The story of MacMahon's Master Theorem

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MacMahon's Master Theorem

Let $A = (a_{ij})_{m\times m}$, $a_{ij} \in \mathbb{C}$, be a complex matrix,
$x_1, \ldots, x_m$ be a set of commuting variables,

$$G(k_1, \ldots, k_m) \;=\; \left[x_1^{k_1} \cdots x_m^{k_m}\right] \;\prod_{i=1}^{m} (a_{i1}x_1 + \ldots + a_{im}x_m)^{k_i}.$$

Let $t_1, \ldots, t_m$ be another set of commutative variables,
and $T = (\delta_{ij} t_i)_{m\times m}$ a diagonal matrix. Then:

$$\sum_{(k_1,\ldots,k_m)} G(k_1, \ldots, k_m)\; t_1^{k_1} \cdots t_m^{k_m} \;=\; \frac{1}{\det(I - TA)},$$

where the summation is over all nonnegative integer vectors $(k_1, \ldots, k_m)$.

Main result:

Let $q_{ij} \in \mathbb{C}$, $q_{ij} \neq 0$, where $1 \le i < j \le m$. Suppose variables $x_1, \ldots, x_m$ are **q**-commuting:

$$x_j x_i \,=\, q_{ij}\, x_i x_j\,, \quad \text{for all} \;\; i < j,$$

Suppose also that the variables a_{ij} **q**-commute within columns:

$$a_{jk} a_{ik} \,=\, q_{ij}\, a_{ik} a_{jk}\,, \quad \text{for all} \;\; i < j,$$

commute with x_s, $1 \le s \le m$, and satisfy:

$$a_{jk}\, a_{il} \,-\, q_{ij}\, a_{ik} a_{jl} \,+\, q_{kl}\, a_{jl} a_{ik} \,-\, q_{kl} q_{ij}\, a_{il} a_{jk} \,=0, \quad \text{for all} \;\; i < j, \; k < l.$$

Define

$$\det_{\mathbf{q}}(I - A) \,=\, \sum_{J \subseteq [m]} (-1)^{|J|} \det_{\mathbf{q}} A_J,$$

where

$$\det_{\mathbf{q}} A \,=\, \sum_{\sigma \in S_m} \left(\prod_{p<r\,:\;\sigma(p)>\sigma(r)} q_{rp}^{-1} \right) a_{\sigma(1)1} \cdots a_{\sigma(k)k}\,.$$

Theorem [Konvalinka–P.] *Let*

$$G(k_1, \ldots, k_m) \,=\, \left[x_1^{k_1} \cdots x_m^{k_m}\right] \; \overrightarrow{\prod_{i=1..m}} \left(a_{i1}x_1 + \ldots + a_{im}x_m\right)^{k_i}.$$

Then

$$\sum_{(k_1,\ldots,k_m)} G(k_1, \ldots, k_m) \,=\, \frac{1}{\det_{\mathbf{q}}(I - A)}\,.$$

This talk

Question: How can prove such a generalization of MMT?

Answer: You need to find a really good proof of MMT – the rest is easy (both the statement and the proof of the theorem).

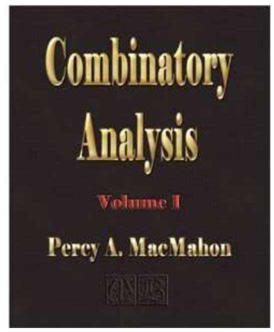
Followup Question: How can one come up with such a proof?

Answer: You need to see and understand the underlying algebra. To do that, I need to tell the full story of MacMahon's Master Theorem, as we understand it now.

Hint: Such proof is part bijective and part algebraic.

Starting point:

Percy A. MacMahon, *Combinatory analysis*, Vol. 1, Cambridge University Press, 1915.



About the author:

Major Percy Alexander MacMahon (1854–1929)

- Second son of Brigadier-General, served in British Army in India
- Taught at Royal Military Academy, retired a Major
- Fellow of the Royal Society
- Sylvester Medal, Morgan Medal, Queen's Medal by the Royal Society
- President of the London Mathematical Society

MacMahon is the father of combinatorics.

