Counting linear extensions and Young tableaux

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The importance of combinatorics

> *"I will now pass on to the subject* [..] *of combinatory analysis, and my selection is influenced by the belief that this subject is not at present upon a proper footing. Its importance is not fully recognised, because much that properly belongs to it appears under other headings in all recent attempts to organize and arrange the various departments of mathematical science."*

Combinatory Analysis: A Review of the Present State of Knowledge,
Outgoing address by the President of the London Mathematical Society,
Major **Percy A. MacMahon**, R.A., F.R.S. (November 12th, 1896).

See also *What is Combinatorics?* page of quotes at: `https://tinyurl.com/combin123`

Linear extensions of finite posets

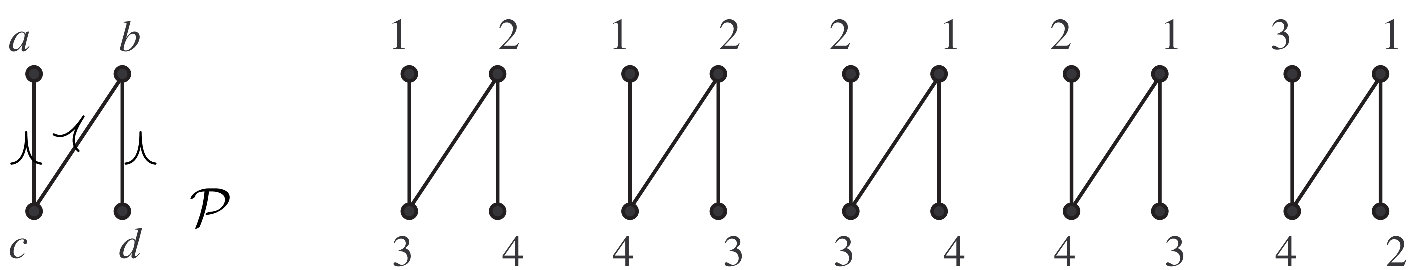
Let $\mathcal{P} = (X, \prec)$, $|X| = n$ be a *poset*.

Bijections $f : X \to \{1, \ldots, n\}$ s.t. $x \prec y \;\Rightarrow\; f(x) < f(y) \;\forall x, y \in X$

are *linear extensions* of $\mathcal{P}$. Let $\mathrm{LE}(\mathcal{P})$ be the set of LE's, and

$e(\mathcal{P}) = \#\mathrm{LE}(\mathcal{P})$ the number of LE's of $\mathcal{P}$.

Example: $X = \{a, b, c, d\}$, $n = 4$, $e(\mathcal{P}) = 5$.



Example: Let $n = a + b$, $\mathcal{P} := C_a \cup C_b$ union of two chains. Then $e(\mathcal{P}) = \binom{a+b}{a}$.

Complexity aspects

Theorem [Brightwell–Winkler'91, formerly *Linial Conjecture*'84]

$\#\mathrm{LE}(\mathcal{P})$ is $\#\mathsf{P}$-complete.

Corollary [Thm + Stanley'86]

Let $P \subset \mathbb{Q}^n$ convex polytope (given by facets). Then $\mathrm{vol}(P)$ is $\#\mathsf{P}$-hard.

Stanley's order polytope: For a poset $\mathcal{P} = (X, \prec)$ on $X = \{1, \ldots, n\}$, define $P \subset \mathbb{R}^n$ by

$$\begin{cases} 0 \leq x_i \leq 1 & \forall 1 \leq i \leq n \\ \quad x_i \leq x_j & \forall 1 \leq i, j \leq n \end{cases}$$

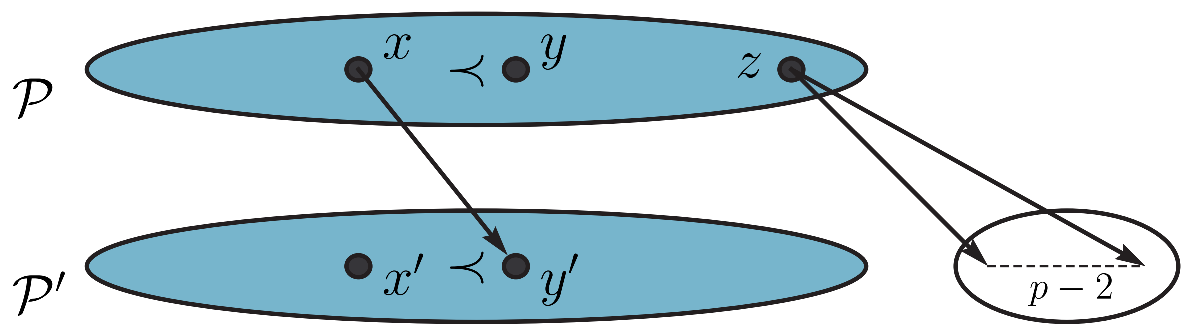
Then $\mathrm{vol}(P) = e(\mathcal{P})/n!$. E.g., when $\mathcal{P}$ is n-antichain, we have $\mathrm{vol}(P) = n!/n! = 1$.

