &= \{\mathbf{123},\mathbf{213},\mathbf{231}\}\,.
\end{aligned}$$

Therefore,

$$B^{(2)} = \{B, B^R, B^C, B^{RC}, B^{-1}, (B^R)^{-1}, (B^C)^{-1}, (B^{RC})^{-1}\}.$$

By the definition of $\sim$, we see that $B^{(2)}$ consists precisely of the sets equivalent to B, as desired. $\square$

As above, when calculating the limiting distribution of $B^{(i)}$ it suffices to analyze the first pair of patterns listed, and then rotate the distribution as needed to calculate for the other pairs of patterns.

As for sets of length two, $\mathcal{S}_n(B)$ was studied and enumerated by Simion and Schmidt[SS].

Theorem 4.1.2 ([SS]). *Let $B^{(1)}, B^{(2)}, B^{(3)}, B^{(4)}$, and $B^{(5)}$ be as above, and let F_n denote the n th Fibonacci number. Then*

$$|\mathcal{S}_n(B^{(1)})| = F_{n+1}\,,$$

while $|\mathcal{S}_n(B^{(2)})| = |\mathcal{S}_n(B^{(3)})| = |\mathcal{S}_n(B^{(4)})| = n$ *and* $|\mathcal{S}_n(B^{(5)})| = 0$, *for* $n \geq 5$.

As in the previous chapter, the size of $\mathcal{S}_n(B)$ depends on which patterns B contains. Again, the fact that $\mathcal{S}_n(B^{(5)}) = \emptyset$ for $n \geq 5$ follows from Erdős and Szekeres's result on longest increasing and decreasing subsequences [ES].

In the following section, we analyze the limit shapes of $\mathcal{S}_n(B)$ for each $B \in \mathcal{B}$.

4.2 Main results

In this section we present the main results of the paper.

4.2.1 Shape of B-avoiding permutations

Let $0 \leq a, b \leq 1$, and $c, d \in \mathbb{R}$ be fixed constants. Let $B_n^{(i)}(j,k)$ denote the probability that a permutation σ chosen uniformly at random from $\mathcal{S}_n(B^{(i)})$ has $\sigma(j) = k$. Define

$$T^i(a,b,c,d) = \sup\left\{ r \in \mathbb{R}_+ \,\middle|\, \lim_{n\to\infty} n^r B_n^{(i)}(an+c, bn+d) < \infty \right\}.$$

Similarly, let

$$M(a,b,c,d) = \lim_{n\to\infty} n^{T(a,b,c,d)} B_n(an-c+1, bn-d)\,,$$

defined for all a, b, c, d as above, for which $T(a,b,c,d) < \infty$; let M be undefined otherwise.

Observe that in the previous chapter, we analyze $A_n^{(i)}(an+cn^\alpha, bn+dn^\beta$, whereas here we are only considering $B_n^{(i)}(an+cn, bn+d)$, removing the potential effect of α and β and fixing them at $\alpha=0$ and $\beta=0$. This change is due to the fact that the limit shapes are simple enough that they do not merit the extra level of complexity in calculation which comes along with allowing α and β to vary.

The following theorems describe the limit shapes for the different classes of permutations, $\mathcal{S}_n(B^{(i)})$.

Theorem 4.2.1. *Let* $0 \le a, b \le 1, c, d \in \mathbb{R}$. *Then we have*

$$T^{(1)}(a,b,c,d) = \begin{cases} \infty & \textit{for } a+b \neq 1\,, \\ 0 & \textit{for } a+b=1, 0 \le c+d \le 2\,, \\ \infty & \textit{if } a+b=1, c+d \notin [0,2]\ . \end{cases}$$

Since $B^{(1)}$ is the only subset whose size grows exponentially with n, its asymptotics are the most interesting. We describe the remaining cases here.

Theorem 4.2.2. *Let* $0 \le a, b \le 1, c, d \in \mathbb{R}$. *Then we have*

$$T^{(2)}(a,b,c,d) = \begin{cases} 1 & \textit{for } a=1, c=0\,, \\ 1 & \textit{for } a=1, b=0, c+d=1\,, \\ 1 & \textit{for } a=0, b=1, c+d=0\,, \\ 0 & \textit{for } a=0, b=1, c+d=1\,, \\ 0 & \textit{for } a=1, b=0, c+d=0\,, \\ 0 & \textit{for } a+b=1, a\in(0,1), c+d \in \{0,1\} \\ \infty & \textit{otherwise}\,. \end{cases}$$

The behavior for $B^{(3)}$ is similar, as explained here.

Theorem 4.2.3. *Let $a, b \in [0,1], c, d \in \mathbb{R}$. Then we have*

$$T^{(3)}(a,b,c,d) = \begin{cases} 0 & \text{for } a+b=1, c+d=1 \text{ and } b>0\,, \\ 1 & \text{for } a>b, a+b\neq 1\,, \\ 1 & \text{for } a>b, a+b=1, c+d\neq 1\,, \\ 1 & \text{for } a=b, c\geq d, a+b\neq 1\,, \\ 1 & \text{for } a=b, c\geq d, a+b=1, c+d\neq 1\,, \\ \infty & \text{otherwise}\,. \end{cases}$$

In this chapter, $T^i(a,b,c,d) = \infty$ means that $B_n^{(i)}(an+c, bn+d) = 0$. In other words, there is no exponential decay in a given square of the matrix unless the matrix does not have any weight in that square at all.

The following theorems give values of $M^i(a,b,c,d)$, for values of a, b, c, d where $T^i(a,b,c,d) \neq \infty$.

Theorem 4.2.4. *Let $0 \leq a \leq 1$ and $c, d \in \mathbb{R}$ such that $0 \leq c+d \geq 2$. Then we have*

$$M^{(1)}(a, 1-a, c, d) = \begin{cases} \frac{1}{\varphi\sqrt{5}} & \text{for } a\in(0,1), c+d\in\{0,2\}\,, \\ \frac{1}{\sqrt{5}} & \text{for } a\in(0,1), c+d=1\,, \\ \varphi^{c-1}F_d & \text{for } a=1, c+d\in\{0,1\}\,, \\ \varphi^{d}F_{1-c} & \text{for } a=1, c+d=2\,, \\ \varphi^{d-1}F_c & \text{for } a=0, c+d\in\{0,1\}\,, \\ \varphi^{d}F_{1-d} & \text{for } a=0, c+d=2\,, \end{cases}$$

where F_n is the n-th Fibonacci number.

Theorem 4.2.5. *Let $a, b \in [0,1], c, d \in \mathbb{R}$. We have*

$$M^{(2)}(a,b,c,d) = \begin{cases} 1 & \text{for } a = 1, b = 0, c + d = 0\,, \\ d - 1 & \text{for } a = 1, b = 0, c + d = 1\,, \\ 1 & \text{for } a = 0, b = 1, c + d = 1\,, \\ c & \text{for } a = 0, b = 1, c + d = 0\,, \\ a & \text{for } a \in (0,1), b = 1 - a, c + d = 1\,, \\ 1 - a & \text{for } a \in (0,1), b = 1 - a, c + d = 0\,. \end{cases}$$

Theorem 4.2.6. *Let $a, b \in [0,1], c, d \in \mathbb{R}$. We have*

$$M^{(3)}(a,b,c,d) = \begin{cases} 1 - a & \text{for } a + b = 1, c + d = 1, a < 1\,, \\ d - 1 & \text{for } a + b = 1, c + d = 1, a = 1\,, \\ 1 & \text{for } a > b, a + b \neq 1 \\ 1 & \text{for } a = b, a + b \neq 1, c \geq d\,, \\ 1 & \text{for } a > \frac{1}{2}, a + b = 1, c + d \neq 1\,, \\ 1 & \text{for } a = \frac{1}{2}, a + b = 1, c + d \neq 1, c \geq d\,. \end{cases}$$

For B_4, the behavior is very straightforward, but for completeness we include the theorem here.

Theorem 4.2.7. *For all $a, b \in [0,1], c, d \in \mathbb{R}$, we have*

$$T^{(4)}(a,b,c,d) = 1\,,$$

and

$$M^{(4)}(a,b,c,d) = 1\,.$$

4.3 Analysis of $B^{(1)}$-avoiding permutations

4.3.1 Symmetry

The permutations $\sigma \in \mathcal{S}_n(B^{(1)})$ obey some simple symmetry.

Lemma 4.3.1. *For all $1 \le j, k \le n$, we have*

$$B_n^{(1)}(j,k) = B_n^{(1)}(k,j) = B_n^{(1)}(n+1-k, n+1-j)\,.$$

Proof. Since $\mathbf{123}^{-1} = \mathbf{123}$, $\mathbf{132}^{-1} = \mathbf{132}$, and $\mathbf{213}^{-1} = \mathbf{213}$, for every $\sigma \in \mathcal{S}_n(B^{(1)})$, we have $\sigma^{-1} \in \mathcal{S}_n(B^{(1)})$ as well. Therefore, $B_n(1,j,k) = B_n(1,k,j)$. Let $\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(n)) \in \mathcal{S}_n(B^{(1)})$, with $\sigma(j) = k$. Then the permutation $\tau = \sigma^{RC} \in \mathcal{S}_n(B^{(1)})$, with $\tau(n+1-k) = n+1-j$.

Here we give a lemma which is the main step towards proving Theorem 4.2.1. Recall that $B_n(1,j,k)$ is the probability that a random $\sigma \in \mathcal{S}_n(B^{(1)})$ with $\sigma(j) = k$.

Lemma 4.3.2. *For all $1 \le j, k \le n$, we have*

$$B_n(1,j,k) = \begin{cases} \frac{F_j F_k}{F_{n+1}} & n \le j+k \le n+1\ , \\ \frac{F_{n+1-j} F_{n+1-k}}{F_{n+1}} & j+k = n+2\ , \\ 0 & \textit{otherwise}\quad . \end{cases}$$

To clarify, permutations in $\mathcal{S}_n(B^{(1)})$ must be almost completely decreasing. For a permutation matrix $B \in \mathcal{S}_n(B^{(1)})$ to have $b_{jk} = 1$, the sum of the indices must satisfy $j + k \in \{n, n+1, n+2\}$.

Proof. We prove the lemma in cases.

First, let $j + k < n$, and let $\sigma \in \mathcal{S}_n(B^{(1)})$ with $\sigma(j) = k$. Let $B_\sigma = \{i < j : \sigma(i) > k\}$ and $C_\sigma = \{i > j : \sigma(i) > k\}$. Together we must have $|B_\sigma| + |C_\sigma| = n - k$.

Also, we know that $|B_\sigma| \leq j-1$. Therefore,

$$|C_\sigma| \geq n-k-(j-1) = n-k-j+1 > 1,$$

since $j+k<n$. Since C_σ is a set, we must have $|C_\sigma| \geq 2$. Let $i_1 < i_2 \in C_\sigma$. Then a **123** or **132** is formed, with $k, \sigma(i_1), \sigma(i_2)$. This is a contradiction, so $B_n(1,j,k) = 0$ for $j+k<n$.

Now suppose $j+k > n+2$. By Lemma 3.4.1, we get $B_n(1,j,k) = B_n(1, n+1-j, n+1-k)$, and since $(n+1-j)+(n+1-k) < n$, we get $B_n(1,j,k) = 0$.

Now, suppose $j+k=n$. The only way to avoid the contradictions in the previous case is to have $|C_\sigma| = 1$, and $|B_\sigma| = j-1$. Suppose $i \in C_\sigma$ with $i > j+1$. Then a **213** is formed with $\sigma(j), \sigma(j+1), \sigma(i)$. Therefore $i = j+1$. Similarly, suppose $\sigma(j+1) > k+1$. Then $\sigma^{-1}(k+1) < j$, so a **213** is formed with $k+1, \sigma(j), \sigma(j+1)$, which is a contradiction. Therefore, we have $\sigma(j+1) = k+1$. The first $j-1$ elements of the permutation must consist of $k+2, k+3, \ldots, n$, and the final elements must consist of $[k-1]$. Within the first $j-1$ elements, there are F_j ways to permute them, since they must be in $\mathcal{S}_{j-1}(B^{(1)})$. Similarly, there are F_k ways to permute the final $k-1$ elements, since they must be in $\mathcal{S}_{k-1}(B^{(1)})$.

Like in the previous case, for $j+k=n+2$, we apply Lemma 3.4.1. The result is that

$$B_n(1,j,k) = \frac{F_{n+1-j}F_{n+1-k}}{F_{n+1}} \; .$$

Our final case is for $j+k=n+1$. Here, suppose $|C_\sigma \geq 1$. Since $|B_\sigma|+|C_\sigma| = n-k$, we get

$$|B_\sigma| \leq n-k-1 = (j-1)-1 = j-2 < j-1.$$

Therefore, there must be some $i<j$ with $\sigma(i) < \sigma(j)$. Together with the element $r \in C_\sigma$, we have a **123**. This is a contradiction, so we must have $|C_\sigma = 0$, and $|B_\sigma = n-k = j-1$. Once we know the first $j-1$ elements of σ are $k+1, k+2, \ldots, n$

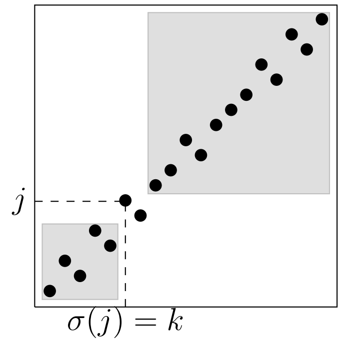


Figure 4.1: Random $\sigma \in \mathcal{S}_n(B^{(1)})$

and the final $n - j$ elements are $[k-1]$, we can form an element of $\mathcal{S}_n(B^{(1)})$ as long as the first $j-1$ elements are in $\mathcal{S}_{j-1}(B^{(1)})$ and the final $n-j$ elements are in $\mathcal{S}_{k-1}(B^{(1)})$. Therefore, there are $F_j F_k$ ways to create the entire permutation, finishing the final case of the lemma. □

A sample element of $\mathcal{S}_n(B^{(1)})$ is pictured here.

4.3.2 Proof of Theorem 4.2.1

The proof follows quickly from the previous lemma, and one other.

Lemma 4.3.3 (First case). *Let $a, b \in [0,1], c, d \in \mathbb{R}$, such that $a + b \neq 1$. Then $T^{(1)}(a,b,c,d) = \infty$. Moreover, for n sufficiently large, we have*

$$B_n^{(1)}(an - c, bn - d) < \varepsilon^n,$$

where ε is independent of n and $0 < \varepsilon < 1$.

Proof. Suppose $a + b < 1$, and $a < b$. (We omit the proofs of the other cases, since they are very similar.) Analyzing $B_n^{(1)}(an - c + 1, bn - d)$, we get

$$B_n^{(1)}(an - c, bn - d) = 0,$$

since $an - c + bn - d = (a+b)n - c - d$, and since $a + b < 1$, we are in the first case of Lemma 3.4.1. Not only do we get the exponential decay described above, we actually have no permutations σ at all in $\mathcal{S}_n(B^{(1)})$ with $\sigma(j) = k$. $\square$

Lemma 4.3.4 (Second case)**.** *Let $a \in (0,1), c, d \in \mathbb{R}$. For $c + d \in \{0, 1, 2\}$, we have*

$$T^{(1)}(a, 1-a, c, d) = 0\,.$$

Furthermore, for these values of c and d we have

$$M^{(1)}(a, 1-a, c, d) = \begin{cases} \frac{1}{\varphi\sqrt{5}} & \textit{for} \;\; a \in (0,1), c+d \in \{0,2\}\,, \\ \frac{1}{\sqrt{5}} & \textit{for} \;\; a \in (0,1), c+d = 1\;. \end{cases}$$

On the other hand, for $c + d \notin \{0, 1, 2\}$, we get $T^{(1)}(a, 1-a, c, d) = \infty$.

Proof. Let $a \in (0,1)$ with $c + d = 0$. The other cases follow by very similar arguments.

Recall that $F_n = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}}$. By Lemma 4.3.2, we have

$$\begin{aligned} B_n^{(1)}(an + c, (1-a)n + d) &= \frac{F_{an+c}F_{(1-a)n+d}}{F_{n+1}}\,, \\ &\sim \frac{\varphi^{an+c}}{\sqrt{5}} \frac{\varphi^{(1-a)n+d}}{\sqrt{5}} \frac{\sqrt{5}}{\varphi^{n+1}}\,, \\ &\sim \frac{\varphi^{n+c+d}}{\varphi^{n+1}\sqrt{5}}\,, \\ &\sim \frac{1}{\varphi\sqrt{5}}\,, \end{aligned}$$

as $n \to \infty$. Here we needed the fact that $a \neq 0, 1$, so that negative powers of φ become negligible as $n \to \infty$.

Since B^n is in fact a constant, we see that $T^{(1)}(a, 1-a, c, d) = 0$, as desired.

For $c + d \notin \{0, 1, 2\}$, we have $j + k \notin \{n, n+1, n+2\}$, so Lemma 4.3.2 gives us $B^{(1)}(an + cn, (1-a)n + d) = 0$ for all n. This gives us $T^{(1)}(a, 1-a, c, d) = \infty$, as desired. $\square$

Lemma 4.3.5. *Let $a \in \{0,1\}, c, d \in \mathbb{R}$ such that $c + d \in \{0,1,2\}$. Then*

$$T^{(1)}(a, 1-a, c, d) = 0\,,$$

and

$$M^{(1)}(a, 1-a, c, d) = \begin{cases} \frac{\varphi^{c-1}F_d}{\sqrt{5}} & \textit{for} \;\; a = 1, c + d \in \{0,1\}\;, \\ \frac{\varphi^{d}F_{1-c}}{\sqrt{5}} & \textit{for} \;\; a = 1, c + d = 2\;, \\ \frac{\varphi^{d-1}F_c}{\sqrt{5}} & \textit{for} \;\; a = 0, c + d \in \{0,1\}\;, \\ \frac{\varphi^{d}F_{1-d}}{\sqrt{5}} & \textit{for} \;\; a = 0, c + d = 2\;. \end{cases}$$

Proof. Suppose $a = 1, c + d = 2$. We prove Lemma 4.3.5 for these values and omit the other cases for brevity. By Lemma 4.3.2, we get

$$\begin{aligned} B_n^{(1)}(an + c, bn + d) &= B_n^{(1)}(n + c, d) \\ &= \frac{F_{n+c}F_d}{F_{n+1}} \\ &\sim \frac{\varphi^{n+c}\sqrt{5}F_d}{\sqrt{5}\varphi^{n+1}} \\ &\sim \varphi^{c-1}F_d\,, \end{aligned}$$

as $n \to \infty$.

Since this is a constant, and the correct constant, we obtain

$$T^{(1)}(a, 1-a, c, d) = 0\,,$$

and

$$M^{(1)}(a, 1-a, c, d) = \varphi^{c-1}F_d\,.$$

This completes the proof of all cases of Theorems 4.2.1 and 4.2.4, as desired. □

4.4 Analysis of $B^{(2)}$-avoiding permutations

Here we analyze the limit shape of $\mathcal{S}_n(B^{(2)})$, where $B^{(2)} = \{\mathbf{123},\ \mathbf{132},\ \mathbf{231}\}$. As described in Proposition 4.1.1, this class of permutations is not self-symmetric, since the *complement, reverse*, and *inverse* operations yield eight distinct sets.

4.4.1 Explicit formulas for $B^{(2)}$-avoiding permutations

In this lemma, we calculate the explicit value of $B_n^{(2)}(j,k)$ for any j,k.

Lemma 4.4.1. *Let $n \in \mathbb{N}$, and $1 \le j,k \le n$. Then we have*

$$B_n^{(2)}(j,k) = \begin{cases} \frac{1}{n} & j = n \ , \\ \frac{j}{n} & j+k = n \ , \\ \frac{k-1}{n} & j+k = n+1 \ \textit{and}\ j < n \ , \\ 0 & j+k < n \ \textit{or}\ (j+k > n+1 \ \textit{and}\ j < n) \ . \end{cases}$$

Proof. As usual, we prove Lemma 4.4.1 case by case. Let $\sigma \in \mathcal{S}_n(B^{(2)})$ with $\sigma(j) = k$. First, let $j = n$. Suppose $i_1 < i_2$ with $\sigma(i_1) < \sigma(i_2)$. If $k < \sigma(i_1)$, then a **231** is formed with $\sigma(i_1), \sigma(i_2), k$. If $\sigma(i_1) < k < \sigma(i_2)$, then a **132** is formed. If $k > \sigma(i_2)$, then a **123** is formed. Therefore, we must have $\sigma(i_1) > \sigma(i_2)$ for all $i_1 < i_2$. In other words, the only permutation with $\sigma(n) = k$ is

$$\sigma = (n, n-1, n-2, \ldots, k+2, k+1, k-1, k-2, \ldots, 3, 2, 1, k),$$

and $B_n^{(2)}(j,k) = \frac{1}{n}$.

Every permutation $\sigma \in \mathcal{S}_n(B^{(2)})$ must have this form for some k. This allows us to calculate the remaining cases without much difficulty. Let

$$\sigma = (n, n-1, n-2, \ldots, t+2, t+1, t-1, t-2, \ldots, 3, 2, 1, t).$$

For $j + k = n$, we have $\sigma(j) = k = n - j$. Because of the form of σ, t must come early enough in the permutation so that $\sigma(j) = n - j$ and not $n - j + 1$. This means that $k < t$. There are $n - k$ such possible values for t, and since $n - k = j$, we get j such possible permutations σ, so

$$B_n^{(2)}(j,k) = \frac{j}{n}.$$

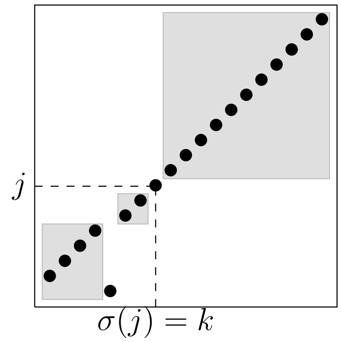


Figure 4.2: Random $\sigma \in \mathcal{S}_n(B^{(2)})$

Now suppose $j+k = n+1$, with $j < n$. We have $\sigma(j) = k = n+1-j$. Unlike the previous case, here we need $k > t$, so that $\sigma(j) = n+1-j$ and not $n-j$. There are $k-1$ possible choices for t, and this is well-defined since $j < n$ implies $k > 1$. Therefore, we have $k-1$ possible permutations σ, and

$$B_n^{(2)}(j,k) = \frac{k-1}{n}.$$

Finally, if $j+k < n$, then no σ will have $\sigma(j) = k$, since that would imply $\sigma(j) < n-k$. Similarly, if $j+k > n+1$, the only way σ will have $\sigma(j) = k$ is if $j = n$, in which case $B_n^{(2)}(j,k) = \frac{1}{n}$, as desired. $\square$

This argument can be visualized with the help of Figure 4.2.

4.4.2 Proof of Theorems 4.2.2 and 4.2.5

The proof follows from a pair of lemmas, one for each case from the theorem.

Lemma 4.4.2. *Let $a = 1$. For $c = 0$, we have*

$$T^{(2)}(1,b,0,d) = 1 \ , \ \textit{and} \ \ M^{(2)}(1,b,0,d) = 1 \, .$$

On the other hand, for $c \neq 0$, we have

$$T^{(2)}(1,0,c,d) = \begin{cases} 0 & c+d=0\,, \\ 1 & c+d=1\,, \\ \infty & c+d \notin \{0,1\}\,, \end{cases}$$

and

$$M^{(2)}(1,0,c,d) = \begin{cases} 1 & c+d=0\,, \\ d-1 & c+d=1\,. \end{cases}$$

Proof. Applying Lemma 4.4.1 with $a = 1$, we get

$$B_n^{(2)}(n, bn+d) = \frac{1}{n}\,.$$

This yields $T^{(2)}(1,b,0,d) = 1$ and $M^{(2)}(1,b,0,d)$ immediately.

For $c \neq 0$, with $c+d = 0$, by Lemma 4.4.1, we have

$$\begin{aligned} B_n^{(2)}(n+c, d) &= \frac{n+c}{n} \\ &\sim 1\,, \end{aligned}$$

as $n \to \infty$. This gives $T^{(2)}(1,0,c,d) = 0$ and $M^{(2)}(1,0,c,d) = 1$, as desired.

For $c \neq 0$ and $c+d = 1$, we have

$$B_n^{(2)}(n+c, d) = \frac{d-1}{n}\,.$$

This implies $T^{(2)}(1,0,c,d) = 1$ and $M^{(2)}(1,0,c,d) = d-1$, as desired.

For $c+d \notin \{0,1\}$, we have $B_n(n+c,d) = 0$ by Lemma 4.4.1, completing the proof of Lemma 4.4.2. □

Lemma 4.4.3. *Let $a \in [0,1)$, $b \in [0,1]$ so that $a+b \neq 1$. Then $T^{(2)}(a,b) = \infty$.*

Proof. By Lemma 4.4.1, we have

$$n^r B_n^{(2)}(an+c, bn+d) = n^r(0)\,,$$

since for n large enough we have $an+c+bn+d \notin \{n, n+1\}$. Therefore, for any $r > 0$ and for n large enough, we have $B_n^{(2)}(an, bn) = 0$, so $T^{(2)}(a,b) = \infty$. $\square$

Lemma 4.4.4. *Let $b = 1, a = 0, c, d \in \{0,1\}$ such that $c+d \in \{0,1\}$. For $c+d = 0$, we have*

$$T^{(2)}(0,1,c,d) = 1 \text{ and } M^{(2)}(0,1,c,d) = c\,.$$

On the other hand, for $c+d = 1$, we have

$$T^{(2)}(0,1,c,d) = 0 \text{ and } M^{(2)}(0,1,c,d) = 1\,.$$

Proof. Each case follows from Lemma 4.4.1, we explain the first carefully and omit the second for brevity. For $c+d = 1$, Lemma 4.4.1 implies that

$$\begin{aligned} B_n^{(2)}(an+c, bn+d) &= B_n^{(2)}(c, n+d) \\ &= \frac{n+d-1}{n} \\ &\sim 1\,, \end{aligned}$$

as $n \to \infty$. Therefore, we have $T^{(2)}(0,1,c,d) = 0$ and $M^{(2)}(0,1,c,d) = 1$, as desired. $\square$

For the final case we consider points on the anti-diagonal of the permutation matrix.

Lemma 4.4.5. *Let $a \in (0,1), c, d \in \mathbb{R}$ such that $c+d \in \{0,1\}$. Then $T^{(2)}(a, 1-a) = 0$, and*

$$M^{(2)}(a, 1-a, c, d) = \begin{cases} a & c+d = 1\,, \\ 1-a & c+d = 0\,. \end{cases}$$

Proof. As before, we rely heavily on Lemma 4.4.1. First, suppose $c+d = 1$. In this case, we see that

$$\begin{aligned} B_n^{(2)}(an+c, (1-a)n+d) &= \frac{an+c}{n} \\ &\sim a\,, \end{aligned}$$

as $n \to \infty$. We see that $T^{(2)}(a, 1-a, c, d) = 0$ and $M^{(2)}(a, 1-a, c, d) = a$.

Similarly, for $c + d = 0$, we get $an + c + (1-a)n + d = n$, so

$$\begin{aligned} B_n^{(2)}(an + c, (1-a)n + d) &= \frac{(1-a)n + d - 1}{n} \\ &\sim (1-a)\,, \end{aligned}$$

as $n \to \infty$. This implies $T^{(2)}(a, 1-a, c, d) = 0$ and $M^{(2)}(a, 1-a, c, d) = 1-a$. This completes the proof of Lemma 4.4.5 and of Theorems 4.2.2 and 4.2.5. □

4.5 Analysis of $B^{(3)}$-avoiding permutations

4.5.1 Explicit formulas for $B^{(3)}$-avoiding permutations

In this section we consider the limit shape of $\sigma \in \mathcal{S}_n(B^{(3)})$, where

$$B^{(3)} = \{\mathbf{132},\ \mathbf{213},\ \mathbf{231}\}\,.$$

We first prove a lemma calculating the explicit value of $B_n^{(3)}(j, k)$.

Lemma 4.5.1. *For all $n \in \mathbb{N}, 1 \le j, k \le n$, we have*

$$B_n^{(3)}(j,k) = \begin{cases} 0 & j < k \ \textit{and}\ j + k \neq n+1\,, \\ \frac{1}{n} & j \ge k \ \textit{and}\ j + k \neq n+1\,, \\ \frac{k-1}{n} & j < k \ \textit{and}\ j + k = n+1\,, \\ \frac{k}{n} & j \ge k \ \textit{and}\ j + k = n+1\,. \end{cases}$$

Proof. Before dealing with a specific case of the lemma, we first examine what permutations in $\mathcal{S}_n(B^{(3)})$ look like. Let $\sigma \in \mathcal{S}_n(B^{(3)})$ and suppose $\sigma(n) = t$. Since σ avoids **213**, all numbers smaller than t must be increasing. Since σ avoids **231**, all numbers larger than t must be decreasing. Since σ avoids **132**, all numbers larger than t must come before all numbers smaller than t. The only permutation

σ which satisfies these criteria is

$$\sigma = (n, n-1, \ldots, t+2, t+1, 1, 2, \ldots, t-2, t-1, t)\,.$$

Permutations of this form are exactly those in $\mathcal{S}_n(B^{(3)})$. These permutations either have $\sigma(j) = n+1-j$ for $j \le n - \sigma(n)$, or $\sigma(j) = j + \sigma(n) - n$ for $j > n - \sigma(n)$.

Now we prove the cases of the lemma. First, suppose $j < k$ with $j+k \ne n+1$. Let $\sigma \in \mathcal{S}_n(B^{(3)})$ with $\sigma(j) = k$. We know that $\sigma(j) \ne n+1-j$, since $k \ne n+1-j$. Therefore, $\sigma(j) = j + \sigma(n) - n = k$. However, this gives $\sigma(n) = n + k - j > n$, which is a contradiction, and we must have $B_n^{(3)}(j,k) = 0$.

Second, suppose $j > k$ with $j + k \ne n+1$. Again, since $j + k \ne n+1$, we must have $\sigma(j) = j + \sigma(n) - n = k$. This gives $\sigma(n) = n + k - j \le n$. For a given j and k, there is only one such permutation, as discussed above. Therefore,

$$B_n^{(3)}(j,k) = \frac{1}{n}\,.$$

Third, suppose $j < k$ with $j + k = n+1$. Let $\sigma \in \mathcal{S}_n(B^{(3)})$ with $\sigma(j) = k$. Since $\sigma(j) = n+1-j$, the permutation σ could have $j \le n - \sigma(n)$, or $\sigma(n) \le n-j$. There are $n - j = k - 1$ possibilities for $\sigma(n)$, and each of these gives a unique permutation. As in the first case, we cannot have $\sigma(j) = j + \sigma(n) - n$, since that would imply $j \ge k$. Therefore, we have

$$B_n^{(3)}(j,k) = \frac{k-1}{n}\,.$$

Finally, suppose $j \ge k$ with $j + k = n+1$. Let $\sigma \in \mathcal{S}_n(B^{(3)})$ with $\sigma(j) = k$. As in the previous case, there are $k-1$ possibilities for $\sigma(n)$ (each of the elements of $[k-1]$) which give $\sigma(j) = k$. Here, we also have another possibility: if $\sigma(n) = n + k - j = 2k + 1$, then we have one more possibility for σ. Therefore, we have

$$B_n^{(3)}(j,k) = \frac{k}{n}\,,$$

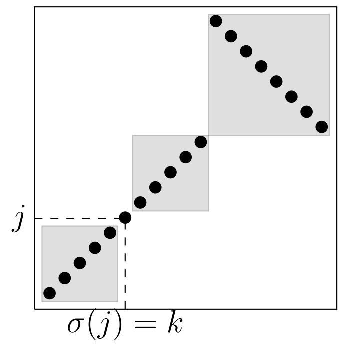


Figure 4.3: Random $\sigma \in \mathcal{S}_n(B^{(3)})$

completing the final case and the proof. $\square$

To help illustrate the limit shape of $\mathcal{S}_n(B^{(3)})$, we include a random permutation in Figure 4.3.

With the help of Lemma 4.5.1, we are ready to prove the main theorem on $\mathcal{S}_n(B^{(3)})$.

4.5.2 Proof of Theorems 4.2.3 and 4.2.6

The proof follows from several lemmas corresponding to each case of the theorem.

Lemma 4.5.2. *Let $a, b \in [0,1]$, with $a + b \neq 1$ and $a < b$. Then $T^{(3)}(a,b,c,d) = \infty$.*

Proof. By Lemma 4.5.1, we have $B_n^{(3)}(an+c, bn+d) = 0$ for n large enough. Since any $r > 0$ will still yield $n^r B_n^{(3)}(an, bn) = 0$, we have $T^{(3)}(a,b) = \infty$. $\square$

Lemma 4.5.3. *Let $a, b \in [0,1]$ with $a+b \neq 1$. For $a > b$, we have $T^{(3)}(a,b,c,d) = 1$ and $M^{(3)}(a,b,c,d) = 1$. On the other hand, for $a = b$, we have*

$$T^{(3)}(a,b,c,d) = \begin{cases} \infty & \text{for } c < d\,, \\ 1 & \text{for } c \geq d\,, \end{cases}$$

and $M^{(3)}(a,a,c,d) = 1$ for $c \geq d$.