— [Jonathan Borwein]

MacMahon's expertise lay in combinatorics, a sort of glorified dicethrowing, and in it he had made contributions original enough to be named a Fellow of the Royal Society.

— [certain MIT professor]

MMT:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mm} \end{pmatrix}, \qquad T = \begin{pmatrix} t_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & t_m \end{pmatrix}$$

$$G(k_1, \ldots, k_m) := \left[x_1^{k_1} \cdots x_m^{k_m}\right] \prod_{i=1}^{m} (a_{i1}x_1 + \ldots + a_{im}x_m)^{k_i}$$

$$(\heartsuit) \qquad G(k_1, \ldots, k_m) = \left[t_1^{k_1} \cdots t_m^{k_m}\right] \frac{1}{\det(I - TA)}$$

First quick example:

$$A = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}, \qquad I - TA = \begin{pmatrix} 1-t_1 & 0 & \dots & 0 \\ 0 & 1-t_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1-t_m \end{pmatrix}$$

Then

$$G(k_1, \dots, k_m) = \left[x_1^{k_1} \dots x_m^{k_m}\right] (x_1)^{k_1} \cdots (x_m)^{k_m} = 1,$$
$$\det(I - TA) = (1-t_1)(1-t_2)\cdots(1-t_m).$$

The MMT now says:

$$\sum_{(k_1,\dots,k_m)} t_1^{k_1} \cdots t_m^{k_m} = \frac{1}{(1-t_1)\cdots(1-t_m)}.$$

Second quick example:

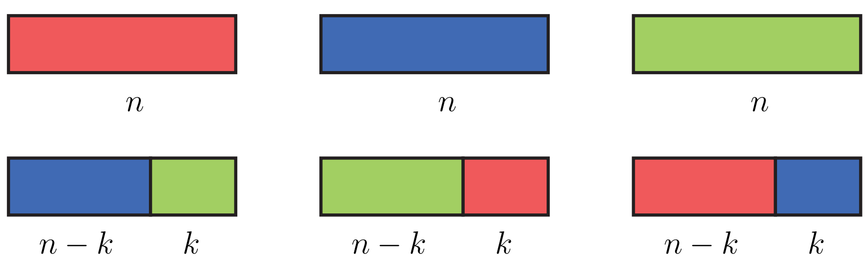
$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Then

$$G(n,n,n) = \left[x_1^n x_2^n x_3^n\right] (x_2 + x_3)^n (x_1 + x_3)^n (x_1 + x_2)^n$$

is the number of *derangements* of $x_1^n x_2^n x_3^n$, i.e. permutations of the letters with no fixed points. Observe (see the figure):

$$G(n,n,n) = \sum_{k=0}^{n} \binom{n}{k}^3.$$



On the other hand, the MMT gives:

$$G(n,n,n) \;=\; \left[t_1^n t_2^n t_3^n\right] \det{}^{-1} \begin{pmatrix} 1 & -t_1 & -t_1 \\ -t_2 & 1 & -t_2 \\ -t_3 & -t_3 & 1 \end{pmatrix}$$

$$=\; \left[t_1^n t_2^n t_3^n\right] \frac{1}{1 - t_1 t_2 - t_1 t_3 - t_2 t_3 - 2\, t_1 t_2 t_3}$$

Third quick example:

$$A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

Then

$$G(n,n,n) = \left[x_1^n x_2^n x_3^n\right] (x_2 - x_3)^n (x_1 - x_3)^n (x_1 - x_2)^n$$

and we similarly have

$$G(n,n,n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}^3,$$

which implies that $G(n,n,n) = 0$ for odd n.