Note: $\exists$ a Hamiltonian cycle is NP-complete, # of Hamiltonian cycles is $\#\mathsf{P}$-complete.

Theorem [Dittmer–P.'18+, formerly *Brightwell–Winkler Conjecture*'91]

#LE($\mathcal{P}$) is #P-complete for $\mathcal{P}$ of height 2.

Proof idea: Take a poset $\mathcal{P} = (X, \prec)$, $|X| = n$ and prime $p > n + 2$. Construct $\mathcal{Q}_p$ with two copies of $\mathcal{P}$ and $n(p-2)$ extra elements, see below. Check: $e(\mathcal{Q}_p) = (-1)^n e(\mathcal{P}) \bmod p$.



Note: #LE($\mathcal{P}$) has FPRAS, i.e. can be approximated up to $(1 \pm \varepsilon)$ factor in time poly$(n, 1/\varepsilon)$.

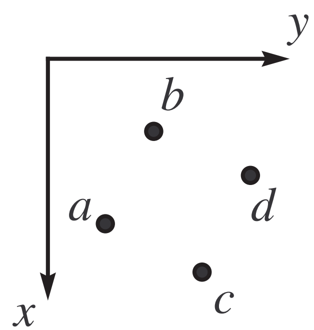
See [Dyer–Frieze–Kannan'89], [Matthews'91], [Karzanov–Khachiyan'91] and [Huber'14].

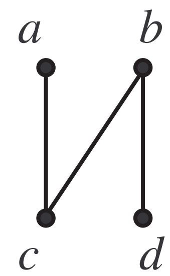
Thus, there is no hope for a parsimonious reduction from #Monotone 3SAT to #LE.

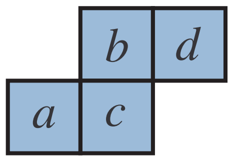
Indeed, both [BW] and [DT] use the Chinese Remainder Theorem.

Two dimensional posets

Let $X \subset \mathbb{R}^2$ with $(x, y) \preccurlyeq (x', y') \Leftrightarrow x \leq x', y \leq y'$. Such $\mathcal{P} = (X, \prec)$ are called 2-*dimensional*.







Theorem [Dittmer–P.'18+, formerly *Möhring Conjecture*'89]

$\#\mathrm{LE}(\mathcal{P})$ is $\#\mathsf{P}$-complete for <u>2-dimensional</u> $\mathcal{P}$.

Corollary [Thm + Björner–Wachs'91, + Felsner–Wernisch'97]

For $\sigma \in S_n$, let $B(\sigma) := \{\omega \preccurlyeq \sigma\} \subseteq S_n$ be the set of permutations ω obtained from σ by *bubble sorting*. E.g. $|B(231)| = 3$, $|B(321)| = 6$. Then $\#B(\sigma)$ is $\#\mathsf{P}$-complete.

Computer Assisted Proof

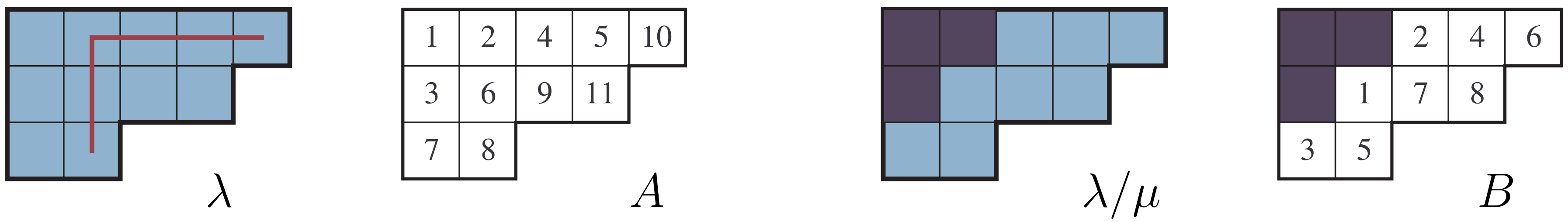
We construct permutations $\sigma \in S_n$ with a 2-diagonal block structure given by logical gates. For every prime $p > 7$ we construct 3 types of gates parameterized by $(z_1, \ldots, z_5) \in \mathbb{N}^5$, which mod p satisfy 3 systems each with 12 equations and inequalities. This is always possible because each system has a solution $(z_1, \ldots, z_5) \in \mathbb{Q}^5$ with denominators ≤ 7.

