Polyhedral Domes

Igor Pak, UCLA

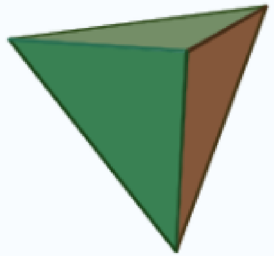
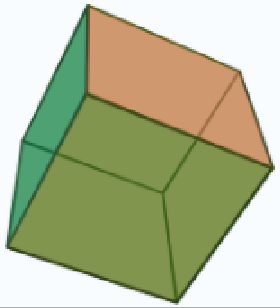
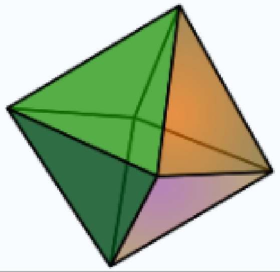
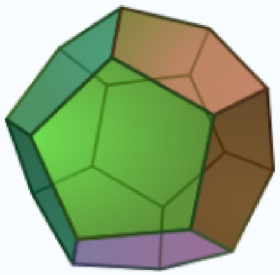
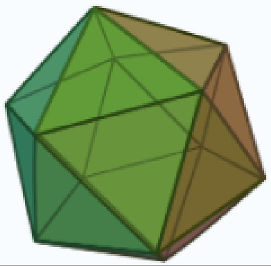
(joint work with Alexey Glazyrin, UTRGV)

Colloquium, King's College London, UK (May 4, 2021)


Platonic solids

ELEMENTS BOOK 13

The Platonic solids†

Tetrahedron	Cube	Octahedron	Dodecahedron	Icosahedron
Four faces	Six faces	Eight faces	Twelve faces	Twenty faces
				

Question 1: How do you know the *icosahedron* really exists?

Question 2: And if it does, how do you know it's *inscribed into a sphere*?

Answer: That depends on how icosahedron is defined!