On the other hand, for $n = 2m$, the MMT gives:

$$\begin{aligned} G(n,n,n) &= \left[t_1^n t_2^n t_3^n\right] \det^{-1} \begin{pmatrix} 1 & -t_1 & t_1 \\ t_2 & 1 & -t_2 \\ -t_3 & t_3 & 1 \end{pmatrix} \\ &= \left[t_1^n t_2^n t_3^n\right] \frac{1}{1 + t_1 t_2 + t_1 t_3 + t_2 t_3} \\ &= \left[t_1^{2m} t_2^{2m} t_3^{2m}\right] (-1)^m \big(t_1 t_2 + t_1 t_3 + t_2 t_3\big)^{3m} \\ &= (-1)^m \binom{3m}{m, m, m} \end{aligned}$$

We obtain *Dixon's identity*:

$$\sum_{k=0}^{2m} (-1)^k \binom{2m}{k}^3 = (-1)^m \binom{3m}{m, m, m}.$$

How about q–MMT?

Finding q–MMT was an open problem for over 50 years.

Here is the q–Dixon identity:

$$\sum_{k=0}^{m} (-1)^k q^{k(3k+1)/2} \begin{bmatrix} 2m \\ m+k \end{bmatrix}_q^3 = \begin{bmatrix} 3m \\ m, m, m \end{bmatrix}_q ,$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{n!_q}{k!_q (n-k)!_q}, \quad n!_q = (n)_q \cdots (1)_q , \quad (r)_q = \frac{q^r - 1}{q - 1} .$$

Lagrange Inversion Theorem (1768)

Let $f(t) = t + a_2 t^2 + a_3 t^3 + a_4 t^4 + \ldots$ *Suppose*

$$f(t) \;=\; t\big(1 \,+\, c_1 f(t) \,+\, c_2 f(t)^2 \,+\, c_3 f(t)^3 \,+\, \ldots\big)$$

Then:

$$a_n \;=\; \frac{1}{n} \cdot \big[z^{n-1}\big]\, \Phi(z)^n\,,$$

where $\Phi(z) = 1 + c_1 z + c_2 z^2 + \ldots.$

Remark: This is a small special case of what is known in the literature as Lagrange Inversion Theorem. Also, $\Phi(z)$ is uniquely determined by $f(t)$.

Quick examples:

(1) $a_n = \#$ binary trees with n vertices, $f(t) = t\big(1 + f(t)\big)^2$,

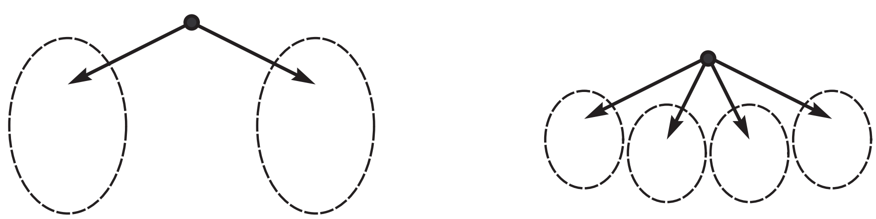
$$\Phi(z) = (1+z)^2, \quad a_n = \frac{1}{n}\left[z^{n-1}\right](1+z)^{2n} \;=\; \frac{1}{n}\binom{2n}{n-1}.$$

[*Catalan numbers*]

(2) $a_n = \frac{1}{n!}\,\#$ rooted labeled trees with n vertices, $f(t) = t \exp f(t)$,

$$\Phi(z) = e^z, \quad a_n = \frac{1}{n}\left[z^{n-1}\right]e^{nz} \;=\; \frac{1}{n}\cdot\frac{n^{n-1}}{(n-1)!} = \frac{1}{n!}\cdot n^{n-1}.$$

[*Cayley formula*]



Life of MMT as an analytic result

I. J. Good observation (1962): MMT immediately follows from the multivariate Lagrange Inversion Theorem.

Fallout: For generations, combinatorialists looked for q–MMT via the q-Lagrange inversion results.

Minor obstacle: There are many inequivalent (multivariate) q–LIT. There are even several non-commutative LIT. None gives a q–MMT.