$2z_5^5 + 10z_4z_5^4 + 10z_3z_5^4 + 7z_2z_5^4 + 4z_1z_5^4 + 20z_5^4 + 20z_4^2z_5^3 + 40z_3z_4z_5^3 + 28z_2z_4z_5^3 + 16z_1z_4z_5^3 + 80z_4z_5^3 + 20z_3^2z_5^3 + 28z_2z_3z_5^3 + 16z_1z_3z_5^3 + 80z_3z_5^3 + 8z_2^2z_5^3 + 10z_1z_2z_5^3 + 50z_2z_5^3 +$
$2z_1^2z_5^3 + 26z_1z_5^3 + 70z_5^3 + 18z_4^3z_5^2 + 54z_3z_4^2z_5^2 + 39z_2z_4^2z_5^2 + 24z_1z_4^2z_5^2 + 114z_4^2z_5^2 + 54z_3^2z_4z_5^2 + 78z_2z_3z_4z_5^2 + 48z_1z_3z_4z_5^2 + 228z_3z_4z_5^2 + 24z_2^2z_4z_5^2 + 30z_1z_2z_4z_5^2 + 147z_2z_4z_5^2 +$
$6z_1^2z_4z_5^2 + 78z_1z_4z_5^2 + 206z_4z_5^2 + 18z_3^3z_5^2 + 39z_2z_3^2z_5^2 + 24z_1z_3^2z_5^2 + 114z_3^2z_5^2 + 24z_2^2z_3z_5^2 + 30z_1z_2z_3z_5^2 + 147z_2z_3z_5^2 + 6z_1^2z_3z_5^2 + 78z_1z_3z_5^2 + 206z_3z_5^2 + 3z_2^3z_5^2 + 6z_1z_2^2z_5^2 + 33z_2^2z_5^2 +$
$3z_1^2z_2z_5^2 + 39z_1z_2z_5^2 + 107z_2z_5^2 + 6z_1^2z_5^2 + 50z_1z_5^2 + 100z_5^2 + 6z_4^4z_5 + 24z_3z_4^3z_5 + 18z_2z_4^3z_5 + 12z_1z_4^3z_5 + 54z_4^3z_5 + 36z_3^2z_4^2z_5 + 54z_2z_3z_4^2z_5 + 36z_1z_3z_4^2z_5 + 162z_3z_4^2z_5 + 18z_2^2z_4^2z_5 +$
$24z_1z_2z_4^2z_5 + 111z_2z_4^2z_5 + 6z_1^2z_4^2z_5 + 66z_1z_4^2z_5 + 160z_4^2z_5 + 24z_3^3z_4z_5 + 54z_2z_3^2z_4z_5 + 36z_1z_3^2z_4z_5 + 162z_3^2z_4z_5 + 36z_2^2z_3z_4z_5 + 48z_1z_2z_3z_4z_5 + 222z_2z_3z_4z_5 + 12z_1^2z_3z_4z_5 +$
$132z_1z_3z_4z_5 + 320z_3z_4z_5 + 6z_2^3z_4z_5 + 12z_1z_2^2z_4z_5 + 60z_2^2z_4z_5 + 6z_1^2z_2z_4z_5 + 72z_1z_2z_4z_5 + 185z_2z_4z_5 + 12z_1^2z_4z_5 + 92z_1z_4z_5 + 172z_4z_5 + 6z_3^4z_5 + 18z_2z_3^3z_5 + 12z_1z_3^3z_5 + 54z_3^3z_5 +$
$18z_2^2z_3^2z_5 + 24z_1z_2z_3^2z_5 + 111z_2z_3^2z_5 + 6z_1^2z_3^2z_5 + 66z_1z_3^2z_5 + 160z_3^2z_5 + 6z_2^3z_3z_5 + 12z_1z_2^2z_3z_5 + 60z_2^2z_3z_5 + 6z_1^2z_2z_3z_5 + 72z_1z_2z_3z_5 + 185z_2z_3z_5 + 12z_1^2z_3z_5 + 92z_1z_3z_5 +$
$172z_3z_5 + 3z_2^3z_5 + 6z_1z_2^2z_5 + 25z_2^2z_5 + 3z_1^2z_2z_5 + 29z_1z_2z_5 + 64z_2z_5 + 4z_1^2z_5 + 28z_1z_5 + 48z_5 = 0$ has a solution $(z_1, z_2, z_3, z_4, z_5) = (-2, -\frac{8}{3}, \frac{5}{3}, -3, 2)$.

Note: Verifying the solution took $<$ 30 sec. in `Macaulay2` on a cheap laptop.
Of course, finding the solution required weeks of CPU time and 9 mo. of research time.

Counting standard Young tableaux

For $\lambda \vdash n$ a partition, we have $e(\lambda) = |\mathrm{SYT}(\lambda)|$. Same for $e(\lambda/\mu) = |\mathrm{SYT}(\lambda/\mu)|$.



Here $\lambda = (5,4,2) \vdash 11$, *hook length* $h(1,2) = 6$, $A \in \mathrm{SYT}(\lambda)$, $\mu = (2,1)$ and $B \in \mathrm{SYT}(\lambda/\mu)$.

Theorem [HLF by Frame–Robinson–Thrall'54; det. formula by Frobenius'1897, Feit'53]

$$\big|\mathrm{SYT}(\lambda)\big| \, = \, n! \prod_{(i,j)\in\lambda} \frac{1}{h(i,j)} \qquad \text{and} \qquad \big|\mathrm{SYT}(\lambda/\mu)\big| \, = \, n! \det\left(\frac{1}{(\lambda_i - \mu_j - i + j)!}\right)$$

Corollary: $\#\mathrm{SYT}(\lambda/\mu) \in \mathsf{FP}$.

Example: Let $n = 2m$, $\lambda = (m,m)$. Then $\mathrm{SYT}(\lambda) = \frac{1}{m+1}\binom{2m}{m}$ Catalan number.