Proof. Suppose $a + b \neq 1$. Again, Lemma 4.5.1 implies

$$B_n^{(3)}(an + c, bn + d) = \frac{1}{n}\,,$$

for $a > b$, or for $a = b$ with $c \geq d$. This yields $T^{(3)}(a, b, c, d) = 1$ and $M^{(3)}(a, b, c, d) = 1$, as desired.

For $a = b$ and $c < d$, we instead have $B_n^{(3)}(an + c, bn + d) = 0$. This gives $T^{(3)}(a, b, c, d) = \infty$, completing the proof. $\square$

Lemma 4.5.4. *Let $a \in [0, 1], c, d \in \mathbb{R}$ with $c + d = 1$. Then*

$$T^{(3)}(a, 1 - a, c, d) = \begin{cases} 0 & \textit{for } a < 1\,, \\ 1 & \textit{for } a = 1\,, \end{cases}$$

and

$$M^{(3)}(a, 1 - a, c, d) = \begin{cases} 1 - a & \textit{for } a < 1 \\ d - 1 & \textit{for } a = 1\,. \end{cases}$$

As in proofs of the previous two lemmas, we consider $B_n^{(3)}(an, (1 - a)n)$.

First, suppose $a = 1$. By Lemma 4.5.1, we have

$$\begin{aligned} B_n^{(3)}(an + c, (1 - a)n + d) &= B_n^{(3)}(n + c, d) \\ &= \frac{d - 1}{n}\,. \end{aligned}$$

This gives $T^{(3)}(a, 1 - a, c, d) = 1$ and $M^{(3)}(a, 1 - a, c, d) = d - 1$.

On the other hand, for $a < 1$, we have

$$\begin{aligned} B_n^{(3)}(an + c, (1 - a)n + d) &= \frac{(1 - a)n + d - 1}{n} \\ &\sim 1 - a\,, \end{aligned}$$

as $n \to \infty$. This clearly gives $T^{(3)}(a, 1 - a, c, d) = 0$ and $M^{(3)}(a, 1 - a, c, d) = 1 - a$, as desired. $\square$

Lemma 4.5.5. *Let $a \in [0,1], c, d \in \mathbb{R}$ with $c + d \neq 1$. Then we have*

$$T^{(3)}(a, 1-a, c, d) = \begin{cases} \infty & \text{for } a < \frac{1}{2}\,, \\ \infty & \text{for } a = \frac{1}{2}, c < d\,, \\ 1 & \text{for } a > \frac{1}{2} \\ 1 & \text{for } a = \frac{1}{2}, c \geq d\,, \end{cases}$$

and

$$M^{(3)}(a, 1-a, c, d) = 1\,,$$

for $a > \frac{1}{2}$ and for $a = \frac{1}{2}, c \geq d$.

Proof. By Lemma 4.5.1, for $a < \frac{1}{2}$, we have $B_n^{(3)}(an + c, (1-a)n + d) = 0$, since $j < k$ for n large enough.

Similarly, for $a = \frac{1}{2}$ with $c < d$, we get

$$B_n^{(3)}\left(\frac{1}{2}n + c, \frac{1}{2}n + d\right) = 0\,,$$

for the same reason.

For $a > \frac{1}{2}$ and for $a = \frac{1}{2}, c \geq d$, Lemma 4.5.1 gives us

$$B_n^{(3))}(an + c, (1-a)n + d) = \frac{1}{n}$$

implying $T^{(3)}(a, 1-a, c, d) = 1$ and $M^{(3)}(a, 1-a, c, d) = 1$, as desired. This completes the proof of Lemma 4.5.5 and of Theorems 4.2.3 and 4.2.6. □

4.6 Analysis of $B^{(4)}$-avoiding permutations

4.6.1 Explicit formulas for $B^{(4)}$-avoiding permutations

In this section, we consider the limit shape of permutations $\sigma \in \mathcal{S}_n(B^{(4)})$, where

$$B^{(4)} = \{\mathbf{123}, \mathbf{231}, \mathbf{312}\}\,.$$

After considering what these permutations must look like, the limit shape is as it turns out very easy to describe. We first prove a lemma which gives the explicit number of such permutations σ with $\sigma(j) = k$.

Lemma 4.6.1. *Let $n \in \mathbb{N}$. For all $1 \leq j, k \leq n$, we have*

$$B_n^{(4)}(j,k) = \frac{1}{n}\,.$$

This lemma implies that the limit shape for $B_n^{(4)}$ is completely flat, or uniform. This matches the limit shape of all permutations $\sigma \in S_n$, since for all $j, \sigma(j) = k$ with probability $p = \frac{1}{n}$ for all k. In some sense, this limit shape is therefore not as interesting, but we describe it here for thoroughness.

Proof. Let $\sigma \in B_n^{(4)}(j,k)$. Suppose $\sigma(j) = n$.

Since σ avoids **123**, we must have $\sigma(r) > \sigma(s)$ for all $r < s < j$. Also, since σ avoids **312**, we must have $\sigma(t) > \sigma(u)$ for all $j < t < u$. Since σ avoids **231**, we must have $\sigma(a) < \sigma(b)$, for $a < j < b$.

These three conditions combined tell us that σ must start with a decreasing subsequence, followed by another decreasing subsequence starting with $\sigma(j) = n$. Also, the first $j-1$ positions of σ must be filled by numbers i with $1 \leq i \leq j-1$, since otherwise σ would contain a **231**. Therefore, σ must equal

$$\sigma = (j-1, j-2, j-3, \ldots, 3, 2, 1, n, n-1, n-2, \ldots, j+2, j+1, j)\,,$$

for some j.

Since there are n possible choices for j, we have verified that

$$|\mathcal{S}_n(B^{(4)})| = n\,.$$

Also, observe that for each σ of this form, $\sigma(j) + j$ is constant modulo n for each j. This allows us to calculate $B_n^{(4)}(j,k)$. Let $\sigma \in \mathcal{S}_n(B^{(4)})$ with $\sigma(j) = k$.

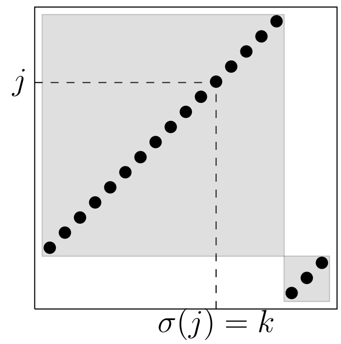


Figure 4.4: Random $\sigma \in \mathcal{S}_{19}(B^{(4)})$

Therefore, $\sigma(i) \equiv j + k \mod n$, and there is only one choice for each i, so one possible permutation σ. The permutation σ must be

$$\sigma = (j+k-1, j+k-2, \ldots, 2, 1, n, n-1, \ldots, j+k+2, j+k+1, j+k)\,,$$

where each of these numbers are written modulo n. Since there is only one possible permutation σ for each j and k, we have

$$B_n^{(4)}(j,k) = \frac{1}{n}\,,$$

for all n, j, k, as desired. $\square$

A random permutation $\sigma \in \mathcal{S}_n(B^{(4)})$ is exhibited in Figure 4.4.

4.6.2 Proof of Theorem 4.2.7

From Lemma 4.6.1, we get the proof of Theorem 4.2.7 almost immediately.

Proof. Let $a, b \in [0,1], c, d \in \mathbb{R}$. By Lemma 4.6.1, we have

$$B_n^{(4)}(an+c, bn+d) = \frac{1}{n}\,.$$

Therefore, $T^{(4)}(a,b,c,d) = M^{(4)}(a,b,c,d) = 1$, as desired. $\square$

CHAPTER 5

Grassmannian permutations

5.1 Introduction

In this chapter, we analyze Grassmannian permutations; permutations with at most one descent. Grassmannian permutations were initially defined in the 1980's by Schützenberger, in the context of Schubert polynomials. Since, they have found to be in bijection with certain classes of Dyck paths, and other combinatorial objects as well. We give a characterization of such permutations in terms of pattern-avoidance, and by analyzing the limit shape, also conclude results about the limit shape of permutations $\sigma \in \mathcal{S}_n(\mathbf{123}, \mathbf{3412})$. In previous chapters we considered permutations which avoid a subset of patterns of length three, while here we have longer patterns to consider.

Other than that distinction, the analysis we undertake is very similar - the enumeration of this permutation class is known, so instead we consider these permutations as n by n $\{0, 1\}$matrices. We let $n \to \infty$, and calculate the limit of the probability of an entry in our matrix in row i and column j being nonzero. By letting i and j grow with n so that $i \sim an + cn^\alpha$ and $j \sim bn + dn^\beta$, for some constants $a, b \in [0, 1], c, d \in \mathbb{R}, \alpha, \beta \in [0, 1)$, we focus on entries in our permutation matrices which move towards specific coordinates in the unit square. We denote this limit probability as $G_n(an + cn^\alpha, bn + dn^\alpha)$. As different values of a, b, c, d, α and β vary, we get probabilities between ε^n and constant values, showing that as $n \to \infty$, permutation matrices in G_n have a very high probability of having a specific shape.

In 1985, Schützenberger defined the Grassmannian permutations as those which have at most one descent. In other words, $\sigma \in \mathcal{S}_n$ is Grassmannian if there exists some index $i \leq n$ such that

$$\sigma(1) < \sigma(2) < \ldots < \sigma(i) > \sigma(i+1) < \sigma(i+2) < \ldots < \sigma(n)\,.$$

Let G_n denote the set of Grassmannian permutations of length n, and let G_n^{-1} denote the set of inverses of Grassmannian permutations of length n. Billey, Jockusch, and Stanley showed in 1993 [BJS] that

$$|\mathcal{S}_n(\mathbf{321}, \mathbf{2143})| = 2^{n+1} - \binom{n+1}{3} - 2n - 1.$$

Vella gave an argument for this formula using Dyck paths in 2003 [Vel], by arguing that $\mathcal{S}_n(\mathbf{321}, \mathbf{2143}) = G_n \cup G_n^{-1}$. An easy counting argument gives

$$|G_n| = 2^n - n = |G_n^{-1}|\,,$$

and $|G_n \cap G_n^{-1}| = \binom{n+1}{3} + 1\,.$

Many researchers have studied Grassmannian permutations in the last few decades; even in the last year, new information on Grassmannians is being submitted [LM].

The resulting limit shape can be visualized with the help of the following figures. In Figure 5.1, we see that for $(a, b) = (0, 1)$ or $(a, b) = (1, 0)$, we have $G_n(an, bn) = \Theta(1)$. For (a, b) on one of the two dashed lines which cross through the square, we get

$$G_n(an, bn) = \Theta(n^{-\frac{1}{2}})\,,$$

as long as $\alpha \leq \frac{1}{2}$ and $\beta \leq \frac{1}{2}$. We in fact go beyond this level of analysis, and include the constant on the leading term $n^{-\frac{1}{2}}$.

If we are not within $c\sqrt{n}$ of one of the two dashed lines, then we instead find that $G_n(an, bn) \sim \varepsilon^{n^{\alpha}}$, for some $\varepsilon \in (0, 1)$, and some $\alpha \in (0, 1)$.

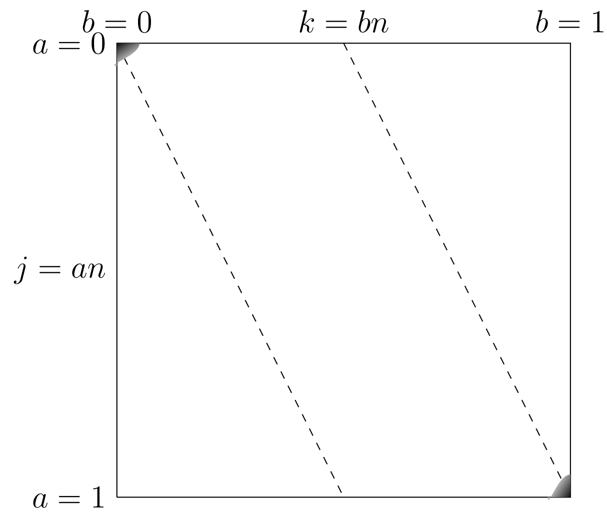


Figure 5.1: Regions within limit shape of $\sigma \in G_n$

In Figure 5.2, we can visualize the probability distribution as a surface.

5.2 Main Results

5.2.1 Shape of Grassmannian permutations

Let $0 \leq a, b \leq 1$, $c, d \in \mathbb{R}$, $\alpha, \beta \in [0,1)$ be fixed constants. Let $G_n(j,k)$ be the proportion of permutations $\sigma \in G_n$ with $\sigma(j) = k$. Define

$$U(a,b,c,d,\alpha,\beta) = \sup \left\{ r \in \mathbb{R}_+ \middle| \lim_{n\to\infty} n^r G_n(an + cn^\alpha, bn + dn^\beta) < \infty \right\}.$$

With this definition, we have the following theorem.

Theorem 5.2.1. *Let $a, b \in [0,1], c, d \in \mathbb{R}, \beta, \alpha \in [0,1)$ be constants, and let*

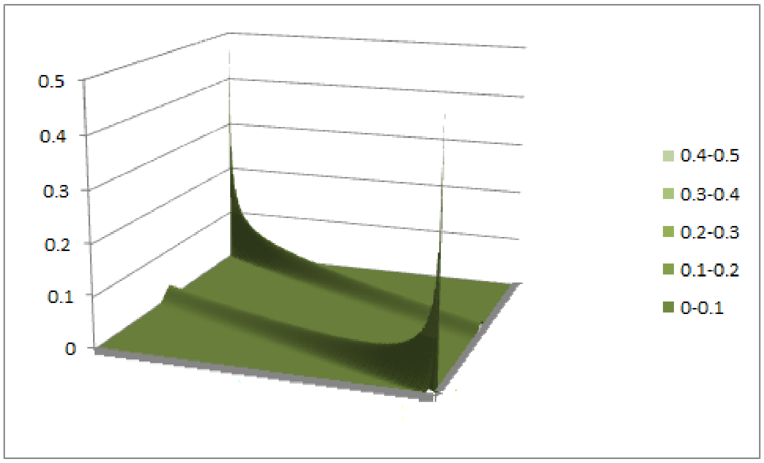


Figure 5.2: Limit shape of $G_{250}(j,k)$

$n \in \mathbb{N}$. *Let* $U(a,b,c,d,\alpha,\beta)$ *be defined as above. Then*

$$U(a,b,c,d,\alpha,\beta) = \begin{cases} \frac{1}{2} & 2a+b \in \{1,2\}, a \notin \{0,1\}, \alpha \leq \frac{1}{2}, \beta \leq \frac{1}{2}\,, \\ \frac{\beta}{2} & a=0, b=1, \alpha=\beta>0, 2c=-d\,, \\ \frac{\beta}{2} & a=1, b=0, \alpha=\beta>0, 2c=-d\,, \\ 0 & a=0, b=1, \alpha=\beta=0, d<0, c+d \leq 1\,, \\ 0 & a=1, b=0, \alpha=\beta=0, c<0, c+d \geq 1\,, \\ \infty & \textit{otherwise.} \end{cases}$$

Here $U(a,b,c,d,\alpha,\beta)=\infty$ means that $G_n(an+cn^\alpha, bn+dn^\beta)=o(n^{-r})$, for all $r>0$. The following result proves the exponential decay of these probabilities.

Theorem 5.2.2. *Let* $a,b \in [0,1], c,d \in \mathbb{R}, \alpha,\beta \in [0,1)$. *Then for* n *large enough,* *we have*

$$G_n(an+cn^\alpha, bn+dn^\beta) < \begin{cases} \varepsilon^n & 2a+b \notin \{1,2\}\,, \\ \varepsilon^{n^{2\alpha-1}} & 2a+b \in \{1,2\}, \alpha \geq \beta, \alpha > \frac{1}{2}\,, \\ \varepsilon^{n^{2\beta-1}} & 2a+b \in \{1,2\}, \alpha < \beta, \frac{1}{2} < \beta\,, \\ \varepsilon^{n^\beta} & a+b=1, a \in \{0,1\}, 0 < \alpha = \beta, 2c \neq -d\,, \\ \varepsilon^{n^\beta} & a+b=1, a\in\{0,1\}, \alpha < \beta\,, \\ \varepsilon^{n^\alpha} & a+b=1, a \in \{0,1\}, \beta < \alpha\,, \end{cases}$$

for some $\varepsilon \in (0,1)$*.*

These theorems analyze the growth of $G_n(an+cn^\alpha, bn+dn^\beta)$. We also obtain the following result detailing the explicit constants on the leading term of the asymptotic growth of $G_n(an+cn^\alpha, bn+dn^\beta)$, for those values of a,b,c,d,α,β where $U(a,b,c,d,\alpha,\beta) < \infty$. Let

$$V(a,b,c,d,\alpha,\beta) = \lim_{n\to\infty} n^{U(a,b,c,d,\alpha,\beta)} G_n(an+cn^\alpha, bn+dn^\beta)\,.$$

Theorem 5.2.3. *Let* $a,b \in [0,1], c,d \in \mathbb{R}, \alpha,\beta \in [0,1)$*. Then we have*

$$V(a,b,c,d,\alpha,\beta) = \begin{cases} \frac{1}{2\sqrt{a\pi}} & a>0, 2a+b=1, \alpha,\beta < \frac{1}{2}\,, \\ \frac{1}{2\sqrt{a\pi}} \exp\left[-\frac{c^2}{a}\right] & a>0, 2a+b=1, \frac{1}{2}=\alpha>\beta\,, \\ \frac{1}{2\sqrt{a\pi}} \exp\left[-\frac{d^2}{4a}\right] & a>0, 2a+b=1, \frac{1}{2}=\beta>\alpha\,, \\ \frac{1}{2\sqrt{a\pi}} \exp\left[-\frac{(d+2c)^2}{4a}\right] & a>0, 2a+b=1, \frac{1}{2}=\beta=\alpha\,, \\ \frac{1}{2\sqrt{(1-a)\pi}} & a<1, 2a+b=2, \alpha,\beta<\frac{1}{2}\,, \\ \frac{1}{2\sqrt{(1-a)\pi}} \exp\left[-\frac{c^2}{(1-a)}\right] & a<1, 2a+b=2, \frac{1}{2}=\alpha>\beta\,, \\ \frac{1}{2\sqrt{(1-a)\pi}} \exp\left[-\frac{d^2}{4(1-a)}\right] & a<1, 2a+b=2, \frac{1}{2}=\beta>\alpha\,, \\ \frac{1}{2\sqrt{(1-a)\pi}} \exp\left[-\frac{(d+2c)^2}{4(1-a)}\right] & a<1, 2a+b=2, \frac{1}{2}=\beta=\alpha\,, \end{cases}$$

and

$$V(a,b,c,d,\alpha,\beta)=\begin{cases}\frac{1}{2\sqrt{c\pi}} & a=0,b=1,\alpha=\beta>0,2c=-d\,,\\ \frac{1}{2\sqrt{-c\pi}} & a=1,b=0,\alpha=\beta>0,-2c=d\,,\\ 2^{d-1}\binom{-d}{c-1} & a=0,b=1,\alpha=\beta=0,d+c\leq 0,c\geq 1\,,\\ 2^{-d}\binom{d-1}{-c} & a=1,b=0,\alpha=\beta=0,d+c\geq 2,c\leq 0\,,\\ 2^{c-1} & a=1,b=0,\alpha=\beta=0,d+c=1,c\leq 0\,,\\ 2^{d-1} & a=0,b=1,\alpha=\beta=0,d+c=1,c\geq 1\,.\end{cases}$$

This theorem shows that even within the regions where the asymptotic growth of $G_n(an+cn^\alpha, bn+dn^\beta)$ is on the same order of magnitude, the constant varies with the choices of parameters a,b,c,d,α, and β.

5.3 Analysis of Grassmannian permutations

5.3.1 Combinatorics of $\sigma\in G_n$

In this section, we prove several lemmas regarding Grassmannian permutations.

Lemma 5.3.1. *For every $n\in\mathbb{N}$, we have*

$$G_n=\mathcal{S}_n(\boldsymbol{123},\boldsymbol{3412},\boldsymbol{2413})\,.$$

This gives a new way of characterizing Grassmannian permutations, based on pattern-avoidance. The proof is not complicated, though we include it here for completeness.

Proof. We prove Lemma 5.3.1 by showing set containment in each direction. First, let $\sigma\in G_n$. Then σ has at most one ascent. If σ has no ascents, then $\sigma=(n,n-1,\dots,2,1)\in\mathcal{S}_n(\mathbf{123},\mathbf{3412},\mathbf{2413})$. If σ has a single ascent, then there exists $1 \leq i < n$ so that $\sigma(1) > \sigma(2) > \ldots > \sigma(i) < \sigma(i+1) > \sigma(i+2) > \ldots > \sigma(n)$.

Suppose σ contains $\mathbf{123}$. Then there are indices $i_1 < i_2 < i_3$ with $\sigma(i_1) < \sigma(i_2) < \sigma(i_3)$. If $i_2 \leq i$, then we have a contradiction, since $\sigma(i_1) > \sigma(i_2)$, Similarly, if $i_2 > i$, then we have a contradiction, since $\sigma(i_2) > \sigma(i_3)$. Therefore, σ avoids $\mathbf{123}$.

Now, suppose that σ contains $\mathbf{3412}$. Then there are indices $i_1 < i_2 < i_3 < i_4$ with $\sigma(i_3) < \sigma(i_4) < \sigma(i_1) < \sigma(i_2)$. First, observe that if $i_2 \leq i$, then we have a contradiction, since $\sigma(i_1) > \sigma(i_2)$. Similarly, since if $i_3 \geq i+1$, then we again have a contradiction, since $\sigma(i_3) > \sigma(i_4)$. Therefore, we have $i_3 \leq i < i_2$, which is also a contradiction. Therefore, σ avoids $\mathbf{3412}$.

Finally, suppose that σ contains $\mathbf{2413}$. Then there are indices $i_1 < i_2 < i_3 < i_4$ with $\sigma(i_3) < \sigma(i_1) < \sigma(i_4) < \sigma(i_2)$. If $i_2 \leq i$, then we have a contradiction, since $\sigma(i_1) > \sigma(i_2)$. If $i_3 \geq i+1$, then we again have a contradiction, since $\sigma(i_3) > \sigma(i_4)$. Therefore, we have $i_3 \leq i < i_2$, which is a contradiction. We see that σ must avoid $\mathbf{2413}$. Therefore $\sigma \in \mathcal{S}_n(\mathbf{123}, \mathbf{3412}, \mathbf{2413})$, so

$$G_n \subset \mathcal{S}_n(\mathbf{123}, \mathbf{3412}, \mathbf{2413})\,.$$

Second, let $\sigma \in \mathcal{S}_n(\mathbf{123}, \mathbf{3412}, \mathbf{2413})$, and suppose for the sake of contradiction that σ has two ascents, so there exist $1 \leq j < k < n$ such that $\sigma(j) < \sigma(j+1)$ and $\sigma(k) < \sigma(k+1)$. First, suppose $\sigma(k+1) > \sigma(j+1)$. Then σ contains $\mathbf{123}$, with $\sigma(j), \sigma(j+1), \sigma(k+1)$. Second, suppose $\sigma(k+1) < \sigma(j)$. Then σ contains $\mathbf{3412}$, with $\sigma(j), \sigma(j+1), \sigma(k), \sigma(k+1)$. Finally, suppose $\sigma(j) < \sigma(k+1) < \sigma(j+1)$. If $\sigma(j) < \sigma(k)$, then σ contains $\mathbf{123}$, with $\sigma(j), \sigma(k), \sigma(k+1)$. If $\sigma(j) > \sigma(k)$, then σ contains $\mathbf{2413}$, with $\sigma(j), \sigma(j+1), \sigma(k), \sigma(k+1)$. Therefore, σ contains either a $\mathbf{123}$, a $\mathbf{3412}$, or a $\mathbf{2413}$, which is a contradiction. Therefore, σ has at most one ascent, so $\sigma \in G_n$,

$$\mathcal{S}_n(\mathbf{123}, \mathbf{3412}, \mathbf{2413}) \subset G_n\,, \text{ and } G_n = \mathcal{S}_n(\mathbf{123}, \mathbf{3412}, \mathbf{2413})\,,$$

as desired. $\square$

The following lemma regarding symmetry of permutations in G_n will be helpful to simplify our arguments in subsequent sections.

Lemma 5.3.2. *Let $n \in \mathbb{N}$, and $1 \leq j, k \leq n$. Then*

$$G_n(j,k) = G_n(n+1-j, n+1-k)\,.$$

Proof. Let $\varphi : G_n \to G_n$ be defined so that $\varphi(\sigma) = \sigma^{RC}$. Since $\mathbf{123}^{RC} = \mathbf{123}, \mathbf{3412}^{RC} = \mathbf{3412}$, and $\mathbf{2413}^{RC} = \mathbf{2413}$, we see that φ does indeed map permutations in G_n to other permutations in G_n. Also, φ is an involution, since $(\sigma^{RC})^{RC} = \sigma$. Therefore, permutations $\sigma \in G_n$ with $\sigma(j) = k$ are equinumerous with permutations $\tau \in G_n$ with $\tau(n+1-j) = n+1-k$, since each permutation $\tau = \varphi(\sigma)$ for some σ. This completes the proof. $\square$

We require one more lemma, in which we exhibit the explicit number of such permutations σ with $\sigma(j) = k$.

Lemma 5.3.3. *Let $j, k, n \in \mathbb{N}$, and $1 \leq j, k \leq n$. Let $G_n(j,k)$ be the proportion of permutations $\sigma \in G_n$ such that $\sigma(j) = k$. Then*

$$G_n(j,k) = \begin{cases} \frac{2^{k-1}\binom{n-k}{j-1}}{2^n-n} & \textit{if } j+k \leq n\,, \\ \frac{2^{n-k}\binom{k-1}{n-j}}{2^n-n} & \textit{if } j+k > n+1\,, \\ \frac{2^{j-1}+2^{k-1}-n}{2^n-n} & \textit{otherwise.} \end{cases}$$

We prove the lemma by considering the structure of a typical permutation $\sigma \in G_n$.

Proof. Let $\sigma \in G_n$ with $\sigma(j) = k$, with $j + k \leq n$, so we are in the first case of Lemma 5.3.3. Since $\sigma \in G_n$, there must be a unique index i which marks the position of an ascent. Suppose $i < j$. Then $\sigma(r) > \sigma(r+1)$ for all $r \geq j$, so $\sigma(n) \leq \sigma(j) - (n-j) = k - n + j \leq 0$, which is a contradiction. Therefore, $i \geq j$, and we must have $\sigma(1) > \sigma(2) > \ldots > \sigma(j)$. There are $\binom{n-k}{j-1}$ ways to choose the $j-1$ elements in these first positions, since they must be larger than k and must be in decreasing order.

Now, since $j+k \leq n$, $j-1$ is strictly less than $n-k$. Therefore, there is some element larger than k which comes after k in σ. We now argue that there are 2^{k-1} ways to complete σ. Once we choose the subset of $[k-1]$ which comes before position i, we have completely determined σ, since the remaining elements after the single ascent at position i must be in increasing order. Since there are 2^{k-1} subsets of $[k-1]$, there are 2^{k-1} ways to complete σ, and $2^{k-1}\binom{n-k}{j-1}$ possible permutations $\sigma \in G_n$ with $\sigma(j) = k$.

By Lemma 5.3.2, we get the second case of Lemma 5.3.3, due to the symmetry of these permutations. Given $\sigma \in G_n$, the permutation σ^{RC} is also in G_n, and the formulas in this case hold as well.

For $j+k = n+1$, there are a couple cases. First, suppose $i < j$. Then we must have $\sigma(j) = k > \sigma(j+1) > \ldots > \sigma(n)$, so $\sigma(n) \leq k-(n-j) = 1$. Therefore $\sigma(n) = 1$, and each of these inequalities must be a difference of only 1, so every number after k is consecutive. There are $2^{j-1}-(j-1)$ ways to choose the first $j-1$ elements of σ, since they can have at most one ascent. The other possibility is that $i \geq j$. Then the first $j-1$ elements must all be bigger than k, and there are only $n-k = j-1$ elements bigger than k, so the permutation must start $(n, n-1, \ldots, k+1, k, \ldots)$. There are $2^{k-1}-(k-1)$ ways to complete the permutation in this case, since there can be at most one ascent in the final $k-1$ elements. To make sure we are not counting any permutation twice, we subtract 1, since the all-decreasing permutation was counted in each case. Therefore, there are $2^{j-1}+2^{k-1}-(j-1)-(k-1)-1 = 2^{j-1}+2^{k-1}-(j+k-1) = 2^{j-1}+2^{k-1}-n$ such permutations, as desired. $\square$

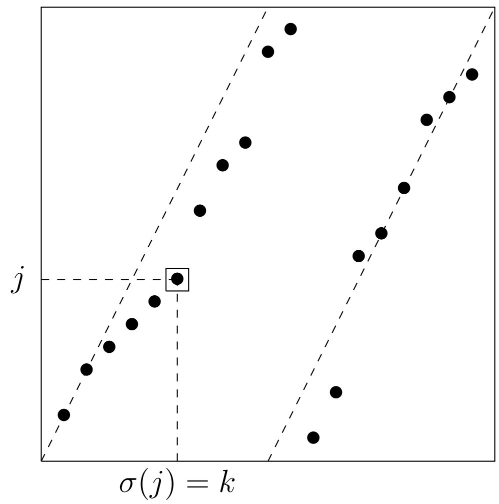


Figure 5.3: Random $\sigma \in G_n$

5.3.2 Random $\sigma \in G_n$

In Figure 5.3, we display a random $\sigma \in G_{19}$, before proving the main theorems.

5.4 Proof of Theorems 5.2.1, 5.2.2, and 5.2.3

The proof of the three theorems relies on several lemmas. We begin with a technical lemma.

Lemma 5.4.1. *Let* $f : [0,1]^2 \to \mathbb{R}$ *be defined so that*

$$f(x,y) = \frac{(x+y)^{(x+y)}}{(2x)^x(2y)^y}.$$

Then $f(x,y)$ *achieves a maximum of 1 at* $x = y$*, for* $x \in [0,1]$.

Proof. First we consider the boundaries of the unit square.

For $x = 0$, we have

$$f(0,y) = \frac{y^y}{(2y)^y} = \left(\frac{1}{2}\right)^y,$$

which is maximized at $y = 0$, with $f(0,0) = 1$.

For $x = 1$, we have

$$f(1, y) = \frac{(1+y)^{(1+y)}}{2(2y)^y} = \frac{1+y}{2}\left(\frac{1+y}{2y}\right)^y.$$

Differentiating with respect to y gives

$$f'(1, y) = \left(\frac{1+y}{2y}\right)^y \left(\frac{1}{2} + \frac{1+y}{2}\left(\ln\frac{1+y}{2y} + \frac{y^2}{1+y}\right)\right),$$

which is always positive since each term is positive.

Therefore, $f(1, y)$ is maximized at $y = 1$, with

$$f(1, 1) = \frac{2^2}{2(2)} = 1.$$

The analysis for $y = 0$ and $y = 1$ yields the same results, since the function is symmetrical in x and y.

Finally, we consider $f(x, y)$ with $x, y \in (0, 1)$. Here we take partial derivatives with respect to x and y. We have

$$\frac{\partial f}{\partial x} = \frac{(2x)^x(2y)^y(x+y)^{x+y}\left(\ln(x+y) - \ln 2x\right)}{(2x)^{2x}(2y)^{2y}},$$

which is 0 when $x = y$.

Due to the symmetry of $f(x, y)$, the other partial derivative has the same behavior. Since

$$f(x, x) = \frac{(2x)^{2x}}{(2x)^x(2x)^x} = 1,$$

we have completed the proof.

