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What is a combinatorial interpretation?

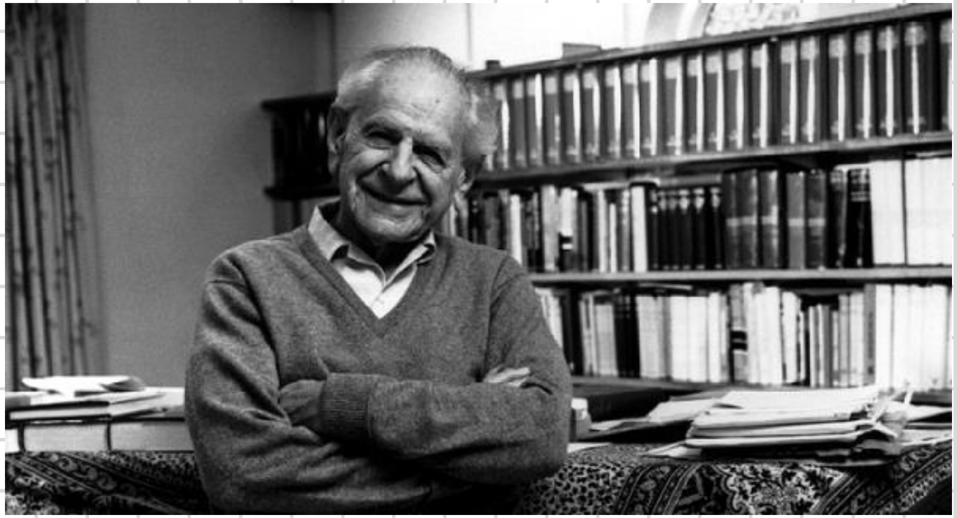
Experimental Mathematics Seminar

Rutgers University

What it does not exist?

"In so far as a scientific statement speaks about reality, it must be falsifiable: and in so far as it is not falsifiable, it does not speak about reality."

― Karl R. Popper, The Logic of Scientific Discovery



Three Deep Problems in Enumerative Combinatorics

(1) ***What is a (good) bijection?***

Rogers-Ramanujan bijection?

Garsia-Milne (1980), Boulet-P. (2006)
based on Bressoud-Zeilberger (1989)

P., asymptotic approach (2004)
Konvalinka-P. (2009)

(2) ***What is a (good) formula?***

Wilf (1982), *What is an answer*?

Garrabrant-P. (2015) ← disproof of the
Noonan-Zeilberger Conjecture,
Negative answer to *Kontsevich problem*

(3) ***What is a (good) combinatorial interpretation?***

P. (2018) ← ICM survey

Super Catalan numbers

$$S(m,n) := \frac{(2m)!\,(2n)!}{m!\,n!\,(m+n)!}$$

defined by E. Catalan in 1874

$$S(m,n) = \sum_k (-1)^k \binom{2m}{m+k}\binom{2n}{n+k}$$

von Szily identity (1894)

$S(1,n)/2 = C_n$ is the usual Catalan number

Ira M. Gessel, Guoce Xin

ABSTRACT. It is well known that the numbers $(2m)!\,(2n)!/m!\,n!\,(m+n)!$ are integers, but in general there is no known combinatorial interpretation for them. When $m=0$ these numbers are the middle binomial coefficients $\binom{2n}{n}$, and when $m=1$ they are twice the Catalan numbers. In this paper, we give combinatorial interpretations for these numbers when $m=2$ or 3.

arXiv:math/0401300

Super Catalan numbers (continued)

$$S(m, m+\ell) = \sum_k 2^{\ell-2k} \binom{\ell}{2k} S(m,k).$$

I. M. Gessel, Super ballot numbers, *J. Symbolic Comput.* **14** (1992), 179–194.

$\exists$ Recurrence of this type

$\Rightarrow$ gives a combinatorial intrepretation

G. Schaeffer: *"What is clearly left open by Gessel type interpretation of Super Catalan numbers is the constructive division issue about their multiplicative factorial form."* (2018)