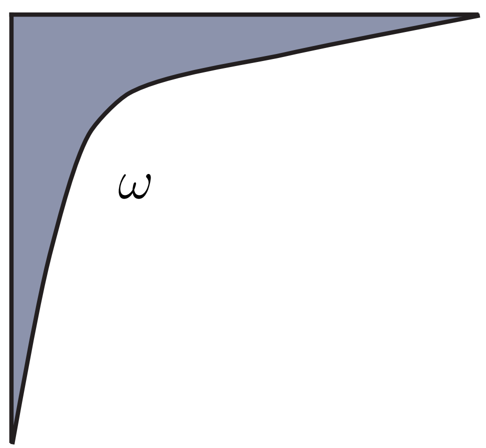
What do we do now?

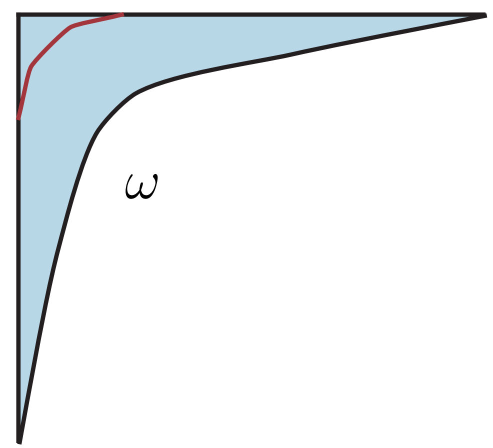
(1) exact product formulas (for some partitions)

(2) positive formulas

(3) asymptotic formulas (for $\lambda/\sqrt{n} \to \omega$ as $n \to \infty$)

(4) limit shapes (as in arctic circle theorem)





Main asymptotic result

Corollary [HLF + Vershik–Kerov'81, or Morales–P.–Panova'18]

Let $\lambda^{(n)}/\sqrt{n} \to \omega$ as $n \to \infty$, and suppose $\omega = \partial\mathcal{C}$. Then:

$$\frac{1}{n}\left[\log\left|\mathrm{SYT}\big(\lambda^{(n)}\big)\right| \,-\, \frac{1}{2}\,n\log n\right] \;\longrightarrow\; -\iint_{\mathcal{C}} \log \hbar(x,y)\,dx\,dy$$

Proposition: Let $\lambda^{(n)}/\sqrt{n} \to \omega$ and $\mu^{(n)}/\sqrt{n} \to \pi$ as $n = \left|\lambda^{(n)}/\mu^{(n)}\right| \to \infty$. Then:

$$\log\left|\mathrm{SYT}\big(\lambda^{(n)}/\mu^{(n)}\big)\right| \,-\, \frac{1}{2}\,n\log n \;=\; O(n)$$

Theorem [Morales–P.–Tassy'18+]

Let $\lambda^{(n)}/\sqrt{n} \to \omega$ and $\mu^{(n)}/\sqrt{n} \to \tau$ as $n = \left|\lambda^{(n)}/\mu^{(n)}\right| \to \infty$. Then:

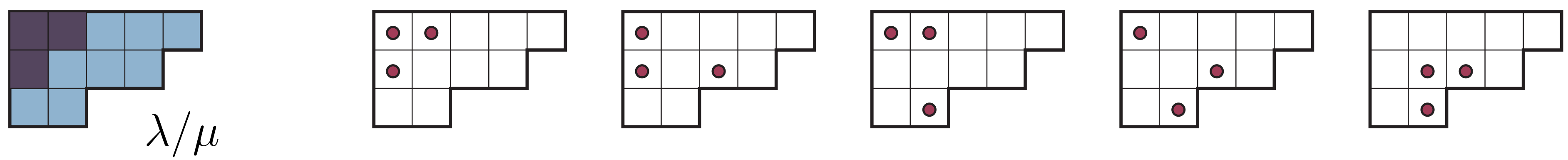
$$\frac{1}{n}\left[\log\left|\mathrm{SYT}\big(\lambda^{(n)}/\mu^{(n)}\big)\right| \,-\, \frac{1}{2}\,n\log n\right] \;\longrightarrow\; \mathbf{c}(\omega/\tau)$$

Key new tool: NHLF (Naruse hook-length formula)

Theorem [Naruse'14 (announced), Morales–P.–Panova'17, '18 (proofs and generalizations)]

$$\big|\mathrm{SYT}(\lambda/\mu)\big| \, = \, |\lambda/\mu|! \sum_{D\in\mathcal{E}(\lambda/\mu)} \prod_{u\in\lambda\setminus D} \frac{1}{h(u)}$$

Here $\mathcal{E}(\lambda/\mu)$ is the set of *excited diagrams*, obtained from λ/μ by shifting particles:



Corollary: ["naive HLF"]

$$F(\lambda/\mu) \, \le \, \big|\mathrm{SYT}(\lambda/\mu)\big| \, \le \, F(\lambda/\mu) \cdot \big|\mathcal{E}(\lambda/\mu)\big|, \quad \text{where} \quad F(\lambda/\mu) \, = \, |\lambda/\mu|! \prod_{u\in\lambda/\mu} \frac{1}{h(u)}$$

Note: The term for $D=\mu\in\mathcal{E}(\lambda/\mu)$ is the largest, and $|\mathcal{E}(\lambda/\mu)|<2^n$, giving $O(n)$ error term for $\log|\mathrm{SYT}(\lambda/\mu)|$.
This positive formula is not useful for exact, but can be useful for asymptotic applications.