$\square$

5.4.1 Proof of Theorems 5.2.1, 5.2.2, and 5.2.3

We prove Theorem 5.2.1 case by case: first, suppose $a + b \leq 1$, with $\alpha \leq \frac{1}{2}$ and $\beta \leq \frac{1}{2}$. First, suppose $b \neq 1$, so $a+b < 1$. This is a position which should be filled by an element in the first decreasing sequence. By Lemma 5.3.3 and Stirling's formula, we see that

$$\begin{aligned} G_n(an+cn^\alpha, bn+dn^\beta) &= \left(\frac{1}{2^n-n}\right) 2^{bn+dn^\beta-1} \binom{n-(bn+dn^\beta)}{an+cn^\alpha-1}, \\ &\sim E_n(a,b,c,d,\alpha,\beta) \times K_n(a,b,c,d,\alpha,\beta), \end{aligned}$$

where

$$\begin{aligned} E_n(a,b,c,d,\alpha,\beta) &= \frac{(an+cn^\alpha)}{((1-b-a)n-dn^\beta-cn^\alpha)} \\ &\times \sqrt{\left[\frac{2\pi\left((1-b)n-dn^\beta\right)}{4\pi^2(an+cn^\alpha)((1-b-a)n-dn^\beta-cn^\alpha)}\right]} \end{aligned}$$

and

$$\begin{aligned} K_n(a,b,c,d,\alpha,\beta) &= \frac{2^{bn+dn^\beta-1}}{(2^n-n)\left(an+cn^\alpha\right)^{an+cn^\alpha}} \\ &\times \frac{\left((1-b)n-dn^\beta\right)^{(1-b)n-dn^\beta}}{\left((1-b-a)n-dn^\beta-cn^\alpha\right)^{(1-b-a)n-dn^\beta-cn^\alpha}} \end{aligned}$$

Simplifying $K_n(a,b,c,d,\alpha,\beta)$, we see that

$$\begin{aligned} K_n(a,b,c,d,\alpha,\beta) &= \frac{2^{bn+dn^\beta-1}(1-b)^{(1-b)n-dn^\beta}}{(2^n-n)a^{an+cn^\alpha}(1-b-a)^{(1-b-a)n-dn^\beta-cn^\alpha}} \\ &\times \left(\frac{1}{\left(1+\frac{c}{a}n^{\alpha-1}\right)^{an+cn^\alpha}}\right) \\ &\times \left(\frac{\left(1-\frac{d}{1-b}n^{\beta-1}\right)^{(1-b)n-dn^\beta}}{\left(1-\frac{c}{1-b-a}n^{\alpha-1}-\frac{d}{1-b-a}n^{\beta-1}\right)^{(1-b-a)n-cn^\alpha-dn^\beta}}\right) \\ &\sim f(a,1-b-a)^n \left(\frac{2^{dn^{\beta-1}-1}(1-b-a)^{dn^{\beta-1}+cn^{\alpha-1}}}{(1-b)^{dn^{\beta-1}}a^{cn^{\alpha-1}}}\right) \\ &\times \exp\left[\frac{d^2}{4a}n^{2\beta-1}-\frac{c^2}{2a}n^{2\alpha-1}-\frac{(cn^{\alpha-1}+dn^{\beta-1})^2 n}{2a}\right] \\ &\sim f(a,1-b-a)^n \left(\frac{2^{dn^{\beta-1}-1}(1-b-a)^{dn^{\beta-1}+cn^{\alpha-1}}}{(1-b)^{dn^{\beta-1}}a^{cn^{\alpha-1}}}\right) \\ &\times \exp\left[-\frac{(2cn^\alpha+dn^\beta)^2}{4an}\right], \end{aligned}$$

using the Taylor expansion for $\ln(1+x)$ and the notation from Lemma 5.4.1.

Applying Lemma 5.4.1, we see that if $a \neq 1-b-a$, we have $f(a, 1-b-a) < 1$. Let

$$\varepsilon = \frac{f(a, 1-b-a)+1}{2}.$$

Since none of the other terms in $G_n(an+cn^\alpha, bn+dn^\beta)$ are exponential in n, we see that for n large enough, we have $G_n(an+cn^\alpha, bn+dn^\beta) < \varepsilon^n$. This fulfills the first case of Theorem 5.2.2.

For $a = 1-b-a$, we instead have

$$\begin{aligned} K_n(a,b,c,d,\alpha,\beta) &\sim f(a,a)^n \left(\frac{2^{dn^{\beta-1}-1} a^{dn^{\beta-1}+cn^{\alpha-1}}}{(2a)^{dn^{\beta-1}} a^{cn^{\alpha-1}}}\right) \times \exp\left[-\frac{(2cn^\alpha + dn^\beta)^2}{4an}\right] \\ &= \frac{1}{2} \exp\left[-\frac{(2cn^\alpha+dn^\beta)^2}{4an}\right], \end{aligned}$$

and

$$\begin{aligned} E_n(a,b,c,d,\alpha,\beta) &\sim \frac{a}{1-b-a}\sqrt{\frac{1-b}{2a(1-b-a)\pi n}} \\ &= n^{-\frac{1}{2}}\sqrt{\frac{1}{a\pi}}. \end{aligned}$$

Therefore,

$$G_n(an+cn^\alpha, bn+dn^\beta) \sim n^{-\frac{1}{2}}\sqrt{\frac{1}{4a\pi}} \exp\left[-\frac{(2cn^\alpha+dn^\beta)^2}{4an}\right].$$

For $\alpha, \beta < \frac{1}{2}$, as $n \to \infty$, we have

$$G_n(an+cn^\alpha, bn+dn^\beta) \sim n^{-\frac{1}{2}}\sqrt{\frac{1}{4a\pi}},$$

proving the first case of Theorem 5.2.3. The next three cases with $\alpha = \frac{1}{2}$ or $\beta = \frac{1}{2}$ follow as well.

Now suppose $\alpha > \frac{1}{2}$, and $\alpha > \beta$. We see that

$$\begin{aligned} G_n(an+cn^\alpha, bn+dn^\beta) &\sim n^{-\frac{1}{2}}\sqrt{\frac{1}{4a\pi}} \exp\left[-\frac{(2cn^\alpha+dn^\beta)^2}{4an}\right] \\ &\sim n^{-\frac{1}{2}}\sqrt{\frac{1}{4a\pi}} \exp -\frac{4c^2n^{2\alpha-1}}{4a}. \end{aligned}$$

For $\varepsilon = \frac{e^{-\frac{c^2}{a}}+1}{2}$, we have

$$G_n(an + cn^{\alpha}, bn + dn^{\beta}) < \varepsilon^n \,,$$

for n large enough, as desired in the second case of Theorem 5.2.2. We omit proofs of the cases with $\alpha = \beta > \frac{1}{2}$ and $\beta > \alpha, \frac{1}{2}$, since they are very similar to this one, .

Again, the cases with $2a + b = 2$ rather than $2a + b = 1$ follow from the fact that $G_n(j, k) = G_n(n + 1 - k, n + 1 - j)$.

Now, suppose $\alpha = \frac{1}{2} > \beta$. Then

$$\begin{aligned} G_n(an + cn^{\alpha}, bn + dn^{\beta}) &\sim n^{-\frac{1}{2}} \sqrt{\frac{1}{4a\pi}} \exp\left[-\frac{2cn^{\alpha} + dn^{\beta})^2}{4an}\right] \\ &\sim n^{-\frac{1}{2}} \sqrt{\frac{1}{4a\pi}} \exp -\frac{c^2}{a} \,, \end{aligned}$$

as $n \to \infty$. This matches the second case of Theorem 5.2.3, and also the first case of Theorem 5.2.1. The results for $\alpha = \beta = \frac{1}{2}$ and $\beta = \frac{1}{2} > \alpha$ are similar.

For cases with $2a + b = 2$ and $\alpha, \beta \le \frac{1}{2}$, we get the analogous results by the symmetry discussed in Lemma 5.3.2.

Now, suppose $a = 0, b = 1$. We must have $d < 0$ and $c > 0$ for $G_n(cn^{\alpha}, n+dn^{\beta})$ to be defined, and we also need $cn^{\alpha} + dn^{\beta} \le 0$ for all n for us to be in the first case of Lemma 5.3.3. This implies we need $\alpha \le \beta$. Calculating, we see that

$$\begin{aligned} G_n(cn^{\alpha}, n + dn^{\beta}) \sim{}& \frac{2^{n+dn^{\beta}-1} cn^{\alpha}}{(2^n - n)(-dn^{\beta} - cn^{\alpha})} \left(\frac{(-dn^{\beta})^{-dn^{\beta}}}{(cn^{\alpha})^{cn^{\alpha}} (-dn^{\beta} - cn^{\alpha})^{dn^{\beta} - cn^{\alpha}}}\right) \\ &\times \sqrt{\frac{-2\pi dn^{\beta}}{4\pi^2 cn^{\alpha}(-dn^{\beta} - cn^{\alpha})}} \,. \end{aligned}$$

For $\alpha < \beta$, we get

$$G_n(cn^{\alpha}, n + dn^{\beta}) \sim \frac{2^{dn^{\beta}-1} cn^{\alpha}}{-dn^{\beta}} \left(e\left(-\frac{d}{c} n^{\beta-\alpha} - 1\right)\right)^{cn^{\alpha}} \,.$$

With $\varepsilon = \frac{1+2^{-d}}{2^{-d+1}}$, we see that $G_n(cn^{\alpha}, n + dn^{\beta}) < \varepsilon^n$ for n large enough, as desired in the final case of Theorem 5.2.2.

Now, suppose $\alpha = \beta$. Then we have

$$G_n(cn^\alpha, n + dn^\beta) \sim \frac{2^{dn^\beta - 1} cn^\beta}{(-d-c)n^\beta} \left(\frac{(-d)^{-d}}{c^c(-d-c)^{-d-c}} \right)^{n^\beta} \sqrt{\frac{-d}{2\pi c(-d-c)n^\beta}}$$

$$\sim f(c, -d-c)^{n^\beta} \frac{c}{-d-c} n^{-\frac{\beta}{2}} \sqrt{\frac{-d}{8\pi c(-d-c)}} \,,$$

where $f(c, -d-c)$ is defined as in Lemma 5.4.1. For $c \neq -d-c$, by Lemma 5.4.1, letting

$$\varepsilon = \frac{f(c, -d-c) + 1}{2} \,,$$

we see that $G_n(cn^\alpha, n + dn^\beta) < \varepsilon^{n^\beta}$, for n large enough, fulfilling the fifth case of Theorem 5.2.2.

On the other hand, for $c = -d - c$, or $2c = -d$, we observe that

$$G_n(cn^\alpha, n + dn^\beta) \sim n^{-\frac{\beta}{2}} \sqrt{\frac{1}{4c\pi}} \,,$$

matching the second case of Theorem 5.2.1.

Finally, consider $a = 1$ and $b = 0$. We see that

$$G_n(an + cn^\alpha, bn + dn^\beta) = G_n(n + cn^\alpha, dn^\beta) \,,$$

and it is only well-defined if $c \leq 0 \leq d$, $\alpha \geq \beta$, and $cn^\alpha + dn^\beta \leq 0$.

Analyzing G_n, we have

$$G_n(n + cn^\alpha, dn^\beta) = \left(\frac{1}{2^n - n} \right) 2^{dn^\beta - 1} \binom{n - dn^\beta}{n + cn^\alpha - 1}$$

$$\sim 2^{dn^\beta - n - 1} \frac{n + cn^\alpha}{-cn^\alpha - dn^\beta - 1}$$

$$\times \left(\frac{(n - dn^\beta)^{n - dn^\beta}}{(n + cn^\alpha)^{n + cn^\alpha} (-cn^\alpha - dn^\beta)^{-cn^\alpha - dn^\beta}} \right)$$

$$\times \sqrt{\frac{2\pi(n - dn^\beta)}{4\pi^2(n + cn^\alpha)(-cn^\alpha - dn^\beta)}} \,.$$

Now, supposing $\alpha > \beta$, and simplifying by using the Taylor expansion for $\ln(1 + x)$, we see that

$$G_n(n + cn^\alpha, dn^\beta) \sim 2^{dn^\beta - n - 1} \frac{n^{1-\alpha}}{-c} \left(\frac{en^{1-\alpha}}{-c} \right)^{-cn^\alpha - dn^\beta} \sqrt{\frac{1}{-2c\pi n^\alpha}} \,.$$

Since the exponent of n only appears on 2 in the denominator, if we let $\varepsilon = \frac{3}{4}$, we see that $G_n(n + cn^\alpha, dn^\beta) < \varepsilon^n$, for n large enough, as desired.

The case with $\alpha = \beta$ is very similar, again we get $G_n(n + cn^\alpha, dn^\beta) < \varepsilon^n$ for n large enough. This completes the proof of Theorems 5.2.1, 5.2.2, and 5.2.3.

$\square$

5.5 Corollaries

Knowing the limit shape of permutations in G_n allows us to calculate limit shapes for permutations in $\mathcal{S}_n(\mathbf{123}, \mathbf{3412})$, as well. Let $R_n(j, k)$ denote the number of permutations $\sigma \in \mathcal{S}_n(\mathbf{123}, \mathbf{3412})$ with $\sigma(j) = k$.

Define

$$Y(a, b, c, d, \alpha, \beta) = \sup \left\{ r \in \mathbb{R}_+ \middle| \lim_{n\to\infty} n^r R_n(an + cn^\alpha, bn + dn^\beta) < \infty \right\}.$$

Similarly, let

$$Z(a, b, c, d, \alpha, \beta) = \lim_{n\to\infty} n^{Y(a,b,c,d,\alpha,\beta)} R_n(an + cn^\alpha, bn + dn^\beta)\,,$$

for values of $a, b, c, d, \alpha, \beta$ with $Y(a, b, c, d, \alpha, \beta) < \infty$.

We have the following corollaries.

Corollary 5.5.1. *Let $0 \le a, b \le 1, c, d \in \mathbb{R}, \alpha, \beta \in [0, 1)$. For $a \notin \{0, 1\}$, we have*

$$Y(a, b, c, d, \alpha, \beta) = \begin{cases} \frac{1}{2} & \textit{if } a + 2b \in \{1, 2\}, \alpha, \beta \le \frac{1}{2}\,, \\ \frac{1}{2} & \textit{if } 2a + b \in \{1, 2\}, \alpha, \beta \le \frac{1}{2}\,. \\ \infty & \textit{otherwise.} \end{cases}$$

For $a \in \{0,1\}$, we have

$$Y(a,b,c,d,\alpha,\beta) = \begin{cases} \frac{\beta}{2} & \textit{if } a+b=1, \alpha=\beta>0, 2c=-d \textit{ or } c=-2d\,, \\ 0 & \textit{if } a+b=1, \alpha=\beta=0\,, \\ \infty & \textit{otherwise.} \end{cases}$$

Corollary 5.5.2. *Let $0 \le a, b \le 1, c, d \in \mathbb{R}, \alpha, \beta \in [0,1)$ be constants. Then for n large enough, we have*

$$H_n(an+cn^\alpha, bn+dn^\beta) < \begin{cases} \varepsilon^n & 2a+b \notin \{1,2\}, \textit{ and } a+2b \notin \{1,2\} \\ \varepsilon^{n^{2\alpha-1}} & 2a+b \in \{1,2\}, \alpha \ge \beta, \alpha > \frac{1}{2}\,, \\ \varepsilon^{n^{2\alpha-1}} & a+2b \in \{1,2\}, \alpha \ge \beta, \alpha > \frac{1}{2} \\ \varepsilon^{n^{2\beta-1}} & 2a+b \in \{1,2\}, \alpha < \beta, \frac{1}{2} < \beta\,, \\ \varepsilon^{n^{2\beta-1}} & a+2b \in \{1,2\}, \alpha < \beta, \frac{1}{2} < \beta\,, \end{cases}$$

and for $a+b=1, a \in \{0,1\}$, we have

$$H_n(an+cn^\alpha, bn+dn^\beta) < \begin{cases} \varepsilon^{n^\beta} & 0 < \alpha = \beta, 2c \notin \{-d, -4d\}\,, \\ \varepsilon^{n^\beta} & \alpha < \beta\,, \\ \varepsilon^{n^\alpha} & \beta < \alpha\,, \end{cases}$$

where $\varepsilon = \varepsilon(a,b,c,d,\alpha,\beta)$ is independent of n, and $0 < \varepsilon < 1$.

Corollary 5.5.3. *Let $0 \le a, b \le 1$, $c, d \in \mathbb{R}$, and $\alpha, \beta \in [0,1)$ s.t.*

$$Y(a,b,c,d,\alpha,\beta) < \infty\,.$$

Then for $a > 0, 2a+b=1, a+2b \ne 1$, we have

$$Z(a,b,c,d,\alpha,\beta) = \begin{cases} \frac{1}{4\sqrt{a\pi}} & \textit{for } \alpha, \beta < \frac{1}{2}\,, \\ \frac{1}{4\sqrt{a\pi}} \exp\left[-\frac{c^2}{a}\right] & \textit{for } \,, \frac{1}{2} = \alpha > \beta\,, \\ \frac{1}{4\sqrt{a\pi}} \exp\left[-\frac{d^2}{4a}\right] & \textit{for } \frac{1}{2} = \beta > \alpha\,, \\ \frac{1}{4\sqrt{a\pi}} \exp\left[-\frac{(d+2c)^2}{4a}\right] & \textit{for } \frac{1}{2} = \beta = \alpha\,, \end{cases}$$

and for $b > 0, a + 2b = 1, 2a + b \neq 1,$ *we have*

$$Z(a,b,c,d,\alpha,\beta) = \begin{cases} \frac{1}{4\sqrt{b\pi}} & \textit{for } \alpha, \beta < \frac{1}{2}\,, \\ \frac{1}{4\sqrt{b\pi}} \exp\left[-\frac{c^2}{b}\right] & \textit{for } \frac{1}{2} = \alpha > \beta\,, \\ \frac{1}{4\sqrt{b\pi}} \exp\left[-\frac{d^2}{4b}\right] & \textit{for } \frac{1}{2} = \beta > \alpha\,, \\ \frac{1}{4\sqrt{b\pi}} \exp\left[-\frac{(2d+c)^2}{4b}\right] & \textit{for } \frac{1}{2} = \beta = \alpha\,. \end{cases}$$

For $2a + b = 2, a < 1, a + 2b \neq 2,$ *we have*

$$Z(a,b,c,d,\alpha,\beta) = \begin{cases} \frac{1}{4\sqrt{(1-a)\pi}} & \textit{for } \alpha, \beta < \frac{1}{2}\,, \\ \frac{1}{4\sqrt{(1-a)\pi}} \exp\left[-\frac{c^2}{(1-a)}\right] & \textit{for } , \frac{1}{2} = \alpha > \beta\,, \\ \frac{1}{4\sqrt{(1-a)\pi}} \exp\left[-\frac{d^2}{4(1-a)}\right] & \textit{for } \frac{1}{2} = \beta > \alpha\,, \\ \frac{1}{4\sqrt{(1-a)\pi}} \exp\left[-\frac{(d+2c)^2}{4(1-a)}\right] & \textit{for } \frac{1}{2} = \beta = \alpha\,, \end{cases}$$

while for $a + 2b = 2, b < 1, 2a + b \neq 2,$ *we have*

$$Z(a,b,c,d,\alpha,\beta) = \begin{cases} \frac{1}{4\sqrt{(1-b)\pi}} & \textit{for } \alpha, \beta < \frac{1}{2}\,, \\ \frac{1}{4\sqrt{(1-b)\pi}} \exp\left[-\frac{c^2}{(1-b)}\right] & \textit{for } \frac{1}{2} = \alpha > \beta\,, \\ \frac{1}{4\sqrt{(1-b)\pi}} \exp\left[-\frac{d^2}{4(1-b)}\right] & \textit{for } \frac{1}{2} = \beta > \alpha\,, \\ \frac{1}{4\sqrt{(1-b)\pi}} \exp\left[-\frac{(2d+c)^2}{4(1-b)}\right] & \textit{for } \frac{1}{2} = \beta = \alpha\,. \end{cases}$$

For $a = b = \frac{1}{3},$ *we have*

$$Z(a,b,c,d,\alpha,\beta) = \begin{cases} \frac{1}{2\sqrt{a\pi}} & \alpha, \beta < \frac{1}{2}\,, \\ \frac{1}{2\sqrt{a\pi}} \exp\left[-\frac{c^2}{a}\right] & \frac{1}{2} = \alpha > \beta\,, \\ \frac{1}{2\sqrt{a\pi}} \exp\left[-\frac{d^2}{4a}\right] & \frac{1}{2} = \beta > \alpha\,, \\ \frac{1}{2\sqrt{a\pi}} \exp\left[-\frac{(d+2c)^2}{4a}\right] & \frac{1}{2} = \beta = \alpha\,. \end{cases}$$

For $a = b = \frac{2}{3}$, we have

$$Z(a,b,c,d,\alpha,\beta) = \begin{cases} \frac{1}{2\sqrt{(1-a)\pi}} & \alpha,\beta < \frac{1}{2}\,, \\ \frac{1}{2\sqrt{(1-a)\pi}} \exp\left[-\frac{c^2}{(1-a)}\right] & \frac{1}{2} = \alpha > \beta\,, \\ \frac{1}{2\sqrt{(1-a)\pi}} \exp\left[-\frac{d^2}{4(1-a)}\right] & \frac{1}{2} = \beta > \alpha\,, \\ \frac{1}{2\sqrt{(1-a)\pi}} \exp\left[-\frac{(d+2c)^2}{4(1-a)}\right] & \frac{1}{2} = \beta = \alpha\,. \end{cases}$$

Finally, for $a + b = 1$, we have

$$Z(a,b,c,d,\alpha,\beta) = \begin{cases} \frac{1}{4\sqrt{c\pi}} & a = 0, \alpha = \beta > 0, 2c = -d\,, \\ \frac{1}{4\sqrt{-d\pi}} & a = 0, \alpha = \beta > 0, -c = 2d\,, \\ \frac{1}{4\sqrt{-c\pi}} & a = 1, \alpha = \beta > 0, -2c = d\,, \\ \frac{1}{4\sqrt{d\pi}} & a = 1, \alpha = \beta > 0, c = -2d\,, \\ 2^{d-2}\binom{-d}{c-1} & a = 0, \alpha = \beta = 0, d + c \le 0, c \ge 1\,, \\ 2^{-c-1}\binom{c-1}{-d} & a = 0, \alpha = \beta = 0, d + c \ge 2, d \le 0\,, \\ 2^{c-2}\binom{-c}{d-1} & a = 1, \alpha = \beta = 0, d + c \le 0, d \ge 1\,, \\ 2^{-d-1}\binom{d-1}{-c} & a = 1, \alpha = \beta = 0, d + c \ge 2, c \le 0\,, \\ 2^{c-1} & a = 1, \alpha = \beta = 0, d + c = 1, c \le 0\,, \\ 2^{d-1} & a = 0, \alpha = \beta = 0, d + c = 1, c \ge 1\,. \end{cases}$$

5.5.1 Proof of Corollaries 5.5.1, 5.5.2, and 5.5.3

The proof here relies crucially on the fact that

$$\mathcal{S}_n(\mathbf{123}, \mathbf{3412}) = G_n \cup G_n^{-1}\,.$$

Because of this, we can calculate $R_n(j,k)$, by

$$R_n(j,k) = \frac{(2^n - n)G_n(j,k) + (2^n - n)G_n(k,j) - D_n(j,k)}{2^{n+1} - 2n - \binom{n+1}{3} - 1}\,,$$

where $D_n(j,k)$ is the number of permutations $\sigma \in G_n \cap G_n^{-1}$ such that $\sigma(j) = k$. Since $D_n(j,k) \leq \binom{n+1}{3}$ for all n, j, k, we have

$$R_n(j,k) \sim \frac{G_n(j,k)}{2} + \frac{G_n(k,j)}{2} + o(G_n(j,k)),$$

as $n \to \infty$. This is because $D_n(j,k) = o(2^n - n)G_n(k,j)$.

Therefore, when analyzing $R_n(an + cn^\alpha, bn + dn^\beta)$, we can take the average of $G_n(an + cn^\alpha, bn + dn^\beta)$ with $G_n(bn + dn^\beta, an + cn^\alpha)$. This has the effect of adding the limit shape to its reflection about the main diagonal of the matrix. Corollaries 5.5.1, 5.5.2, and 5.5.3 now follow from Theorems 5.2.1, 5.2.2, and 5.2.3.

□

This behavior is exhibited in Figure 5.4. The shape that appears comes from superimposing the limit shape in Figure 5.2 with a version of it which was reflected about the main diagonal of the square. We describe it as a "longboat" here, to distinguish it from the "canoe" in Figure 8.5. By requiring permutations in $\mathcal{S}_n(\mathbf{123})$ to also avoid permutations in $\mathcal{S}_n(\mathbf{3412})$, we find that the canoe becomes wider. In fact, as $n \to \infty$, the sides of the canoe no longer get narrower and narrower, instead the relative dimensions of the longboat remain constant as $n \to \infty$.

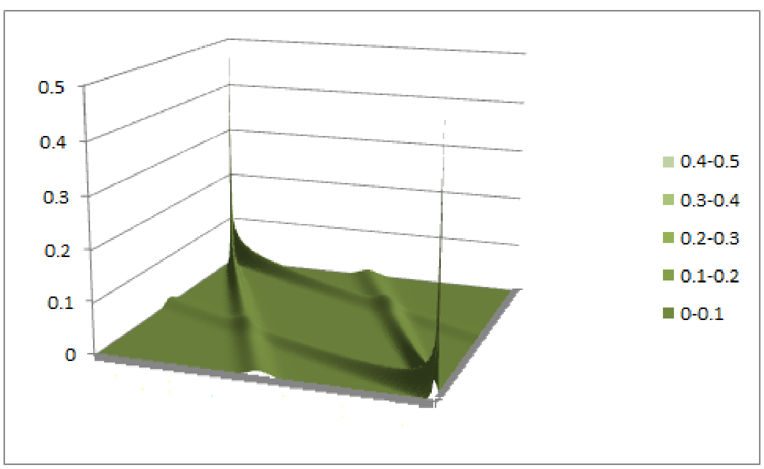


Figure 5.4: Limit shape of $R_{250}(j,k)$

5.6 Results on permutation statistics in $\mathcal{S}_n(\mathbf{123}, \mathbf{3412})$

5.6.1 Fixed points in $\mathcal{S}_n(\mathbf{123}, \mathbf{3412})$

As in Chapter 2.9.9, we calculate the expected number of fixed points in our permutation classes.

Theorem 5.6.1. *Let $\sigma \in G_n, \tau \in \mathcal{S}_n(\mathbf{123}, \mathbf{3412})$ be chosen uniformly at random. Then*

$$\mathbf{E}[fp_n(\sigma)] \to \frac{2}{3}$$

and

$$\mathbf{E}[fp_n(\tau)] \to \frac{2}{3} \quad as \quad n \to \infty\,.$$

Theorem 5.6.2. *Let $\sigma \in G_n^R, \tau \in \mathcal{S}_n(\mathbf{321}, \mathbf{2143})$ be chosen uniformly at random. Then*

$$\mathbf{E}[fp_n(\sigma)], \mathbf{E}[fp_n(\tau)] \to 2, \quad as \quad n \to \infty\,.$$

For both these permutation classes, the expected number of fixed points is finite. Perhaps counterintuitively, requiring a permutation in $\mathcal{S}_n(\mathbf{123})$ to additionally avoid **3412** increases the expected number of fixed points from $\frac{1}{2}$ to $\frac{2}{3}$. Similarly, requiring a permutation to avoid **2143** as well as **321** increases the expected number of fixed points from 1 to 2.

5.6.2 Proof of Theorem 5.6.1

Let σ be chosen uniformly at random from G_n. Then

$$\mathbf{E}[\mathrm{fp}_n(\sigma)] = \sum_{j=1}^{n} G_n(j,j)\,.$$

Letting $j = an + cn^\alpha$, we can apply Theorems 5.2.1, 5.2.2, and 5.2.3. By Theorem 5.2.2, we see that

$$G_n(an + cn^\alpha, an + cn^\alpha) < \varepsilon^n\,,$$

unless $a \in \{\frac{1}{3}, \frac{2}{3}\}$. In fact, even if $a \in \{\frac{1}{3}, \frac{2}{3}\}$, we need $\alpha \leq \frac{1}{2}$, for $G_n(an+cn^\alpha, an+cn^\alpha)$ to contribute something bigger than $\varepsilon^{n^{2\alpha-1}}$, for some $\varepsilon \in (0,1)$.

By Theorems 5.2.1 and 5.2.2, we obtain

$$G_n(\frac{n}{3} + c\sqrt{n}, \frac{n}{3} + c\sqrt{n}) \sim \frac{1}{2\sqrt{\frac{1}{3}\pi n}} \exp\left[-\frac{(3c)^2}{\frac{4}{3}}\right].$$

Summing from $j = \frac{n}{3} - c\sqrt{n}$ to $j = \frac{n}{3} + c\sqrt{n}$, and interpreting the sum as a Riemann sum, we get

$$\begin{aligned}\sum_{j=\frac{n}{3}-c\sqrt{n}}^{j=\frac{n}{3}+c\sqrt{n}} G_n(j,j) &\sim \frac{1}{2\sqrt{\frac{1}{3}\pi n}}\sqrt{n}\int_{-c}^{c} \exp\left[-\frac{27x^2}{4}\right] dx \\ &\sim \frac{1}{2\sqrt{\frac{1}{3}\pi}}\frac{2}{3\sqrt{3}}\int_{-c}^{c} e^{-x^2} dx\,.\end{aligned}$$

We can choose c to be as large as we want, giving

$$\begin{aligned}\sum_{j=1}^{\frac{n}{2}} G_n(j,j) &\geq \frac{1}{3\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-x^2} dx \\ &= \frac{1}{3}\,.\end{aligned}$$

A similar argument shows that summing values of j near $\frac{2}{3}n$ also yields $\frac{1}{3}$. Since adding values of $G_n(j,j)$ with $|j - \frac{n}{3}| > c\sqrt{n}$ does not contribute more than a positive power of ε, we have

$$\frac{2}{3} \leq \sum_{j=1}^{n} G_n(j,j) \leq \frac{2}{3} + n\varepsilon^{n^{2\alpha-1}},$$

as $n \to \infty$, where α can be any exponent slightly larger than $\frac{1}{2}$. Since $n\varepsilon^{n^{2\alpha-1}} \to 0$ as $n \to \infty$, we get

$$\mathbf{E}[\mathrm{fp}_n(\sigma)] \to \frac{2}{3}$$

as $n \to \infty$, as desired.

Observe that an argument which is almost equivalent works for $\mathbf{E}[\mathrm{fp}_n(\tau)]$ as well, since the values of $H_n(j,j)$ which are greater than a power of ε have exactly the same limiting behavior as $n \to \infty$. Again, for j near $\frac{n}{3}$ or $\frac{2n}{3}$, each term is on the order of $n^{-\frac{1}{2}}$. When the number of terms we sum is on the order of $n^{\frac{1}{2}}$, we end up getting a constant, which happens to be $\frac{2}{3}$, in this situation.

To make this more precise, we have

$$\begin{aligned}
\mathbf{E}[\mathrm{fp}_n(\tau)] &= \sum_{j=1}^{n} H_n(j,j) \\
&= \sum_{|j-\frac{n}{3}|\le c\sqrt{n}} H_n(j,j) + \sum_{|j-\frac{2n}{3}|\le c\sqrt{n}} H_n(j,j) + \sum_{\text{all other } j} H_n(j,j)\,, \\
&\sim \frac{1}{3\sqrt{\pi}} \int_{-c}^{c} e^{-x^2}dx + \frac{1}{3\sqrt{\pi}} \int_{-c}^{c} e^{-x^2}dx + o(1)\,.
\end{aligned}$$

Letting $c \to \infty$ gives us

$$\mathbf{E}[\mathrm{fp}_n(\tau)] \to \frac{2}{3} \quad \text{as} \quad n \to \infty\,,$$

as desired. $\square$

5.6.3 Proof of Theorem 5.6.2

Let ρ be chosen uniformly at random from G_n. Let $\sigma = \rho^R$. Observe that σ is now uniform in G_n^R. Therefore, $\mathbf{E}[\mathrm{fp}_n(\sigma)] = \mathbf{E}[\mathrm{afp}_n(\rho)]$.

Applying this logic, we get

$$\mathbf{E}[\mathrm{afp}_n(\rho)] = \sum_{j=1}^{n} G_n(j, n+1-j)\,.$$

By Lemma 5.3.3, for all $1 \le j \le n$, we have

$$G_n(j, n+1-j) = \frac{2^{j-1} + 2^{n-j+1-1} - n}{2^n - n}\,.$$

Summing over j, we get

$$\begin{aligned}
\sum_{j=1}^{n} G_n(j, n+1-j) &= \frac{1}{2^n - n} \sum_{j=1}^{n} 2^j + 2^{n-j} - n \\
&= \frac{1}{2^n - n} \left(2^n - 1 + 2^n - 1 - n^2\right) \\
&\sim 2\,,
\end{aligned}$$

as $n \to \infty$. Therefore, we get

$$\mathbf{E}[\mathrm{fp}_n(\sigma)] = \mathbf{E}[\mathrm{afp}_n(\rho)] \sim \frac{2^{n+2}}{2^{n+1}} = 2\,,$$

as $n \to \infty$, as desired.