Unimodality of q-binomial coefficients

A sequence $(a_1, a_2, \ldots, a_n)$ is called *unimodal*, if for some k we have

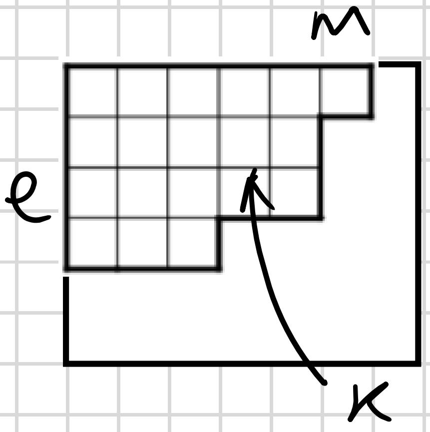
$$a_1 \le a_2 \le \ldots \le a_k \ge a_{k+1} \ge \ldots \ge a_n$$

The *q-binomial* (Gaussian) *coefficients* are defined as:

$$\binom{m+\ell}{m}_q = \frac{(q^{m+1}-1)\cdots(q^{m+\ell}-1)}{(q-1)\cdots(q^{\ell}-1)} = \sum_{k=0}^{\ell m} p_k(\ell,m)\, q^k$$

Sylvester's theorem establishes unimodality of the sequence

$$p_0(\ell,m),\ p_1(\ell,m),\ \ldots,\ p_{\ell m}(\ell,m). \qquad (1878).$$



Question: *Is there a combinatorial interpretation of*

$$p_k(\ell,m) - p_{k-1}(\ell,m),\ 1 \le k \le \ell m/2 \quad ??$$

$\|$

$[m \times \ell]$ # partitions $\lambda \vdash k$ which fit

Unimodality of q-binomial coefficients (continued)

Theorem (Pak–Panova, 2015). *Fix $\ell, m \geq 1$. The sequence*

$$\{p_k(\ell, m) - p_{k-1}(\ell, m),\ 1 \leq k \leq \ell m/2\}$$

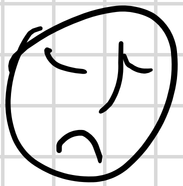
has a combinatorial interpretation

Partition tree $T(m, \ell, k)$: vertices labelled by (a, b, λ, j), $j \leq ab$, $\lambda \vdash b$.
Leaves – $b = 1$, labels (a, i) with $0 \leq i \leq a$.
Root – (m, ℓ, λ, k) for some $\lambda \vdash \ell$.
Local conditions on vertices and children:
If label – (a, b, λ, j), with $\lambda = (1^{b_1}, \ldots, n^{b_n})$, then $\leq n$ children, labels – $(a_1, b_1, \lambda^1, j_1), \ldots, (a_n, b_n, \lambda^n, j_n)$. s.t.

- $a_i = i(a + 2) - 2(\lambda'_1 + \cdots + \lambda'_i)$ for all $i = 1, \ldots, n$.
- $j_1 + \cdots + j_n = j - 2n(\lambda)$.

All leaves $(a_0, i_0), \ldots, (a_t, i_t)$, satisfy:

- For each $r < t$: $i_r \geq 2(i_{r+1} + \cdots + i_t) - (a_{r+1} + \cdots + a_t)$.



Theorem $\sum_{i \geqslant 1} \lambda_i = n$. Set $Y_i := \sum_{j=1}^{i} \lambda_j$ for all $i \geqslant 1$, and $Y_0 := 0$.

$$\binom{a+b}{a}_q = \sum_{\lambda \vdash b} q^{2\sum_{i \geqslant 1}\binom{\lambda_i}{2}} \prod_{j \geqslant 1} \binom{j(a+2) - Y_{j-1} - Y_{j+1}}{\lambda_j - \lambda_{j+1}}_q .$$

