The Discrete Square Peg Problem

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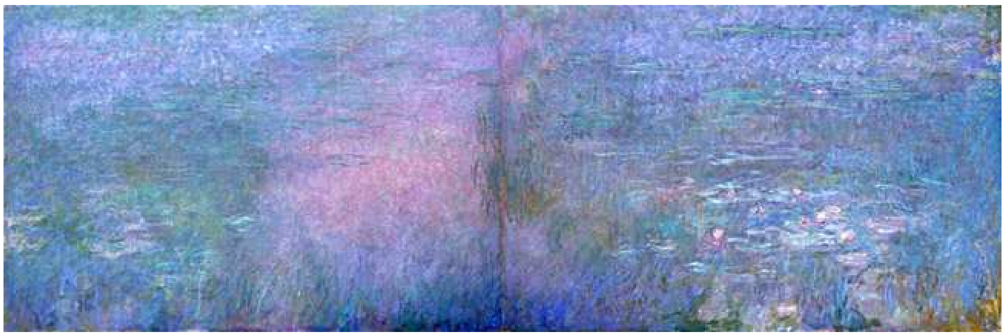
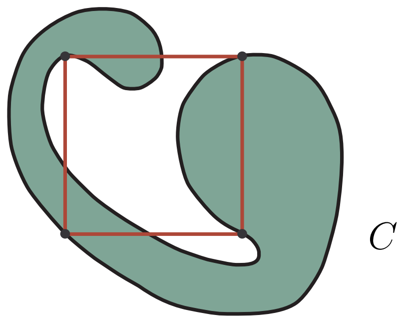


Conjecture (Square Peg Problem)
Every Jordan curve $C \subset \mathbb{R}^2$ has four points which form a square.



History outline

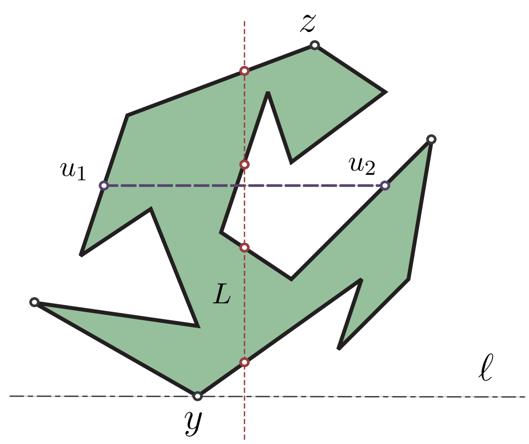
1. O. Toeplitz, 1911 (convex case, claimed)
2. A. Emch, 1913 (convex case, proved)
3. L. G. Shnirelman, 1929, 1944 (continuous curvature, incomplete)
4. C. S. Ogilvy, 1959 (most general, totally wrong)
5. R. P. Jerrard, 1960 (analytic)
6. H. Guggenheimer, 1965 (Shnirelman corrected, +bounded var.)

History outline (continued)

7. R. Fenn, 1970 (the table theorem)
8. E. H. and P. B. Kronheimer, 1981 (more table theorem)
9. W. Stromquist, 1989 (local monotone)
10. H. B. Griffiths, 1991 (C^1 curves)
11. M. J. Nielsen, 1995 (rectangles in Jordan curves)
12. V. V. Makeev, 2005 (rhombi in space curves, sort of)

Ogilvy's "proof" (find three mistakes even in the PL case)

1) Fix line ℓ. Move $(u_1u_2)\|\ell$ from y to z.
2) Take intersections of $L\perp\ell$ through midpoint of $[u_1, u_2]$.
   Do this until you have an inscribed rhombus.
3) Rotate ℓ continuously from 0 to $\pi/2$. Wait for a square.



Three mistakes

1) the “meandering” issue

2) the “connectivity” issue

3) the “uniqueness” issue

Theorem (Emch, 1913)
Every convex curve has an inscribed square.

Idea: the above scheme works for convex curves.
The uniqueness follows by elementary arguments.

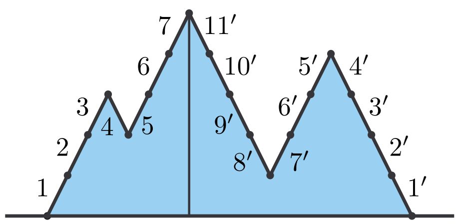
Theorem (Nielsen, 1995)
Every Jordan curve in the plane has an inscribed rhombus with a diagonal $\| \ell$.

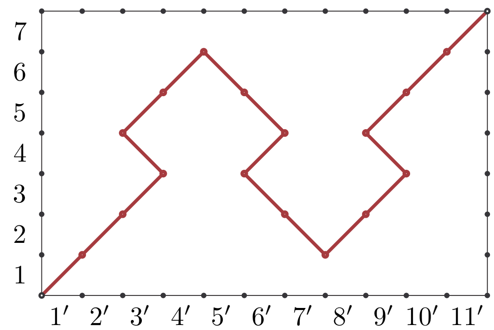
Idea: use the *mountain climbing lemma*.