Similarly, since

$$H_n(j, n+1-j) \sim \frac{G_n(j, n+1-j) + G_n(n+1-j, j)}{2}\,,$$

when we sum these values from $j = 1$ to $j = n$, we get

$$\begin{aligned}\sum_{j=1}^{n} H_n(j, n+1-j) &\sim \sum_{j=1}^{n} \frac{G_n(j, n+1-j) + G_n(n+1-j, j)}{2} \\ &\sim \frac{\mathbf{E}[\mathrm{fp}_n(\sigma)] + \mathbf{E}[\mathrm{fp}_n(\sigma)]}{2} = \mathbf{E}[\mathrm{fp}_n(\sigma)] = 2\,,\end{aligned}$$

by the first part of our argument. This completes the proof. $\square$

CHAPTER 6

Avoiding 132 and 4231 simultaneously

6.1 Introduction

We analyze permutations which simultaneously avoid the patterns **132** and **4231**. These permutations were enumerated by Julian West in 1995 [West]. West showed that

$$\mathcal{S}_n(\mathbf{132}, \mathbf{4231}) = 1 + (n-1)2^{n-2}\,.$$

In this chapter, we calculate the number of permutations $\sigma \in \mathcal{S}_n(\mathbf{132}, \mathbf{4231})$ with $\sigma(j) = k$, for all $1 \le j, k \le n$. We then use this information to calculate what the "expected" permutation in $\mathcal{S}_n(\mathbf{132}, \mathbf{4231})$ looks like, when viewed as a permutation matrix. As in our previous chapters, we obtain a limit shape for such permutation matrices. We do this by taking permutations uniformly at random from $\mathcal{S}_n(\mathbf{132}, \mathbf{4231})$, letting $n \to \infty$ while rescaling the matrices so they always remain the same size, and considering the resulting probability distribution.

In Figure 6.1, we depict the regions of the limiting distribution: for $j = an$ and $k = bn$, we let a, b vary between 0 and 1. The regions without shading have exponentially small probabilities of having any nonzero entries in them. Darker shading implies a region is more likely to have a nonzero entry in a given point. The shaded regions in the figure correspond to probabilities asymptotically equal to $\frac{1}{n}$, $\frac{2}{n}$, and $\frac{3}{n}$, as $n \to \infty$. Entries with $j = k$ with $j = n - c$ for some constant c have probabilities tending to $\frac{1}{2^{c+1}}$ as $n \to \infty$.

After analyzing the limiting distribution in some detail, we prove a result on

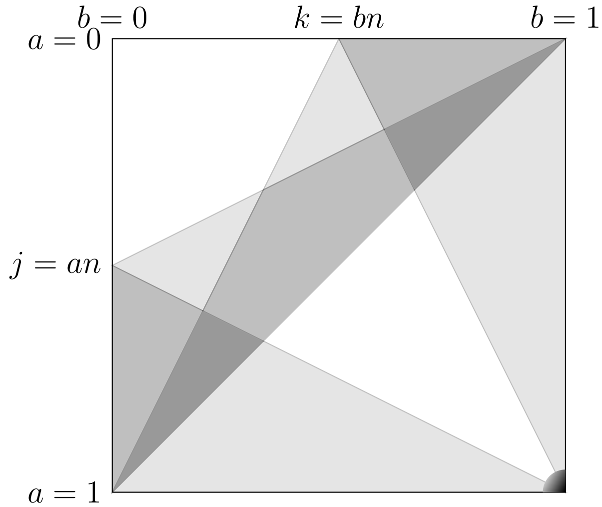


Figure 6.1: Regions within limit shape of $\sigma \in \mathcal{S}_n(\mathbf{132}, \mathbf{4231})$

the expected number of fixed points and anti-fixed points of a permutation chosen uniformly at random from $\mathcal{S}_n(\mathbf{132}, \mathbf{4231})$.

6.2 Main Results

6.2.1 Shape of $\{\mathbf{132}, \mathbf{4231}\}$-avoiding permutations

Let $a, b \in [0,1], c, d \in \mathbb{R}, \alpha, \beta \in [0,1)$ be fixed constants. Let $H_n(j,k)$ be the proportion of permutations $\sigma \in \mathcal{S}_n(132, 4231)$ with $\sigma(j) = k$. Let $J_n(j,k)$ be the number of such permutations σ.

Define

$$W(a,b,c,d,\alpha,\beta) = \sup\left\{ d \in \mathbb{R}_+ \,\middle|\, \lim_{n\to\infty} n^d H_n(an + cn^\alpha, bn + dn^\beta) < \infty \right\}.$$

Theorem 6.2.1. *Let $a, b \in [0,1], c, d \in \mathbb{R}, \alpha, \beta \in [0,1)$. Then we have*

$$W(a,b,c,d,\alpha,\beta) = \begin{cases} 0 & \text{for } a=1, b=1, c=d, \alpha=\beta=0\,, \\ \infty & \text{for } a+2b<1 \text{ and } 2a+b<1\,, \\ \infty & \text{for } 2a-b<1, a+b>1, \text{ and } 2b-a<1\,. \end{cases}$$

Also, for $2b-a=1$, we have

$$W(a,b,c,d,\alpha,\beta)=\begin{cases}\infty & \textit{for } 2a-b<1\,, a+b>1\,, \alpha>\beta\,, \alpha>\frac{1}{2}\,, c>0\ ,\\ \infty & \textit{for } 2a-b<1\,, a+b>1\,, \beta>\alpha\,, \beta>\frac{1}{2}\,, d<0\ ,\\ \infty & \textit{for } 2a-b<1\,, a+b>1\,, \alpha=\beta>\frac{1}{2}\,, 2d-c<0\ ,\end{cases}$$

and for $2a-b=1$, we have

$$W(a,b,c,d,\alpha,\beta)=\begin{cases}\infty & \textit{for } a+b>1\,, 2b-a<1\,, \alpha>\beta, \alpha>\frac{1}{2}\,, c<0\ ,\\ \infty & \textit{for } a+b>1\,, 2b-a<1\,, \beta>\alpha, \beta>\frac{1}{2}\,, d>0\ ,\\ \infty & \textit{for } a+b>1\,, 2b-a<1\,, \alpha=\beta>\frac{1}{2}\,, 2c-d<0\ .\end{cases}$$

For $a+b=1$, we have

$$W(a,b,c,d,\alpha,\beta)=\begin{cases}\infty & \textit{for } 2a-b<1\,, 2b-a<1\,, \alpha>\beta\,, c>0\ ,\\ \infty & \textit{for } 2a-b<1\,, 2b-a<1\,, \beta>\alpha\,, d>0\ ,\\ \infty & \textit{for } 2a-b<1\,, 2b-a<1\,, \alpha=\beta>0\,, c+d>0\ ,\\ \infty & \textit{for } 2a-b<1\,, 2b-a<1\,, \alpha=\beta=0\,, c+d>1\ ,\\ \infty & \textit{for } 2a-b=1\,, 2b-a<1\,, \beta>\alpha\,, \beta>\frac{1}{2}\,, d>0\ ,\\ \infty & \textit{for } 2a-b=1\,, 2b-a<1\,, \beta=\alpha>\frac{1}{2}\,, -d<c<\frac{d}{2}\ ,\\ \infty & \textit{for } 2a-b<1\,, 2b-a=1\,, \alpha>\beta, \alpha>\frac{1}{2}\,, c>0\ ,\\ \infty & \textit{for } 2a-b<1\,, 2b-a=1\,, \alpha=\beta>\frac{1}{2}\,, -c<d<\frac{c}{2}\ ,\end{cases}$$

while for $2a+b=1$, we have

$$W(a,b,c,d,\alpha,\beta)=\begin{cases}\infty & \textit{for } a+2b<1\,, \alpha>\beta\,, \alpha>\frac{1}{2}\,, c<0\ ,\\ \infty & \textit{for } a+2b<1\,, \beta>\alpha\,, \beta>\frac{1}{2}\,, d<0\ ,\\ \infty & \textit{for } a+2b<1\,, \alpha=\beta>\frac{1}{2}\,, 2c+d<0\ ,\\ \infty & \textit{for } a+2b=1\,, \alpha>\beta\,, \alpha>\frac{1}{2}\,, c<0\ ,\\ \infty & \textit{for } a+2b=1\,, \beta>\alpha\,, \beta>\frac{1}{2}\,, d<0\ ,\\ \infty & \textit{for } a+2b=1\,, \alpha=\beta>\frac{1}{2}\,, c+2d<0\,, 2c+d<0\ .\end{cases}$$

We also have

$$W(a,b,c,d,\alpha,\beta)=\begin{cases}\infty & \textit{for } a+2b=1\,,2a+b<1\,,\alpha>\beta\,,\alpha>\frac{1}{2}\,,c<0\ ,\\ \infty & \textit{for } a+2b=1\,,2a+b<1\,,\beta>\alpha\,,\beta>\frac{1}{2}\,,d<0\ ,\\ \infty & \textit{for } a+2b=1\,,2a+b<1\,,\alpha=\beta>\frac{1}{2}\,,c+2d<0\ ,\\ \infty & \textit{for } a=b=1\,,c=d\,,\alpha=\beta>0\ ,\\ \infty & \textit{for } a=b=1\,,2d<c<d\,,\alpha=\beta>0\ ,\\ \infty & \textit{for } a=b=1\,,2c<d<c\,\alpha=\beta>0\ ,\\ 1 & \textit{otherwise}\ .\end{cases}$$

Here $W(a,b,c,d,\alpha,\beta)=\infty$ means that $H_n(an+cn^{\alpha},bn+dn^{\beta})=o(n^{-s})$, for all $s>0$. The following result proves the exponential decay of these probabilities.

Theorem 6.2.2. *Let $a,b\in[0,1],c,d\in\mathbb{R},\alpha,\beta\in[0,\frac{1}{2})$ s.t. $a+2b<1,2a+b<1$ or $2a-b<1,a+b>1,2b-a<1$. Then, for n large enough, we have*

$$H_n(an+cn^{\alpha},bn+dn^{\beta})<\varepsilon^n\,,$$

where $\varepsilon=\varepsilon(a,b,c,d,\alpha,\beta)$ is independent of n, and $0<\varepsilon<1$. Similarly, for $2b-a=1,a+b>1$, we have

$$H_n(an+cn^{\alpha},bn+dn^{\beta})<\begin{cases}\varepsilon^{n^{2\alpha-1}} & \textit{for } b<1\,,\alpha>\beta\,,\alpha>\frac{1}{2}\,,c>0\ ,\\ \varepsilon^{n^{2\beta-1}} & \textit{for } b<1\,,\beta>\alpha\,,\beta>\frac{1}{2}\,,d<0\ ,\\ \varepsilon^{n^{2\alpha-1}} & \textit{for } b<1\,,\alpha=\beta>\frac{1}{2}\,,2d-c<0\ ,\\ \varepsilon^{n^{2\alpha-1}} & \textit{for } a<1\,,\alpha>\beta,\alpha>\frac{1}{2}\,,c<0\ ,\\ \varepsilon^{n^{2\beta-1}} & \textit{for } a<1\,,\beta>\alpha,\beta>\frac{1}{2}\,,d>0\ ,\\ \varepsilon^{n^{2\alpha-1}} & \textit{for } a<1\,,\alpha=\beta>\frac{1}{2}\,,2c-d<0\ ,\end{cases}$$

and for $a+b=1, 2b-a<1$*, we have*

$$H_n(an+cn^\alpha, bn+dn^\beta) < \begin{cases} \varepsilon^n & \text{for } 2a-b<1\,,\alpha>\beta\,,c>0\ , \\ \varepsilon^n & \text{for } 2a-b<1\,,\beta>\alpha\,,d>0\ , \\ \varepsilon^n & \text{for } 2a-b<1\,,\alpha=\beta>0\,,c+d>0\ , \\ \varepsilon^n & \text{for } 2a-b<1\,,\alpha=\beta=0\,,c+d>1\ , \\ \varepsilon^{n^{2\beta-1}} & \text{for } 2a-b=1\,,\beta>\alpha\,,\beta>\frac{1}{2}\,,d>0\ , \\ \varepsilon^{n^{2\alpha-1}} & \text{for } 2a-b=1\,,\beta=\alpha>\frac{1}{2}\,,-d<c<\frac{d}{2}\ . \end{cases}$$

For $a=\frac{1}{3}, b=\frac{2}{3}$*, we have*

$$H_n(an+cn^\alpha, bn+dn^\beta) < \begin{cases} \varepsilon^{n^{2\alpha-1}} & \text{for } \alpha>\beta, \alpha>\frac{1}{2}\,,c>0\ , \\ \varepsilon^{n^{2\alpha-1}} & \text{for } \alpha=\beta>\frac{1}{2}\,,-c<d<\frac{c}{2}\ . \end{cases}$$

For $2a+b=1, a+2b<1$*, we also have*

$$H_n(an+cn^\alpha, bn+dn^\beta) < \begin{cases} \varepsilon^{n^{2\alpha-1}} & \text{for } \alpha>\beta\,,\alpha>\frac{1}{2}\,,c<0\ , \\ \varepsilon^{n^{2\beta-1}} & \text{for } \beta>\alpha\,,\beta>\frac{1}{2}\,,d<0\ , \\ \varepsilon^{n^{2\alpha-1}} & \text{for } \alpha=\beta>\frac{1}{2}\,,2c+d<0\ , \end{cases}$$

while for $a=b=\frac{1}{3}$*, we have*

$$H_n(an+cn^\alpha, bn+dn^\beta) < \begin{cases} \varepsilon^{n^{2\alpha-1}} & \text{for } \alpha>\beta\,,\alpha>\frac{1}{2}\,,c<0\ , \\ \varepsilon^{n^{2\beta-1}} & \text{for } \beta>\alpha\,,\beta>\frac{1}{2}\,,d<0\ , \\ \varepsilon^{n^{2\alpha-1}} & \text{for } \alpha=\beta>\frac{1}{2}\,,c+2d<0\,,2c+d<0\ , \end{cases}$$

Finally, for $a+2b=1, 2a+b<1$*, we have*

$$H_n(an+cn^\alpha, bn+dn^\beta) < \begin{cases} \varepsilon^{n^{2\alpha-1}} & \text{for } \alpha>\beta\,,\alpha>\frac{1}{2}\,,c<0\ , \\ \varepsilon^{n^{2\beta-1}} & \text{for } \beta>\alpha\,,\beta>\frac{1}{2}\,,d<0\ , \\ \varepsilon^{n^{2\alpha-1}} & \text{for } \alpha=\beta>\frac{1}{2}\,,c+2d<0\ , \end{cases}$$

and for $a = b = 1$, we have

$$H_n(an + cn^\alpha, bn + dn^\beta) < \begin{cases} \varepsilon^{n^{2\alpha-1}} & \textit{for } c = d\,, \alpha = \beta > 0\ , \\ \varepsilon^{n^{2\alpha-1}} & \textit{for } 2d < c < d\,, \alpha = \beta > 0\ , \\ \varepsilon^{n^{2\alpha-1}} & \textit{for } 2c < d < c\, \alpha = \beta > 0\ , \end{cases}$$

for n large enough.

For values of $a, b, c, d, \alpha, \beta$ with $W(a, b, c, d, \alpha, \beta) < \infty$, we give the following result on the constant on $H_n(an + cn^\alpha, bn + dn^\beta)$. Let

$$X(a, b, c, d, \alpha, \beta) = \lim_{n\to\infty} n^{W(a,b,c,d,\alpha,\beta)} H_n(an + cn^\alpha, bn + dn^\beta)\,.$$

Theorem 6.2.3. *Let $a, b \in [0, 1], c, d \in \mathbb{R}, \alpha, \beta \in [0, 1)$. For values of $a, b, c, d, \alpha,$ and β such that $W(a, b, c, d, \alpha, \beta) < \infty$, we have*

$$X(a, b, c, d, \alpha, \beta) = \begin{cases} \frac{1}{2} & \textit{for } a = b = 1, c = d = 0, \alpha = \beta = 0\,, \\ 1 & \textit{for } a + b > 1, 2b - a > 1\,,\ \textit{or } a + b > 1, 2a - b > 1\,, \\ & \quad \textit{or}\ \ 2a + b > 1, a + 2b < 1, 2a - b < 1\,, \\ & \quad \textit{or}\ \ a + 2b > 1, 2a + b < 1, -a + 2b < 1\,, \\ 3 & \textit{for } a + b < 1, a + 2b > 1, 2a - b > 1 \\ & \quad \textit{or } a + b < 1, 2a + b > 1, 2b - a > 1\,, \\ 2 & \textit{for } a + 2b < 1, 2a - b > 1\,,\ \textit{or } 2a + b < 1, 2b - a > 1\,, \\ & \quad \textit{or}\ \ a + b < 1, a + 2b > 1, 2a + b > 1\,, \\ & \qquad \textit{and } 2a - b < 1, 2b - a < 1\,. \end{cases}$$

Note that here we do not mention the behavior of $X(a, b, c, d, \alpha, \beta)$ for a, b on any of the lines $a + b = 1, a + 2b = 1, 2a + b = 1, 2a - b = 1, -a + 2b = 1$. There are many distinct cases, which are listed in the Appendix.

These theorems analyze the growth of $H_n(an + cn^\alpha, bn + dn^\beta)$.

6.3 Analysis of $\{\mathbf{132}, \mathbf{4231}\}$-avoiding permutations

6.3.1 Combinatorics of $\sigma \in \mathcal{S}_n(\mathbf{132}, \mathbf{4231})$

As stated in the introduction, West showed in 1995 that

$$|\mathcal{S}_n(\mathbf{132}, \mathbf{4231})| = 1 + (n-1)2^{n-2}.$$

We first state a lemma exhibiting the explicit number of such permutations σ with $\sigma(j) = k$. Let $D_{n,k} = \binom{n-1}{n-k} + \sum_{i=2}^{k-1} 2^{i-2}\binom{n-i}{n-k}$.

Lemma 6.3.1. *Let $j, k, n \in \mathbb{N}$, and $1 \leq j, k \leq n$. Let $J_n(j,k)$ be the number of permutations $\sigma \in \mathcal{S}_n(\mathbf{132}, \mathbf{4231})$ such that $\sigma(j) = k$. Then for $j + k > n + 1$, we have*

$$J_n(j,k) = \begin{cases} 1 + (j-2)2^{j-3} & \text{for } \ j = k\,, \\ 2^{j-2} D_{n-j+1,k-j+1} & \text{for } \ j < k\,, \\ 2^{k-2} D_{n-k+1,j-k+1} & \text{for } \ j > k,\,, \end{cases}$$

and for $j + k \leq n + 1$, we have

$$J_n(j,k) = \begin{cases} D_{n,k} & \text{for } \ j = 1\,, \\ D_{n,j} & \text{for } \ k = 1\,, \\ 1 + (j-2)2^{j-3} + 2^{j-2}D_{n-j+1,k} & \\ \qquad\qquad + 2^{k-2}D_{n-k+1,j} - 2^{j+k-4} & \text{for } \ j = k \neq 1\,, \\ 2^{j-2}D_{n-j+1,k-j+1} + 2^{j-2}D_{n-j+1,k} & \\ \qquad\qquad + 2^{k-2}D_{n-k+1,j} - 2^{j+k-4} & \text{for } \ 1 < j < k\,, \\ 2^{k-2}D_{n-k+1,j-k+1} + 2^{j-2}D_{n-j+1,k} & \\ \qquad\qquad + 2^{k-2}D_{n-k+1,j} - 2^{j+k-4} & \text{for } \ j > k > 1\,. \end{cases}$$

Proof. First, observe that since $\mathbf{132}^{-1} = \mathbf{132}$, and $\mathbf{4231}^{-1} = \mathbf{4231}$, we must have $J_n(j,k) = J_n(k,j)$ for all j, k. This allows us to exclusively focus on cases where $j \leq k$, since the cases where $k < j$ follow by symmetry.

We prove the lemma one case at a time.

First, let $\sigma \in \mathcal{S}_n(\mathbf{132}, \mathbf{4231})$, with $\sigma(1) = k$. Since σ avoids $\mathbf{132}$, the elements larger than k must be in increasing order. The elements smaller than k must avoid $\mathbf{231}$, since σ avoids $\mathbf{4231}$.

We separate these permutations σ based on the value smaller than k with the largest index. For example, suppose $\max\{r : \sigma(r) < k\} = t$, and $\sigma(t) = i$. I claim that for all $u \in [i-1]$, we have $\sigma(t-u) < i$. Suppose for contradiction that $\sigma(t-u) > i$ for some $u \in [i-1]$. Since there are $i-1$ elements smaller than i, at least one of them, say v, must have $\sigma^{-1}(v) < t-(i-1)$. But then a $\mathbf{132}$ is formed, with $v, \sigma(t-u)$, and i.

Now, consider elements r with $i < r < k$. Since $\sigma(t) = i$, we must have $\sigma^{-1}(r) \leq t - i$. Suppose we have some r_1, r_2 with $i < r_1 < r_2 < k$, and $\sigma^{-1}(r_1) < \sigma^{-1}(r_2)$. This gives a contradiction, since we have a $\mathbf{4231}$, with k, r_1, r_2, s. Therefore, we must have all the elements r between i and k be in decreasing order.

We are now ready to count the permutations σ. First, suppose $i = 1$. Then the elements from 1 to $k-1$ are in decreasing order, while the elements from $k+1$ to n are in increasing order. These two subsequences can be combined in any order, so to count the possibilities, we simply need to choose the positions of the elements larger than k. There are $n-k$ elements and $n-1$ positions, so there are $\binom{n-1}{n-k}$ ways to complete the permutations.

Now, suppose $i > 1$. The elements from $i+1$ to $k-1$ are still in decreasing order, while the elements from $k+1$ to n are still in increasing order. Now, however, there are 2^{i-2} ways to order the elements smaller than i, since they must be in $\mathcal{S}_{i-1}(A^{(2)})$. When combining the elements of our permutation together, these elements from 1 to i must be consecutive, so we are essentially considering them as one block of elements, which cannot be separated. Therefore, we have $n-k$ elements larger than n, $k-i-1$ elements between i and k, and 1 element consisting of the elements smaller than s. Overall, we have $n-i$ elements to order, and once we choose the positions of the elements larger than k, the rest must be in decreasing order. So there are $\binom{n-i}{n-k}$ ways to order the blocks, and $2^{i-2}\binom{n-i}{n-k}$ permutations.

Considering all possible values of i between 1 and $k-1$, we get

$$J_n(1,k) = \binom{n-1}{n-k} + \sum_{i=2}^{k-1} 2^{i-2}\binom{n-i}{n-k} = D_{n,k}\,,$$

as desired.

This takes care of the case when $k=1$ as well, by symmetry.

Next, let $\sigma \in \mathcal{S}_n(\mathbf{132},\mathbf{4231})$, with $\sigma(j)=k$, $j \le k$, and $j+k > n+1$. There are $n-k$ positions after k, and $j-1$ numbers smaller than j. Since $j+k>n+1$, we get $j-1 > n-k$, so not every number smaller than k can fit after j. There must be some $i<j$ with $\sigma(i)<k$. Suppose there is also some $r<j$ with $\sigma(r)>k$. If $i<r$, then we have a **132** with $\sigma(i),\sigma(r)$, and k. So we must have $r<i$. Let's count the numbers r which have $r>j$ and $\sigma(r)>\sigma(j)$. There can be at most $n-k-1$ of these, since there are k values larger than $\sigma(j)$, and one of them is taken by $\sigma(r)$. There are $n-j$ positions after j, so for $j \le k$, we must have some $s>j$ with $\sigma(s)<k$, since $n-j>n-k-1$. Now we compare $\sigma(i)$ to $\sigma(s)$. If $\sigma(i)<\sigma(s)$, then we have a **132** with $\sigma(i),k$, and $\sigma(s)$. If $\sigma(i)>\sigma(s)$, then we have a **4231** with $\sigma(r),\sigma(i),k$, and $\sigma(s)$. Therefore, we cannot have any $r<j$ with $\sigma(r)>j$. Instead, we must have $\sigma(r)<j$ for all $r<j$.

Now suppose $j=k$. In this case, σ must consist of the numbers smaller than j, followed by j, followed by the numbers larger than j. Since σ avoids **132**, the numbers after j must be in increasing order. The numbers before j can be in any order, as long as they avoid **132** and **4231**. There are $1+(j-2)2^{j-3}$ such permutations, as desired.

Instead, for $j < k$, not all the elements smaller than k fit before k, so we must have some $r > j$ with $\sigma(r) < k$. Suppose some such r exists with $r > j$ and $k-j < \sigma(r) < k$. This means there can be at most $j-2$ elements s between $k-j$ and k with $\sigma^{-1}(s) < j$. Since there are $j-1$ elements t overall with $\sigma^{-1}(t) < j$, we must have at least one of them with $\sigma^{-1}(t) < j$ and $t \leq k-j$. However, then we have a **132** with t, k, and $\sigma(r)$. Therefore, we must have $\sigma^{-1}(r) < j$ for all r with $k-j < r < k$. Consider these first $j-1$ elements, consisting of values between $k-j+1$ and $k-1$. Since $\sigma^{-1}(k-j) > j$, these first $j-1$ elements must avoid **312**, and **132**. By the enumeration of $\mathcal{S}_n(A_2)$ in Theorem 6.2.2, there are 2^{j-2} such subpermutations. We now know that σ consists of the elements between $k-j+1$ and $k-1$, followed by k, followed by everything else. As long as the first j elements avoid **132** and **312**, they cannot contribute to a bad pattern, since the first j elements are consecutive. Once we make this distinction, it remains to complete the permutation, and we can consider the first j elements as a single block, since two elements from within the first j can never each be part of the same bad pattern. Therefore, once we order the first j elements, we essentially have $n-(j-1)$ elements to order, with the $k-(j-1)^{st}$ element in the first position. From our proof of the first case, there are $D_{n-j+1,k-j+1}$ ways to complete the permutation, and 2^{j-2} ways to order the first $j-1$ elements. Together, we get $2^{j-2}D_{n-j+1,k-j+1}$ permutations, as desired.

Now, let $\sigma \in \mathcal{S}_n(\mathbf{132}, \mathbf{4231})$, with $\sigma(j) = k, j \leq k$, and $j+k \leq n+1$.

First, suppose there exist r, s with $r < s < j$ with $\sigma(r) < \sigma(j) < \sigma(s)$. We see immediately that a **132** is formed, so this cannot happen. Second, suppose there exist r, s with $r < s < j$ with $\sigma(s) < \sigma(j) < \sigma(r)$. We claim that this cannot happen as well. There are $n-j$ positions after $\sigma(j) = k$, and $n-k$ elements larger than k. Since $\sigma(r) > \sigma(j) = k$, and $r < j$, we actually have at most $n-k-1$ elements larger than k which can occur after k. Since $j \leq k$, we have $j < k+1$, so $n-j > n-k-1$. This means that we don't have enough elements larger than k to fill the positions after k, so there must be some $t > j$ with $\sigma(t) < k$. Suppose $\sigma(t) > \sigma(s)$. Then we have a **132**, with $\sigma(s), \sigma(j) = k$, and $\sigma(t)$, which is a contradiction. We must then have $\sigma(t) < \sigma(s)$. But this gives us a **4231**, with $\sigma(r), \sigma(s), \sigma(j) = k$, and $\sigma(t)$, which is again a contradiction. Therefore, if $r < s < j$, then $\sigma(r) < \sigma(j)$ if and only if $\sigma(s) < j$.

To count the possible permutations σ, first suppose $\sigma(r) > \sigma(j)$ for all $r < j$.

For $j \leq k$ and $j + k \leq n + 1$, let

$$\begin{aligned} P_{n,j,k} = \{\sigma \in \mathcal{S}_n(\mathbf{132}, \mathbf{4231}) \text{ such that } \sigma(j) = k\,, \\ \text{and for all } \ r, s \ \text{ with } \ r < j < s \ \text{ and } \ \sigma(j) < \sigma(s)\,, \\ \text{we have } \sigma(j) < \sigma(r) < \sigma(s)\}\,, \end{aligned}$$

and let

$$\begin{aligned} Q_{n,j,k} = \{\sigma \in \mathcal{S}_n(\mathbf{132}, \mathbf{4231}) \text{ such that } \sigma(j) = k\,, \\ \text{and for all } \ p, q \ \text{ with } \ j < p < q \ \text{ and } \ \sigma(q) < \sigma(j)\,, \\ \text{we have } \sigma(p) < \sigma(j)\}\,. \end{aligned}$$

We now prove a lemma describing the permutations of interest using the forementioned sets.

Lemma 6.3.2. *Let $j, k, n \in \mathbb{N}$, and $1 \leq j, k \leq n$, with $j \leq k$ and $j + k \leq n + 1$. Then*

$$\begin{aligned} \{\sigma \in \mathcal{S}_n(\mathbf{132}, \mathbf{4231}) \text{ such that } \sigma(j) = k \text{ and } \sigma(r) > k \text{ for some } r < j\} \\ = P_{n,j,k} \cup Q_{n,j,k}\,. \end{aligned}$$

Proof. We prove the lemma by contradiction. Suppose there is some permutation $\sigma \in \mathcal{S}_n(\mathbf{123}, \mathbf{4231})$ such that $\sigma(j) = k$ and $\sigma(r) > k$ for some $r < j$, and $\sigma \notin P_{n,j,k} \cup Q_{n,j,k}$. Then there must exist r, s with $r < j < s$ and $\sigma(j) < \sigma(s) < \sigma(r)$, and there must also exist p, q with $j < p < q$ and $\sigma(q) < \sigma(j) < \sigma(p)$. We will prove that no such permutations σ can exist.

Suppose $s < q$. Then we have a **4231** with $\sigma(r), \sigma(j), \sigma(s)$ and $\sigma(q)$, which is a contradiction. . Therefore, we must have $r < j < p < q < s$. Now suppose $\sigma(p) > \sigma(s)$. Then we have a **132** with $\sigma(j), \sigma(p)$, and $\sigma(s)$, which is again a contradiction. Therefore, we must have $\sigma(q) < \sigma(j) < \sigma(p) < \sigma(s) < \sigma(r)$. However, we now have a **4231**, with $\sigma(r), \sigma(j), \sigma(p)$, and $\sigma(q)$, which is a contradiction. So no such σ can exist, completing the proof. □

It now remains to calculate $|P_{n,j,k} \cup Q_{n,j,k}|$. By the inclusion-exclusion principle, we know that

$$|P_{n,j,k} \cup Q_{n,j,k}| = |P_{n,j,k}| + |Q_{n,j,k}| - |P_{n,j,k} \cap Q_{n,j,k}|.$$

First, we count $P_{n,j,k}$. Let $\sigma \in P_{n,j,k}$. Suppose there exists $i < j - 1$ such that $\sigma(i) \geq k + j$. Since $\sigma \in P_{n,j,k}$, for every $s > j$ with $\sigma(s) > k$, we must have $\sigma(s) > \sigma(i)$. There can be at most $n - (k + j)$ such positions s. Also there are at most $k - 1$ positions $t > j$ with $\sigma(t) < k$ since there are only $k - 1$ elements smaller than k. Together, we have at most $n - (k + j) + k - 1 = n - j - 1$ positions after j filled. However, there are $n - j$ positions after j, so we have a contradiction. Therefore, for every $i < j - 1$, we must have $\sigma(j) \geq k + j$.

Now, for all $i \leq j - 1$, we must have $k < \sigma(i) < k + j$. Since each of these first $j - 1$ elements in the permutation comes before k and is larger than k, they must avoid **312**, in order to avoid forming a **4231** with k as the 1. Therefore, by the enumeration of $\mathcal{S}_n(A_2)$, there are at most 2^{j-2} orderings for the first $j - 1$ elements of σ.

Now, since for all $i \leq j$, we have $k \leq \sigma(i) \leq k + j - 1$, we can view these first j elements of σ as a contiguous block. As long as the permutation of length $n - j + 1$ consisting of this block, and the remaining $n - j$ elements of σ avoids **132** and **4231**, we must have $\sigma \in P_{n,j,k}$. By the first case of this lemma, we know there are exactly $D_{n-j+1,k}$ such ways to complete σ, given the first $j - 1$ elements.

Observe that as long as the first $j - 1$ elements avoid **312** and **132**, we can combine them with any of our our $D_{n-j+1,k}$ permutations of length $n-j+1$ to form σ. The only thing that could go wrong is if a **132** or **4231** is formed within σ, with more than one element coming within the initial block of length j. Since the block of length j consists of a connected block of j numbers, and it is at the start of our permutation, the only possibilities are than the block contains the $1, 3$, and 2 in a **132**, or they contain the 4,2, and 3 or 4,2,3, and 1 in a **4231**. We know the block avoids **132**, so the first case cannot occur. Also, since they avoid **312**, they cannot contain the 4,2, and 3, let alone the 4,2,3 and 1 in a **4231**. Therefore, any such ordering of the elements in the block which avoid **132** and **312** are legitimate, and do not affect the remaining $n-j$ elements of σ.