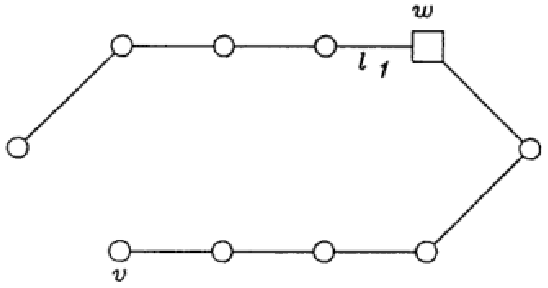
Kathy O'Hara (1990)

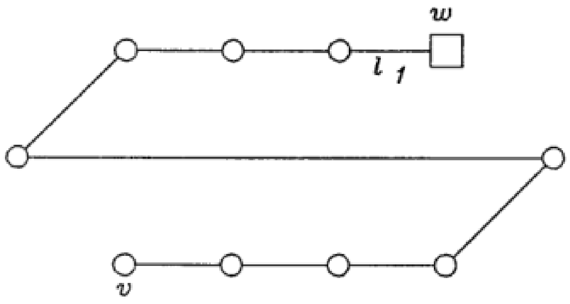
Hamiltonian cycles in cubic graphs

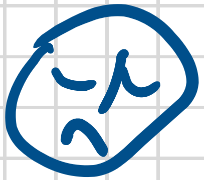
Theorem (Smith and Tutte 1946). *Let e be an edge in a cubic graph G. Then the number $N_e(G)$ of Hamiltonian cycles in G containing e, is always even.*

Open Problem *Find a combinatorial interpretation for $N_e(G)/2$.*

Price - Thomason Algorithm: (1977, 1978)







THEOREM *For any $n \geqslant 1$, there exists a graph G_n with $8n+2$ vertices, an edge e of G, and an initial Hamiltonian cycle C in G containing e, for which Thomason's algorithm makes 2^n steps.*

Krawczyk (1999)

Gessel Sequence (2002)

$$b_n := 2\cdot 5^n - (3+4i)^n - (3-4i)^n, \quad \text{where } i = \sqrt{-1}.$$

A250102 16, 64, 16, 2304, 5776, 7744, 309136, 451584, 2062096, 38837824, 27920656

$$b_n = 2\cdot 5^n - 2\sum_r (-1)^r \binom{n}{2r} 3^{n-2r}\, 4^{2r}$$

$b_i \geq 0$ since $|3 \pm 4i| = 5$.

$$B(t) := \sum_{n=0}^{\infty} b_n t^n = \frac{16t(1+5t)}{(1-5t)(1+6t+25t^2)}$$

$\{b_n\}$ is a C-recursive nonnegative sequence

Th $\{b_n\}$ is NOT $\mathbb{N}$-rational

Question: *Is there a combinatorial interpretation?*

Remark The (metamathematical) *Schützenberger principle* states that all combinatorial sequences with rational GFs must be $\mathbb{N}$-rational

J. Berstel and C. Reutenauer, 2011.

Kronecker Coefficients

$$\chi^\lambda \cdot \chi^\mu = \sum_{\nu \vdash n} g(\lambda,\mu,\nu)\, \chi^\nu,$$

where $\lambda, \mu \vdash n$,

and $\chi^\lambda, \chi^\mu, \chi^\nu$ are irreducible characters of S_n.

$$g(\lambda,\mu,\nu) := \langle \chi^\lambda \chi^\mu, \chi^\nu \rangle = \frac{1}{n!} \sum_{\sigma \in S_n} \chi^\lambda(\sigma)\, \chi^\mu(\sigma)\, \chi^\nu(\sigma),$$

$$\sum_{\lambda,\mu,\nu} g(\lambda,\mu,\nu)\, s_\lambda s_\mu s_\nu = \prod_{i,j,k} \frac{1}{1 - x_i y_j z_k}$$

Schur's theorem

$$g(\lambda,\mu,\nu) = \sum_{\omega \in S_\ell} \sum_{\pi \in S_m} \sum_{\tau \in S_r} \operatorname{sign}(\omega\pi\tau) \cdot \mathrm{CT}\big(\lambda + 1_\ell - \omega,\, \mu + 1_m - \pi,\, \lambda + 1_r - \tau\big)$$

where $\mathrm{CT}(\alpha,\beta,\gamma) = \#\big[\text{3-dim contingency tables with marginals } \alpha,\beta,\gamma\big]$.