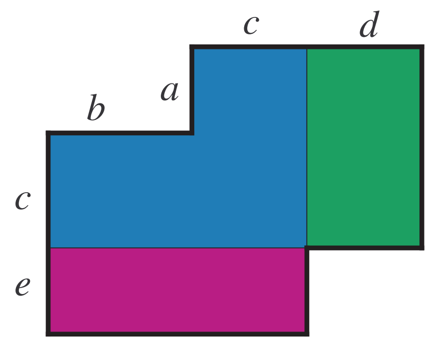
Combinatorial proof of the NHLF was found by Konvalinka'17+. Two q-analogues in MPP'17.

New exact formulas

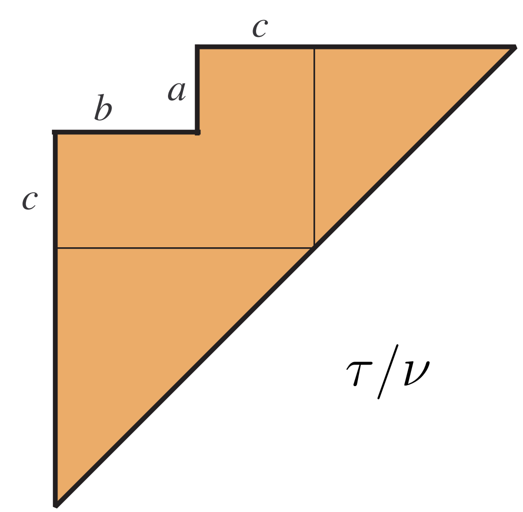
Theorem [Kim–Oh'17, DeWitt'12, Morales–P.–Panova'17+ (6 par. family)]
For $|\lambda/\mu| = n$, $\Phi(n) = 1!\cdot 2!\cdots(n-1)!$, $\Psi(n) = 1!!\cdot 3!!\cdots(2n-3)!!$, we have:

$$\left|\mathrm{SYT}(\lambda/\mu)\right| = n!\,\frac{\Phi(a)\,\Phi(b)\,\Phi(c)\,\Phi(d)\,\Phi(e)\,\Phi(a+b+c)\,\Phi(c+d+e)\,\Phi(a+b+c+d+e)}{\Phi(a+b)\,\Phi(d+e)\,\Phi(a+c+d)\,\Phi(b+c+e)\,\Phi(a+b+2c+d+e)}$$

$$\left|\mathrm{SYT}(\tau/\nu)\right| = n!\,\frac{\Phi(a)\,\Phi(b)\,\Phi(c)\,\Phi(a+b+c)\,\cdot\,\Psi(c)\,\Psi(a+b+c)}{\Phi(a+b)\,\Phi(b+c)\,\Phi(a+c)\,\cdot\,\Psi(a+c)\,\Psi(b+c)\,\Psi(a+b+2c)}$$



λ/μ



τ/ν

Applications of exact formulas

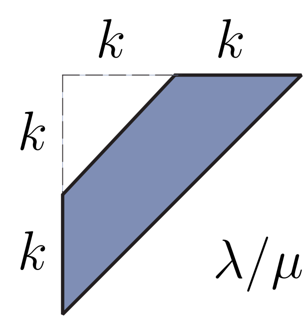
Let $a = \alpha k$, $b = \beta k$, $\ldots$, and $k = \sqrt{n} \to \infty$. Use

$$\log \Phi(k) = \frac{1}{2} k^2 \log k - \frac{3}{4} k^2 + 2k \log k + O(k)$$

$$\log \Psi(k) = k^2 \log k + \left(\log 2 - \frac{3}{2} \right) k^2 + \frac{5}{2} k \log k + O(k)$$

This gives an explicit formula for $\mathbf{c} = \mathbf{c}(\alpha, \beta, \ldots)$.

Question: What is $\mathbf{c}$ for shape in the figure? Best known estimates: $-0.22 < \mathbf{c} < -0.14$ [MPP'18, '19+, P.–Panova–Yeliussizov'19]. Computer calculations give $\mathbf{c} \approx -0.18$ (Pantone)
Note: in this case $|\mathcal{E}(\lambda/\mu)|$ has a product formula (MPP'18 + Proctor'90).



Interlude: Coxeter's curious identity (1935)

$$\sum_{n=1}^{\infty} \frac{\phi^n}{n^2} \cos\left(\frac{2\pi n}{5}\right) = \frac{\pi^2}{100}, \qquad \text{where} \qquad \phi = \frac{\sqrt{5}-1}{2}$$

Proof idea:

(1) Spherical volume $\mathrm{vol}(\mathbb{S}^3) = 2\pi^2 R^3$. Volume of the regular tetrahedron $\Delta(\theta)$ with dihedral angle θ as a formula involving $\mathrm{Li}_2\left(e^{\pm i\theta}\right)$, where $\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} z^n/n^2$

(2) The 600-cell with edge lengths 1 has circumradius $R = \phi$, and $\mathrm{vol}(\mathbb{S}^3) = 600 \cdot \mathrm{vol}\,\Delta(2\pi/5)$ by the symmetry.