Because of this, there are $2^{j-2}D_{n-j+1,k}$ possibilities for σ, and

$$|P_{n,j,k}| = 2^{j-2}D_{n-j+1,k}\,.$$

Now, to calculate the size of $Q_{n,j,k}$, let $\sigma \in Q_{n,j,k}$. Suppose for the sake of contradiction that there exists i with $j < i < j+k$ such that $\sigma(i) > k$. There are at least $n-(j+k)+1$ positions t greater j. Since $\sigma \in Q_{n,j,k}$, none of these positions t can have $\sigma(t) < k$, since this would contradict the definition of $Q_{n,j,k}$. Also, each of the $j-1$ positions s with $s < j$ must have $\sigma(s) > k$, again by the definition of $Q_{n,j,k}$. Therefore, we have at least $n-(j+k)+1+j-1 = n-k$ positions which each must have $\sigma(s) > k$. However, since $\sigma(i) > k$, there are only $n-k-1$ remaining elements which can fill the $n-k$ positions. This is a contradiction, so we must have $\sigma(i) < k$ for all i with $j < i < j+k$.

When ordering these $k-1$ elements, we realize that they must avoid **231**, since otherwise a **4231** would be formed with k as the 4. Also, they must avoid **132**. By the enumeration of $\mathcal{S}_n(A_2)$, there are 2^{k-2} ways to order these $k-1$ elements. Once these numbers are ordered, they form a block of size k, occupying positions j through $j+k-1$. When combining this block with the remaining $n-k$ elements, we have a permutation which must avoid **132** and **4231**, and which has the smallest element in position j. By the first case of this lemma, we know that there are exactly $D_{n-k+1,j}$ such ways to complete σ. Observe now that as long as the elements in positions j through $j+k-1$ avoid **132** and **231**, we will have $\sigma \in Q_{n,j,k}$. Therefore, we have

$$|Q_{n,j,k}| = 2^{k-2} D_{n-k+1,j} .$$

The final step is calculating $|P_{n,j,k} \cap Q_{n,j,k}|$. Let $\sigma \in P_{n,j,k} \cap Q_{n,j,k}$. Then for $s < j$, we have $k < \sigma(s) < j+k$. Similarly, for t with $j < t < j+k$, we have $\sigma(t) < k$. Finally, for $r > j+k$, we must have $\sigma(r) = r$, since these numbers are all greater than k and after k, and must be in increasing order. There are 2^{j-2} ways to order the $j-1$ numbers before j, since they must avoid both **132** and **312**. There are 2^{k-2} ways to order the $k-1$ elements after k, since they must avoid both **132** and **231**. Any pair of these combinations works, so in all, we have

$$|P_{n,j,k} \cap Q_{n,j,k}| = 2^{j-2} 2^{k-2} = 2^{j+k-4} .$$

Our final type of permutation σ is that with $\sigma(r) < k$ for some $r < k$, and therefore for all $r < k$. First, suppose $j = k$. Suppose we have some $s > j$ with $\sigma(s) < k$. Then, the $n-k$ elements larger than k cannot all fit after j, since there are normally $n-j = n-k$ positions after j, but now one of them is filled by an element smaller than j. This means there must be some $t < j$ with $\sigma(t) > k$, which is a contradiction. Therefore, we must have $\sigma(s) > k$ for all $s > k$. Since σ avoids **132**, these numbers must be in increasing order. Therefore, the only choice we have in determining σ is the order of the first $j-1$ elements. Since elements $\sigma(j), \sigma(j+1), \ldots, \sigma(n)$ are now determined and are in increasing order, none of them can be part of a **132** or a **4231** within σ, since they would have to be the largest element in the pattern, and none of the patterns end with the largest element. Therefore, as long as the first $j-1$ elements avoid **132** and **4231**, σ will as well. There are precisely $1 + (j-2)2^{j-3}$ such permutations of length

$j-1$. Combining this with the number of permutations σ with $\sigma(r) > \sigma(j)$ for all $r < j$, we get

$$1 + (j-2)2^{j-3} + 2^{j-2}D_{n-j+1,k} + 2^{k-2}D_{n-k+1,j} - 2^{j+k-4},$$

as desired.

Now, instead of having $j = k$, suppose $j < k$. Suppose $\sigma(s) \leq k-j$, for some $s < j$. There are $n-j$ positions $t > j$, and $n-k$ values greater than k, so there are still $k-j$ positions $t > j$ such that $\sigma(t) < k$. Since $\sigma(s) \leq k-j$, there can be at most $k-j-1$ positions $r > j$ such that $\sigma(r) \leq k-j$. Therefore, there must be some position $t > j$ with $\sigma(t) > k-j$. However, this forms a **132** with $\sigma(s), \sigma(j) = k$, and $\sigma(t)$. Therefore, $\sigma(s) > k-j$ for all $s < j$. These initial $j-1$ values must avoid **132** and **312**, since they would form a **4231** with k as the 1 otherwise. Therefore, there are at most 2^{j-2} orderings for the first $j-1$ elements of σ. Once this ordering is chosen, the first j elements form a block, since they are consecutive from $k-j+1$ to k. Viewing these j elements as a block, and each of the remaining $n-j$ elements separately, we essentially have $n-j+1$ elements now, which need to avoid **132** and **4231**, with the first element having exactly $k-j$ elements smaller than it. There are precisely $D_{n-j+1,k-j+1}$ such elements, so again, overall, there are $2^{j-2}D_{n-j+1,k-j+1}$ possible permutations σ. In all, this gives

$$J_n(j,k) = 2^{j-2}D_{n-j+1,k-j+1} + 2^{j-2}D_{n-j+1,k} + 2^{k-2}D_{n-k+1,j} - 2^{j+k-4},$$

as desired. This completes the final case of Lemma 6.3.1. □

6.3.2 Example of $\sigma \in \mathcal{S}_n(\mathbf{132}, \mathbf{4231})$

As in previous sections, we present an extended example to clarify our reasoning. Suppose $n = 8$, with $j = 3$ and $k = 4$. We will exhibit here each of the permutations $\sigma \in \mathcal{S}_n(\mathbf{132}, \mathbf{4231})$ with $\sigma(j) = k$. Since $j \leq k$ and $j + k \leq n+1$, we know how many permutations there should be. We also know that σ either has

$$\{\sigma(1), \sigma(2)\} = \{2, 3\}\,,$$

or $\sigma(r) > 4$ for $r \in \{1, 2\}$. First, suppose $\{\sigma(1), \sigma(2)\} = \{2, 3\}$. There are $2^{j-2} = 2^{3-2} = 2$ ways to order the first j elements of the permutation, shown here:

$$(324 * * * **)\ (234 * * * **)\,.$$

We know that there are $D_{n-j+1,k-j+1} = D_{8-3+1,4-3+1} = D_{6,2}$ ways to complete σ. By the expression for $D_{6,2}$, we see that there are $\binom{6-1}{6-2} = \binom{5}{4} = 5$ ways to complete σ, since the elements larger than 4 must be in increasing order, and must fill four of the five positions r with $r > j$. Therefore, our permutations of this type must have the form:

$$(**415678)\ (**451678)\ (**456178)\ (**456718)\ (**456781)\ .$$

In all, there are 10 such permutations. Now, we need to count permutations in $P_{n,j,k}$ and $Q_{n,j,k}$. Permutations in $P_{n,j,k}$ must start with the elements between k and $k + j - 1$, shown here:

$$(564 * * * **)\ (654 * * * **)\ .$$

There are $D_{n-j+1,k}$ ways to complete σ. First, suppose $1 = \sigma(\max\{r : \sigma(r) < k\})$. Then we must have the elements after 4 consisting of a decreasing sequence 3,2, and 1, interlaced with an increasing sequence 7,8. There are 10 such interlacings, shown here:

$$(**432178)\ (**432718)\ (**432781)\ (**437218)\ (**437281)$$

$$(**437821)\ (**473218)\ (**473281)\ (**473821)\ (**478321)\,.$$

Second, suppose $2 = \sigma(\max\{r : \sigma(r) < k\})$. Then we must be interlacing the increasing 7,8 with the decreasing sequence 3, followed by a block of (1,2), which must be consecutive. There are $\binom{4}{2} = 6$ such interlacings, shown here:

$$(**431278)\ (**437128)\ (**437812)\ (**473128)\ (**473812)\ (**478312)\,.$$

Finally, suppose $3 = \sigma(\max\{r : \sigma(r) < k\})$. Here, we must interlace the sequence 7,8 with a block which ends in 3, which there are $2^{3-2} = 2$ ways to begin, either **123** or **213**. The interlacings are shown here:

$$(**412378)\ (**421378)\ (**471238)\ (**472138)\ (**478123)\ (**478213)\,.$$

In all, we get $|P_{8,3,4}| = 2(10+6+6) = 44$.

Now, suppose $\sigma \in Q_{8,3,4}$. First, we order the elements smaller than $k = 4$. Since they all must come consecutively after $k = 4$, and must avoid **132** and **231**, we see the possibilities are

$$(**4123**)\ (**4213**)\ (**4312**)\ (**4321**)\,.$$

To complete σ, we need a permutation in $\mathcal{S}_5(\mathbf{132}, \mathbf{4231})$ with $\sigma(3) = 1$. There are $D_{5,3}$ of these permutations, so we can count them based on whether the first two elements are increasing or decreasing. If they are decreasing, then we simply need the final two elements to be increasing. There are $\binom{4}{2}$ ways to choose the final elements, shown here:

$$(654***78)\ (754***68)\ (764***58)\ (854***67)\ (864***57)\ (874***56)\,.$$

If instead, the first two elements are increasing, then they must be consecutive, in order to avoid a **132**. There are $\binom{3}{1}$ ways to choose these, shown here:

$$(564***78)\ (674***58)\ (784***56)\,.$$

Therefore, overall we have $|Q_{8,3,4}| = 4(6+3) = 36$.

Finally, to avoid double-counting, we need to enumerate permutations in $P_{8,3,4} \cap Q_{8,3,4}$. Let $\sigma \in P_{8,3,4} \cap Q_{8,3,4}$. Then there are $2^{3-2} = 2$ ways to choose $\sigma(t)$ for $t < j = 3$, either

$$(564*****) \text{ or } (654*****)\,.$$

Also, there are 2^{4-2} ways to choose $\sigma(s)$ for $j < s \le j+k-1$, shown here:

$$(**4123**)\ (**4213**)\ (**4312**)\ (**4321**)\,.$$

The final two elements of σ must be increasing, so σ is completely determined. There are $2(4) = 8$ such possible permutations $\sigma \in P_{8,3,4} \cap Q_{8,3,4}$, shown here:

$$(56412378)\ (56421378)\ (56431278)\ (56432178)$$

$$(65412378)\ (65421378)\ (65431278)\ (65432178)\,.$$

6.4 Proof of theorems 6.2.1 and 6.2.2

The proof of the two theorems relies on several lemmas - two lemmas which analyze the asymptotic behavior of $D_{sn+tn^\gamma, un+vn^\gamma+wn^\delta}$, and one more for each case of Theorem 6.2.1. The first lemma analyzes how binomial coefficients behave as their inputs grow to infinity.

Lemma 6.4.1. *Let $a \in (0,\infty), b \in \mathbb{R}, \alpha \in [0,1]$. Suppose $x \sim bn^\alpha$, as $n \to \infty$. Then we have*

$$\binom{2an-x}{an} \sim \begin{cases} \frac{2^{2an-x}}{\sqrt{a\pi n}} & \alpha \in [0,\frac{1}{2})\,, \\ \frac{2^{2an-x}}{\sqrt{a\pi n}} \exp\left[-\frac{b^2}{4a}\right] & \alpha = \frac{1}{2}\,, \\ \frac{2^{2an-x}}{\sqrt{a\pi n}} \varepsilon^{n^{2\alpha-1}} & \alpha \in (\frac{1}{2}, 1]\,, \end{cases}$$

as $n \to \infty$, where $\varepsilon = \varepsilon(a,b,\alpha) \in (0,1)$.

Proof. By Stirling's formula, we have

$$\begin{aligned} \binom{2an-x}{an} &\sim \sqrt{\frac{2\pi(2an-x)}{(2\pi an)(2\pi)(an-x)}} \left(\frac{(2an-x)^{2an-x}}{(an)^{an}(an-x)^{an-x}}\right)\,, \\ &\sim \sqrt{\frac{1}{a\pi n}} \left(\frac{(2an)^{2an-x}}{(an)^{an}(an)^{an-x}}\right) \frac{(1-\frac{x}{2an})^{2an-x}}{(1-\frac{x}{an})^{an-x}}\,, \\ &\sim \frac{2^{2an-x}}{\sqrt{a\pi n}} \frac{(1-\frac{x}{2an})^{2an-x}}{(1-\frac{x}{an})^{an-x}}\,, \end{aligned}$$

as $n \to \infty$.

Using the Taylor expansion for $\ln(1+x)$, we see that

$$\begin{aligned}
\ln\left[\frac{(1-\frac{x}{2an})^{2an-x}}{(1-\frac{x}{an})^{an-x}}\right] &\sim (2an-x)\left(-\frac{x}{2an}-\frac{x^2}{8a^2n^2}-O\left(\frac{x^3}{n^3}\right)\right) \\
&\quad -(an-x)\left(-\frac{x}{an}-\frac{x^2}{2a^2n^2}-O\left(\frac{x^3}{n^3}\right)\right), \\
&\sim \left(-x-\frac{x^2}{4an}+\frac{x^2}{2an}+O\left(\frac{x^3}{n^2}\right)\right) \\
&\quad -\left(-x-\frac{x^2}{2an}+\frac{x^2}{an}+O\left(\frac{x^3}{n^2}\right)\right), \\
&\sim -\frac{x^2}{4an}, \\
&\sim -\frac{b^2}{4a}n^{2\alpha-1},
\end{aligned}$$

as $n \to \infty$. Plugging into our first equation, we get

$$\binom{2an-x}{an} \sim \frac{2^{2an-x}}{\sqrt{a\pi n}}\exp\left[-\frac{b^2}{4a}n^{2\alpha-1}\right],$$

as $n \to \infty$. Finally, considering the various cases of α completes the proof. □

This second lemma allows us to the calculate the asymptotic behavior of $H_n(an+cn^\alpha, bn+dn^\beta)$ for various values of a, b, c, d, α, and β.

Lemma 6.4.2. *Let $s, u \in [0,1], t, v, w \in \mathbb{R}, \gamma, \delta \in [0,1)$. Then as $n \to \infty$, we have*

$$D_{sn+tn^\gamma, un+vn^\gamma+wn^\delta} \sim 2^{sn+tn^\gamma-1},$$

for $0 < 2u-s < s$.

Furthermore, for $2u-s<0$, we have

$$D_{sn+tn^\gamma, un+vn^\gamma+wn^\delta} < \varepsilon^n,$$

and for $2u - s = 0$, *we have*

$$\frac{D_{sn+tn^\gamma, un+vn^\gamma+wn^\delta}}{2^{sn+tn^\gamma-2}} \sim \begin{cases} 1 & \gamma, \delta < \frac{1}{2}\,, \\ 1 - erf(t-2v) & \gamma = \frac{1}{2} > \delta\,, \\ 1 - erf(-2w) & \delta = \frac{1}{2} > \gamma\,, \\ 1 - erf(t-2v-2w) & \gamma = \delta = \frac{1}{2}\,, \end{cases}$$

as well as

$$\frac{D_{sn+tn^\gamma, un+vn^\gamma+wn^\delta}}{2^{sn+tn^\gamma-2}} \sim \begin{cases} 2 & \delta > \frac{1}{2}, \delta > \gamma, w > 0\,, \\ \varepsilon^{n^{2\delta-1}} & \delta > \frac{1}{2}, \delta > \gamma, w < 0\,, \\ 2 & \delta > \gamma > \frac{1}{2}, w = 0, 2v - t > 0\,, \\ 1 & \delta > \gamma > \frac{1}{2}, w = 0, 2v - t = 0\,, \\ \varepsilon^{n^{2\gamma-1}} & \delta > \gamma > \frac{1}{2}, w = 0, 2v - t < 0\,, \\ 1 - erf(t-2v) & \delta > \gamma = \frac{1}{2}, w = 0\,, \\ 1 & \delta > \frac{1}{2} > \gamma, w = 0\,, \end{cases}$$

and

$$\frac{D_{sn+tn^\gamma, un+vn^\gamma+wn^\delta}}{2^{sn+tn^\gamma-2}} \sim \begin{cases} 2 & \gamma > \frac{1}{2}, \gamma > \delta, 2v - t > 0\,, \\ \varepsilon^{n^{2\gamma-1}} & \gamma > \frac{1}{2}, \gamma > \delta, 2v - t < 0\,, \\ 2 & \gamma > \delta > \frac{1}{2}, 2v - t = 0, w > 0\,, \\ 1 & \gamma > \delta > \frac{1}{2}, 2v - t = 0, w = 0\,, \\ \varepsilon^{n^{2\delta-1}} & \gamma > \delta > \frac{1}{2}, 2v - t = 0, w < 0\,, \\ 1 - erf(-2w) & \gamma > \delta = \frac{1}{2}, 2v - t = 0\,, \\ 1 & \gamma > \frac{1}{2} > \delta, 2v - t = 0\,, \\ 2 & \gamma = \delta > \frac{1}{2}, 2v - t + 2w > 0\,, \\ 1 & \gamma = \delta > \frac{1}{2}, 2v - t + 2w = 0\,, \\ \varepsilon^{n^{2\gamma-1}} & \gamma = \delta > \frac{1}{2}, 2v - t + 2w < 0 \end{cases}$$

Similarly, for $s = u > 0$, we have

$$\frac{D_{sn+tn^{\gamma},un+vn^{\gamma}+wn^{\delta}}}{2^{sn+tn^{\gamma}-2}} \sim \begin{cases} 1+erf(t-v) & \gamma = \frac{1}{2} > \delta\,, \\ 1+erf(-w) & \delta = \frac{1}{2} > \gamma\,, \\ 1+erf(t-v-w) & \delta = \gamma = \frac{1}{2}\,, \\ 1 & \delta, \gamma < \frac{1}{2}\,, \\ 2 & \gamma > \frac{1}{2}, \gamma > \delta, v-t < 0\,, \\ \varepsilon^{n^{2\gamma-1}} & \gamma > \frac{1}{2}, \gamma > \delta, v-t > 0\,, \\ 2 & \gamma > \delta > \frac{1}{2}, v-t = 0, w < 0\,, \\ 1 & \gamma > \delta > \frac{1}{2}, v-t = 0, w = 0\,, \\ \varepsilon^{n^{2\delta-1}} & \gamma > \delta > \frac{1}{2}, v-t = 0, w > 0\,, \\ 1+erf(-w) & \gamma > \delta = \frac{1}{2}, v-t = 0\,, \\ 1 & \gamma > \frac{1}{2} > \delta, v-t = 0\,, \end{cases}$$

and

$$\frac{D_{sn+tn^{\gamma},un+vn^{\gamma}+wn^{\delta}}}{2^{sn+tn^{\gamma}-2}} \sim \begin{cases} 2 & \delta > \gamma, \delta > \frac{1}{2}, w < 0\,, \\ \varepsilon^{n^{2\delta-1}} & \delta > \gamma, \delta > \frac{1}{2}, w > 0\,, \\ 2 & \delta > \gamma > \frac{1}{2}, w = 0, v-t < 0\,, \\ 1 & \delta > \gamma > \frac{1}{2}, w = 0, v-t = 0\,, \\ \varepsilon^{n^{2\gamma-1}} & \delta > \gamma > \frac{1}{2}, w = 0, v-t > 0\,, \\ 1+erf(t-v) & \delta > \gamma = \frac{1}{2}, w = 0\,, \\ 1 & \delta > \frac{1}{2} > \gamma, w = 0\,, \\ 2 & \delta = \gamma > \frac{1}{2}, t-v-w > 0\,, \\ 1 & \delta = \gamma > \frac{1}{2}, t-v-w = 0\,, \\ \varepsilon^{n^{2\gamma-1}} & \delta = \gamma > \frac{1}{2}, t-v-w < 0\,, \end{cases}$$

where each $\varepsilon = \varepsilon(s,t,u,v,w,\gamma,\delta) \in (0,1)$, *and*

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy\,.$$

Proof. Let $s,u \in [0,1], t,v,w \in \mathbb{R}, \gamma,\delta \in [0,1)$. For the remainder of the lemma, let

$$D = D_{sn+tn^\gamma, un+vn^\gamma+wn^\delta}\,.$$

Then, we have

$$D = \binom{sn+tn^\gamma - 1}{(s-u)n + (t-v)n^\gamma - wn^\delta} + \sum_{i=2}^{un+vn^\gamma+wn^\delta-1} g_i\,,$$

where

$$g_i = 2^{i-2}\binom{sn+tn^\gamma - i}{(s-u)n + (t-v)n^\gamma - wn^\delta}\,.$$

We now analyze g_i carefully. Let

$$r = (2u-s)n + (2v-t)n^\gamma + 2wn^\delta\,,$$

and let $x = i - r$. where $x \sim yn^z$ for some $y \in \mathbb{R}, z \in [0,1]$. By Lemma 6.4.1, we see that

$$g_i \sim \begin{cases} \frac{2^{sn+tn^\gamma-2}}{\sqrt{(s-u)\pi n}} & z \in [0,\frac{1}{2})\,, \\ \frac{2^{sn+tn^\gamma-2}}{\sqrt{(s-u)\pi n}} \exp\left[-\frac{y^2}{4(s-u)}\right] & z = \frac{1}{2}\,, \\ \frac{2^{sn+tn^\gamma-2}}{\sqrt{(s-u)\pi n}}\varepsilon^{n^{2z-1}} & \alpha \in (\frac{1}{2},1]\,. \end{cases}$$

This lemma says that terms g_i only ever contribute to the total sum when $x \sim yn^z$ with $z \leq \frac{1}{2}$. For $z > \frac{1}{2}$, the power of ε means that even if we sum over such terms g_i, they will not contribute more than a power of $\varepsilon^{n^{2z-1}}$ to the total sum. Therefore, we need i to be relatively close to r to have terms which contribute to the sum.

Note that r needs to be within $y\sqrt{n}$ of the allowable values of i, in order for a term g_i to contribute more than a power of ε.

We are now ready to consider cases.

For $0 < 2u - s < s$, we see that $r - x$ is a valid choice for i for any z up to 1. In other words, we have

$$
\begin{aligned}
D &\sim \sum_{i=2}^{r-y\sqrt{n}-1} g_i + \sum_{i=r-y\sqrt{n}}^{r+y\sqrt{n}} g_i + \sum_{i=r+y\sqrt{n}+1}^{un+vn^\gamma+wn^\delta-1} g_i\,, \\
&\sim (r-y\sqrt{n})\varepsilon^n + \frac{2^{sn+tn^\gamma-2}}{\sqrt{(s-u)\pi n}} \sum_{x=-y\sqrt{n}}^{y\sqrt{n}} \exp\left[-\frac{y^2}{4(s-u)}\right] + (un-r)\varepsilon^n\,, \\
&\sim \frac{2^{sn+tn^\gamma-2}}{\sqrt{(s-u)\pi n}} \sum_{x=-y\sqrt{n}}^{y\sqrt{n}} \exp\left[-\frac{y^2}{4(s-u)}\right]\,.
\end{aligned}
$$

Interpreting this sum as a Riemann sum, and letting $y \to \infty$, we have

$$
\begin{aligned}
D &\sim \frac{2^{sn+tn^\gamma-2}}{\sqrt{(s-u)\pi n}}\sqrt{n} \int_{-\infty}^{\infty} \exp\left[-\frac{y^2}{4(s-u)}\right] dy\,, \\
&\sim \frac{2^{sn+tn^\gamma-2}}{\sqrt{(s-u)\pi}} 2\sqrt{(s-u)\pi}\,, \\
&\sim 2(2^{sn+tn^\gamma-2})\,,
\end{aligned}
$$

as desired.

On the other extreme, let $2u - s < 0$. The allowable i which is closest to r is 0, but unfortunately this gives $r - i \sim (2u - s)n$. By Lemma 6.4.1, we have

$$
\begin{aligned}
D &\sim 2^{2n+tn^\gamma-2} \sum_{i=2}^{un+vn^\gamma+wn^\delta} \varepsilon^n\,, \\
&< 2^{2n+tn^\gamma-2}\varepsilon^n\,,
\end{aligned}
$$

for some other $\varepsilon \in (0, 1)$.

We now prove one more specific case, omitting proofs of most cases for brevity, since they are all very similar.

Let $2u - s = 0, \gamma = \frac{1}{2} > \delta$. Then

$$r = (2v - t)n^{\frac{1}{2}} + 2wn^\delta\,.$$

We see that g_i for $i \in [2, y\sqrt{n}]$ will contribute more than a power of ε to D. Letting $y \to \infty$, we see that

$$
\begin{aligned}
D &\sim \sum_{i=2}^{y\sqrt{n}} g_i \,, \\
&\sim \frac{2^{sn+tn^\gamma-2}}{\sqrt{(s-u)\pi n}} \sqrt{n} \int_0^\infty \exp\left[-\frac{(y-(2v-t))^2}{4(s-u)}\right] dy \,, \\
&\sim 2^{sn+tn^\gamma-2} \frac{2}{\sqrt{\pi}} \int_{t-2v}^\infty \exp\left[-y^2\right] dy \,, \\
&\sim 2^{sn+tn^\gamma-2} \left(1 - erf(t-2v)\right) \,,
\end{aligned}
$$

as $n \to \infty$, as desired. □

6.4.1 Proof of Theorems 6.2.1 and 6.2.2

We prove Theorem 6.2.1 case by case: first, let $a = 1$ and $b = 1$ and consider $H_n(an+c, bn+c)$. We are in the case of Lemma 6.3.1 with $j = k, j+k > n+1$, so we know

$$
H_n(an+c, bn+c) = \frac{1 + (n+c-2)2^{n+c-3}}{1 + (n-1)2^{n-2}}.
$$

We see that

$$
\lim_{n\to\infty} n^d H_n(n,n) = \lim_{n\to\infty} 2^{c-1} n^d \,,
$$

which means that $W(1,1,c,c,0,0) = 0$, and $L(1,1,c,c,0,0) = 2^{c-1}$, as desired.

We now consider cases where $W(a,b,c,d,\alpha,\beta) = \infty$. For the next two lemmas, we prove cases of Theorem 6.2.2, as well as Theorem 6.2.1. The next case requires a lemma.

Lemma 6.4.3. *Let $a, b \in [0,1]$ such that $a + 2b < 1$ and $2a + b < 1$. Then $W(a,b,c,d,\alpha,\beta) = \infty$.*

Proof. Let $a, b \in [0,1]$ satisfying the conditions of the lemma, with $a = b$. Suppose $a = 0$, so $b = 0$, Then $J_n(an + cn^\alpha, bn + dn^\beta) = 1$, since there is only one permutation σ with $\sigma(1) = 1$, namely $\sigma = (1, 2, 3, \ldots, n-1, n)$. Clearly,

$$n^d H_n(an + cn^\alpha, bn + dn^\beta) \sim \frac{n^d}{n2^{n-2}}\,,$$

and $W(0, 0, c, d, \alpha, \beta) = \infty$. The conclusion of Theorem 6.2.2 is satisfied as well, with $\varepsilon = \frac{1}{2}$.

Now, suppose $a > 0$. We must have $a < \frac{1}{3}$, since otherwise $a + 2b \geq 1$. We are in the case of Lemma 6.3.1 with $j + k \leq n + 1$ and $j = k$.

By Lemma 6.3.1, we have

$$\begin{aligned} J_n(an + cn^\alpha, bn + dn^\beta) = 1 + (an-2)2^{an-3} + 2^{an-2} D_{n-an+1,an} \\ + 2^{an-2} D_{n-an+1,an} - 2^{an+an-4}\,. \end{aligned}$$

Since $2an < (1-a)n$, by applying Lemma 6.4.2 to $D_{n-an+1,an}$, we see that

$$\begin{aligned} n^d R_n(an, an) &< \frac{n^d 2^{an-2}\left(\frac{1}{2}an - 1 + 2\varepsilon^{n-an+1} - 2^{an-2}\right)}{1 + (n-1)2^{n-2}} \\ &= \Theta\left(n^{d-1}\left(\frac{\varepsilon}{2}\right)^{n-an+1}\right)\,, \end{aligned}$$

implying that $F(a, a) = \infty$, and $R_n(an, an) < \frac{\varepsilon}{2}^{(1-a)n}$, as desired.

Next, we consider $a < b$. For $a = 0$, we see that $J_n 1, bn = D_{n,bn}$. Since $a + 2b < 1$, we must have $b < \frac{1}{2}$. By Lemma 6.4.2, we obtain $H_n 1, bn < \varepsilon^n$, for some $\varepsilon < 1$. Therefore, we get

$$n^d R(1, bn) < n^d \left(\frac{\varepsilon}{2}\right)^n\,,$$

so $F(0, b) = \infty$ and $R_n(an, bn) < \frac{\varepsilon}{2}^n$, as desired.

Finally, we suppose $0 < a < b$. We are now in the case where $j + k \leq n + 1$, and $j < k$. By Lemma 6.3.1, we see that

$$J_n(an, bn) = 2^{an-2} D_{n-an+1,bn-an+1} + 2^{an-2} D_{n-an+1,bn} + 2^{bn-2} D_{n-bn+1,an} - 2^{an+bn-4}\,.$$

To analyze the asymptotics of $H_n(an, bn)$, we need to first consider the appropriate $D_{x,y}$ terms. Since $2b < 1 - a < 1 + a$, we have $2b - 2a < 1 - a$, so by

Lemma 6.4.2, we see that

$$D_{n-an+1,bn-an+1} = D_{n(1-a)+1,n(b-a)+1} < \varepsilon_1^{n(1-a)+1},$$

for some $\varepsilon_1 < 1$. Similarly, we know $2b < 1-a$, so by Lemma 6.4.2, we observe

$$D_{n(1-a)+1,bn} < \varepsilon_2^{n(1-a)+1},$$

for some $\varepsilon_2 < 1$. Finally, $2a+b < 1$ implies $2a < 1-b$, so by Lemma 6.4.2, we have

$$D_{n(1-b)+1,an} < \varepsilon_3^{n(1-b)+1},$$

for some $\varepsilon_3 < 1$. Let $\varepsilon = \max\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$. Applying these results, we have

$$n^d R_n(an,bn) < \frac{n^d}{2}\left(\frac{\varepsilon}{2}\right)^{n-an+1} + \frac{n^d}{2}\left(\frac{\varepsilon}{2}\right)^{n-an+1} + \frac{n^d}{2}\left(\frac{\varepsilon}{2}\right)^{n-bn+1} - \frac{n^d}{2^{(1-a-b)n+4}}.$$

Therefore, $W(a,b,c,d,\alpha,\beta) = \infty$, and $R_n(an,bn) < \left(\frac{\varepsilon}{2}\right)^{(1-a)n}$ for n large enough, as desired.

The only cases we have not yet considered are for $a > b$, but these follow by symmetry, so the proof is complete. □

We prove another lemma for the second region where $W(a,b,c,d,\alpha,\beta) = \infty$.

Lemma 6.4.4. *Let $a,b \in [0,1]$ such that $2a-b<1, a+b>1$, and $2b-a<1$. Then $W(a,b,c,d,\alpha,\beta)=\infty$.*

Proof. Let a,b satisfy the conditions of the lemma, and consider $J_n(an,bn)$. First, suppose $a=b$. The case where $a=b=1$ is dealt with in the first case of Theorem 6.2.1, so we assume here $a<1$. We are in the case of Lemma 6.3.1 with $j=k$ and $j+k>n+1$, so

$$n^d H_n(an,an) = n^d \frac{J_n(an,an)}{1+(n-1)2^{n-2}} = \frac{n^d\left(1+(an-2)2^{an-3}\right)}{1+(n-1)2^{n-2}} \sim an^d 2^{n(a-1)},$$

so $F(a,a)=\infty$, and $H_n(an+cn^\alpha, an+dn^\beta) < 2^{(a-1)n}$, for n large enough.

Now, suppose $a < b$. The case with $a > b$ follows by symmetry. For $a < b$, we are in the case of Lemma 6.3.1 with $j < k$ and $j + k > n + 1$. We now have $J_n(an, bn) = 2^{an-2} D_{n-an+1, bn-an+1}$. Since $2b - a < 1$, we have $2b - 2a < 1 - a$. By Lemma 6.4.2, we see that as $n \to \infty$, we get

$$D_{n-an+1, bn-an+1} < \varepsilon^{n-an+1},$$

for some $\varepsilon < 1$. Therefore, we have

$$n^d H_n(an + cn^\alpha, bn + dn^\beta) = \frac{n^d}{2} \left(\frac{\varepsilon}{2}\right)^{n-an+1}.$$

This immediately gives $W(a, b, c, d, \alpha, \beta) = \infty$, and $H_n(an + cn^\alpha, bn + dn^\beta) < \left(\frac{\varepsilon}{2}\right)^n$, as desired. $\square$

The remaining values of $a, b, c, d, \alpha, \beta$ have $W(a, b, c, d, \alpha, \beta) = 1$. We prove some of these cases, and omit proofs of the others, since they are very similar to these. on the cases of Theorem 6.3.1.

