Can irrational tilings give Catalan numbers?

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(joint work with Scott Garrabrant)

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Biographical tidbit

My best investment ever: 2.38 roubles $\Longrightarrow$ [new world].

Extraterrestrial research: a modest proposal

Carl Sagan: We should communicate with aliens using prime numbers.

SETI: Systematically sends prime number sequence to outer space.

My proposal: Start sending Catalan numbers!

Hey, You Never Know!

Difficult Question:

Can there be a world without Catalan numbers?

In other words, maybe there is a model of computation which is powerful enough, yet is unable to count any Catalan objects?

Answer: Maybe! Consider the world of 1-dimensional irrational tilings!

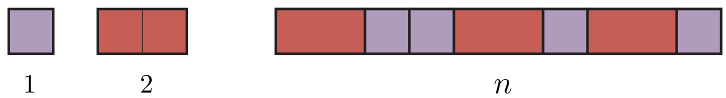
Tilings of $[1 \times n]$ rectangles

Fix a finite set $T = \{\tau_1, \ldots, \tau_k\}$ of *rational* tiles of height 1.

Let $a_n = a_n(T)$ the number of tilings of $[1 \times n]$ with T.

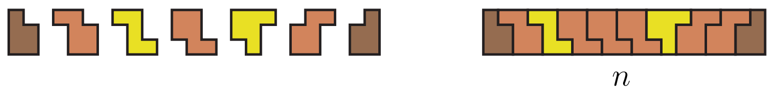
Transfer-matrix Method: $A_T(t) = \sum_n a_n t^n = P(t)/Q(t)$, where $P, Q \in \mathbb{Z}[t]$.

Therefore, *NO* Catalan numbers!



$a_n = F_n$

$A(t) = \frac{1}{1-t-t^2}$



$a_n = \binom{n-2}{2}$

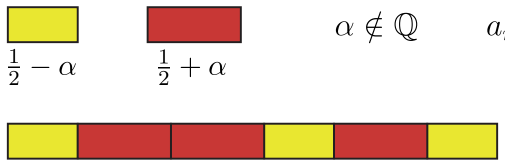
$A(t) = \frac{t^4}{(1-t)^3}$

Irrational Tilings of $[1 \times (n+\varepsilon)]$ rectangles

Fix $\varepsilon > 0$ and a finite set $T = \{\tau_1, \ldots, \tau_k\}$ of *irrational tiles* of height 1.

Let $a_n = a_n(T, \varepsilon)$ the number of tilings of $[1 \times (n+\varepsilon)]$ with T.

Observe: we can get *algebraic* g.f.'s $A_T(t)$.



$\frac{1}{2} - \alpha$ $\quad$ $\frac{1}{2} + \alpha$ $\quad$ $\alpha \notin \mathbb{Q}$ $\quad$ $a_n = \binom{2n}{n}$ $\quad$ $A(t) = \frac{1}{\sqrt{1-4t}}$

$[1 \times n]$

Main Conjecture:

Let $\mathcal{F}$ denote the class of g.f. $A_T(t)$ enumerating irrational tilings.

Then:

$$C(t) \notin \mathcal{F}\,, \qquad \textit{where} \qquad C(t) = \frac{1-\sqrt{1-4t}}{2t}\,.$$

In other words, there is no set T of irrational tiles and $\varepsilon > 0$, s.t.

$$a_n(T,\varepsilon) \,=\, C_n \quad \text{for all} \;\; n \geq 1, \quad \text{where} \quad C_n \,=\, \frac{1}{n+1}\binom{2n}{n}.$$

Diagonals of Rational Functions

Let $G \in \mathbb{Z}[[x_1,\ldots,x_k]]$. A *diagonal* is a g.f. $B(t) = \sum_n b_n t^n$, where

$$b_n = \left[x_1^n,\ldots,x_k^n\right] G(x_1,\ldots,x_k).$$

Theorem 1. *Every $A(t) \in \mathcal{F}$ is a diagonal of a rational function P/Q, for some polynomials $P, Q \in \mathbb{Z}[x_1,\ldots,x_k]$.*

