# Kronecker coefficients: bounds and complexity

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## Basic Definitions

Let $\chi^\lambda$ denote *character* of $S_n$ associated with $\lambda \vdash n$.

*Kronecker coefficients* $g(\lambda, \mu, \nu)$ are defined by:

$$\chi^\lambda \cdot \chi^\mu \;=\; \sum_{\nu \vdash n} g(\lambda, \mu, \nu)\, \chi^\nu \;, \quad \text{where} \quad \lambda, \mu, \nu \vdash n.$$

$\Rightarrow \quad g(\lambda, \mu, \nu) \in \mathbb{N}$. Also:

$$g(\lambda, \mu, \nu) \;=\; \langle \chi^\lambda \cdot \chi^\mu, \chi^\nu \rangle \;=\; \frac{1}{n!} \sum_{\sigma \in S_n} \chi^\lambda(\sigma)\, \chi^\mu(\sigma)\, \chi^\nu(\sigma)$$

$\Rightarrow \quad g(\lambda, \mu, \nu) = g(\mu, \lambda, \nu) = g(\lambda, \nu, \mu) = \ldots \quad \leftarrow$ *symmetries*

$\Rightarrow \quad g(\lambda, \mu, \nu) = g(\lambda', \mu', \nu) = g(\lambda, \mu', \nu') = g(\lambda', \mu, \nu') \quad \leftarrow$ *conjugations*

**Example:** $n = 3$, partitions $\{3, 21, 1^3 \vdash n\}$

Characters: $\chi^{(3)} = (1, 1, 1)$, $\chi^{(21)} = (2, 0, 1)$, $\chi^{(1^3)} = (1, -1, 1)$

$$\chi^{(21)} \cdot \chi^{(21)} \;=\; (4, 0, 1) \;=\; \chi^{(3)} + \chi^{(21)} + \chi^{(1^3)} \quad \Longrightarrow \quad g(21, 21, 21) \,=\, 1$$

# Main Problems

(1) ***Compute*:** $g(\lambda, \mu, \nu)$ $\leftarrow$ find formulas, complexity aspects

(2) ***Decide*:** $g(\lambda, \mu, \nu) >^? 0$ $\leftarrow$ *vanishing problem*

(3) ***Estimate*:** $g(\lambda, \mu, \nu)$ $\leftarrow$ even in some special cases

(4) ***Give*:** *combinatorial interpretation* for $g(\lambda, \mu, \nu)$ $\leftarrow$ classical open problem

## History:

- [Murnaghan, 1937], [Murnaghan, 1956] $\leftarrow$ definition, stability, generalizations of LR–coefficients
- [Mulmuley, 2011] $\leftarrow$ connections to the *Geometric Complexity Theory*

## Complexity of Computing

$\lambda = (\lambda_1, \ldots, \lambda_\ell) \vdash n$ given in *binary* $\to$ size $\phi(\lambda) := \log_2(\lambda_1) + \ldots + \log_2(\lambda_\ell)$

*unary* $\to$ size $\phi(\lambda) := n$.

KRON $\leftarrow$ the problem of computing $g(\lambda, \mu, \nu)$

LR $\leftarrow$ the problem of computing $c^{\lambda}_{\mu\nu}$

**Theorem** [binary ← Narayanan'06] ⇐ [unary ← P.–Panova'20+]

LR is #P-complete.

**Theorem** [binary ← Bürgisser–Ikenmeyer'08] ⇐ [unary ← Ikenmeyer–Mulmuley–Walter'17]

KRON is #P-hard.

**Theorem** [Christandl–Doran–Walter'12], [P.–Panova'17]:

Let $\ell = \ell(\lambda)$, $m = \ell(\mu)$, $r = \ell(\nu)$. Then: KRON $\in$ FP for $\ell, m, r = O(1)$.