Lemma 6.4.5. *Let $a, b \in [0, 1]$ such that $a + b > 1$, and $2a - b \geq 1$ or $2b - a \geq 1$. Then $W(a, b, c, d, \alpha, \beta) = 1$. For $2a - b > 1$ or $2b - a > 1$, we have $L(a, b, c, d, \alpha, \beta) = 1$. For $2a - b = 1$ or $2b - a = 1$ and $\alpha, \beta < \frac{1}{2}$, we have $L(a, b, c, d, \alpha, \beta) = \frac{1}{2}$.*

Proof. Let $a, b \in [0, 1]$ such that $a + b > 1$ and $2b - a > 1$. The proof of the case for $2a - b \geq 1$ will follow by symmetry. By Lemma 6.3.1, we see that

$$W_n(an + cn^\alpha, bn + dn^\beta) = 2^{an+cn^\alpha-2} D_{n-an-cn^\alpha+1, bn+dn^\beta-an-cn^\alpha+1}.$$

Since $2b - a > 1$ implies $2(b - a) > (1 - a)$, by Lemma 6.4.2, we see that

$$D_{n-an-cn^\alpha+1, bn+dn^\beta-an-cn^\alpha+1} \sim 2 * 2^{(n-an-cn^\alpha)-1} \quad \text{as } n \to \infty.$$

Therefore,

$$n^d W_n(an + cn^\alpha, bn + dn^\beta) \sim \frac{n^d 2^{an+cn^\alpha-2} 2^{n-an-cn^\alpha}}{1 + (n-1)2^{n-2}} \sim n^{d-1}.$$

Clearly, $W(a,b,c,d,\alpha,\beta) = 1$ and $L(a,b) = 1$.

For $2b - a = 1$, we instead have $D_{n-an-cn^\alpha+1, bn+dn^\beta-an-cn^\alpha+1}$ depending on c, d, α, and β, as $n \to \infty$. By Lemma 6.4.2, for $\alpha, \beta < \frac{1}{2}$, then we get

$$D \sim 2^{n-an-cn^\alpha+1-2},$$

implying that

$$n^d H_n(an, bn) \sim \frac{n^d 2^{an+cn^\alpha-2} 2^{n-an-cn\alpha-1}}{1 + (n-1)2^{n-2}} \sim \frac{n^{d-1}}{2}.$$

We see that $W(a,b,c,d,\alpha,\beta) = 1$ and $L(a,b) = \frac{1}{2}$, as desired. For other values of α, β, the asymptotics work differently, but they can be checked by using Lemma 6.4.2.

The cases where $2a - b \geq 1$ follow by symmetry, so we have completed the proof. □

Lemma 6.4.6. *Let $a, b \in [0,1]$ such that $a + 2b \geq 1, 2a + b \leq 1, 2b - a \leq 1$, or $a + 2b \leq 1, 2a + b \geq 1, 2a - b \leq 1$. Then $W(a,b,c,d,\alpha,\beta) = 1$. Also, $L(a,b,c,d,\alpha,\beta) = 1$ if all the inequalities are strict.*

Proof. Let $a, b \in [0,1]$ satisfy the conditions of the first case of the lemma, with the inequalities strict. Therefore, we have $a < b$. By Lemma 6.3.1, we have

$$\begin{aligned} J_n(an + cn^\alpha, bn + dn^\beta) = 2^{an-2}(&D_{n-an+1, bn-an+1} + D_{n-an+1, bn}) \\ &+ 2^{bn-2} D_{n-bn+1, an} - 2^{an+bn-4}. \end{aligned}$$

Since $2b - a < 1$, we have $2b - 2a < 1 - a$. By Lemma 6.4.2, we have

$$D_{n-an+1, bn-an+1} < \varepsilon_1^{n-an+1},$$

for some $\varepsilon_1 < 1$. Also, since $2a + b < 1$, we have $2a < 1 - b$. Again, by Lemma 6.4.2, we have $D_{n-bn+1, bn-an+1} < \varepsilon_2^{n-bn+1}$, for some $\varepsilon_2 < 1$. Also, since $a + b < 1$, we have $2^{an+bn-4} = o(2^n)$.

On the other hand, since $2b+a>1$, we get $2b>1-a$, so $D_{n-an+1,bn} \sim 2^{n-an}$ by Lemma 6.4.2. Applying these facts, we get

$$J_n(an+cn^\alpha, bn+dn^\beta) \sim 2^{an+cn^\alpha-2}2^{n-an-cn^\alpha} = 2^{n-2}\,,$$

as $n \to \infty$. This immediately gives $n^d H_n(an+cn^\alpha, bn+dn^\beta) \sim n^{d-1}$, so $W(a,b,c,d,\alpha,\beta)=1$ and $L(a,b,c,d,\alpha,\beta)=1$, as desired. □

Lemma 6.4.7. *Let $a,b \in [0,1]$ such that $a+b \le 1, a+2b \ge 1, 2a-b \ge 1$, or $a+b \le 1, 2a+b \ge 1, 2b-a \ge 1$. Then $W(a,b,c,d,\alpha,\beta)=1$. Also, $L(a,b)=3$ if the inequalities are strict.*

Proof. Let $a,b \in [0,1]$ such that $a+b \le 1, 2a+b \ge 1, 2b-a \ge 1$. By Lemma 6.3.1, we have

$$J_n(an,bn) = 2^{an-2}D_{n-an+1,bn-an+1} + 2^{an-2}D_{n-an+1,bn} + 2^{bn-2}D_{n-bn+1,an} - 2^{an+bn-4}\,.$$

Since $2b-a \ge 1$, we have $2b-2a \ge 1-a$. Therefore, by Lemma 6.4.2, we have

$$D_{n-an+1,bn-an+1} \sim 2^{n-an}\,,$$

as $n \to \infty$. Also, since $2b-a \ge 1, 2b \ge 1-a$, and

$$D_{n-an+1,bn} \sim 2^{n-an}\,,$$

as $n \to \infty$. Finally, since $2a+b \ge 1$, we have $2a \ge 1-b$, so

$$D_{n-bn+1,an} \sim 2^{n-bn}\,,$$

as $n \to \infty$. Since $a+b \le 1$, we have $2^{an+bn-4} = o(2^n)$.

Applying these facts, we see that

$$J_n(an,bn) \sim 2^{an-2}2^{n-an} + 2^{an-2}2^{n-an} + 2^{bn-2}2^{n-bn} = 3(2^{n-2})\,.$$

Therefore,

$$n^d H_n(an+cn^\alpha, bn+dn^\beta) \sim 3n^{d-1}\,,$$

giving $W(a,b,c,d,\alpha,\beta)=1$ and $L(a,b)=3$, as desired. □

Lemma 6.4.8. *Let $a, b \in [0,1]$ such that $2a - b \geq 1, 2b + a \leq 1$, or $2b - a \geq 1, 2a + b \leq 1$. Then $W(a,b,c,d,\alpha,\beta) = 1$. Also, if the inequalities are strict, we have $L(a,b) = 2$.*

Proof. Let $a, b \in [0,1]$ such that $2b - a \geq 1, 2a + b \leq 1$. The other case follows by symmetry. By Lemma 6.3.1, we see that

$$J_n(an + cn^\alpha, bn + dn^\beta) = 2^{an-2}\left(D_{n-an+1,bn-an+1} + D_{n-an+1,bn}\right) + 2^{bn-2}D_{n-bn+1,an} - 2^{an+bn-4}\,.$$

Since $2b - a \geq 1$, we get $2b - 2a \geq 1 - a$. By Lemma 6.4.2, we have

$$D_{n-an+1,bn-an+1} \sim 2^{n-an}\,,$$

as $n \to \infty$.

Also, since $2a + b \leq 1$, we get $2a \leq 1 - b$. By Lemma 6.4.2, we have

$$D_{n-bn+1,an} < \varepsilon^{n-bn+1}\,,$$

for some $\varepsilon < 1$.

Finally, since $2b - a \geq 1$, we also have $2b + a \geq 1$, so $2b \geq 1 - a$. Again, by Lemma 6.4.2, we have

$$D_{n-an+1,bn} \sim 2^{n-an}\,,$$

as $n \to \infty$.

Since $2a + b \leq 1$, we have $a + b \leq 1$, so $2^{an+bn-4} = o(2^n)$.

Therefore, we have

$$\begin{aligned} n^d H_n(an + cn^\alpha, bn + dn^\beta) &= \frac{n^d J_n(an + cn^\alpha, bn + dn^\beta)}{1 + (n-1)2^{n-2}} \\ &\sim \frac{n^{d-1}}{2^{n-2}}\left(2^{an-2}2^{n-an} + 2^{an-2}2^{n-an}\right) \\ &\sim 2n^{d-1}\,. \end{aligned}$$

We clearly see that $W(a,b,c,d,\alpha,\beta) = 1$, and when these inequalities are strict, $L(a,b) = 2$, as desired. $\square$

Lemma 6.4.9. *Let $a, b \in [0, 1]$ with $a + 2b \geq 1, 2a + b \geq 1, 2a - b \leq 1, 2b - a \leq 1, a + b \leq 1$. Then $W(a, b, c, d, \alpha, \beta) = 1$. When these inequalities are strict, we have $L(a, b) = 2$.*

Proof. First, suppose $a = b$. Then we have $\frac{1}{3} \leq a \leq \frac{1}{2}$. We are in the case of Lemma 6.3.1 with $j = k, j + k \leq n + 1$, so

$$J_n(an + cn^\alpha, bn + dn^\beta) = 1 + (an - 2)2^{an-3} + 2^{an-2}D_{n-an+1,an} \\ + 2^{an-2}D_{n-an+1,an} - 2^{an+an-4} .$$

Since $3a \geq 1$, we have $2a \geq 1 - a$. By Lemma 6.4.2, we have

$$D_{n-an+1,an} \sim 2^{n-an} ,$$

as $n \to \infty$.

On the other hand, we have $(an - 2)2^{an-3} = o(2^n)$, and $2^{an+an-4} = o(2^n)$ as well, as long as $a < \frac{1}{2}$.

Therefore, we have

$$J_n(an + cn^\alpha, bn + dn^\beta) \sim 2\left(2^{an-2}2^{n-an}\right) = 2^{n-1} ,$$

and $n^d H_n(an, bn) \sim 2n^{d-1}$. We see that $W(a, b, c, d, \alpha, \beta) = 1$ and $L(a, b) = 2$, as desired.

Second, suppose $a < b$. The case with $a > b$ will hold by symmetry. By Lemma 6.3.1, we have

$$J_n(an + cn^\alpha, bn + dn^\beta) = 2^{an-2}\left(D_{n-an+1,bn-an+1} + D_{n-an+1,bn}\right) \\ + 2^{bn-2}D_{n-bn+1,an} - 2^{an+bn-4} .$$

Since $2b - a \leq 1$, we have $2b - 2a \leq 1 - a$. By Lemma 6.4.2, we see that

$$D_{n-an+1,bn-an+1} < \varepsilon^{n-an+1} ,$$

for some $\varepsilon < 1$.

Since $2b + a \geq 1$, we have $2b \geq 1 - a$. By Lemma 6.4.2, we see that

$$D_{n-an+1,bn} \sim 2^{n-an},$$

as $n \to \infty$.

Since $2a + b \geq 1$, we have $2a \geq 1 - b$. By Lemma 6.4.2, we see that

$$D_{n-bn+1,an} \sim 2^{n-bn}$$

as $n \to \infty$.

Finally, we know that $2^{an+bn-4} = o(2^n)$, for $a + b < \frac{1}{2}$.

Therefore, we get

$$n^d H_n(an, bn) \sim \frac{n^{d-1}}{2^{n-2}} \left(2^{an-2} 2^{n-an} + 2^{bn-2} 2^{n-bn}\right) = 2n^{d-1},$$

yielding $W(a, b, c, d, \alpha, \beta) = 1$ and $L(a, b) = 2$, as desired.

$\square$

The following figure shows a sample permutation in $\mathcal{S}_{19}(\mathbf{132}, \mathbf{4231})$. The second figure displays the limit shape for $n = 100$, and the third shows the shape for $n = 100$ with the peak at $a = b = 1$ ignored.

6.5 Results on permutation statistics in $\mathcal{S}_n(\mathbf{132}, \mathbf{4231})$

6.5.1 Fixed points in $\mathcal{S}_n(\mathbf{132}, \mathbf{4231})$

As in Chapter 2.9.9, we calculate the expected number of fixed points in our permutation classes.

Theorem 6.5.1. *Let $\sigma \in \mathcal{S}_n(\mathbf{132}, \mathbf{4231})$ be chosen uniformly at random. Then*

$$\mathbf{E}[fp_n(\sigma)] \to \frac{4}{3}, \quad as \quad n \to \infty.$$

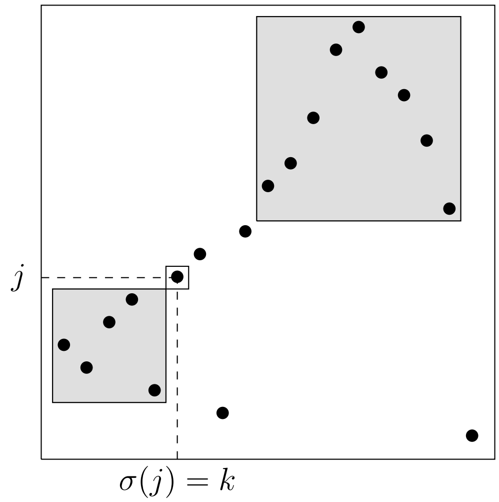


Figure 6.2: Random $\sigma \in \mathcal{S}_n(\mathbf{132}, \mathbf{4231})$

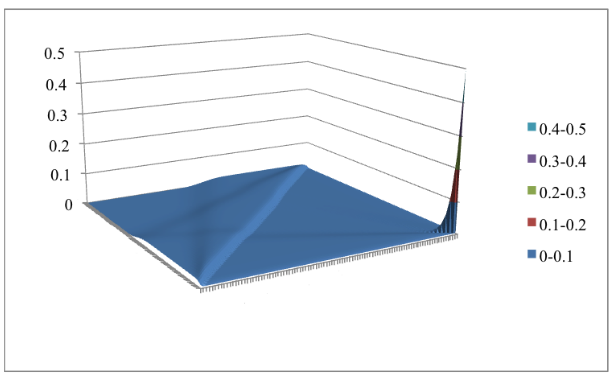


Figure 6.3: Limit shape of $H_{100}(j, k)$

Theorem 6.5.2. *Let $\sigma \in \mathcal{S}_n(\boldsymbol{231}, \boldsymbol{1324})$ be chosen uniformly at random. Then*

$$\mathbf{E}[fp_n(\sigma)] \to \frac{7}{6}, \quad as \quad n \to \infty\,.$$

6.5.2 Proof of Theorem 6.5.1

Let $\sigma \in \mathcal{S}_n(\mathbf{132}, \mathbf{4231})$ be chosen uniformly at random. Clearly,

$$\mathbf{E}[\mathrm{fp}_n(\sigma)] = \sum_{j=1}^{n} H_n(j, j) = \frac{1}{1 + (n-1)2^{n-2}} \sum_{j=1}^{n} J_n(j, j)\,.$$

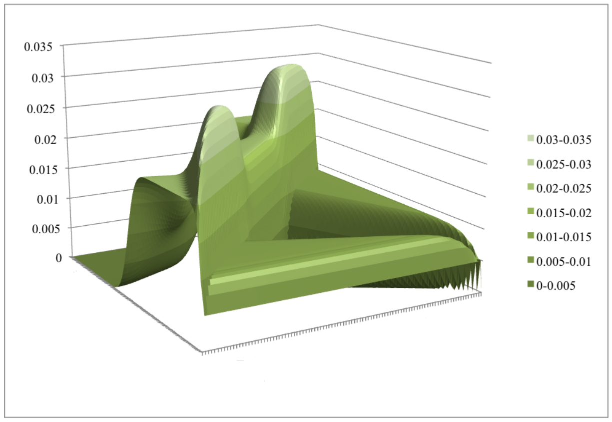


Figure 6.4: Limit shape of $H_{100}(j,k)$, with the corner at $j = k = 100$ ignored

We therefore analyze the sum of $J_n(j,j)$. We see that

$$\begin{aligned}\sum_{j=1}^{n} J_n(j,j) &= D_{n,1} + \sum_{j=2}^{\lfloor\frac{n+1}{2}\rfloor} 1 + (j-2)2^{j-3} + 2^{j-1}D_{n-j+1,j} - 2^{2j-4} \\ &\quad + \sum_{j=\lfloor\frac{n+1}{2}\rfloor+1}^{n} 1 + (j-2)2^{j-3} \\ &= 1 + \sum_{j=2}^{n} 1 + (j-2)2^{j-3} + \sum_{j=2}^{\lfloor\frac{n+1}{2}\rfloor} 2^{j-1}D_{n-j+1,j} - 2^{2j-4}\,.\end{aligned}$$

Observe that $\sum_{j=2}^{n} 1 + (j-2)2^{j-3} = (n-3)2^{n-2} + n$, by induction. Also,

$$\sum_{j=2}^{\lfloor\frac{n+1}{2}\rfloor} -2^{2j-4} = -\frac{1}{3}2^{2\lfloor\frac{n+1}{2}\rfloor-2} = o(n2^n)\,,$$

so it suffices to calculate

$$\sum_{j=2}^{\lfloor\frac{n+1}{2}\rfloor} 2^{j-1}D_{n-j+1,j}\,.$$

Let $j = an$. As seen in Lemmas 6.4.2 and 6.4.2, we know that

$$D_{n-j+1,j} < \varepsilon^{n-j+1} \text{ for } a < \frac{1}{3}\,,$$

and

$$D_{n-j+1,j} \sim 2^{n-j} \text{ for } \frac{1}{3} < a < \frac{1}{2}\,,$$

as $n \to \infty$.

Therefore, we see that

$$\sum_{j=2}^{\lfloor \frac{n+1}{2} \rfloor} 2^{j-1} D_{n-j+1,j} \sim \frac{n}{6} 2^{j-1} 2^{n-j} = \frac{n}{3} 2^{n-2} .$$

Plugging into our complete sum, we have

$$\sum_{j=1}^{n} J_n(j,j) \sim (n-3)2^{n-2} + \frac{n}{3} 2^{n-2} \sim \frac{4}{3} n 2^{n-2} .$$

Therefore, we have

$$\mathbf{E}[\mathrm{fp}_n(\sigma)] \sim \frac{1}{n2^{n-2}} \left(\frac{4}{3} n 2^{n-2} \right) = \frac{4}{3} ,$$

as $n \to \infty$, as desired.

$\square$

6.5.3 Proof of Theorem 6.5.2

Let τ be chosen uniformly at random from $\mathcal{S}_n(\mathbf{132}, \mathbf{4231})$. Let $\sigma = \tau^r$. Observe that σ is now uniform in $\mathcal{S}_n(\mathbf{231}, \mathbf{4231})$. Therefore, $\mathbf{E}[\mathrm{fp}_n(\sigma)] = \mathbf{E}[\mathrm{afp}_n(\tau)]$.

Applying this logic, we get

$$\mathbf{E}[\mathrm{afp}_n(\tau)] = \sum_{j=1}^{n} H_n(j, n+1-j) = \frac{1}{1+(n-1)2^{n-2}} \sum_{j=1}^{n} J_n(j, n+1-j) ,$$

so it suffices to consider $J_n(j, n+1-j)$ for all j.

By Lemma 6.3.1, for $j < n+1-j$, we have

$$\begin{aligned} J_n(j, n+1-j) &= 2^{j-2} D_{n-j+1,(n+1-j)-j+1} + 2^{j-2} D_{n-j+1,n-j+1} \\ &\qquad + 2^{(n-j+1)-2} D_{j,j} - 2^{j+(n-j+1)-4} \\ &= 2^{j-2} D_{n-j+1,n-2j+2} + 2^{j-2} D_{n-j+1,n-j+1} + 2^{n-j-1} D_{j,j} - 2^{n-3} . \end{aligned}$$

A quick calculation shows us that $D_{x,x} = 2^{x-2}$, so by plugging this in, we get

$$\begin{aligned} J_n(j, n+1-j) &= 2^{j-2} D_{n-j+1,n-2j+2} + 2(2^{j-2}2^{n-j-1}) - 2^{n-3} \\ &= 2^{j-2} D_{n-j+1,n-2j+2} + 2^{n-3}\,. \end{aligned}$$

Let $a \in [0, \frac{1}{2})$. Let $j = an$. By Lemmas 6.4.2 and 6.4.2, we know that there exists $\varepsilon < 1$ such that

$$D_{n-j+1,n-2j+2} < \varepsilon^{n-j+1}\,, \text{ for } \frac{1}{2} > a > \frac{1}{3}\,,$$

and that

$$D_{n-j+1,n-2j+2} \sim 2^{n-j} \text{ for } a < \frac{1}{3}\,, \text{ as } n \to \infty\,.$$

Applying this to our sum where $j < n + 1 - j$, we see that

$$\sum_{j=1}^{\frac{n}{2}} 2^{j-2} D_{n-j+1,n-2j+2} + 2^{n-3} \sim \frac{n}{2} 2^{n-3} + \frac{n}{3} 2^{j-2} 2^{n-j} \sim \frac{7}{12} n 2^{n-2}\,,$$

as $n \to \infty$. Since $J_n(j,k) = J_n(j,k)$, when we sum over all values of j from 1 to n, we get

$$\sum_{j=1}^{n} J_n(j, n+1-j) \sim \frac{7}{6} n 2^{n-2}\,, \text{ as } n \to \infty\,.$$

Since

$$\mathbf{E}[\mathrm{fp}_n(\sigma)] \sim \frac{1}{n 2^{n-2}} \left(\frac{7}{6} n 2^{n-2} \right) \text{ as } n \to \infty\,,$$

we get $\mathbf{E}[\mathrm{fp}_n(\sigma)] \to \frac{7}{6}$, as desired. $\square$

CHAPTER 7

Combinatorial proofs of results by Liouville and Andrews

7.1 Introduction

In this chapter, we consider the natural problem of counting the number of positive integer solutions to $x_1x_2 + x_2x_3 + \ldots + x_kx_{k+1} = n$. Let $L_k(n)$ be defined as

$$L_k(n) := |\{(x_1, \ldots, x_{k+1}) \in \mathbb{Z}^{k+1}|$$

such that

$$x_1x_2 + x_2x_3 + \ldots + x_kx_{k+1} = n \ , \ \text{and} \ \ x_1, x_2, \ldots, x_{k+1} > 0 \, .$$

Somewhat surprisingly, even today, the values of this function are only known for $1 \leq k \leq 5$. The first instance of this problem appearing in the literature is by Liouville, in the 1860's [Lou].

Let

$$D_i(n) = \sum_{d|n} d^i \, ,$$

and

$$D_{i_1,i_2,\ldots,i_r}(n) = \sum_{\substack{j_1+j_2+\ldots+j_r=n \\ j_1,j_2,\ldots,j_r \geq 1}} D_{i_1}(j_1)D_{i_2}(j_2)\ldots D_{i_r}(j_r) \, .$$

After defining these functions, we state Liouville's result.

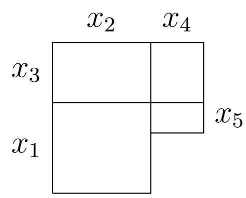


Figure 7.1: Area is $x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 = n$

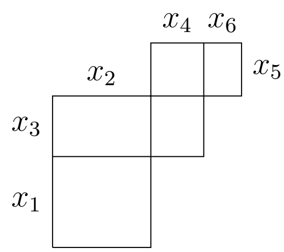


Figure 7.2: Area is $x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_6 = n$

Theorem 7.1.1. *(Liouville) Let $n \in \mathbb{N}$. Then*

$$L_4(n) = D_2(n) - nD_0(n) - D_{0,0}(n)\,.$$

This can be visualized in Figure 7.1; the total area of the four rectangles is n, and the number of such configurations with area n is $L_4(n)$.

In 1998, Andrews [And] gave the following formula for $L_5(n)$.

Theorem 7.1.2. *(Andrews) Let $n \in \mathbb{N}$. Then*

$$L_5(n) = \frac{1}{6}D_{0,0,0}(n) + D_{0,0}(n) + \frac{1}{2}D_{0,1}(n) + \left(2n - \frac{1}{6}\right)D_0(n) - \frac{11}{6}D_2(n)\,.$$

A visualization of the situation in Theorem 7.1.2 is given in Figure 7.2; as Andrews mentions in [And], the rectangles start to form a "staircase" configuration.

In this chapter, we give combinatorial proofs of Theorems 7.1.1 and 7.1.2. In Section 7.2, we prove formulae for $L_1(n), L_2(n)$, and $L_3(n)$. Section 7.3 contains the proof of Theorem 7.1.1, and Section 7.4 the proof of Theorem 7.1.2.

7.2 Smaller values of k

For completeness, we state and prove expressions for $L_k(n)$ with $k < 4$, all three of which are old results.

Theorem 7.2.1. *Let $L_k(n)$ be defined as above. Then*

$$\begin{aligned} L_1(n) &= D_0(n)\,, \\ L_2(n) &= D_1(n) - D_0(n) \\ L_3(n) &= \frac{1}{2}D_{0,0}(n) - \frac{1}{2}D_1(n) + \frac{1}{2}D_0(n)\,. \end{aligned}$$

We now prove the preceding theorem.

Proof. For $k = 1$, we have $L_k(n) = D_0(n)$ by definition of each function.

For $k = 2$, we see that the number of solutions to $x_1x_2 + x_2x_3 = n$ is equal to the number of solutions to $(x_1 + x_3)x_2 = n$. Each solution of this form comes from splitting an integer divisor of n into two positive integers. Therefore, the number of such solutions is

$$L_2(n) = \sum_{d|n} d - 1 = \sum_{d|n} d - \sum_{d|n} 1 = D_1(n) - D_0(n)\,,$$

as desired.

For $k = 3$, we are counting solutions to $x_1x_2 + x_2x_3 + x_3x_4 = n$, or $(x_1 + x_3)x_2 + x_3x_4 = n$. In other words, we want to split n into the sum of two smaller integers, each of which is written as a product of two divisors. We also want the first divisor of the first integer to be larger than the first divisor of the second integer. We see that

$$L_3(n) = \sum_{i=1}^{n-1} \sum_{\substack{dx_2 = i \\ x_3x_4 = n-i \\ d > x_3}} 1\,.$$

Clearly, by symmetry, $L_3(n)$ is also equal to the same double sum, where $d < x_3$. If we consider the double sum with $d = x_3$, we are calculating the number of solutions to $dx_2 + dx_4 = n$, which is $L_2(n)$. Summing all three of these cases gives us $D_{0,0}(n)$, by definition, so we have

$$D_{0,0}(n) = 2L_3(n) + L_2(n)\,,$$

implying that

$$L_3(n) = \frac{D_{0,0}(n) - L_2(n)}{2} = \frac{D_{0,0}(n)}{2} + \frac{D_0(n)}{2} - \frac{D_1(n)}{2}\,,$$

as desired.

This completes the proof of Theorem 7.2.1. $\square$

7.3 Proof of Theorem 7.1.1

The proof depends heavily on a lemma, which we present here.

Lemma 7.3.1. *We have*

$$\sum_{\substack{wx+xy+yz=n \\ w,x,y,z\geq 1}} y = \frac{1}{2}D_2(n) + \frac{1}{2}D_1(n) - nD_0(n)\,.$$

Proof. First, observe that

$$\sum_{\substack{wx+xy+yz=n \\ w,x,y,z\geq 1}} y \qquad = \sum_{\substack{wx+xy+yz=n \\ w,x,y,z\geq 1}} w + y - z\,,$$

since as we sum over all solutions to $wx + xy + yz = n$, w and z range over the same values. In other words, we have an involution φ which acts on solutions by mapping $(w_0, x_0, y_0, z_0) \mapsto (z_0, y_0, x_0, w_0)$. This can be visualized in Figure 7.3.

Now, suppose we have a solution (w, x, y, z) where $z \nmid x$. We can write $x = qz + r$ in a unique way, where $q \geq 0$ and $0 < r < z$. Observe that $(z - r, (q +$

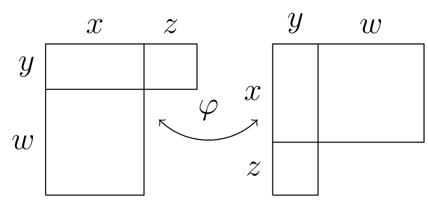


Figure 7.3: Summing y is equivalent to summing $y+w-z$, as shown by the above involution φ.

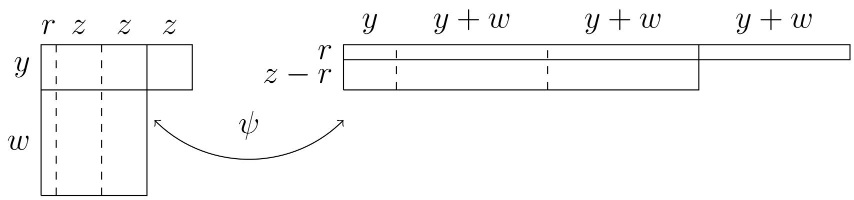


Figure 7.4: For each $z \nmid x$, we have an involution ψ which maps $w+y-z$ to $-(w+y-z)$, showing that these configurations contribute nothing to the total sum.

$1)y+qw, r, w+y)$ is also a solution, with

$$\begin{aligned}(z-r)((q+1)y+qw)+((q+1)y+qw)r+r(w+y)\\ &= z(q+1)y+zqw+rw+ry\\ &= zy+(qz+r)(w+y)\\ &= zy+xw+xy=n\,,\end{aligned}$$

as desired. Also, this solution has the same property as the original (w,x,y,z), since $w+y$ does not divide $(q+1)y+qw$. The first solution contributes $w+y-z$ to our sum, while the second contributes $z-r+r-(w+y)=-(w+y-z)$. We essentially have an involution ψ on solutions where $z \nmid x$ which changes positive values of $w+y-z$ to negative values, so summing over all of these solutions will contribute nothing. In Figure 7.4, we depict the involution ψ.

We can therefore restrict our sum to solutions where z divides x, as shown here.

$$\sum_{\substack{wx+xy+yz=n\\ w,x,y,z\geq 1}} w+y-z \quad = \sum_{\substack{wx+xy+yz=n\\ w,x,y,z\geq 1\\ z\mid x}} w+y-z \quad + \sum_{\substack{wx+xy+yz=n\\ w,x,y,z\geq 1\\ z\nmid x}} w+y-z$$

$$= \sum_{\substack{wqz+qzy+yz=n\\ w,y,z,q\geq 1}} w+y-z \quad +0\,,$$

where $qz = x$. Our equation becomes $z(wq + qy + y) = z(q(w + y) + y) = n$. Therefore, z is a divisor of n, and $w + y$ is some integer smaller than n/z but which does not divide n/z, since we have y as a remainder.

We can therefore express the sum as

$$\sum_{z(q(w+y)+y)=n} w+y-z \quad = \sum_{z\delta=n} \; \sum_{\substack{1\leq r\leq\delta\\ r\nmid\delta}} r-z\,. \qquad (**)$$

We now argue that summing over values of r which do divide δ does not contribute to the overall sum. In other words, showing that

$$\sum_{z\delta=n} \; \sum_{r\mid\delta} r-z=0$$

allows us to write the sum on the right side of (**). To do this, consider some z such that $z\delta = n$ and some r which divides δ, so $\delta = sr$. We essentially have a triple (z, r, s) with $zrs = n$, and each such triple contributes $r - z$. Given such a triple, we can switch the entries in the first two coordinates, giving (r, z, s), which will contribute $z - r = -(r - z)$ instead. This gives a sign-reversing involution, and its only fixed points are triples with $r = z$ which contribute $r - z = 0$ to the sum as well.

Therefore,

$$\sum_{z\delta=n} \; \sum_{\substack{1\leq r\leq\delta\\ r\nmid\delta}} r-z \quad = \sum_{z\delta=n} \; \sum_{r=1}^{\delta} r-z\,,$$

giving us a sum which we can now evaluate.

Evaluating the sum on the right, we see that

$$\begin{aligned}\sum_{z\delta=n}\sum_{r=1}^{\delta} r-z &= \sum_{z\delta=n}\binom{\delta+1}{2}-z\delta\\ &= \sum_{z\delta=n}\frac{\delta^2}{2}+\frac{\delta}{2}-n\\ &= \frac{1}{2}D_2(n)+\frac{1}{2}D_1(n)-nD_0(n)\,,\end{aligned}$$

completing the proof of Lemma 7.3.1. □

7.3.1 Proof of Theorem 7.1.1

We are now ready to complete the proof of Theorem 7.1.1.