For example,

$$\binom{2n}{n} = [x^n y^n] \, \frac{1}{1-x-y}\,.$$

Proof idea: Say, $\tau_i = [1 \times \alpha_i]$, $\alpha_i \in \mathbb{R}$. Let $V = \mathbb{Q}\langle \alpha_1,\ldots,\alpha_k\rangle$, $d = \dim(V)$.
We have natural maps $\varepsilon \mapsto (c_1,\ldots,c_d)$, $\alpha_i \mapsto v_i \in \mathbb{Z}^d \subset V$.
Interpret irrational tilings as walks $O \to (n+c_1,\ldots,n+c_d)$ with steps $\{v_1,\ldots,v_k\}$.

Properties of Diagonals of Rational Functions

(1) must be *D-finite*, see [Stanley, 1980], [Gessel, 1981].

(2) when $k = 2$, must be *algebraic*, and

(2′) every algebraic $B(t)$ is a diagonal of $P(x, y)/Q(x, y)$, see [Furstenberg, 1967].

No surprise now that Catalan g.f. $C(t)$, $tC(t)^2 - C(t) + 1 = 0$, is a diagonal:

$$C_n = [x^n y^n] \frac{y(1 - 2xy - 2xy^2)}{1 - x - 2xy - xy^2},$$

see e.g. [Rowland–Yassawi, 2014].

Moral: Theorem 1 is not strong enough to prove the Main Conjecture.

$\mathbb{N}$-Rational Functions $\mathcal{R}_k$

Definition: Let $\mathcal{R}_k$ be the smallest set of functions $F(x_1, \ldots, x_k)$ which satisfies

(1) $1, x_1, \ldots, x_k \in \mathcal{R}_k\,,$

(2) $F, G \in \mathcal{R}_k \implies F + G, F \cdot G \in \mathcal{R}_k\,,$

(3) $F \in \mathcal{R}_k,\ F(0) = 0 \implies 1/(1 - F) \in \mathcal{R}_k\,.$

Note that all $F \in \mathcal{R}_k$ satisfy: $F \in \mathbb{N}[[x_1, \ldots, x_k]]$, and $F = P/Q$, for some $P, Q \in \mathbb{Z}[x_1, \ldots, x_k]$.

Let $\mathcal{N}$ be a class of diagonals of $F \in \mathcal{R}_k$, for some $k \geq 1$. For example,

$$\sum_n \binom{2n}{n} t^n \in \mathcal{N} \qquad \text{because} \qquad \frac{1}{1 - x - y} \in \mathcal{R}_2\,.$$

$\mathbb{N}$-rational functions of one variable:

Word of caution: $\mathcal{R}_1$ is already quite complicated, see [Gessel, 2003].

For example, take the following $F, G \in \mathbb{N}[[t]]$:

$$F(t) \;=\; \frac{t+5t^2}{1+t-5t^2-125t^3}\,, \qquad G(t) \;=\; \frac{1+t}{1+t-2t^2-3t^3}\,.$$

Then $F \notin \mathcal{R}_1$ and $G \in \mathcal{R}_1$; neither of these are obvious.

The proof follows from results in [Berstel, 1971] and [Soittola, 1976], see also [Katayama–Okamoto–Enomoto, 1978].

Main Theorem: $\mathcal{F} = \mathcal{N}$.

In other words, every tile counting function $A_T \in \mathcal{F}$ is a diagonal of an $\mathbb{N}$-rational function $F \in \mathcal{R}_k$, $k \geq 1$, and vice versa.