Kronecker Coefficients

$$\chi^\lambda \cdot \chi^\mu = \sum_{\nu \vdash n} g(\lambda, \mu, \nu)\, \chi^\nu,$$

where $\lambda, \mu \vdash n$,

and $\chi^\lambda, \chi^\mu, \chi^\nu$ are irreducible characters of S_n.

Open Problem *Find a combinatorial interpretation for the Kronecker coefficients*

Murnaghan (1938)

$$\{g(\lambda, \mu, \nu),\ \lambda, \mu, \nu \vdash n\}.$$

Lemma *Let $n = \ell m$, $\tau_k = (n-k, k)$, where $0 \le k \le n/2$ and set $p_{-1}(\ell, m) = 0$. Then*

$$g(m^\ell, m^\ell, \tau_k) = p_k(\ell, m) - p_{k-1}(\ell, m).$$

Theoren *For all $\ell, m \ge 8$, we have the following strict inequalities:*

$$(\circ) \qquad p_1(\ell, m) < \ldots < p_{\lfloor \ell m/2 \rfloor}(\ell, m) = p_{\lceil \ell m/2 \rceil}(\ell, m) > \ldots > p_{\ell m-1}(\ell, m).$$

Note: Many other special cases are known.

Answers:

3 principles of comb. interpretation

1) function $\in$ $\#P - \#P$ $\qquad \#P / \#P$

$$\frac{\#P - \#P}{\#P - \#P}$$

Ex $\quad$ $\#HC$ in G $-$ $\#$sp trees in G

$$\left\{ \frac{\#HC \text{ in cubic graph}}{2} \right\} \in \frac{\#P}{2}$$

2^{λ} , $\quad g(\lambda, \mu, \nu)$

Answers:

2) function ← #P-hard.

$$f(n,k) = \underline{\binom{n}{k}} - \underline{\binom{n}{k-1}} \qquad 1 < k \le \frac{n}{2}$$

$P_k(\ell, m)$ ← easy to compute

$$\underbrace{p(n)^2 - p(n+1)\,p(n-1)}_{\text{comb. int?}} \geq 0 \qquad n > 25$$

Answers:

3) $\underline{\text{function} \in \#P}$ $\in \#P - \#P$

$\underline{\text{Ex}}$

$$a_\lambda := \sum_{\mu \vdash n} \chi^\lambda[\mu]$$

$$b_\lambda := \sum_{\mu \vdash n} \chi^\mu[\lambda]$$

$\Big\} \in \#P$ — $\underline{\text{Th}}$

open problem by Stanley

$$\#\{w \in S_n : w^2 = \sigma\}$$

$$\sigma \in [\lambda]$$

Answers:

Ex G - graph

$a(k) := \#$ k-forests in G

/forests w/ k edges/

Th $a(k) \in \#P$-hard

Th [HK], [AHK]

$$a(k)^2 \geq a(k-1)\,a(k+1)$$

$\in \#P - \#P$

$\in \#P - h$

$a(k)^2 - a(k-1)\,a(k+1) \leftarrow$ Does this have comb. int.?

Answers:

<u>Ex</u> G - graph

$a(k) = \#$ k-matchings in G

/ k disjoint edges /

pairwise

<u>Th</u> $a(k)^2 - a(k-1)\,a(k+1) \geqslant 0$

<u>Th</u> [Krattenthaler] ∃ a combinat. interpret.

1996

Answers:

Th (FKG inequality)

[# subgraphs of G w/ HC]
$\times$[— // — which are planar]

$\geq$ [# subgraphs of G w/ HC and planar]
$\cdot \times$ [# all subgraphs]
$\backslash\backslash$
$2^{\#E}$

Th FKG ineq has a comb. intpr.

Thank you!