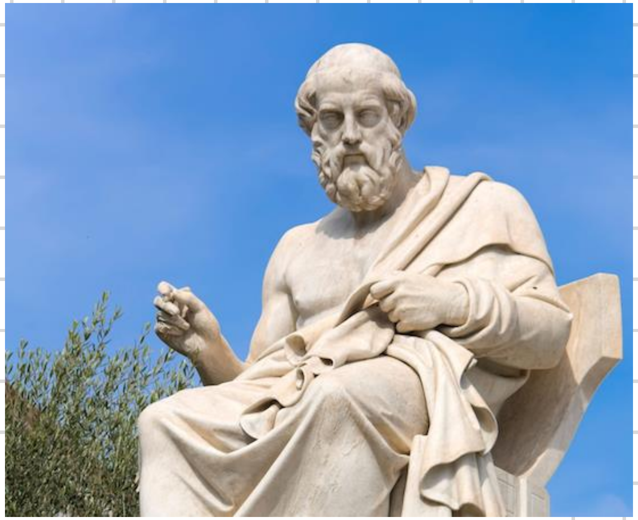
MAGAZINE OPINIONS ATHENS LITERATURE PLATO

Plato got virtually everything wrong

Big brain, big mistakes

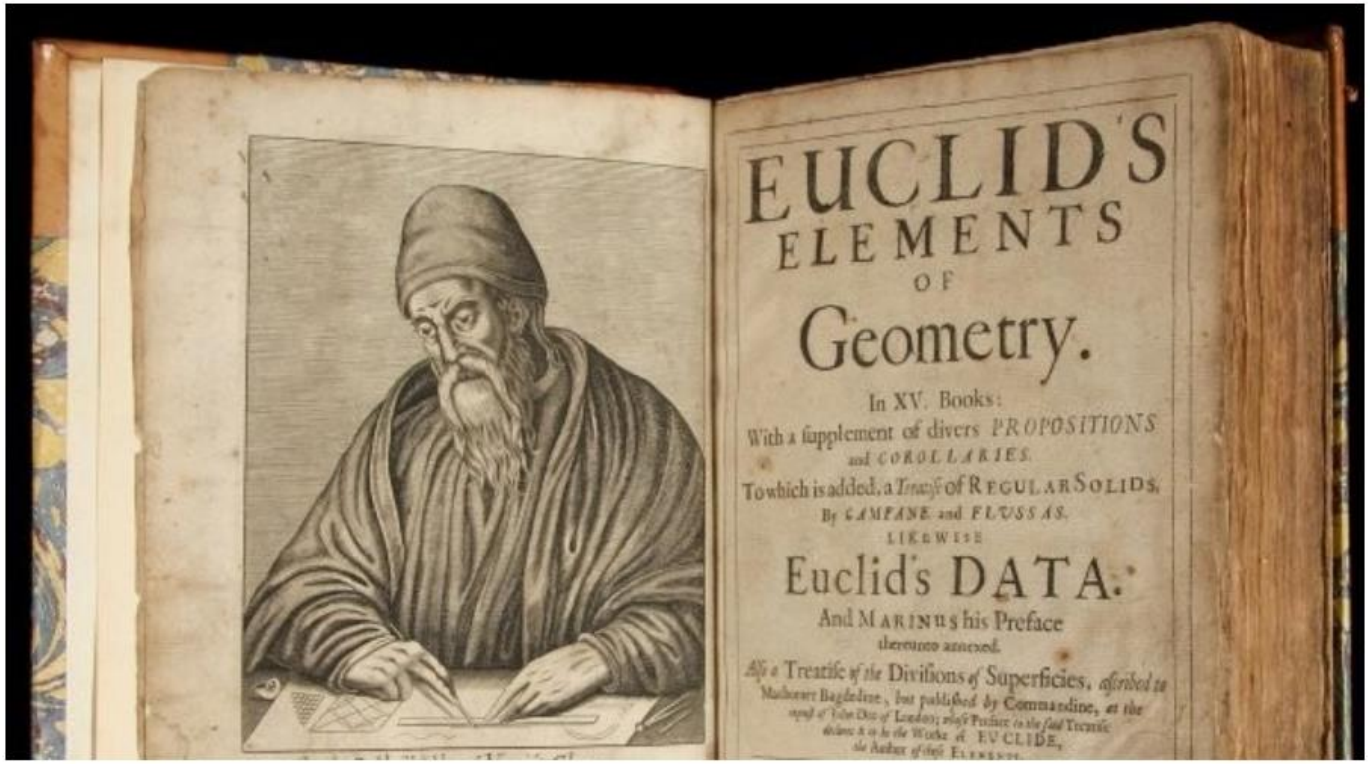
Prospect

By Julian Baggini September 20, 2018 OCTOBER 2018



Euclid Was Wrong

AUG 12, 2015 JOHN-ERIK PERSSON MATHEMATICS, PHILOSOPHY 0 COMMENT



JOURNAL ARTICLE

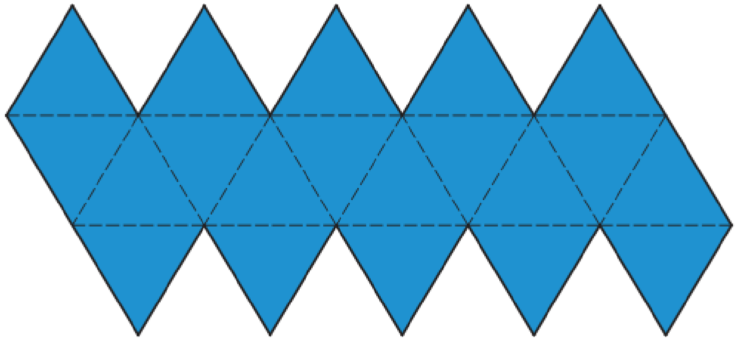
What is wrong with Euclid?

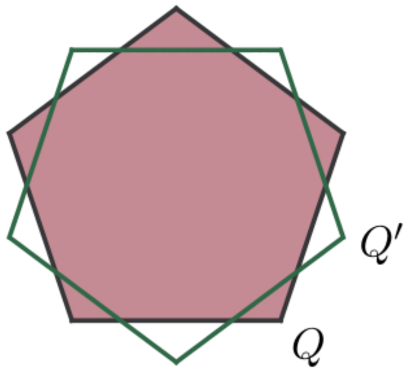
A. E. MEDER JR.

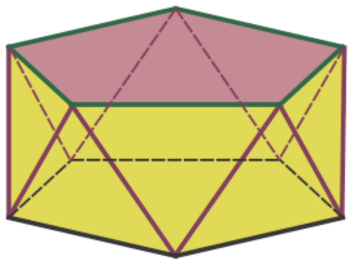
The Mathematics Teacher
Vol. 51, No. 8 (December 1958), pp. 578-584 (7

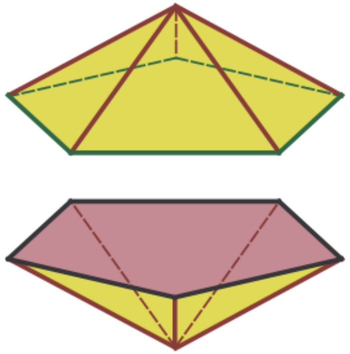
Platonic solids

Definition 1: *Regular polytopes* = convex polytopes where all sides are regular polygons with the same number of sides, and where every vertex has the same degree