(3) Easy calculations with dilogarithms $\mathrm{Li}_2(z)$.

Very vague proof idea of NHLF and exact formulas

(1) Macdonald's (multiparameter) *factorial Schur functions* describe equivariant cohomology of the Grassmannian (structural results + technical properties)

(2) hidden symmetry of factorial Schur functions

(3) careful evaluation (+ some computation)

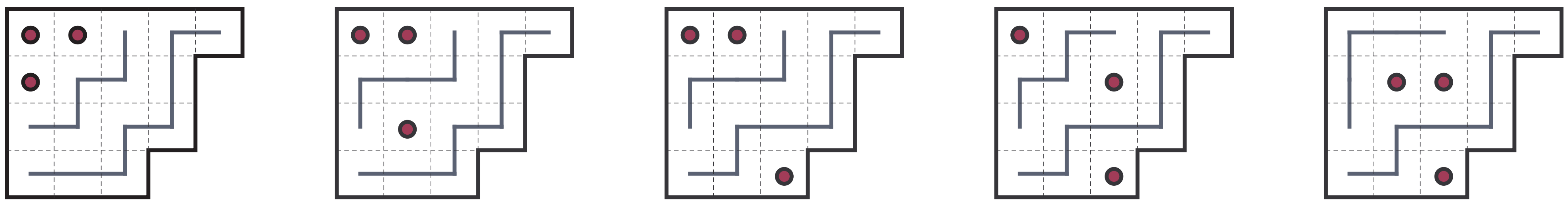
For NHLF: skip step (2).

For exact formulas: 6-parameter family is the largest that survives under symmetry in step (2).

We replace our result with a much clearer result which has a similar proof.

From excited diagrams to non-intersecting paths

Lemma The complement to excited diagrams can be partitioned into non-intersecting paths with fixed start and end points.



Note: This implies that $\left|\mathcal{E}(\lambda/\mu)\right|$ = explicit determinant

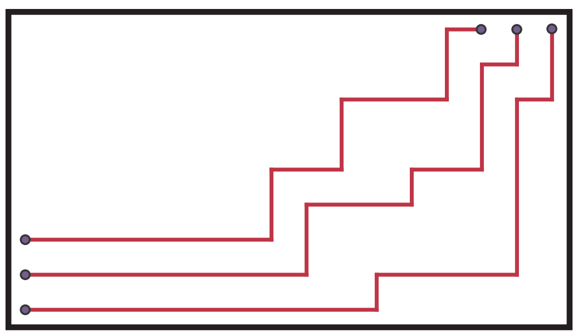
(see MPP'18 via Lindström-Gessel-Viennot lemma, cf. Kreiman'05)

First fundamental symmetry (for factorial Schur functions)

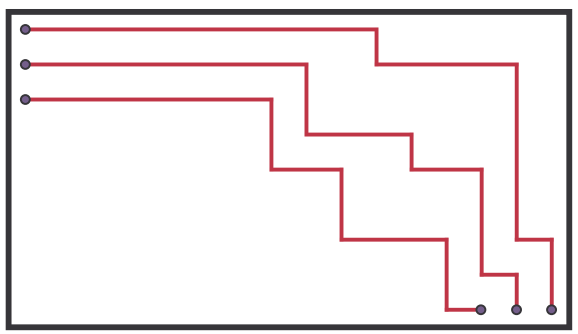
Let $x_1, \ldots, x_{a+c}, y_1, \ldots, y_{b+c}$ be variables, $\Upsilon = (\gamma_1, \ldots, \gamma_c)$ be a c-tuple of non-intersecting NE paths in $(a+c) \times (b+c)$ rectangle, and $\Upsilon' = (\gamma'_1, \ldots, \gamma'_c)$ the same for SE paths.

Theorem [Morales–P.–Panova'17+]

$$\sum_{\Upsilon} \prod_{(i,j)\in\Upsilon} \frac{1}{x_i - y_j} = \sum_{\Upsilon'} \prod_{(i,j)\in\Upsilon'} \frac{1}{x_i - y_j}$$



Υ



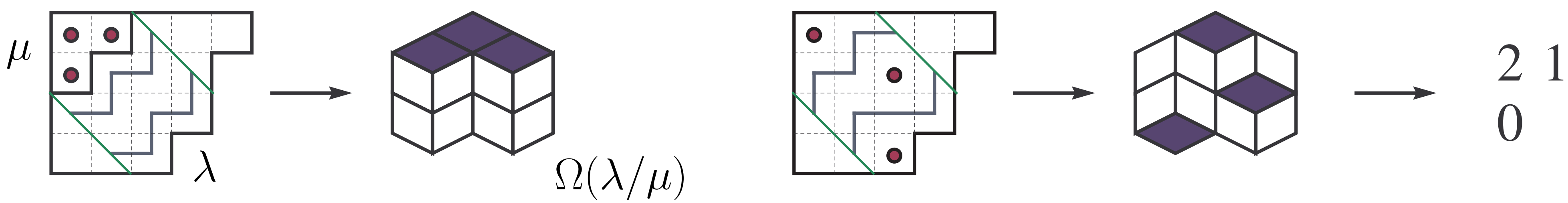
Υ'

Observe: Fix p, q and let $x_i := p + i$, $y_j := q - j$. Then the terms in the RHS are equal.