Mountain climbing lemma

Let $f_1, f_2 : [0,1] \to [0,1]$ be two continuous piecewise linear functions with $f_1(0) = f_2(0) = 0$ and $f_1(1) = f_2(1) = 1$. Then there exist two continuous piecewise linear functions $g_1, g_2 : [0,1] \to [0,1]$, such that $g_1(0) = g_2(0) = 0$, $g_1(1) = g_2(1) = 1$, and

$$f_1(g_1(t)) = f_2(g_2(t)) \quad \text{for every} \quad t \in [0,1].$$





Main theorem

Every simple polygon in the plane has an inscribed square.

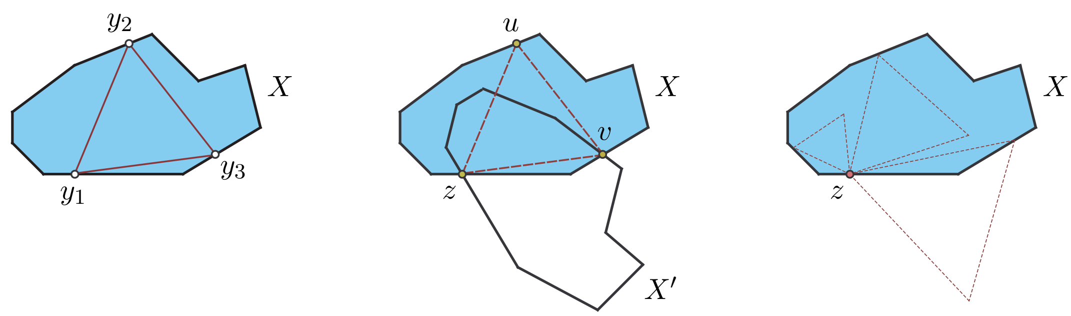
We give two new proofs of this result. Some ideas are classical. Most details are new.

(1) Proof via inscribed triangles (based on Jerrard's approach)

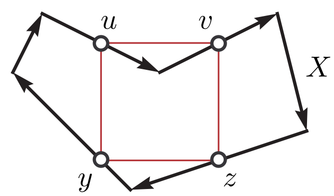
(2) Proof by deformation (based on Shnirelman's approach)

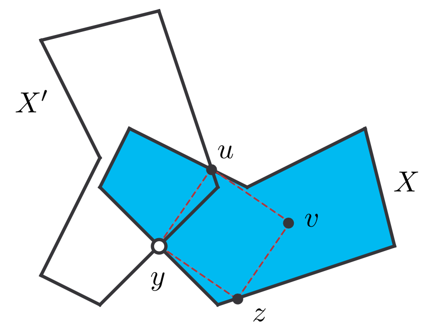
Inscribed triangles

Theorem *For every simple polygon $X \subset \mathbb{R}^2$ and a point z in the interior of an edge in X, there exists an equilateral triangle inscribed into X with z as a vertex.*



Proof via inscribed squares





Proof steps

0) It suffices to proves the result for *generic polygons* with angles between $\pi/2$ and $3\pi/2$. Use the limit argument.

1) Let $U = \{(y, z) \subset C^2$ such that $u \in C\}$.
   Let $V = \{(y, z) \subset C^2$ such that $v \in C\}$.
   We need to prove that $U \cap V \neq \varnothing$.

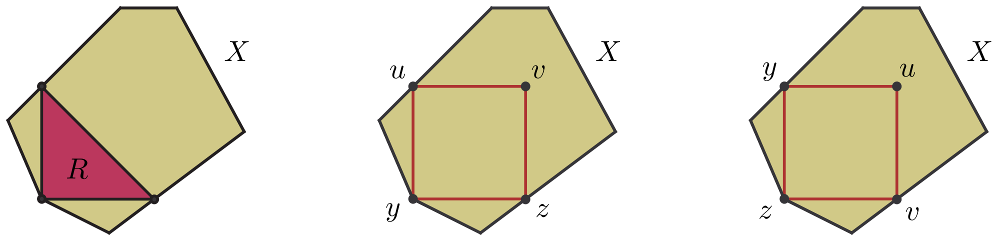
2) Check that sets U and V are disjoint unions of polygons.

3) Use a parity argument to find curves $U^\circ \subset U$ and $V^\circ \subset U$.

4) Describe the topology of $U^\circ, V^\circ \subset T$ when $U^\circ \cap V^\circ = \varnothing$

5) Find the smallest inscribed right isosceles triangle.
   Obtain a contradiction.

Set U on a torus T and the sequence of regions $A, B, C \subset T$.