Proof. Let $A(n) = \{(x_1,x_2,x_3,x_4,x_5) \in \mathbb{Z}^5 \text{ s.t. } x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 = n, x_1,x_2,x_4,x_5 > 0, x_3 \geq 0\}$. Since

$$\begin{aligned}D_{0,0}(n) &= \sum_{i=1}^{n-1} D_0(i)D_0(n-i)\\ &= \sum_{i=1}^{n-1} |\{(x_1,x_2,x_4,x_5)\in\mathbb{Z}^4\\ &\qquad\qquad \text{such that } x_1x_2 = i, x_4x_5 = n-i, x_1,x_2,x_4,x_5 > 0\}|\\ &= |\{(x_1,x_2,x_4,x_5)\in\mathbb{Z}^4 \text{ s.t. } x_1x_2+x_4x_5 = n, x_1,x_2,x_4,x_5 > 0\}|\,,\end{aligned}$$

we see that

$$|A(n)| = L_4(n) + D_{0,0}(n)\,.$$

Figure 7.5 helps visualize the argument.

We can partition $A(n)$ by comparing x_1 to x_5, so $A(n) = A_<(n) + A_>(n) + A_=(n)$, where

$$A_<(n) = \{(x_1,x_2,x_3,x_4,x_5) \in A(n) | x_1 < x_5\}\,,$$

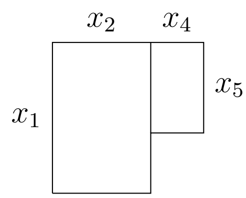


Figure 7.5: If $x_3 = 0$, the area is $x_1x_2 + x_4x_5 = n$, and there are $D_{0,0}(n)$ solutions.

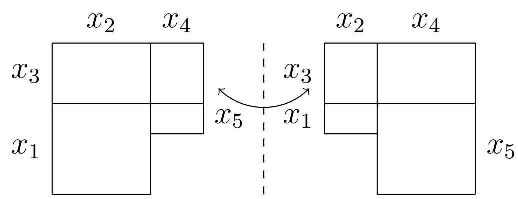


Figure 7.6: The number of solutions with $x_1 < x_5$ is equal to the number with $x_1 > x_5$, due to this bijection.

and $A_>(n)$ and $A_=(n)$ are defined similarly. We can biject between $A_<(n)$ and $A_>(n)$ by mapping a solution $(x_1, x_2, x_3, x_4, x_5) \mapsto (x_5, x_4, x_3, x_2, x_1)$, as shown in Figure 7.6.

Therefore $|A(n)| = 2|A_<(n)| + |A_=(n)|$.

To calculate $|A_=(n)|$, realize that we are counting solutions to

$$x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1 = (x_1 + x_3)(x_2 + x_4) = n\,.$$

The number of such solutions is

$$\sum_{\substack{d\delta=n \\ d=x_1+x_3,\delta=x_2+x_4}} d(\delta - 1) \quad = \sum_{d\delta=n} n - d = nD_0(n) - D_1(n)\,.$$

Note that there are d choices for x_3 since $0 \le x_3 < d$, while there are $\delta - 1$ choices for x_2, since $0 < x_2 < \delta$. This can be visualized with the help of Figure 7.7.

Therefore, to complete the proof, we need to show that

$$\begin{aligned} 2|A_<(n)| &= L_4(n) + D_{0,0}(n) - |A_=(n)| \\ &= (D_2(n) - nD_0(n) - D_{0,0}(n)) + D_{0,0}(n) - (nD_0(n) - D_1(n)) \\ &= D_2(n) + D_1(n) - 2nD_0(n)\,. \end{aligned}$$

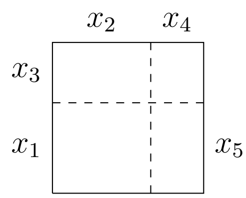


Figure 7.7: The number of solutions with $x_1 = x_5$ is $nD_0(n) - D_1(n)$, since for each choice of $d = x_1 + x_3$ there are d options for x_3 and $(n/d - 1)$ options for x_2. Summing over d which are divisors of n gives us $nD_0(n) - D_1(n)$.

Since

$$\begin{aligned}
|A_<(n)| &= |\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{Z}^5 | x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 = n\,, \\
&\qquad \text{such that } x_5 > x_1 > 0, x_2, x_4 > 0, x_3 \geq 0\}| \\
&= |\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{Z}^5 \text{ such that} \\
&\qquad (x_5 - x_1)x_4 + x_4(x_1 + x_3) + (x_1 + x_3)x_2 = n\,, \\
&\qquad \text{and } x_5 > x_1 > 0, x_2, x_4 > 0, x_3 \geq 0\}| \\
&= \sum_{\substack{(x_5-x_1)x_4+x_4(x_1+x_3)+(x_1+x_3)x_2=n \\ x_5-x_1, x_4, x_1, x_2 \geq 1 \\ x_3 \geq 0}} 1 \\
&= \sum_{\substack{wx+xy+yz=n \\ w,x,y,z \geq 1 \\ x_1+x_3=y \\ x_1 \geq 1 \\ x_3 \geq 0}} 1\,.
\end{aligned}$$

Since for each such solution w, x, y, z we have y options for x_3, we in fact have

$$|A_<(n)| = \sum_{\substack{wx+xy+yz=n \\ w,x,y,z \geq 1}} y\,.$$

Now applying Lemma 7.3.1 gives us exactly what we need, and the proof is complete. $\square$

7.4 Proof of Theorem 7.1.2

The proof will heavily involve analyzing positive integer solutions to

$$d_1\delta_1 + d_2\delta_2 + d_3\delta_3 = n\,. \tag{*}$$

We first need two lemmas which classify and count the number of such solutions. Clearly, $D_{0,0,0}(n)$ is defined as the number of solutions to (*). We can split this into cases based on the relationship between d_1, d_2 and d_3.

Lemma 7.4.1. *Let $B_1(n)$ represent the number of solutions to (*) where $d_1 = d_2 = d_3$. Let $B_2(n)$ represent the number of solutions to (*) where $d_1 = d_2$, but neither are equal to d_3. Let $B_3(n)$ represent the number of solutions to (*) where $d_1 \neq d_2 \neq d_3$. Then*

$$D_{0,0,0}(n) = B_1(n) + 3B_2(n) + B_3(n)\,.$$

Proof. This fact is almost immediate - the only step to verify is why the coefficient 3 appears on $B_2(n)$. The coefficient appears because $B_2(n)$ could also count the number of solutions to (*) with $d_1 = d_3 \neq d_2$, or with $d_2 = d_3 \neq d_1$, due to the symmetry of the expression $d_1\delta_1 + d_2\delta_2 + d_3\delta_3 = n$. □

Lemma 7.4.2. *Let $n \in \mathbb{N}$. Then*

$$B_1(n) = \frac{1}{2}D_2(n) - \frac{3}{2}D_1(n) + D_0(n)\,, \text{ and}$$
$$B_2(n) = D_{0,1}(n) - D_{0,0}(n) - B_1(n)\,.$$

Proof. The function $B_1(n)$ counts the number of solutions to

$$d_1\delta_1 + d_1\delta_2 + d_1\delta_3 = n\,, \text{ or } d_1(\delta_1 + \delta_2 + \delta_3) = n\,.$$

Since we need $\delta_1, \delta_2, \delta_3 > 0$, for each d_1, there are $\binom{n/d_1 - 1}{2}$ possible ways to define

$\delta_1, \delta_2, \delta_3$. Therefore,

$$B_1(n) = \sum_{d\delta | n} \binom{\delta - 1}{2} = \sum_{d\delta|n} \frac{\delta^2}{2} - \frac{3\delta}{2} + \delta = \frac{1}{2} D_2(n) - \frac{3}{2} D_1(n) + D_0(n)\,,$$

as desired.

The function $B_2(n)$ counts the number of solutions to

$$d_1\delta_1 + d_1\delta_2 + d_3\delta_3 = n\,, \text{ or } d_1(\delta_1 + \delta_2) + d_3\delta_3 = n\,,$$

where $d_1 \neq d_3$.

Temporarily ignoring the condition that $d_1 \neq d_3$, we see that the number of solutions to $d_1(\delta_1 + \delta_2) + d_3\delta_3 = n$ is

$$\sum_{\substack{d_1 r + d_3\delta_3 = n \\ \delta_1 + \delta_2 = r}} 1 = \sum_{d_1 r + d_3\delta_3 = n} r - 1 = \sum_{d_1 r + d_3\delta_3 = n} r - \sum_{d_1 r + d_3\delta_3 = n} 1 = D_{0,1}(n) - D_{0,0}(n)\,.$$

The case where $d_1 = d_3$ is counted by $B_1(n)$, so if we want to ensure that $d_1 \neq d_3$, we see that $B_2(n) = D_{0,1}(n) - D_{0,0}(n) - B_1(n)$, as desired.

$\square$

7.4.1 Proof of Theorem 7.1.2

By definition,

$$L_5(n) = \sum_{\substack{d_1\delta_1 + d_2\delta_2 + d_3\delta_3 = n \\ d_1 = x_1 + x_3 \\ d_2 = x_3 + x_5 \\ d_3 = x_5 \\ \delta_1 = x_2 \\ \delta_2 = x_4 \\ \delta_3 = x_6}} 1\,.$$

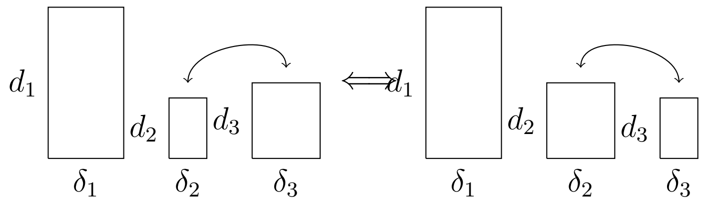


Figure 7.8: Solutions to (*) with $d_2 < d_3$ are equinumerous with solutions to (*) with $d_2 > d_3$.

Clearly, we need $d_2 > d_3$ and $d_1 + d_3 > d_2$. Changing variables, we can calculate $L_5(n)$ by looking at all the solutions to (*), and excluding those with $d_2 \le d_3$ or $d_1 + d_3 \le d_2$.

By definition, the number of solutions to (*) with no restrictions is $D_{0,0,0}(n)$. As in the previous proof, we can partition these solutions based on whether $d_2 < d_3$.

Let $C_<(n)$ be the number of solutions to (*) with $d_2 < d_3$, and let $C_>(n)$ and $C_=(n)$ be defined similarly. Given a solution with $d_2 < d_3$ counted by $C_<(n)$, we can get a solution with $d_2 > d_3$ by switching δ_2 and δ_3, as seen in Figure 7.8. Since this map is an involution, we get $C_<(n) = C_>(n)$.

For $d_2 = d_3$, since d_1 can either equal d_2 or not, we have $C_=(n) = B_1(n) + B_2(n)$. By Lemma 7.4.2, this is $D_{0,1}(n) - D_{0,0}(n)$. Therefore, we have

$$D_{0,0,0}(n) = 2C_<(n) + D_{0,1}(n) - D_{0,0}(n)\,,$$

and

$$C_<(n) = \frac{D_{0,0,0}(n) - D_{0,1}(n) + D_{0,0}(n)}{2}\,.$$

Let $F(n)$ be the number of solutions to (*), with $d_1 + d_3 \le d_2$. Then we have $L_5(n) = D_{0,0,0}(n) - C_<(n) - C_=(n) - F(n)$. We can separate solutions counted by $F(n)$ into those where $d_1 + d_3 = d_2$ and those where $d_1 + d_3 < d_2$, so $F(n) = F_<(n) + F_=(n)$, as seen in Figure 7.9.

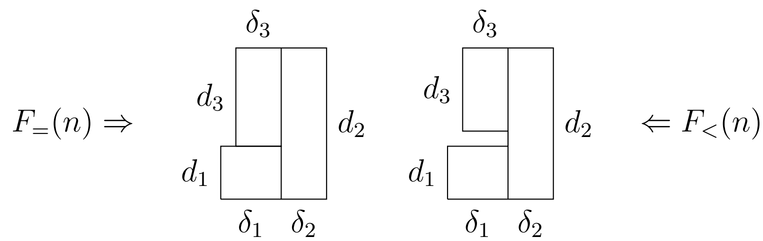


Figure 7.9: Solutions to (*) with $d_1 + d_3 \leq d_2$ are counted either by $F_=(n)$ or $F_<(n)$.

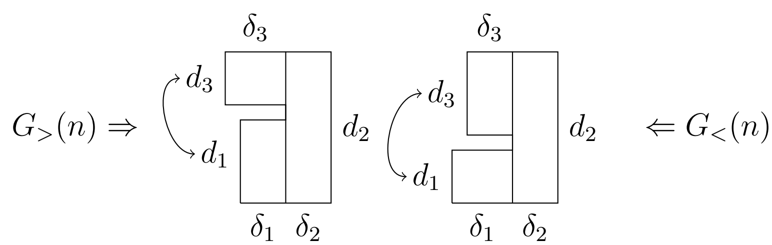


Figure 7.10: Solutions to (*) counted by $G_<(n)$ are equinumerous with solutions to (*) counted by $G_>(n)$.

We see that

$$F_=(n) = \sum_{\substack{d_1\delta_1+d_2\delta_2+d_3\delta_3=n \\ d_1+d_3=d_2}} 1 = \sum_{\delta_1 d_1+d_1\delta_2+\delta_2 d_3+d_3\delta_3=n} 1 = L_4(n)\,,$$

by definition of $L_4(n)$.

On the other hand, we can split $F_<(n)$ into terms based on how δ_1 and δ_3 compare to each other. Let $G_<(n)$ be the number of solutions to (*), with both $d_1 + d_3 < d_2$ and $\delta_1 < \delta_3$, and let $G_=(n)$ and $G_>(n)$ be defined similarly. Then $F_<(n) = G_<(n) + G_=(n) + G_>(n)$.

Consider the map $(d_1, \delta_1, d_2, \delta_2, d_3, \delta_3) \mapsto (d_3, \delta_3, d_2, \delta_2, d_1, \delta_1)$. Here solutions counted by $G_<(n)$ are in one-to-one correspondence with to those counted by $G_>(n)$, as seen in Figure 7.10, so $G_<(n) = G_>(n)$.

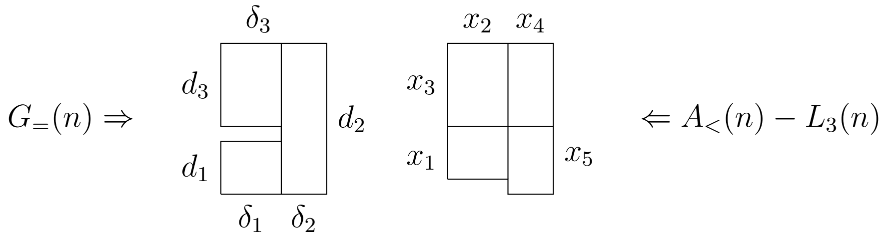


Figure 7.11: Solutions to (*) counted by $G_=(n)$ are equinumerous with $A_<(n) - L_3(n)$, with $x_3 = d_3, x_1 = d_1, x_2 = \delta_3, x_4 = \delta_2$, and $x_5 = d_2 - d_1$.

We see that

$$
\begin{aligned}
G_=(n) &= \sum_{\substack{d_1\delta_1+d_2\delta_2+d_3\delta_3=n \\ d_1+d_3<d_2 \\ \delta_1=\delta_3}} 1 \\
&= \sum_{\substack{d_1\delta_1+d_3\delta_1+d_1\delta_2+d_3\delta_2+r\delta_2=n \\ r=d_2-d_1-d_3}} 1 \\
&= \sum_{\substack{\delta_1 s+s\delta_2+\delta_2 r=n \\ s=d_1+d_3}} 1 \\
&= \sum_{\delta_1 s+s\delta_2+\delta_2 r=n} s-1 \\
&= \sum_{\delta_1 s+s\delta_2+\delta_2 r=n} s \quad - \sum_{\delta_1 s+s\delta_2+\delta_2 r=n} 1 \\
&= A_<(n) - L_3(n)\,,
\end{aligned}
$$

as seen in Figure 7.11.

By Lemma 7.3.1 and Theorem 7.2.1, we have

$$
G_=(n) = \left(\frac{1}{2}D_2(n) + \frac{1}{2}D_1(n) - nD_0(n)\right) - \left(\frac{D_{0,0}(n)}{2} + \frac{D_0(n)}{2} - \frac{D_1(n)}{2}\right).
$$

It suffices to calculate $G_<(n)$. By the definition of this value, we have

$$\begin{aligned}
G_<(n) &= \sum_{\substack{d_1\delta_1+d_2\delta_2+d_3\delta_3=n\\ d_1+d_3<d_2\\ \delta_1<\delta_3}} 1\\
&= \sum_{\substack{d_3t+u\delta_1+d_2\delta_2=n\\ t=\delta_3-\delta_1\\ u=d_1+d_3<d_2}} 1\\
&= \sum_{\substack{d_3t+u\delta_1+d_2\delta_2=n\\ d_3<u<d_2}} 1\\
&= \frac{B_3(n)}{6}.
\end{aligned}$$

The last equality above comes from the fact that we are now imposing a strict order condition on the values of d_3, u, and d_2. Since these elements play the same role in the equation $d_3t + u\delta_1 + d_2\delta_2 = n$, the total number of solutions with $d_3 \neq u \neq d_2$ is given by $B_3(n)$, but these solutions are partitioned equally into six parts, one corresponding to any specific order of d_3, u, and d_2. Since here we are requiring $d_3 < u < d_2$, the number of such solutions is $G_<(n) = \frac{1}{6}B_3(n)$.

Solving for $\frac{B_3(n)}{6}$ in Lemma 7.4.1 and applying Lemma 7.3.1 gives

$$\begin{aligned}
G_<(n) &= \frac{D_{0,0,0}(n)}{6} - \frac{B_1(n)}{6} - \frac{B_2(n)}{2}\\
&= \frac{D_{0,0,0}(n)}{6} - \frac{B_1(n)}{6} - \frac{D_{0,1}(n) - D_{0,0}(n) - B_1(n)}{2}\\
&= \frac{D_{0,0,0}(n)}{6} + \frac{B_1(n)}{3} - \frac{D_{0,1}(n)}{2} + \frac{D_{0,0}(n)}{2}\\
&= \frac{D_{0,0,0}(n)}{6} + \frac{D_2(n)}{6} - \frac{D_1(n)}{2} + \frac{D_0(n)}{3} - \frac{D_{0,1}(n)}{2} + \frac{D_{0,0}(n)}{2}
\end{aligned}$$

We are finally ready to compute $L_5(n)$. Plugging in all our results, we get

$$\begin{aligned}
L_5(n) &= D_{0,0,0}(n) - C_<(n) - C_=(n) - F(n) \\
&= D_{0,0,0}(n) - \left(\frac{D_{0,0,0}(n) - D_{0,1}(n) + D_{0,0}(n)}{2}\right) \\
&\quad - \left(\frac{D_{0,1}(n)}{2} - \frac{D_{0,0}(n)}{2}\right) - (F_<(n) + F_=(n)) \\
&= \frac{D_{0,0,0}(n)}{2} - \frac{D_{0,1}(n)}{2} + \frac{D_{0,0}(n)}{2} - (2G_<(n) + G_=(n) + L_4(n)) \\
&= \frac{D_{0,0,0}(n)}{2} - \frac{D_{0,1}(n)}{2} + \frac{D_{0,0}(n)}{2} \\
&\quad - 2\left(\frac{D_{0,0,0}(n)}{6} + \frac{D_2(n)}{6} - \frac{D_1(n)}{2} + \frac{D_0(n)}{3} - \frac{D_{0,1}(n)}{2} + \frac{D_{0,0}(n)}{2}\right) \\
&\quad - \left(\left(\frac{1}{2}D_2(n) + \frac{1}{2}D_1(n) - nD_0(n)\right) - \left(\frac{D_{0,0}(n)}{2} + \frac{D_0(n)}{2} - \frac{D_1(n)}{2}\right)\right) \\
&\quad - (D_2(n) - nD_0(n) - D_{0,0}(n)) \\
&= \frac{1}{6}D_{0,0,0}(n) + \frac{1}{2}D_{0,1}(n) + D_{0,0}(n) - \frac{11}{6}D_2(n) + (2n - \frac{1}{6})D_0(n)\,,
\end{aligned}$$

as desired.

□

CHAPTER 8

Appendix: Numerical calculations and extended Theorem 6.2.3

8.1 Extended Theorem 6.2.3

Here we include the extended version of Theorem 6.2.3, which has too many cases to fit comfortably in the body of Chapter 6.

Theorem 8.1.1. *Let $a, b \in [0,1], c, d \in \mathbb{R}, \alpha, \beta \in [0,1)$. For values of a, b, c, d, α, and β such that $W(a,b,c,d,\alpha,\beta) < \infty$, we have*

$$X(a,b,c,d,\alpha,\beta) = \begin{cases} \frac{1}{2} & \textit{for } a = b = 1, c = d = 0, \alpha = \beta = 0\,, \\ 1 & \textit{for } a+b > 1, 2b-a > 1\,,\ \textit{or } a+b > 1, 2a-b > 1\,, \\ & \quad \textit{or } 2a+b > 1, a+2b < 1, 2a-b < 1\,, \\ & \quad \textit{or } a+2b > 1, 2a+b < 1, -a+2b < 1\,, \\ 3 & \textit{for } a+b < 1, a+2b > 1, 2a-b > 1 \\ & \quad \textit{or } a+b < 1, 2a+b > 1, 2b-a > 1\,, \\ 2 & \textit{for } a+2b < 1, 2a-b > 1\,,\ \textit{or } 2a+b < 1, 2b-a > 1\,, \\ & \quad \textit{or } a+b < 1, a+2b > 1, 2a+b > 1\,, \\ & \qquad \textit{and } 2a-b < 1, 2b-a < 1\,. \end{cases}$$

For $2a-b=1, a+b>1, a<1,$ *we have*

$$X(a,b,c,d,\alpha,\beta)=\begin{cases} 1 & \mathit{for}\ \alpha>\beta, \alpha>\frac{1}{2}, c>0\,, \\ 1 & \mathit{for}\ \alpha>\beta>\frac{1}{2}, c=0, d<0\,, \\ 1 & \mathit{for}\ \alpha=\beta>\frac{1}{2}, 2c-d>0\,, \\ 1 & \mathit{for}\ \beta>\alpha>\frac{1}{2}, d=0, c>0\,, \\ 1 & \mathit{for}\ \beta>\alpha, \beta>\frac{1}{2}, d<0\,, \\ \frac{1}{2} & \mathit{for}\ \frac{1}{2}>\alpha,\beta\,, \\ \frac{1}{2} & \mathit{for}\ \alpha>\frac{1}{2}>\beta\,, c=0\,, \\ \frac{1}{2} & \mathit{for}\ \beta>\frac{1}{2}>\alpha\,, d=0\,, \\ \frac{1}{2} & \mathit{for}\ \alpha>\beta>\frac{1}{2}\,, c=d=0\,, \\ \frac{1}{2} & \mathit{for}\ \beta>\alpha>\frac{1}{2}\,, c=d=0\,, \\ \frac{1}{2} & \mathit{for}\ \alpha=\beta>\frac{1}{2}\,, 2c-d=0\,, \\ \frac{1-erf(d)}{2} & \mathit{for}\ \beta=\frac{1}{2}>\alpha\,, \\ \frac{1-erf(-2c)}{2} & \mathit{for}\ \alpha=\frac{1}{2}>\beta\,, \\ \frac{1-erf(d-2c)}{2} & \mathit{for}\ \alpha=\beta=\frac{1}{2}\,, \\ \frac{1-erf(d)}{2} & \mathit{for}\ \alpha>\beta=\frac{1}{2}\,, c=0\,, \\ \frac{1-erf(-2c)}{2} & \mathit{for}\ \beta>\alpha=\frac{1}{2}\,, d=0\,. \end{cases}$$

For $2a-b=1, a+b<1, a+2b>1,$ *we have*

$$X(a,b,c,d,\alpha,\beta)=\begin{cases} 3 & \textit{for } \alpha>\beta, \alpha>\frac{1}{2}, c>0\,, \\ 3 & \textit{for } \alpha>\beta>\frac{1}{2}, c=0, d<0\,, \\ 3 & \textit{for } \alpha=\beta>\frac{1}{2}, 2c-d>0\,, \\ 3 & \textit{for } \beta>\alpha>\frac{1}{2}, d=0, c>0\,, \\ 3 & \textit{for } \beta>\alpha, \beta>\frac{1}{2}, d<0\,, \\ \frac{5}{2} & \textit{for } \frac{1}{2}>\alpha,\beta\,, \\ \frac{5}{2} & \textit{for } \alpha>\frac{1}{2}>\beta\,, c=0\,, \\ \frac{5}{2} & \textit{for } \beta>\frac{1}{2}>\alpha\,, d=0\,, \\ \frac{5}{2} & \textit{for } \alpha>\beta>\frac{1}{2}\,, c=d=0\,, \\ \frac{5}{2} & \textit{for } \beta>\alpha>\frac{1}{2}\,, c=d=0\,, \\ \frac{5}{2} & \textit{for } \alpha=\beta>\frac{1}{2}\,, 2c-d=0\,, \\ \frac{5-erf(d)}{2} & \textit{for } \beta=\frac{1}{2}>\alpha\,, \\ \frac{5-erf(-2c)}{2} & \textit{for } \alpha=\frac{1}{2}>\beta\,, \\ \frac{5-erf(d-2c)}{2} & \textit{for } \alpha=\beta=\frac{1}{2}\,, \\ \frac{5-erf(d)}{2} & \textit{for } \alpha>\beta=\frac{1}{2}\,, c=0\,, \\ \frac{5-erf(-2c)}{2} & \textit{for } \beta>\alpha=\frac{1}{2}\,, d=0\,. \end{cases}$$

For $2a-b=1, a+2b<1, a>0,$ *we have*

$$X(a,b,c,d,\alpha,\beta)=\begin{cases} 2 & for\ \alpha>\beta, \alpha>\frac{1}{2}, c>0\,, \\ 2 & for\ \alpha>\beta>\frac{1}{2}, c=0, d<0\,, \\ 2 & for\ \alpha=\beta>\frac{1}{2}, 2c-d>0\,, \\ 2 & for\ \beta>\alpha>\frac{1}{2}, d=0, c>0\,, \\ 2 & for\ \beta>\alpha, \beta>\frac{1}{2}, d<0\,, \\ \frac{3}{2} & for\ \frac{1}{2}>\alpha,\beta\,, \\ \frac{3}{2} & for\ \alpha>\frac{1}{2}>\beta\,, c=0\,, \\ \frac{3}{2} & for\ \beta>\frac{1}{2}>\alpha\,, d=0\,, \\ \frac{3}{2} & for\ \alpha>\beta>\frac{1}{2}\,, c=d=0\,, \\ \frac{3}{2} & for\ \beta>\alpha>\frac{1}{2}\,, c=d=0\,, \\ \frac{3}{2} & for\ \alpha=\beta>\frac{1}{2}\,, 2c-d=0\,, \\ \frac{3-erf(d)}{2} & for\ \beta=\frac{1}{2}>\alpha\,, \\ \frac{3-erf(-2c)}{2} & for\ \alpha=\frac{1}{2}>\beta\,, \\ \frac{3-erf(d-2c)}{2} & for\ \alpha=\beta=\frac{1}{2}\,, \\ \frac{3-erf(d)}{2} & for\ \alpha>\beta=\frac{1}{2}\,, c=0\,, \\ \frac{3-erf(-2c)}{2} & for\ \beta>\alpha=\frac{1}{2}\,, d=0\,. \end{cases}$$

For $2b - a = 1, a + b > 1, b < 1,$ *we have*

$$X(a,b,c,d,\alpha,\beta) = \begin{cases} 1 & \textit{for } \alpha > \beta, \alpha > \frac{1}{2}, c < 0\,, \\ 1 & \textit{for } \alpha > \beta > \frac{1}{2}, c = 0, d > 0\,, \\ 1 & \textit{for } \alpha = \beta > \frac{1}{2}, 2d - c > 0\,, \\ 1 & \textit{for } \beta > \alpha > \frac{1}{2}, d = 0, c < 0\,, \\ 1 & \textit{for } \beta > \alpha, \beta > \frac{1}{2}, d > 0\,, \\ \frac{1}{2} & \textit{for } \frac{1}{2} > \alpha, \beta\,, \\ \frac{1}{2} & \textit{for } \alpha > \frac{1}{2} > \beta\,, c = 0\,, \\ \frac{1}{2} & \textit{for } \beta > \frac{1}{2} > \alpha\,, d = 0\,, \\ \frac{1}{2} & \textit{for } \alpha > \beta > \frac{1}{2}\,, c = d = 0\,, \\ \frac{1}{2} & \textit{for } \beta > \alpha > \frac{1}{2}\,, c = d = 0\,, \\ \frac{1}{2} & \textit{for } \alpha = \beta > \frac{1}{2}\,, 2d - c = 0\,, \\ \frac{1 - erf(-2d)}{2} & \textit{for } \beta = \frac{1}{2} > \alpha\,, \\ \frac{1 - erf(c)}{2} & \textit{for } \alpha = \frac{1}{2} > \beta\,, \\ \frac{1 - erf(-2d + c)}{2} & \textit{for } \alpha = \beta = \frac{1}{2}\,, \\ \frac{1 - erf(-2d)}{2} & \textit{for } \alpha > \beta = \frac{1}{2}\,, c = 0\,, \\ \frac{1 - erf(c)}{2} & \textit{for } \beta > \alpha = \frac{1}{2}\,, d = 0\,. \end{cases}$$

For $2b-a=1, a+b<1, 2a+b>1,$ *we have*

$$X(a,b,c,d,\alpha,\beta)=\begin{cases} 3 & \textit{for } \alpha>\beta, \alpha>\frac{1}{2}, c<0\,,\\ 3 & \textit{for } \alpha>\beta>\frac{1}{2}, c=0, d>0\,,\\ 3 & \textit{for } \alpha=\beta>\frac{1}{2}, 2d-c>0\,,\\ 3 & \textit{for } \beta>\alpha>\frac{1}{2}, d=0, c<0\,,\\ 3 & \textit{for } \beta>\alpha, \beta>\frac{1}{2}, d>0\,,\\ \frac{5}{2} & \textit{for } \frac{1}{2}>\alpha,\beta\,,\\ \frac{5}{2} & \textit{for } \alpha>\frac{1}{2}>\beta\,, c=0\,,\\ \frac{5}{2} & \textit{for } \beta>\frac{1}{2}>\alpha\,, d=0\,,\\ \frac{5}{2} & \textit{for } \alpha>\beta>\frac{1}{2}\,, c=d=0\,,\\ \frac{5}{2} & \textit{for } \beta>\alpha>\frac{1}{2}\,, c=d=0\,,\\ \frac{5}{2} & \textit{for } \alpha=\beta>\frac{1}{2}\,, 2d-c=0\,,\\ \frac{5-erf(-2d)}{2} & \textit{for } \beta=\frac{1}{2}>\alpha\,,\\ \frac{5-erf(c)}{2} & \textit{for } \alpha=\frac{1}{2}>\beta\,,\\ \frac{5-erf(d-2c)}{2} & \textit{for } \alpha=\beta=\frac{1}{2}\,,\\ \frac{5-erf(c)}{2} & \textit{for } \alpha>\beta=\frac{1}{2}\,, c=0\,,\\ \frac{5-erf(-2d)}{2} & \textit{for } \beta>\alpha=\frac{1}{2}\,, d=0\,. \end{cases}$$

For $2b-a=1, 2a+b<1, a>0,$ *we have*

$$X(a,b,c,d,\alpha,\beta)=\begin{cases} 2 & \textit{for } \alpha>\beta, \alpha>\frac{1}{2}, c<0\,, \\ 2 & \textit{for } \alpha>\beta>\frac{1}{2}, c=0, d>0\,, \\ 2 & \textit{for } \alpha=\beta>\frac{1}{2}, 2d-c>0\,, \\ 2 & \textit{for } \beta>\alpha>\frac{1}{2}, d=0, c<0\,, \\ 2 & \textit{for } \beta>\alpha, \beta>\frac{1}{2}, d>0\,, \\ \frac{3}{2} & \textit{for } \frac{1}{2}>\alpha,\beta\,, \\ \frac{3}{2} & \textit{for } \alpha>\frac{1}{2}>\beta\,, c=0\,, \\ \frac{3}{2} & \textit{for } \beta>\frac{1}{2}>\alpha\,, d=0\,, \\ \frac{3}{2} & \textit{for } \alpha>\beta>\frac{1}{2}\,, c=d=0\,, \\ \frac{3}{2} & \textit{for } \beta>\alpha>\frac{1}{2}\,, c=d=0\,, \\ \frac{3}{2} & \textit{for } \alpha=\beta>\frac{1}{2}\,, 2d-c=0\,, \\ \frac{3-erf(-2d)}{2} & \textit{for } \beta=\frac{1}{2}>\alpha\,, \\ \frac{3-erf(c)}{2} & \textit{for } \alpha=\frac{1}{2}>\beta\,, \\ \frac{3-erf(c-2d)}{2} & \textit{for } \alpha=\beta=\frac{1}{2}\,, \\ \frac{3-erf(-2d)}{2} & \textit{for } \alpha>\beta=\frac{1}{2}\,, c=0\,, \\ \frac{3-erf(c)}{2} & \textit{for } \beta>\alpha=\frac{1}{2}\,, d=0\,. \end{cases}$$