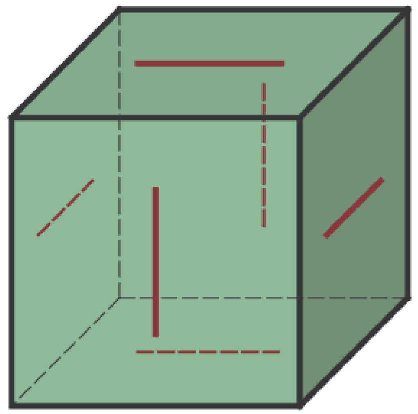


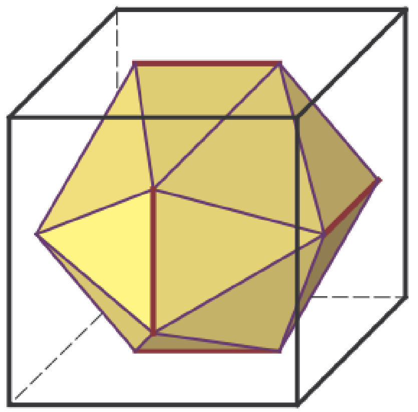
Need: *Alexandrov Unfolding Theorem*

Works by continuity. **Question:** Is it inscribed into a sphere?

Platonic solids

Definition 2: *Regular polytopes* = convex polytopes *P* whose group of symmetries acts transitively on complete flags of *P.*





$(0, \pm 1, \pm\phi), (\pm 1, \pm\phi, 0)$, and $(\pm\phi, 0, \pm 1)$,

$\phi = \frac{-1+\sqrt{5}}{2}$ is the *golden ratio*

Note: Still need a calculation to check Def. 2

Adrien-Marie Legendre

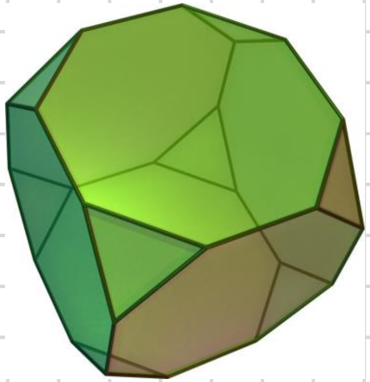
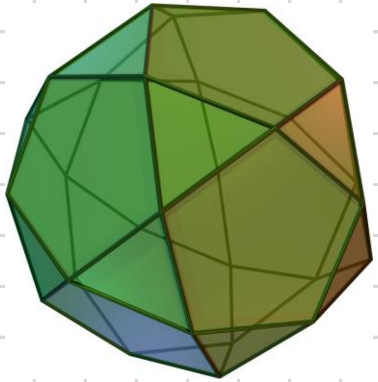
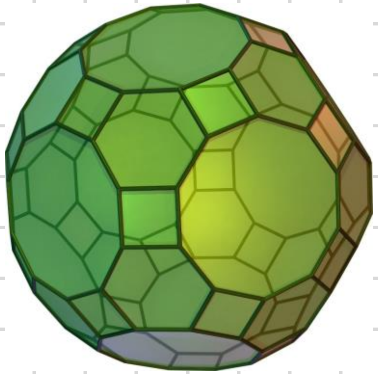
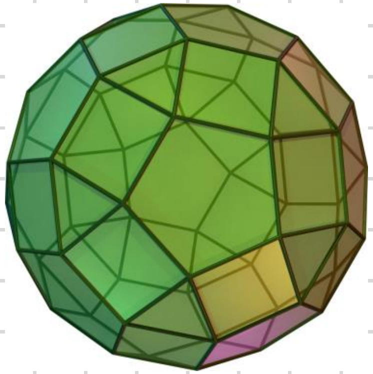
Discovered and fixed the mistake in his translation of

Éléments de géométrie, 1794

Modern day approach:

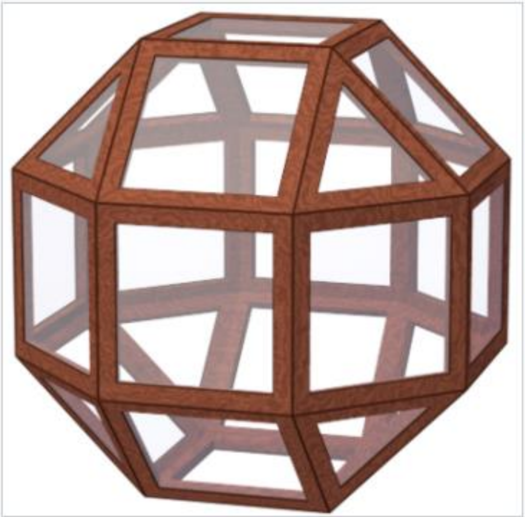
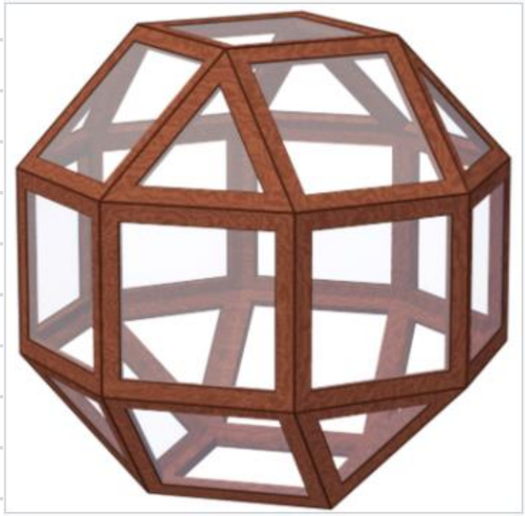
Cauchy Rigidity Theorem (1813)