Mail Lemma: Both $\mathcal{F}$ and $\mathcal{N}$ coincide with a class of g.f. $F(t) = \sum_n f(n)t^n$, where $f : \mathbb{N} \to \mathbb{N}$ is given as finite sums $f = \sum g_j$, and each g_j is of the form

$$g_j(m) \;=\; \begin{cases} \displaystyle\sum_{v \in \mathbb{Z}^{d_j}} \prod_{i=1}^{r_j} \binom{\alpha_{ij}(v,n)}{\beta_{ij}(v,n)} & \text{if} \quad m = p_j n + k_j\,, \\ \\ & 0 \;\; \text{otherwise}, \end{cases}$$

for some $\alpha_{ij} = a_{ij}v + a'_{ij}n + a''_{ij}$, $\beta_{ij} = b_{ij}v + b'_{ij}n + b''_{ij}$, and $p_j, k_j, r_j, d_j \in \mathbb{N}$.

Asymptotic applications

Corollary 2. There exist $\sum_n f_n, \sum_n g_n \in \mathcal{F}$, s.t.

$$f_n \sim \frac{\sqrt{\pi}}{\Gamma\left(\frac{5}{8}\right)\Gamma\left(\frac{7}{8}\right)}\, 128^n, \qquad g_n \sim \frac{\Gamma\left(\frac{3}{4}\right)^3}{\sqrt[3]{2}\,\pi^{5/2}}\, n^{-3/2}\, 384^n$$

Proof idea: Take

$$f_n := \sum_{k=0}^{n} 128^{n-k} \binom{4k}{k} \binom{3k}{k}.$$

Note: We have $b_n \sim \mathrm{B}\, n^{\beta} \gamma^n$, where $\beta \in \mathbb{N}$, and $\mathrm{B}, \gamma \in \mathbb{A}$, for all $\sum_n b_n t^n = P/Q$.

Conjecture 3. *For every $\sum_n f_n \in \mathcal{F}$, we have $f_n \sim \mathrm{B} n^{\beta} \gamma^n$, where $\beta \in \mathbb{Z}/2, \gamma \in \mathbb{A}$, and B is spanned by values of ${}_p\Phi_q(\cdot)$ at rational points, cf.* [Kontsevich–Zagier, 2001].

Back to Catalan numbers

Recall
$$C_n \sim \frac{1}{\sqrt{\pi}} n^{-3/2}\, 4^n.$$

Corollary 4. *There exists* $\sum_n f_n t^n \in \mathcal{F}$, *s.t.* $f_n \sim \frac{3\sqrt{3}}{\pi} C_n$. *Furthermore,* $\forall \epsilon > 0$, *there exists* $\sum_n f_n t^n \in \mathcal{F}$, *s.t.* $f_n \sim \lambda C_n$ *for some* $\lambda \in [1-\epsilon, 1+\epsilon]$.

Moral: Main Conjecture cannot be proved via rough asymptotics. However:

Conjecture 5. *There is no* $\sum_n f_n t^n \in \mathcal{F}$, *s.t.* $f_n \sim C_n$.

Note: Conj. 5 does not follow from Conj. 3; probably involves deep number theory.

Bonus applications

Proposition 6: *For every $m \geq 2$, there is* $\sum_n f_n t^n \in \mathcal{F}$, *s.t.*

$$f_n = C_n \mod m, \quad \textit{for all} \ \ n \geq 1.$$

Proposition 7: *For every prime $p \geq 2$, there is* $\sum_n g_n t^n \in \mathcal{F}$, *s.t.*

$$\operatorname{ord}_p(g_n) = \operatorname{ord}_p(C_n), \quad \textit{for all} \ \ n \geq 1,$$

where $\operatorname{ord}_p(N)$ *is the largest power of p which divides N.*

Moral: Elementary number theory doesn't help either to prove the Main Conjecture.

Note: For $\operatorname{ord}_p(C_n)$, see [Kummer, 1852], [Deutsch–Sagan, 2006].

Proof idea: Take

$$f_n = \binom{2n}{n} + (m-1)\binom{2n}{n-1}.$$

In summary:

As promised, we created a rich world of tile counting functions, which may have Catalan objects, but probably not!

Happy Birthday, Richard!