[unary ← easy, binary ← *Barvinok's Algorithm* to #CT's]

# Complexity Classes

**Major Open Problem:** $\textsc{Kron} \in \#\mathsf{P}$ ($\exists$ combinatorial interpretation)

**Theorem** [Bürgisser–Ikenmeyer, 2008]:

$\textsc{Kron} \in \mathsf{GapP} := \#\mathsf{P} - \#\mathsf{P}$ (both binary and unary)

$$g(\lambda,\mu,\nu) \ = \sum_{\omega\in S_\ell}\sum_{\pi\in S_m}\sum_{\tau\in S_r} \operatorname{sign}(\omega\pi\tau)\cdot \mathrm{CT}\big(\lambda+1_\ell-\omega,\, \mu+1_m-\pi,\, \lambda+1_r-\tau\big)$$

where $\mathrm{CT}(\alpha,\beta,\gamma) \ = \ \#\big[\text{3-dim contingency tables with marginals } \alpha,\beta,\gamma\big]$.

**For comparison:**

$\mathrm{LR} \in \#\mathsf{P}$ unary $\leftarrow$ LR–rule, binary $\leftarrow$ GT–patterns

$\textsc{Schubert} \in \mathsf{GapP}_{\geq 0}$ $\leftarrow$ [Postnikov–Stanley'09]

$\big[\chi^\lambda(\mu)\big]^2 \in \mathsf{GapP}_{\geq 0}$ $\leftarrow$ Murnaghan–Nakayama rule (unary only)

$\sum_{\mu\vdash n} \chi^\lambda(\mu) \in \mathsf{GapP}_{\geq 0}$ $\leftarrow$ self-adjoint multiplicities

# Easy Bounds

**Proposition 1.** $g(\lambda,\mu,\nu) \le \min\{f^\lambda, f^\mu, f^\nu\}$, where $f^\lambda := \chi^\lambda(1)$.

$$g(\lambda,\mu,\nu) \le \frac{f^\lambda f^\mu}{f^\nu} \le f^\lambda, \quad \text{for all } f^\lambda \le f^\mu \le f^\nu$$

**Proposition 2.** $g(\lambda,\mu,\nu) \le \mathrm{CT}(\lambda,\mu,\nu)$

$$\mathrm{CT}(\alpha,\beta,\gamma) = \sum_{\lambda,\mu,\nu\vdash n} g(\lambda,\mu,\nu)\cdot K_{\lambda\alpha} K_{\mu\beta} K_{\nu\gamma}$$

where $K_{\lambda\alpha}$ is the *Kostka number* = #SSYT of shape $\lambda$ and weight $\alpha$

**Proposition 2′.** [Vallejo'00]

$g(\lambda,\mu,\nu) \le \mathrm{BCT}(\lambda',\mu',\nu')$ $\leftarrow$ 0/1 contingency tables

Used by [Ikenmeyer–Mulmuley–Walter'17] via matching lower bound in some cases.

# More Bounds

**Theorem** [P.-Panova, 2020] Let $\ell(\lambda) = \ell$, $\ell(\mu) = m$, and $\ell(\nu) = r$. Then:

$$g(\lambda,\mu,\nu) \;\le\; \left(1+\frac{\ell m r}{n}\right)^{n}\left(1+\frac{n}{\ell m r}\right)^{\ell m r}$$

Uses Prop. 2, [Barvinok'09] and *majorization* over reals.

**Example:** $\lambda = \mu = \nu = (\ell^2)^\ell$. Then Prop. 1 gives $g(\lambda,\mu,\nu) \le f^\lambda = \sqrt[3]{n!}$.
Thm gives $g(\lambda,\mu,\nu) \le 4^n$. We conjecture: $g(\lambda,\mu,\nu) = 4^{n-o(n)}$.

**Theorem:**

(1) $$\max_{\lambda,\mu,\nu\vdash n} g(\lambda,\mu,\nu) = \sqrt{n!}\, e^{-O(\sqrt{n})}$$ [Stanley'16]

(2) $$\max_{\nu\vdash n} g(\lambda,\mu,\nu) \ge \frac{f^\lambda f^\mu}{\sqrt{p(n)\, n!}}$$ [P.–Panova–Yeliussizov'19]

For (1), use: $$\sum_{\lambda,\mu,\nu\vdash n} g(\lambda,\mu,\nu)^2 = \sum_{\alpha\vdash n} z_\alpha \ge n!$$