Archimedean solids

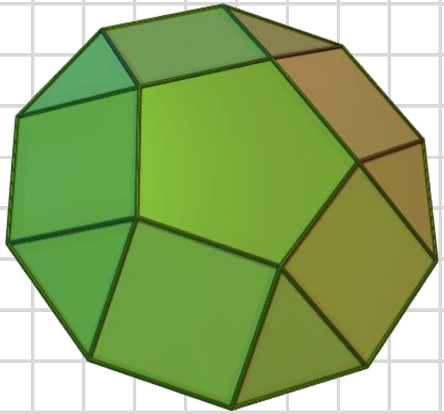
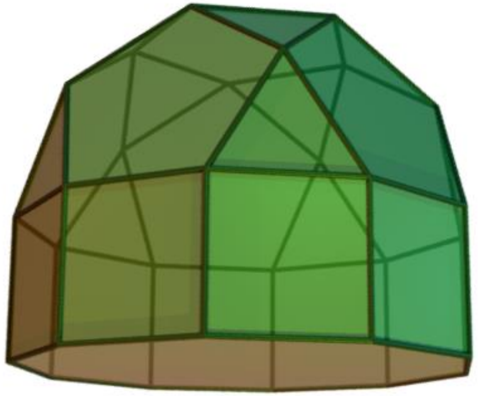
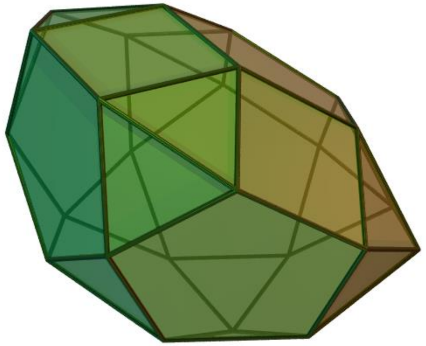
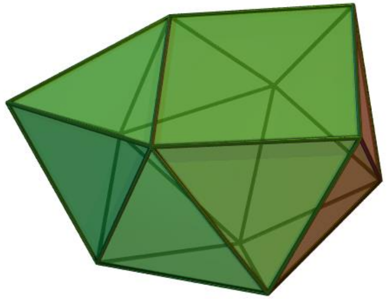
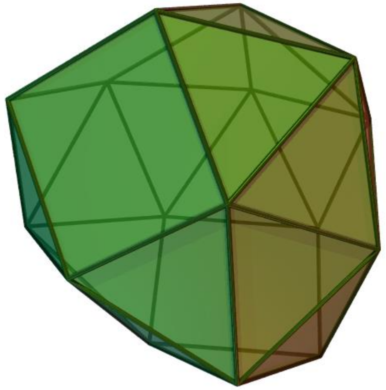






Definition: all faces are regular polygons, and symmetry group acts transitively on vertices

Note: Archimedes did not miss one!
(don't trust *Wikipedia*)