For $a+b=1, 2a-b<1, -a+2b<1,$ *we have*

$$X(a,b,c,d,\alpha,\beta)=\begin{cases} 2 & for\ \alpha>\beta, \alpha>\frac{1}{2}, c<0\,,\\ 2 & for\ \alpha>\beta>\frac{1}{2}, c=0, d<0\,,\\ 2 & for\ \alpha=\beta>\frac{1}{2}, c+d<0\,,\\ 2 & for\ \beta>\alpha>\frac{1}{2}, d=0, c<0\,,\\ 2 & for\ \beta>\alpha, \beta>\frac{1}{2}, d<0\,,\\ 1 & for\ \frac{1}{2}>\alpha,\beta\,,\\ 1 & for\ \alpha>\frac{1}{2}>\beta\,, c=0\,,\\ 1 & for\ \beta>\frac{1}{2}>\alpha\,, d=0\,,\\ 1 & for\ \alpha>\beta>\frac{1}{2}\,, c=d=0\,,\\ 1 & for\ \beta>\alpha>\frac{1}{2}\,, c=d=0\,,\\ 1 & for\ \alpha=\beta>\frac{1}{2}\,, c+d=0\,,\\ 1+erf(-c) & for\ \beta=\frac{1}{2}>\alpha\,,\\ 1+erf(-d) & for\ \alpha=\frac{1}{2}>\beta\,,\\ 1+erf(-c-d) & for\ \alpha=\beta=\frac{1}{2}\,,\\ 1+erf(-d) & for\ \alpha>\beta=\frac{1}{2}\,, c=0\,,\\ 1+erf(-c) & for\ \beta>\alpha=\frac{1}{2}\,, d=0\,. \end{cases}$$

For $a+b=1, 2a-b>1, a<1,$ *we have*

$$X(a,b,c,d,\alpha,\beta)=\begin{cases} 3 & \mathit{for}\ \alpha>\beta, \alpha>\frac{1}{2}, c<0\,,\\ 3 & \mathit{for}\ \alpha>\beta>\frac{1}{2}, c=0, d<0\,,\\ 3 & \mathit{for}\ \alpha=\beta>\frac{1}{2}, c+d<0\,,\\ 3 & \mathit{for}\ \beta>\alpha>\frac{1}{2}, d=0, c<0\,,\\ 3 & \mathit{for}\ \beta>\alpha, \beta>\frac{1}{2}, d<0\,,\\ 2 & \mathit{for}\ \frac{1}{2}>\alpha,\beta\,,\\ 2 & \mathit{for}\ \alpha>\frac{1}{2}>\beta\,, c=0\,,\\ 2 & \mathit{for}\ \beta>\frac{1}{2}>\alpha\,, d=0\,,\\ 2 & \mathit{for}\ \alpha>\beta>\frac{1}{2}\,, c=d=0\,,\\ 2 & \mathit{for}\ \beta>\alpha>\frac{1}{2}\,, c=d=0\,,\\ 2 & \mathit{for}\ \alpha=\beta>\frac{1}{2}\,, c+d=0\,,\\ 2+erf(-c) & \mathit{for}\ \beta=\frac{1}{2}>\alpha\,,\\ 2+erf(-d) & \mathit{for}\ \alpha=\frac{1}{2}>\beta\,,\\ 2+erf(-c-d) & \mathit{for}\ \alpha=\beta=\frac{1}{2}\,,\\ 2+erf(-d) & \mathit{for}\ \alpha>\beta=\frac{1}{2}\,, c=0\,,\\ 2+erf(-c) & \mathit{for}\ \beta>\alpha=\frac{1}{2}\,, d=0\,, \end{cases}$$

For $a+b=1, -a+2b>1, b<1,$ *we have*

$$X(a,b,c,d,\alpha,\beta)=\begin{cases} 3 & for\ \alpha>\beta, \alpha>\frac{1}{2}, c<0\,, \\ 3 & for\ \alpha>\beta>\frac{1}{2}, c=0, d<0\,, \\ 3 & for\ \alpha=\beta>\frac{1}{2}, c+d<0\,, \\ 3 & for\ \beta>\alpha>\frac{1}{2}, d=0, c<0\,, \\ 3 & for\ \beta>\alpha, \beta>\frac{1}{2}, d<0\,, \\ 2 & for\ \frac{1}{2}>\alpha,\beta\,, \\ 2 & for\ \alpha>\frac{1}{2}>\beta\,, c=0\,, \\ 2 & for\ \beta>\frac{1}{2}>\alpha\,, d=0\,, \\ 2 & for\ \alpha>\beta>\frac{1}{2}\,, c=d=0\,, \\ 2 & for\ \beta>\alpha>\frac{1}{2}\,, c=d=0\,, \\ 2 & for\ \alpha=\beta>\frac{1}{2}\,, c+d=0\,, \\ 2+erf(-c) & for\ \beta=\frac{1}{2}>\alpha\,, \\ 2+erf(-d) & for\ \alpha=\frac{1}{2}>\beta\,, \\ 2+erf(-c-d) & for\ \alpha=\beta=\frac{1}{2}\,, \\ 2+erf(-d) & for\ \alpha>\beta=\frac{1}{2}\,, c=0\,, \\ 2+erf(-c) & for\ \beta>\alpha=\frac{1}{2}\,, d=0\,, \end{cases}$$

For $a+2b=1, 2a-b>1, a<1,$ *we have*

$$X(a,b,c,d,\alpha,\beta)=\begin{cases} 3 & for\ \alpha>\beta, \alpha>\frac{1}{2}, c>0\,, \\ 3 & for\ \alpha>\beta>\frac{1}{2}, c=0, d>0\,, \\ 3 & for\ \alpha=\beta>\frac{1}{2}, c+2d>0\,, \\ 3 & for\ \beta>\alpha>\frac{1}{2}, d=0, c>0\,, \\ 3 & for\ \beta>\alpha, \beta>\frac{1}{2}, d>0\,, \\ \frac{5}{2} & for\ \frac{1}{2}>\alpha,\beta\,, \\ \frac{5}{2} & for\ \alpha>\frac{1}{2}>\beta\,, c=0\,, \\ \frac{5}{2} & for\ \beta>\frac{1}{2}>\alpha\,, d=0\,, \\ \frac{5}{2} & for\ \alpha>\beta>\frac{1}{2}\,, c=d=0\,, \\ \frac{5}{2} & for\ \beta>\alpha>\frac{1}{2}\,, c=d=0\,, \\ \frac{5}{2} & for\ \alpha=\beta>\frac{1}{2}\,, c+2d=0\,, \\ \frac{5}{2}+erf(2d) & for\ \beta=\frac{1}{2}>\alpha\,, \\ \frac{5}{2}+erf(c) & for\ \alpha=\frac{1}{2}>\beta\,, \\ \frac{5}{2}+erf(c+2d) & for\ \alpha=\beta=\frac{1}{2}\,, \\ \frac{5}{2}+erf(c) & for\ \alpha>\beta=\frac{1}{2}\,, c=0\,, \\ \frac{5}{2}+erf(2d) & for\ \beta>\alpha=\frac{1}{2}\,, d=0\,, \end{cases}$$

For $a+2b=1, 2a-b<1, 2a+b>1,$ *we have*

$$X(a,b,c,d,\alpha,\beta)=\begin{cases} 2 & \text{for } \alpha>\beta, \alpha>\frac{1}{2}, c>0\,, \\ 2 & \text{for } \alpha>\beta>\frac{1}{2}, c=0, d>0\,, \\ 2 & \text{for } \alpha=\beta>\frac{1}{2}, c+2d>0\,, \\ 2 & \text{for } \beta>\alpha>\frac{1}{2}, d=0, c>0\,, \\ 2 & \text{for } \beta>\alpha, \beta>\frac{1}{2}, d>0\,, \\ \frac{3}{2} & \text{for } \frac{1}{2}>\alpha,\beta\,, \\ \frac{3}{2} & \text{for } \alpha>\frac{1}{2}>\beta\,, c=0\,, \\ \frac{3}{2} & \text{for } \beta>\frac{1}{2}>\alpha\,, d=0\,, \\ \frac{3}{2} & \text{for } \alpha>\beta>\frac{1}{2}\,, c=d=0\,, \\ \frac{3}{2} & \text{for } \beta>\alpha>\frac{1}{2}\,, c=d=0\,, \\ \frac{3}{2} & \text{for } \alpha=\beta>\frac{1}{2}\,, c+2d=0\,, \\ \frac{3}{2}+erf(2d) & \text{for } \beta=\frac{1}{2}>\alpha\,, \\ \frac{3}{2}+erf(c) & \text{for } \alpha=\frac{1}{2}>\beta\,, \\ \frac{3}{2}+erf(c+2d) & \text{for } \alpha=\beta=\frac{1}{2}\,, \\ \frac{3}{2}+erf(c) & \text{for } \alpha>\beta=\frac{1}{2}\,, c=0\,, \\ \frac{3}{2}+erf(2d) & \text{for } \beta>\alpha=\frac{1}{2}\,, d=0\,, \end{cases}$$

For $a+2b=1, a>0, 2a+b<1,$ *we have*

$$X(a,b,c,d,\alpha,\beta)=\begin{cases} 1 & \textit{for } \alpha>\beta, \alpha>\frac{1}{2}, c>0\,, \\ 1 & \textit{for } \alpha>\beta>\frac{1}{2}, c=0, d>0\,, \\ 1 & \textit{for } \alpha=\beta>\frac{1}{2}, c+2d>0\,, \\ 1 & \textit{for } \beta>\alpha>\frac{1}{2}, d=0, c>0\,, \\ 1 & \textit{for } \beta>\alpha, \beta>\frac{1}{2}, d>0\,, \\ \frac{1}{2} & \textit{for } \frac{1}{2}>\alpha,\beta\,, \\ \frac{1}{2} & \textit{for } \alpha>\frac{1}{2}>\beta\,, c=0\,, \\ \frac{1}{2} & \textit{for } \beta>\frac{1}{2}>\alpha\,, d=0\,, \\ \frac{1}{2} & \textit{for } \alpha>\beta>\frac{1}{2}\,, c=d=0\,, \\ \frac{1}{2} & \textit{for } \beta>\alpha>\frac{1}{2}\,, c=d=0\,, \\ \frac{1}{2} & \textit{for } \alpha=\beta>\frac{1}{2}\,, c+2d=0\,, \\ \frac{1}{2}+erf(2d) & \textit{for } \beta=\frac{1}{2}>\alpha\,, \\ \frac{1}{2}+erf(c) & \textit{for } \alpha=\frac{1}{2}>\beta\,, \\ \frac{1}{2}+erf(c+2d) & \textit{for } \alpha=\beta=\frac{1}{2}\,, \\ \frac{1}{2}+erf(c) & \textit{for } \alpha>\beta=\frac{1}{2}\,, c=0\,, \\ \frac{1}{2}+erf(2d) & \textit{for } \beta>\alpha=\frac{1}{2}\,, d=0\,, \end{cases}$$

For $2a+b=1, 2b-a>1, b<1,$ *we have*

$$X(a,b,c,d,\alpha,\beta)=\begin{cases} 3 & \textit{for } \alpha>\beta, \alpha>\frac{1}{2}, c>0\,, \\ 3 & \textit{for } \alpha>\beta>\frac{1}{2}, c=0, d>0\,, \\ 3 & \textit{for } \alpha=\beta>\frac{1}{2}, 2c+d>0\,, \\ 3 & \textit{for } \beta>\alpha>\frac{1}{2}, d=0, c>0\,, \\ 3 & \textit{for } \beta>\alpha, \beta>\frac{1}{2}, d>0\,, \\ \frac{5}{2} & \textit{for } \frac{1}{2}>\alpha,\beta\,, \\ \frac{5}{2} & \textit{for } \alpha>\frac{1}{2}>\beta\,, c=0\,, \\ \frac{5}{2} & \textit{for } \beta>\frac{1}{2}>\alpha\,, d=0\,, \\ \frac{5}{2} & \textit{for } \alpha>\beta>\frac{1}{2}\,, c=d=0\,, \\ \frac{5}{2} & \textit{for } \beta>\alpha>\frac{1}{2}\,, c=d=0\,, \\ \frac{5}{2} & \textit{for } \alpha=\beta>\frac{1}{2}\,, 2c+d=0\,, \\ \frac{5}{2}+erf(d) & \textit{for } \beta=\frac{1}{2}>\alpha\,, \\ \frac{5}{2}+erf(2c) & \textit{for } \alpha=\frac{1}{2}>\beta\,, \\ \frac{5}{2}+erf(2c+d) & \textit{for } \alpha=\beta=\frac{1}{2}\,, \\ \frac{5}{2}+erf(2c) & \textit{for } \alpha>\beta=\frac{1}{2}\,, c=0\,, \\ \frac{5}{2}+erf(d) & \textit{for } \beta>\alpha=\frac{1}{2}\,, d=0\,, \end{cases}$$

For $2a+b=1, 2b-a<1, a+2b>1,$ *we have*

$$X(a,b,c,d,\alpha,\beta)=\begin{cases} 2 & for\ \alpha>\beta, \alpha>\frac{1}{2}, c>0\,, \\ 2 & for\ \alpha>\beta>\frac{1}{2}, c=0, d>0\,, \\ 2 & for\ \alpha=\beta>\frac{1}{2}, 2c+d>0\,, \\ 2 & for\ \beta>\alpha>\frac{1}{2}, d=0, c>0\,, \\ 2 & for\ \beta>\alpha, \beta>\frac{1}{2}, d>0\,, \\ \frac{3}{2} & for\ \frac{1}{2}>\alpha,\beta\,, \\ \frac{3}{2} & for\ \alpha>\frac{1}{2}>\beta\,, c=0\,, \\ \frac{3}{2} & for\ \beta>\frac{1}{2}>\alpha\,, d=0\,, \\ \frac{3}{2} & for\ \alpha>\beta>\frac{1}{2}\,, c=d=0\,, \\ \frac{3}{2} & for\ \beta>\alpha>\frac{1}{2}\,, c=d=0\,, \\ \frac{3}{2} & for\ \alpha=\beta>\frac{1}{2}\,, 2c+d=0\,, \\ \frac{3}{2}+erf(d) & for\ \beta=\frac{1}{2}>\alpha\,, \\ \frac{3}{2}+erf(2c) & for\ \alpha=\frac{1}{2}>\beta\,, \\ \frac{3}{2}+erf(2c+d) & for\ \alpha=\beta=\frac{1}{2}\,, \\ \frac{3}{2}+erf(2c) & for\ \alpha>\beta=\frac{1}{2}\,, c=0\,, \\ \frac{3}{2}+erf(d) & for\ \beta>\alpha=\frac{1}{2}\,, d=0\,, \end{cases}$$

For $2a+b=1, b>0, a+2b<1,$ *we have*

$$X(a,b,c,d,\alpha,\beta)=\begin{cases} 1 & \mathit{for}\ \alpha>\beta, \alpha>\frac{1}{2}, c>0\,, \\ 1 & \mathit{for}\ \alpha>\beta>\frac{1}{2}, c=0, d>0\,, \\ 1 & \mathit{for}\ \alpha=\beta>\frac{1}{2}, 2c+d>0\,, \\ 1 & \mathit{for}\ \beta>\alpha>\frac{1}{2}, d=0, c>0\,, \\ 1 & \mathit{for}\ \beta>\alpha, \beta>\frac{1}{2}, d>0\,, \\ \frac{1}{2} & \mathit{for}\ \frac{1}{2}>\alpha,\beta\,, \\ \frac{1}{2} & \mathit{for}\ \alpha>\frac{1}{2}>\beta\,, c=0\,, \\ \frac{1}{2} & \mathit{for}\ \beta>\frac{1}{2}>\alpha\,, d=0\,, \\ \frac{1}{2} & \mathit{for}\ \alpha>\beta>\frac{1}{2}\,, c=d=0\,, \\ \frac{1}{2} & \mathit{for}\ \beta>\alpha>\frac{1}{2}\,, c=d=0\,, \\ \frac{1}{2} & \mathit{for}\ \alpha=\beta>\frac{1}{2}\,, 2c+d=0\,, \\ \frac{1}{2}+erf(d) & \mathit{for}\ \beta=\frac{1}{2}>\alpha\,, \\ \frac{1}{2}+erf(2c) & \mathit{for}\ \alpha=\frac{1}{2}>\beta\,, \\ \frac{1}{2}+erf(2c+d) & \mathit{for}\ \alpha=\beta=\frac{1}{2}\,, \\ \frac{1}{2}+erf(2c) & \mathit{for}\ \alpha>\beta=\frac{1}{2}\,, c=0\,, \\ \frac{1}{2}+erf(d) & \mathit{for}\ \beta>\alpha=\frac{1}{2}\,, d=0\,, \end{cases}$$

8.2 Figures and numerical calculations

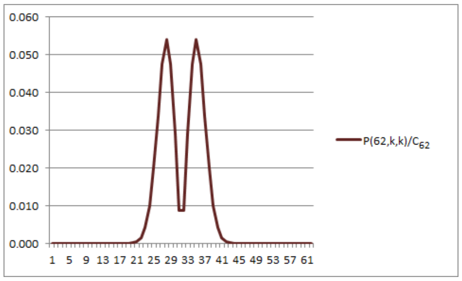


Figure 8.1: Values of $P_{62}(k,k)$.

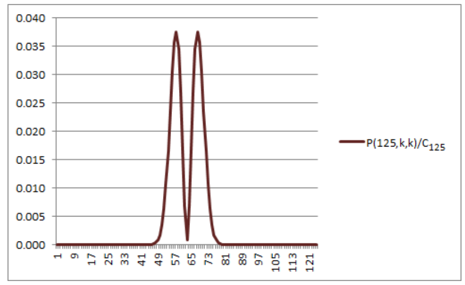


Figure 8.2: Values of $P_{125}(k,k)$.

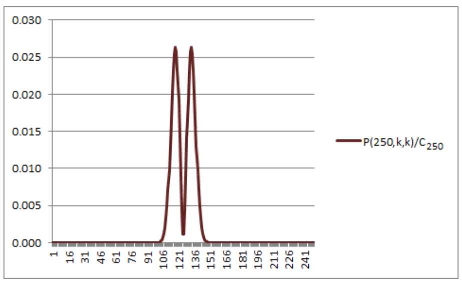


Figure 8.3: Values of $P_{250}(k,k)$.

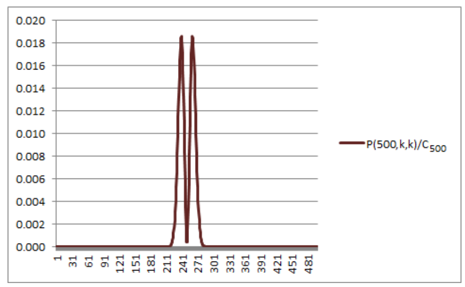


Figure 8.4: Values of $P_{500}(k,k)$.

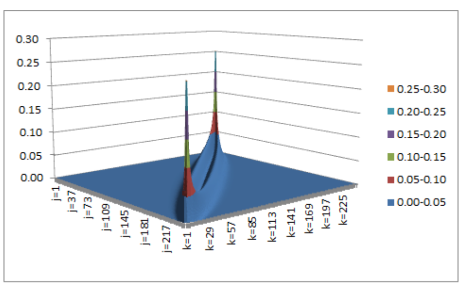


Figure 8.5: Surface of $P_{250}(j,k)$.

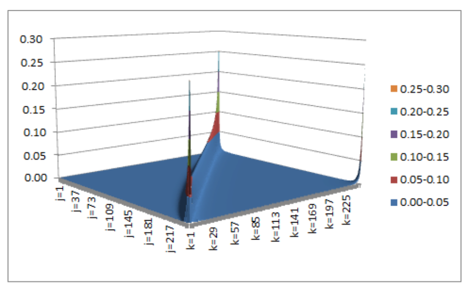


Figure 8.6: Surface of $Q_{250}(j,k)$.

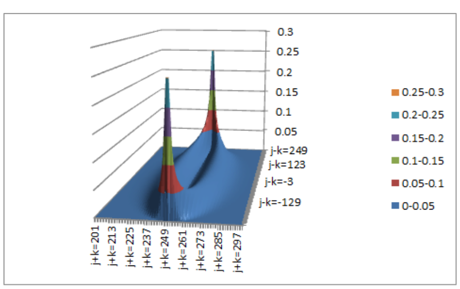


Figure 8.7: A closer look at $P_{250}(j,k)$, $201 \leq j+k \leq 301$.

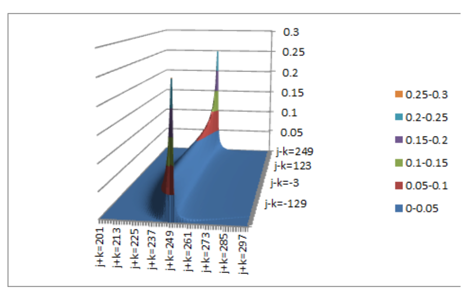


Figure 8.8: A closer look at $Q_{250}(j,k)$, $201 \leq j+k \leq 301$.

References

[AS] M. Abramowitz and I. Stegun, *Handbook of mathematical functions*, Dover, New York, 1965.

[A1] D. Aldous, The continuum random tree. I, *Ann. Probab.* **19** (1991), 1–28.

[A2] D. Aldous, Recursive self-similarity for random trees, random triangulations, and Brownian excursion, *Ann. Probab.* **22** (1994), 527–545.

[A3] D. Aldous, Triangulating the circle, at random, *Amer. Math. Monthly* **101** (1994), 223–233.

[AD] D. Aldous and P. Diaconis, Longest increasing subsequences: from patience sorting to the Baik-Deift-Johansson theorem, *Bull. AMS* **36** (1999), 413–432.

[And] G. Andrews, Stacked lattice boxes, *Ann. Comb.* **3** (1999), 115–130.

[AM] M. Atapour and N. Madras, Large Deviations and Ratio Limit Theorems for Pattern-Avoiding Permutations, unpublished preprint.

[BDJ] J. Baik, P. Deift and K. Johansson, On the distribution of the length of the longest increasing subsequence of random permutations, *J. AMS* **12** (1999), 1119–1178.

[Be] E. T. Bell, The form $wx + xy + yz + zu$, *Bull. Amer. Math. Soc.* **42** (1936), 377-380.

[BPS] N. Bernasconi, K. Panagiotou and A. Steger, On properties of random dissections and triangulations, *Combinatorica* **30** (2010), 627–654.

[Bev] D. Bevan, Permutations avoiding 1324 and patterns in Lukasiewicz paths, arXiv:1406.2890 (2014).

[BJS] S. Billey, W. Jockusch and R. Stanley, Some combinatorial properties of Schubert polynomials, *J. Alg. Combin.* **2.4** (1993), 345–374.

[B4] M. Bóna, Exact enumeration of 1342-avoiding permutations, a close link with labeled trees and planar maps, *J. Combin. Ser. A* **80** (1997), 257–272.

[B5] M. Bóna, A new record for 1324-avoiding permutations, *Eur. J. Math.* **1** (2015), 198–206.

[B1] M. Bóna, Permutations avoiding certain patterns: the case of length 4 and some generalizations, *Discrete Math.* **175** (1997), 55–67.

[B2] M. Bóna, *Combinatorics of Permutations*, CRC Press, Boca Raton, FL, 2012.

[B3] M. Bóna, Surprising symmetries in objects counted by Catalan numbers, *Electron. J. Combin.* **19** (2012), no. 1, Paper 62, 11 pp.

[Chu] K. L. Chung, Excursions in Brownian motion, *Ark. Mat.* **14** (1976), 155–177.

[CK] A. Claesson and S. Kitaev, Classification of bijections between 321-and 132-avoiding permutations, *Sém. Loth. Comb.* **60** (2008), Art. B60d.

[CG] A. R. Conway and A. J. Guttmann, On the growth rate of 1324-avoiding permutations, *Adv. Appl. Math* **64** (2015), 50–69.

[CS] R. Cori and G. Schaeffer, Description trees and Tutte formulas, *Theor. Comp. Sci.* **292** (2003), 165–183.

[DHW] E. Deutsch, A. Hildebrand and H. Wilf, Longest increasing subsequences in pattern-restricted permutations, *Electron. J. Combin.* **9** (2002), R 12, 8 pp.

[Dev] L. Devroye, Branching processes in the analysis of the heights of trees, *Acta Inform.* **24** (1987), 277–298.

[DFHNS] L. Devroye, P. Flajolet, F. Hurtado, M. Noy and W. Steiger, Properties of random triangulations and trees, *Discrete Comput. Geom.* **22** (1999), 105–117.

[DP] T. Dokos and I. Pak, The expected shape of random doubly alternating Baxter permutations, *Onl. J. Anal. Combin.* **9** (2014), 12 pp.

[Drm] M. Drmota, *Random Trees*, Springer, Vienna, 2009.

[DG] M. Drmota and B. Gittenberger, On the profile of random trees, *Random Struct. Alg.* **10** (1997), 421–451.

[DI] R. T. Durrett and D. L. Iglehart, Functionals of Brownian meander and Brownian excursion, *Ann. Probab.* **5** (1977), 130-135.

[E1] S. Elizalde, *Statistics on pattern-avoiding permutations*, Ph.D. Thesis, MIT, 2004, 116 pp.

[E2] S. Elizalde, Fixed points and excedances in restricted permutations, *Electron. J. Combin.* **18** (2011), Paper 29, 17 pp.

[EP] S. Elizalde and I. Pak, Bijections for refined restricted permutations, *J. Combin. Theory, Ser. A* **105** (2004), 207–219.

[ES] P. Erdős and Gy. Szekeres, A combinatorial problem in geometry, *Compositio Math.* **2** (1935), 463–470.

[Eul] L. Euler, Summary of "Enumeratio modorum quibus figurae planae..., by I. A. de Segner", *Novi Commentarii Academiae Scientiarum Imperialis Petropolitanae* **7** (1758/59), 13–15; English translation available at http://tinyurl.com/bhvsht2

[FO] P. Flajolet and A. Odlyzko, The average height of binary trees and other simple trees, *J. Comput. System Sci.* **25** (1982), 171–213.

[FS] P. Flajolet and R. Sedgewick, *Analytic Combinatorics*, Cambridge Univ. Press, Cambridge, 2009.

[GW] Z. Gao and N. C. Wormald, The distribution of the maximum vertex degree in random planar maps, *J. Combin. Theory, Ser. A* **89** (2000), 201–230.

[Gar] M. Gardner, Mathematical Games, Catalan numbers: an integer sequence that materializes in unexpected places, *Scientific Amer.* **234**, No. 6 (June, 1976), 120–125, 132.

[Ges] I. Gessel, Symmetric functions and P-recursiveness, *J. Combin. Theory Ser. A* **53** (1990), 257–285.

[Gou] H. W. Gould, *Bell and Catalan Numbers: A Research Bibliography of Two Special Number Sequences* (Revised 2007); available at http://www.math.wvu.edu/~gould/BibGood%20final.pdf

[GP] W. Gutjahr and G. Pflug, The asymptotic distribution of leaf heights in binary trees, *Graphs and Combin.* **8** (1992), 243–251.

[HRS] C. Hoffman, D. Rizzolo and E. Slivken, Pattern avoiding permutations and Brownian excursion, arXiv preprint arXiv:1406.5156 (2014).

[Hom] C. Homberger, Expected patterns in permutation classes, *Electron. J. Combin.* **19** (2012), no. 3, Paper 43, 12 pp.

[KP] M. Kauers and P. Paule, *The concrete tetrahedron*, Springer, Vienna, 2011.

[Kim] H. S. Kim, Interview with Professor Stanley, *MIT Math Majors Magazine* **1**, No. 1 (2008), 28–33.

[Kit] S. Kitaev, *Patterns in Permutations and Words*, Springer, Berlin, 2011.

[KMV] S. Kitaev, T. Mansour and A. Vella, Pattern avoidance in matrices, *J. Integer Seq.* **8** (2005), no. 2, Art. 05.2.2, 16 pp.

[Knu] D. Knuth, *The Art of Computer Programming*, vol. 1, Addison-Wesley, Reading, MA, 1968.

[Kra] C. Krattenthaler, Permutations with restricted patterns and Dyck paths, *Adv. Appl. Math.* **27** (2001), 510–530.

[LM] J. B. Lewis and A. H. Morales, Combinatorics of diagrams of permutations, arXiv:1405.1608 (2014).

[Lou] J. Liouville, $x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5$, *Comptes Rendus* **62** (1866), 714.

[Mac] P. A. MacMahon, *Combinatory Analysis*, vol. 1, Cambridge Univ. Press, Cambridge, 1915.

[ML] N. Madras and H. Liu, Random Pattern-Avoiding Permutations, in *Algorithmic Probability and Combinatorics*, AMS, Providence, RI, 2010, 173–194.

[MP2] N. Madras and L. Pehlivan, Structure of random 312-avoiding permutations, arXiv preprint arXiv:1401.6230, 2014.

[MV1] T. Mansour and A. Vainshtein, Restricted 132-avoiding permutations, *Adv. Appl. Math* **26** (2001), 258–269.

[MV2] T. Mansour and A. Vainshtein, Counting occurences of 132 in a permutation, *Adv. Appl. Math* **28** (2002), 185–195.

[MP] S. Miner and I. Pak, The shape of random pattern-avoiding permutations, *Adv. Appl. Math* **55** (2014), 86–130.

[Ort] J. Ortmann, Large deviations for non-crossing partitions, *Electron. J. Probab.* **17** (2012), 1–25.

[P1] I. Pak, The nature of partition bijections II. Asymptotic stability, preprint, 32 pp.; available at http://www.math.ucla.edu/~pak/papers/stab5.pdf

[P2] I. Pak, Catalan Numbers Page, http://www.math.ucla.edu/~pak/lectures/Cat/pakcat.htm

[P3] I. Pak, History of Catalan numbers, preprint, 10 pp.; to appear as an appendix to Richard Stanley's book *Catalan Numbers*.

[Pep] T. Pepin, Sur quelques formules d'Analyse utiles dans la Théorie des nombres, *J. de Math. Pures et Appl.* **4** (1888), 83–127.

[Pit] J. Pitman, Combinatorial stochastic processes, *Lecture Notes in Mathematics* **1875**, Springer, Berlin, 2006, 256 pp.

[Rob] A. Robertson, Restricted permutations from Catalan to Fine and back, *Sém. Loth. Comb.* **50** (2004), Art. B50g.

[Rus] F. Ruskey, On the average shape of binary trees, *SIAM J. Alg. Disc. Meth.* **1** (1980), 43–50.

[RSZ] A. Robertson, D. Saracino and D. Zeilberger, Refined restricted permutations, *Ann. Comb.* **6** (2002), 427–444.

[Rota] G.-C. Rota, Ten lessons I wish I had been taught, *Notices AMS* **44** (1997), 22–25.

[SS] R. Simion and F. Schmidt, Restricted permutations, *Eur. J. Comb.* **6** (1985), 383–406.

[Spi] A. Spiridonov, *Pattern-avoidance in binary fillings of grid shapes*, Ph.D. Thesis, MIT, 2009, 88 pp.

[Slo] N. J. A. Sloane, A000108, in *Online Encyclopedia of Integer Sequences*, http://oeis.org/A000108

[S1] R. Stanley, *Enumerative Combinatorics*, vols. 1 and 2, Cambridge Univ. Press, Cambridge, 2011.

[S2] R. Stanley, Catalan addendum, available at http://www-math.mit.edu/~rstan/ec/catadd.pdf

[Tak] L. Takács, A Bernoulli excursion and its various applications, *Adv. Appl. Probab.* **23** (1991), 557–585.

[Tut] W. T. Tutte, A census of planar maps, *Can. J. Math.* **33** (1963), 249–271.

[Vel] A. Vella, Pattern avoidance in permutations: linear and cyclic orders, *Electron. J. Combin.* **10** (2003), R18,

[Ver] A. M. Vershik, Statistical mechanics of combinatorial partitions, and their limit configurations, *Funct. Anal. Appl.* **30** (1996), 90–105.

[VK] A. Vershik and S. Kerov, Asymptotics of the Plancherel measure of the symmetric group and the limiting form of Young tables, *Soviet Math. Dokl.* **18** (1977), 527–531.

[West] J. West, Generating trees and the Catalan and Schröder numbers, *Discrete Math.* **146** (1995), 247–262.