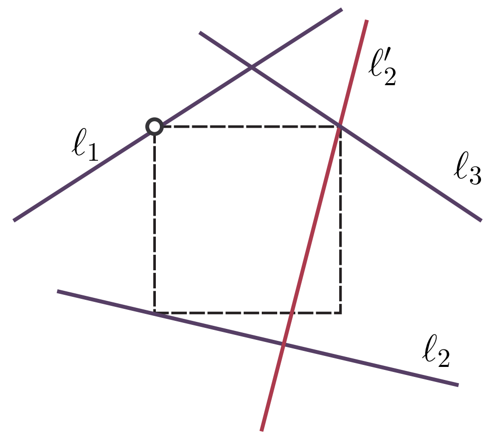
The smallest inscribed right isosceles triangle and its two labelings.

Minor extension

Theorem. *Every* generic *simple polygon has an odd number of inscribed squares.*

Lemma (Hebbert, 1914) *Let ℓ_1, ℓ_2, ℓ_3 and ℓ_4 be four lines in $\mathbb{R}^2$ in general position. Then there exists a unique square $A = [a_1 a_2 a_3 a_4]$ such that $x_i \in \ell_i$ and A is oriented clockwise. Moreover, the map $(\ell_1, \ell_2, \ell_3, \ell_4) \to (a_1, a_2, a_3, a_4)$ is continuously differentiable, where defined.*

Proof of the Lemma



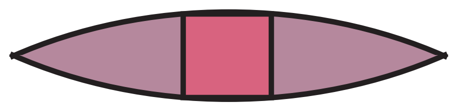
Proof by deformation

1) Subdivide the edges of a polygon so that all inscribed squares must have vertices on different edges.

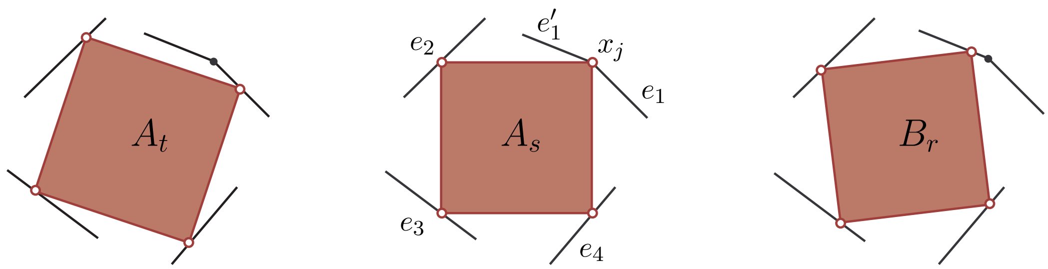
2) Show that a polygon can be deformed to the near-interval so that all intermediate polygons satisfy 1).

3) Show that the parity of the number of inscribed squares is unchanged under the deformation.

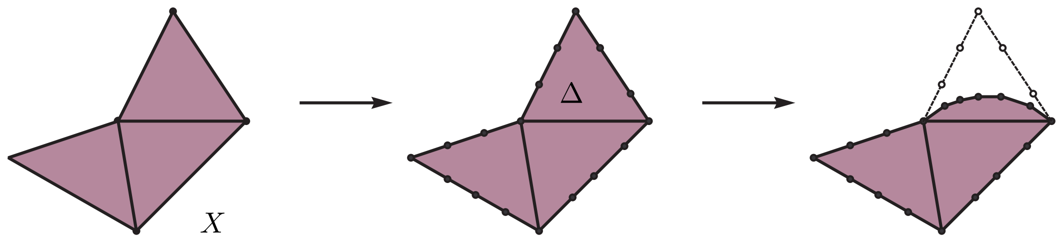
4) Check that near-interval convex polygons have a unique inscribed square.



Deformation of a square



Deformation of a polygon



Extensions and generalizations (polygons)

(1) (Kakeya, 1916) *Every convex polygon has at least two inscribed rectangles with given aspect ratio* $\neq 1$.

(2) (Griffiths, 1991) *Every simple polygon has an inscribed rectangle with given aspect ratio.*

(3) (Conjecture) *Every simple polygon has an inscribed trapezoid similar to a given isosceles trapezoid.*

Note: only isosceles trapezoids can be inscribed into both cycles and near intervals.

(4) (Stromquist, 1989) *Every space polygon has an inscribed quadrilateral with equal angles and edge lengths.*

(5) (Makeev, 2005) *Every space polygon has an inscribed flat rhombus.*

Note: this implies the square peg problem for simple polygons in the plane.

Extensions and generalizations (polyhedra)

(1) (Kakutani, 1942) *Every convex body in* $\mathbb{R}^3$ *has an outscribed cube.*

(2) (Guggenheimer, 1965) *Prove that every convex polytope* $P \subset \mathbb{R}^3$ *has an inscribed regular octahedron.*

(3) (Kramer, 1980) *Every convex polytope in* $\mathbb{R}^3$ *win general position with respect to the orthogonal axes has an inscribed equihedral octahedron with its diagonals parallel to the axes.*

(4) (Modified table theorem) *Every convex polytope in* $\mathbb{R}^3$ *symmetric with respect to a plane* H *has an inscribed cube symmetric with respect to* H.