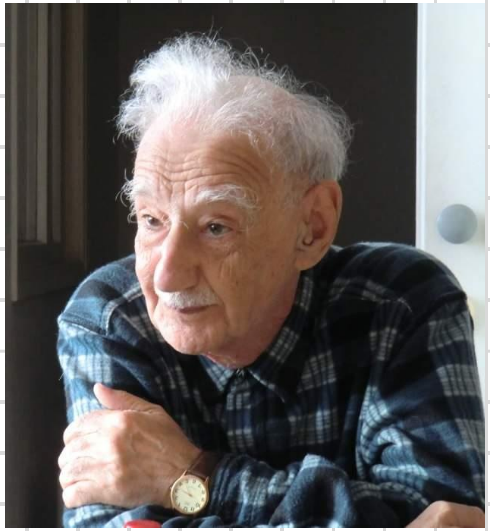
Johnson solids

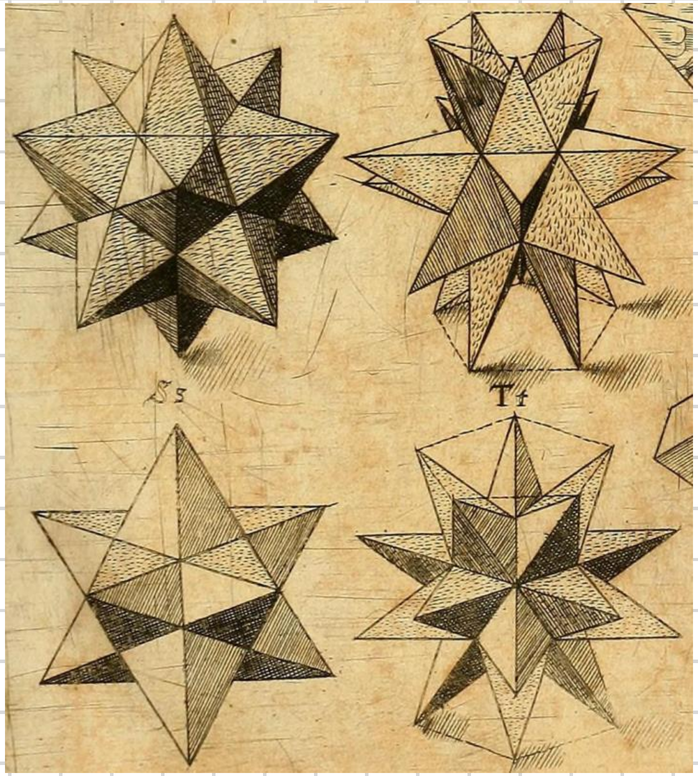
Norman Johnson 1966

Victor Zalgaller 1969 (218 pages, habilitation)

Proof ingredients:
Alexandrov unfolding theorem, variation on Volkov stability theorem, heavy computer calculations (BESM-3M at S.Pb. University)

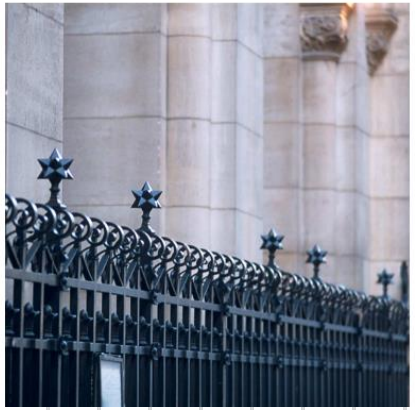
Kepler–Poinsot polyhedra



Johannes Kepler

Harmonices Mundi 1619

The church Sant'Ivo alla Sapienza in Rome



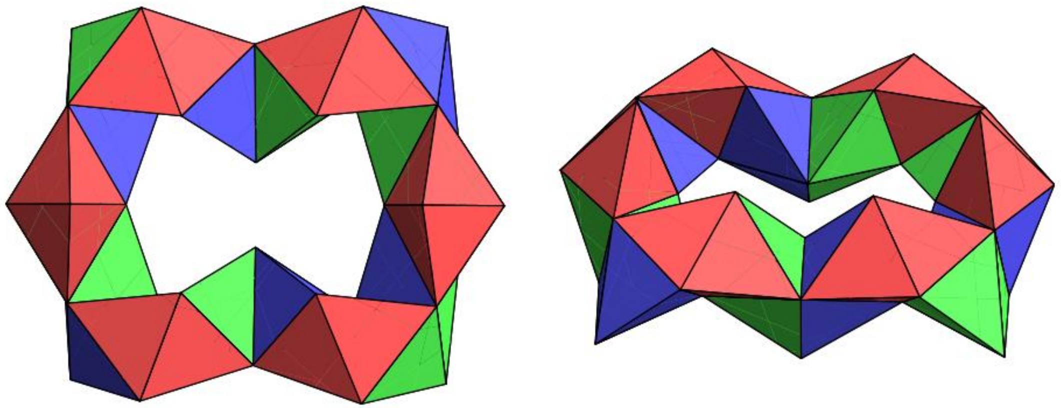
Grande Synagogue de Paris, rue de la Victoire,

The Exhibition Centre in Beijing

Steinhaus problem (*Scottish book*, 1957)

(1) Does there exist a closed tetrahedral chain? $\longleftarrow$ *Coxeter helix*

(2) Are the end-triangles dense in the space of all triangles?



A length-36 fake tetratorus with a final gap of about 0.0005 cm.

Art Tower Mito

Part (1) was resolved negatively by Świerczkowski (1959)

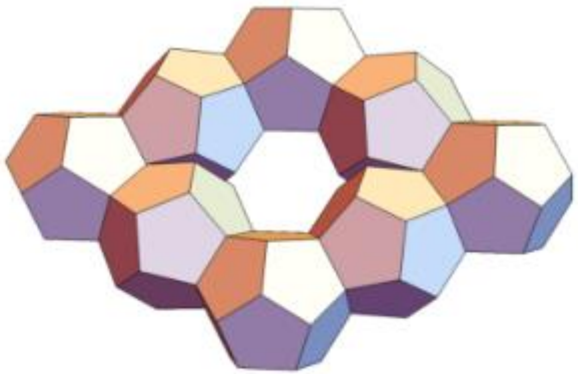
Part (2) was partially resolved by Elgersma–Wagon (2015) and Stewart (2019)