Timeline of q–Lagrange Inversion theorems:

- L. Carlitz (1974, stated as open problem)
- G. Andrews (1980)
- I. Gessel (1980, + a non-commutative version)
- A. Garsia (1981)
- J. Hofbauer (1984)
- C. Krattenthaler (1984)
- A. Garsia, J. Remmel (1986)
- I. Gessel, D. Stanton (1986)
- D. Singer (1995, unification result)

Further generalizations:

- IP, A. Postnikov, V. Retakh (1995, non-commutative version via quasi-determinants)
- I. Gessel, G. Labelle (1995, Lagrange inversion for species)
- C. Lenart (2000, symmetric functions and q-version)
- J.-C. Novelli, J.-Y. Thibon (2008, non-commutative symmetric functions + unification)

One application

For a tree τ in K_n with root at 1, define the *number of inversions*

$$\mathrm{inv}(\tau) \;=\; \#(i,j) \quad \text{such that} \quad \big[i \to_\tau j \to_\tau 1\big],\;\; 1 < i < j \le n.$$

Let

$$J_n(q) = \sum_{T \in K_n} q^{\mathrm{inv}(T)}, \quad \text{so that} \quad J_n(1) = n^{n-2}.$$

Theorem [Mallows, Riordan, 1968]

$$\sum_{n=1}^{\infty} (q-1)^{n-1} J_n(q)\, \frac{z^n}{n!} \;=\; \log \left[\sum_{k=0}^{\infty} q^{\binom{k}{2}} \frac{z^k}{k!} \right]$$

Proof idea: Use recurrence relations to show that $J_n(q) = T_{K_n}(1,q)$. Use that $T_{K_n}(1, 1{+}y)$ counts connected subgraphs of K_n by the number of edges.

Theorem [Gessel, 1982]

$$\sum_{n=1}^{\infty} J_n(q)\,\frac{z^n}{n!} \;=\; \frac{\sum_{k=0}^{\infty} q^{-\binom{k+1}{2}}\,(1+q+\cdots+q^k)^k\,\frac{z^k}{k!}}{\sum_{k=0}^{\infty} q^{-\binom{k+1}{2}}\,(1+q+\cdots+q^{k-1})^k\,\frac{z^k}{k!}}$$

Proof idea: Use Gessel's q–Lagrange inversion applied to

$$\mathbf{J}(z) \;=\; \sum_{k=1}^{\infty} q^{\binom{k}{2}}\,\mathbf{J}(z)\cdot\mathbf{J}(qz)\cdots\mathbf{J}(q^{k-1}z)\,,$$

where

$$\mathbf{J}(z) \;=\; \sum_{n=1}^{\infty} J_n(q)\,\frac{z^n}{n!}\,.$$

Theorem [P., Postnikov, Retakh, 1995]

Let $a_1, a_2, a_3, \dots$ be non-commuting formal variables, and let R be the ring generated by them. Suppose $f \in R$ satisfies

$$f \,=\, a_0 + a_1 f + a_2 f^2 + a_3 f^3 + \dots$$

Then $f = P \cdot Q^{-1}$, where

$$P \,=\, a_0 + \sum_{i_1+i_2+\cdots+i_{\ell+1}=\ell} a_{i_1} a_{i_2} \cdots a_{i_{\ell+1}}$$

and

$$Q \,=\, 1 + \sum_{i_1+i_2+\cdots+i_\ell=\ell} a_{i_1} a_{i_2} \cdots a_{i_\ell}\,.$$

Remark: To obtain Gessel's formula, use $a_i = g_i x^i y$, where g_i commute with everything, and $xy = qyx$.

Proof idea:

Take a nearly upper triangular *Toeplitz matrix*:

$$A := \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & \cdots \\ a_0 & a_1 & a_2 & a_3 & \cdots \\ 0 & a_0 & a_1 & a_2 & \cdots \\ 0 & 0 & a_0 & a_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Prove:

$$f = \left(\frac{1}{I - A}\right)_{11} a_0 = \left(I + A + A^2 + A^3 + \ldots\right)_{11} a_0$$

For that, use tools for working with non-commutative (quasi-)determinants and explicit combinatorial calculations.

Back to MMT

- P. Cartier, D. Foata (1968, partially commutative matrix entries)
- D. Foata, D. Zeilberger (1988, powers of matrices)
- C. Krattenthaler, M. Schlosser (1999, q-multiplying operators)

Breakthrough:

- S. Garoufalidis, T. Lê, D. Zeilberger (2006, quantum MMT)

Remark: In 1980, Zeilberger introduced a highly technical "operator elimination" technique which he used to prove MMT. In [GLZ], the authors extend this technique to get a quantum extension.