MacMahon's box formula

Lemma [MPP'17+] There is a bijection between $\mathcal{E}(\lambda/\mu)$ and lozenge tilings of $\Omega(\lambda/\mu)$.

For $\lambda = (b+c)^{(a+c)}$, $\mu = (b^a)$, we have $\Omega(\lambda/\mu)$ is a $\langle a \times b \times c \rangle$ symmetric hexagon.
Thus $\#\{\Upsilon\} = |\mathcal{E}(\lambda/\mu)| = |\mathrm{PP}(a,b,c)|$.

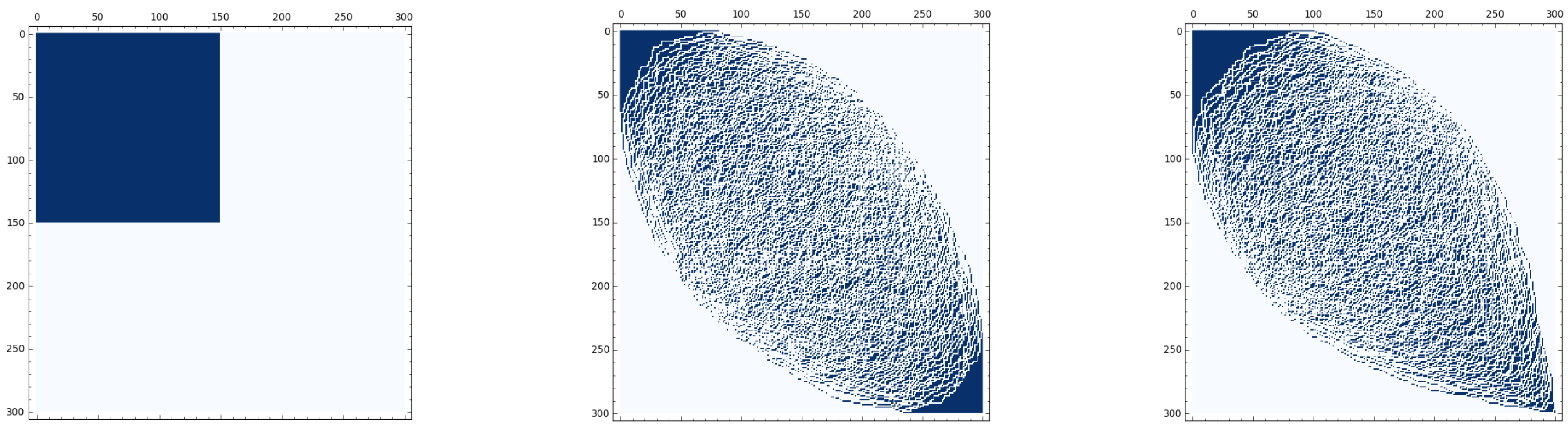


Theorem [MacMahon, 1915]
Number of solid (plane) partitions which fit $a \times b \times c$ box is equal to

$$\big|\mathrm{PP}(a,b,c)\big| \;=\; \prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2} \;=\; \frac{\Phi(a+b+c)\,\Phi(a)\,\Phi(b)\,\Phi(c)}{\Phi(a+b)\,\Phi(b+c)\,\Phi(a+c)}$$

Random excited diagrams

Let $a = b = c = 150$, so $\lambda = (300^{150}, 150^{150})$ and $\mu = (150^{150})$. Here is what *uniform* and *hook weighted* excited diagrams looks like (after 10^{10} steps of Metropolis algorithm).



Problem: Clearly, the limit shape is emerging in both cases. What's going on?

Arctic circle

Back to lozenge tilings – in the uniform case the limit shape is the *Arctic circle* for random lozenge tilings of a region. The circle was discovered in Cohn–Larsen–Propp'98 (for the hexagon), by Kenyon'08 (for general regions), Kenyon–Okounkov'07 (explicit formulas).