Idea: The group of face reflections is isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ which is dense in $O(3, \mathbb{R})$

General surfaces with regular polyhedral faces

Square surfaces: Dolbilin–Shtanko–Shtogrin (1997)

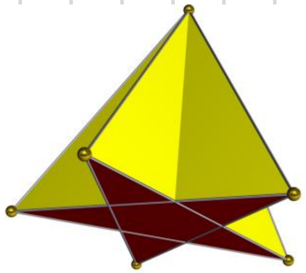
(no new polyhedra of this kind)

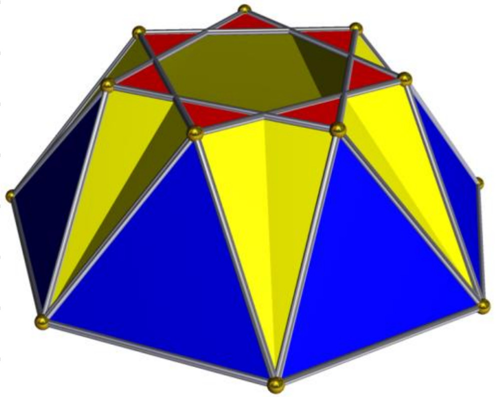


Pentagonal surfaces: Alevy (2018+)

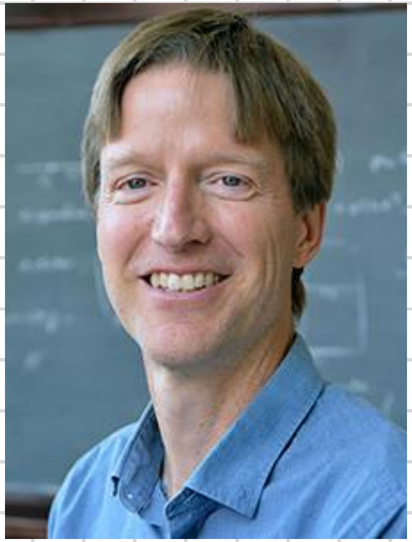
(for small genus all such polyhedra are comprised of dodecahedra
attached along faces)

Many ad hoc examples:





Integral curves

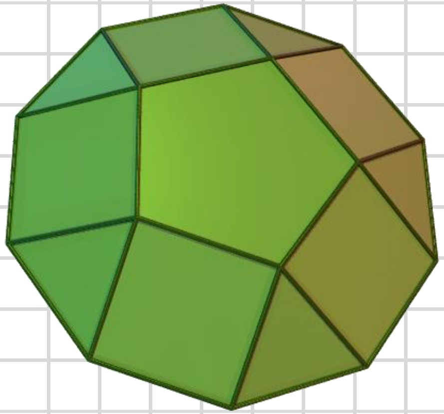


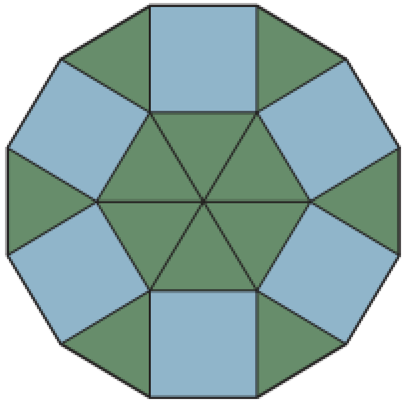
A PL-curve $\gamma \subset \mathbb{R}^3$ is called *integral* if comprised of unit length intervals.

A *dome* is a 2-dim PL-surface $S \subset \mathbb{R}^3$ comprised of unit equilateral triangles.

Integral curve γ *can be domed* if there is a dome S s.t. $\partial S = \gamma$.

Problem [Kenyon, c. 2005]: Can every closed integral curve be domed?





Dome constructions over regular *n*-gons, *n*=4, 5, 10, 12

Other domes

Question: Is this a dome over a planar n-gon?

Buckminster Fuller's real dome and his sketch of the *Dome over Manhattan* (1960)

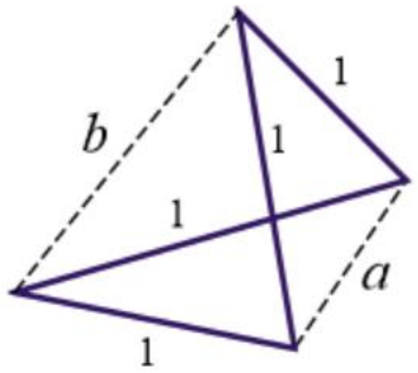
Positive results:

Theorem 1 [Glazyrin–P., 2020+]
For every integral curve $\gamma \subset \mathbb{R}^3$ and $\varepsilon > 0$, there is an integral curve $\gamma' \subset \mathbb{R}^3$, such that $|\gamma| = |\gamma'|$, $|\gamma, \gamma'|_F < \varepsilon$ and the curve γ' can be domed.