Back story:

- J. Bernstein, M. Sato (1971, independently, D-modules and holonomic systems)
- V. Jones (1983, Jones polynomial of knots)
- L. Faddeev, M. Jimbo, V. Drinfeld, etc. (late 1980's, quantum groups)
- N. Reshetikhin, V. Turaev (1990, Jones polynomial from quantum groups)
- D. Zeilberger (1990, holonomic systems and binomial identities)
- S. Garoufalidis, T. Lê (2005, colored Jones polynomial is q-holonomic)

The algebraic proof of MMT:

Observe that:

$$\sum_{k_1+\ldots+k_m=n} G(k_1,\ldots,k_m) \;=\; \operatorname{tr} S^n A\,,$$

Now MMT can be reduced to a basic result in linear algebra:

$$(MMT) \qquad \sum_{n=0}^{\infty} \operatorname{tr} S^n A \;=\; \frac{1}{\det(I-A)}$$

Follow the standard scheme:

1) MMT is trivial for diagonal matrices (see first quick example).
2) Check that (MMT) is invariant under conjugation BAB^{-1}.
3) Extend to general matrices by continuity.

Life of MMT as an algebraic result

- D. Foata, G.-N. Han (2007-8, 3 papers, 2 new algebraic proofs, various quantum extensions)
- P. Hai and M. Lorenz (2007, Koszul duality proof)
- M. Konvalinka, IP (2007, quantum extensions to Manin matrices)
- P. Etingof, IP (2008, extension to Berger's N-homogeneous algebras, proof uses known generalized Koszul duality)
- P. Hai, B. Kriegk, M. Lorenz (2008, extension to N-homogeneous superalgebras)
- A. Molev, E. Ragoucy (2009, right quantum superalgebras)

Combinatorial proof of MMT:

Recall that:

$$\left(B^{-1}\right)_{11} = \frac{\det B^{11}}{\det B}.$$

Use $B = I - A$ to rewrite the r.h.s. of (MMT):

$$\frac{1}{\det(I-A)} = \frac{\det\left(I - A^{11}\right)}{\det(I-A)} \cdot \frac{\det\left(I - A^{12,12}\right)}{\det(I-A^{11})} \cdot \frac{\det\left(I - A^{123,123}\right)}{\det(I-A^{12,12})} \cdots$$

$$= \left(\frac{1}{I-A}\right)_{11} \left(\frac{1}{I-A^{11}}\right)_{22} \left(\frac{1}{I-A^{12,12}}\right)_{33} \cdots \frac{1}{1-a_{mm}}$$

$$= \left(I + A + A^2 + \ldots\right)_{11} \left(I + A^{11} + (A^{11})^2 + \ldots\right)_{22}$$
$$\times \left(I + A^{12,11} + (A^{12,12})^2 + \ldots\right)_{33} \cdot \ldots$$

Now both l.h.s. and r.h.s. of (MMT) are positive sums of monomials in $\{a_{ij}\}$. Konvalinka and IP provide an explicit bijection proving MMT.

Remark: To obtain the proof of quantum generalizations we extend this bijection and use the theory of non-commutative and quantum determinants by Gelfand-Retakh, Etingof-Retakh and Manin.

One final application

Denote by $L_{m,k}(n)$ the number of sequences $(i_1 \ldots i_n)$, $i_r \in \{1, \ldots, m\}$, such that no k subsequent indices are strictly decreasing.

Corollary [Etingof, P.]

$$1 + \sum_{n=1}^{\infty} L_{m,k}(n)\, t^n \;=$$

$$\left(1 - m\,t + \binom{m}{k} t^k - \binom{m}{k+1} t^{k+1} + \binom{m}{2k} t^{2k} - \binom{m}{2k+1} t^{2k+1} + \ldots\right)^{-1}$$

Question: An easy combinatorial proof for $k > 2$?

Thank you!