# Harder Bounds

*Reduced Kronecker coefficients:* $\overline{g}(\alpha,\beta,\gamma) := \lim_{n\to\infty} g\big(\alpha[n],\beta[n],\gamma[n]\big)$,
where $\alpha[n] := (n-|\alpha|,\alpha_1,\alpha_2,\ldots)$, and $n \geq |\alpha|+\alpha_1$

**Theorem** [P.–Panova'20]

$$\max_{|\alpha|+|\beta|+|\gamma|\leq 3n} \overline{g}(\alpha,\beta,\gamma) \;=\; \sqrt{n!}\, e^{O(n)}$$

The proof is based on the following identity in [Bowman – De Visscher – Orellana, 2015]

$$\overline{g}(\alpha,\beta,\gamma) \;=\; \sum_{m=0}^{\lfloor k/2\rfloor} \sum_{\pi\vdash q+m-b} \sum_{\rho\vdash q+m-a} \sum_{\sigma\vdash m} \sum_{\lambda,\mu,\nu\vdash k-2m} c^{\alpha}_{\nu\pi\rho}\, c^{\beta}_{\mu\pi\sigma}\, c^{\gamma}_{\lambda\rho\sigma}\; g(\lambda,\mu,\nu)$$

where $a=|\alpha|$, $b=|\beta|$, $q=|\gamma|$, $k=a+b-q$, and

$$c^{\lambda}_{\alpha\beta\gamma} \;=\; \sum_{\tau} c^{\lambda}_{\alpha\tau}\, c^{\tau}_{\beta\gamma}.$$

combined with bounds in [P.–Panova–Yeliussizov'19].

# Conjectural Bounds

*Staircase shape* $\delta_k := (k-1, k-2, \ldots, 2, 1) \vdash n = \binom{k}{2}$

**Conjecture:** $g(\delta_k, \delta_k, \delta_k) = \sqrt{n!}\, e^{-O(n)}$

**Theorem** [Bessenrodt–Behns'04]: $g(\delta_k, \delta_k, \delta_k) \geq 1$

**Theorem** [P.–Panova'20+]: $g(\delta_k, \delta_k, \delta_k, \delta_k) = n!\, e^{-O(n)}$,

where $g(\lambda, \mu, \nu, \tau) := \langle \chi^\lambda \chi^\mu \chi^\nu \chi^\tau, 1 \rangle$.

**Saxl Conjecture:** $g(\delta_k, \delta_k, \mu) > 0$ for all $\mu \vdash \binom{k}{2}$.

Remains open. Known for:

[Ikenmeyer'15], [P.–Panova–Vallejo'16] $\leftarrow$ various families of $\mu$

[Luo–Sellke'17] $\leftarrow$ random $\mu \vdash \binom{k}{2}$

[Bessenrodt–Bowman–Sutton] $\leftarrow$ $\mu \vdash \binom{k}{2}$ s.t. $f^\mu$ is odd

# Explicit Constructions

## Open Problem:

Give an explicit construction of $\lambda, \mu, \nu \vdash n$, s.t. $g(\lambda, \mu, \nu) = \sqrt{n!}\, e^{-O(n)}$.

Here an *explicit construction* means $\lambda, \mu, \nu \vdash n$ can be computed in polynomial time.

**Note 1.** Similar "derandomization problems" are classical, e.g. to find explicit construction of *Ramsey graphs*.

**Note 2.** It follows from [PPY'19] that one can take $\lambda, \mu, \nu \vdash n$ to have *VKLS shape* so that $g(\lambda, \mu, \nu) = \sqrt{n!}\, e^{-O(\sqrt{n})}$. This is NOT an explicit construction.

**Theorem** [P.–Panova'20]:

There is an explicit construction of $\lambda, \mu, \nu \vdash n$, s.t. $g(\lambda, \mu, \nu) = e^{\Omega(n^{2/3})}$.

**Proof idea:** Use $\lambda = \mu = \nu := \left(\binom{k}{2}, \binom{k-1}{2}, \ldots, \binom{2}{2}\right)$, and Prop. $2'$.

**Theorem** [P.–Panova'20+]: $g(k^k, k^k, k^k) = e^{\Omega(n^{1/4})}$.

The proof used [P.–Panova'17] and the *semigroup property*.

# Thank you!