Here $|\gamma, \gamma'|_F$ is the *Fréchet distance* $|\gamma, \gamma'|_F = \max_{1 \leq i \leq n} |v_i, v_i'|$.

Theorem 2 [Glazyrin–P., 2020+]
Every regular integral n-gon in the plane can be domed.

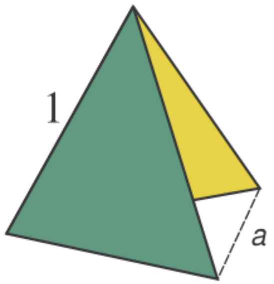
How to prove positive results?

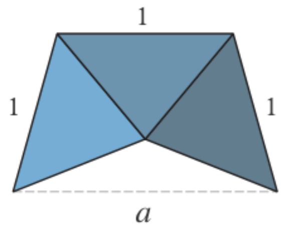


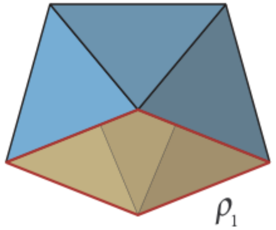
Definition: $\rho(a,b)$ is a rhombus with unit sides and diagonals *a* and *b*.

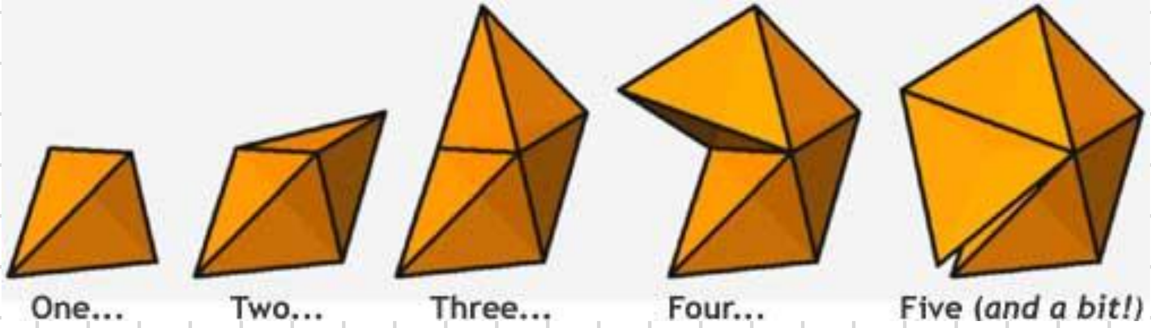
Rhombus Lemma

Fix $a \notin \overline{\mathbb{Q}}$. The set of b for which rhombus $\rho(a,b)$ which can be domed is dense in $\left(0, \sqrt{4-a^2}\right)$.







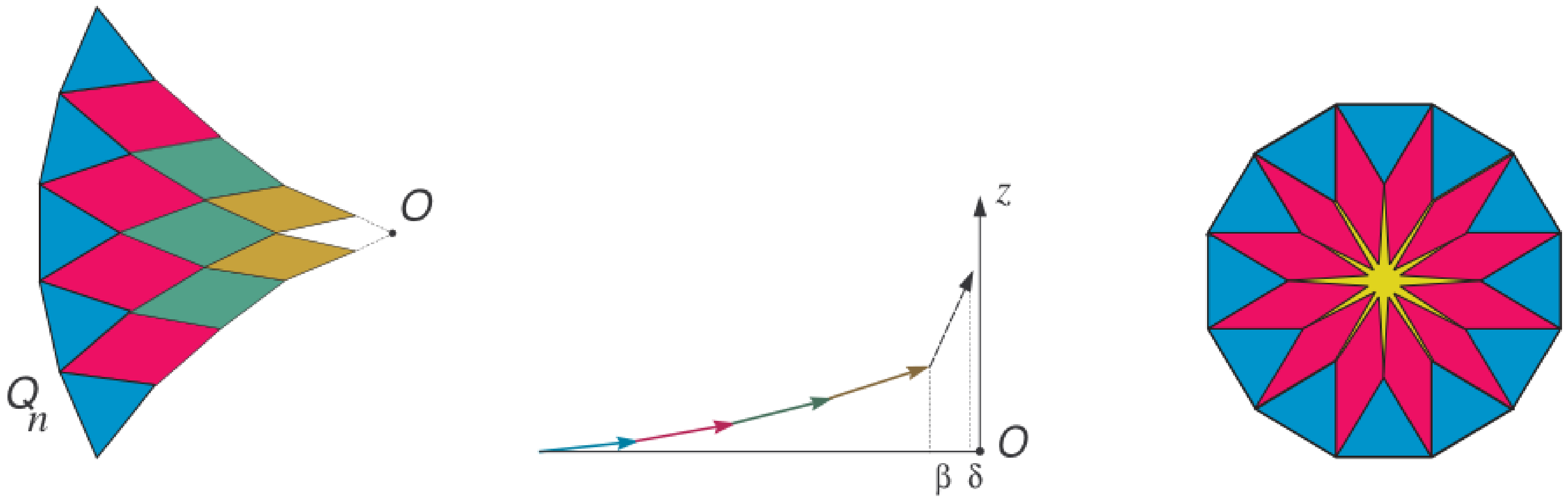


Open: Can all planar unit rhombi $\rho(a,b)$ be domed?