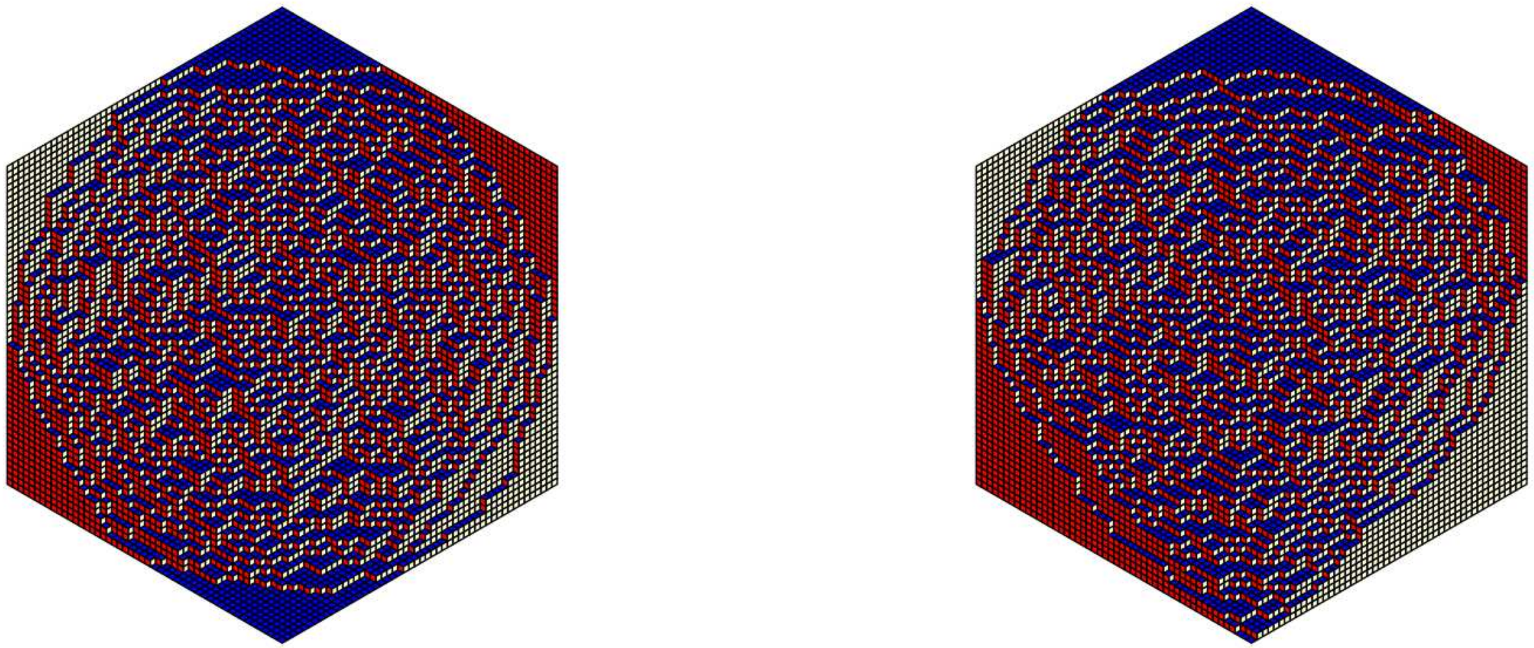


Figure: Uniform vs. hook weighted lozenge tilings.

Theorem [Morales–P.–Tassy'18+]

Let $\{\lambda^{(n)}\}$ and $\{\mu^{(n)}\}$ be two partition sequences with strongly stable limit shapes ω and τ, and s.t. $|\lambda^{(n)}/\mu^{(n)}| = n + o(n/\log n)$. Then

$$\frac{1}{n}\left[\log\left|\mathrm{SYT}\big(\lambda^{(n)}/\mu^{(n)}\big)\right| - \frac{1}{2}\,n\log n\right] \longrightarrow \mathbf{c}(\omega/\tau) = F(\tau) + G(\omega,\tau),$$

where

$$G(\omega,\tau) = \max_{f\in \mathrm{Lip}_{[0,1]}} \iint_{\mathcal{C}(\omega/\tau)} \Big(\sigma(\nabla f) + (1-\partial_x f - \partial_y f)\log \hbar(x,y)\Big)\,dx\,dy,$$

$$\sigma(s,t) = \frac{1}{\pi}\bigg(\Lambda(\pi s) + \Lambda(\pi t) + \Lambda\big(\pi(1-s-t)\big)\bigg),$$

$$F(\tau) = \iint_{\mathcal{C}(\pi)} \hbar(x,y)\,dx\,dy \qquad \text{and} \qquad \Lambda(\theta) = \frac{1}{2}\sum_{n=1}^{\infty}\frac{\sin(2n\theta)}{n^2}$$

Proof idea: We prove the variational principle (and thus the limit shape) for hook weighted lozenge tilings. Then we use the NHLF (very carefully).

Note: Wangru Sun'18+ proved variational principle for SYT's of skew shape.

Open problems:

(1) Prove that $\#\mathrm{SYT}(\lambda)$ is $\#\mathsf{P}$-complete for 3-dim Young diagrams $\lambda \subset \mathbb{N}^3$.

I conjecture that λ of height 2 suffices.

(2) Prove that for sequences $\lambda^{(n)}$ of 3-dim Young diagrams with a limit surface, we have:

$$\frac{1}{n}\left[\log\left|\mathrm{SYT}\big(\lambda^{(n)}\big)\right| - \frac{2}{3}\, n\log n\right] \longrightarrow \mathbf{c}$$

(3) There are two probabilistic algorithms for *exact* sampling from $\mathrm{SYT}(\lambda)$, see Greene–Nijenhuis–Wilf'79 and Novelli–P.–Stoyanovskii'97. Can one use the NHLF (or some other approach) to get a natural random process to sample from $\mathrm{SYT}(\lambda/\mu)$?

Thank you!