Domes over regular polygons

Wayman AME Church
Minneapolis, MN

Construction sketch:



Tilt blue triangles by $\angle\theta$. Make near-planar rhombi until the center is overshot.

Use continuity to find θ for which the tip of the slice is on the vertical axis.

Negative results:

Theorem 3 [Glazyrin–P., 2020+]

Let $\rho(a,b) \subset \mathbb{R}^3$ be a unit rhombus with diagonals $a, b > 0$. Suppose $\rho(a,b)$ can be domed.

Then there is a nonzero polynomial $P \in \mathbb{Q}[x,y]$, such that $P(a^2, b^2) = 0$.

Theorem 4 [Glazyrin–P., 2020+]

Let $\rho(a,b) \subset \mathbb{R}^3$ be a unit rhombus with diagonals $a, b > 0$.

If $a \notin \overline{\mathbb{Q}}$ and $a/b \in \overline{\mathbb{Q}}$, then $\rho(a,b)$ cannot be domed.

Examples:

$\rho\big(\frac{1}{\pi}, \frac{e^\pi}{\sqrt{97}}\big) \ \leftarrow$ Thm 3,

$\rho\big(\frac{1}{\pi}, \frac{1}{\pi}\big)$ and $\rho\big(\frac{e}{\sqrt{17}}, \frac{e}{\sqrt{19}}\big) \ \leftarrow$ Thm 4.

Proof ingredients:

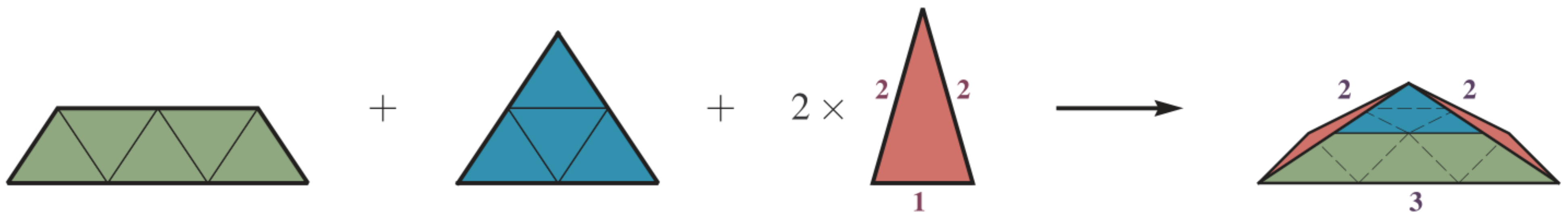
- heavy use of *theory of places*
- technical inductive topological argument

[Conelly–Sabitov–Walz, 1997], [Connelly, 2009], [Gaifullin–Gaifullin, 2014]

Conjectures and open problems

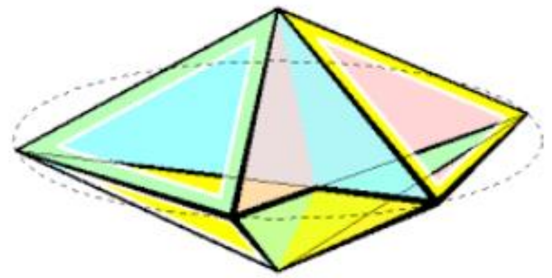
Conjecture 1. An isosceles triangle $\Delta = (2, 2, 1)$ cannot be domed.

Proposition: Conjecture 1 false $\Rightarrow$ every triangle $\Delta = (p, q, r)$ can be domed.



Conjecture 2. Every non-degenerate closed dome is rigid.

Proposition: Conjecture 1 is false $\Rightarrow$ Conjecture 2 is false.



Conjectures and open problems

Conjecture 3:

The set of a, s.t. planar rhombus $\rho\left(a, \sqrt{4-a^2}\right)$ can be domed, is countable.

Conjecture 4:

There are unit triangles $\Delta_1, \Delta_2 \subset \mathbb{R}^3$, such that $\Delta_1 \cup \Delta_2$ cannot be domed.

Conjecture 5 [“cobordism for domes”]:

For every integral curve $\gamma \in \mathbb{R}^3$, there is a unit rhombus ρ, and a dome over $\gamma \cup \rho$.

Thank you!