This a summary of Segner's article which appeared in the beginning of the same journal volume. It is presumed to have been written by Leonhard Euler.

VI. THE NUMBER OF TRIANGULATIONS OF A POLYGON

Author: I.A. de Segner, page 203.

In geometry, when studying a polygon, a triangulation is useful, as this way each triangle is easy to solve from its sides. As the number of sides of the polygon grows, so does the number of ways to triangulate it, and it becomes difficult to compute. For example, there are two ways to triangulate a quadrilateral. Similarly, there are 5 ways to triangulate a pentagon, 14 ways to triangulate a hexagon, 42 ways to triangulate a heptagon, 132 ways to triangulate an 8-gon, and 432 ways to triangulate a 9-gon. Beyond that, enumeration of the number of triangulation becomes more complicated and unenlightening.

That is why computing the number of triangulations of an n-gon becomes a very interesting geometric question, worthy of attention. The author answers this question in an unusual and clever way, by finding a recurrence relation for the number of triangulations of n-gons in terms of the number of triangulations of polygons with fewer sides. By using the recurrence relation, we can start with the number of triangulations of polygons with few sides and compute the number of triangulations of n-gons with larger n.

The author includes a table at the end, in which he computes the number of triangulations of an n-gon, for all $n \leq 20$. Unfortunately, this amazing geometer makes an error in calculations, and the table is correct only for $n < 15$, as the number of triangulations of 15-gon is not 751,900, but 742,900. We find that the numbers of triangulations in the table are too many for $15 \leq n \leq 17$, while the number of triangulations for $18 \leq n \leq 20$ are too few, as there are exactly 477,638,700 triangulations of a 20-gon.

The recurrence relation allows an easy calculation of the number of triangulations of an n-gon in term of those numbers for smaller values of n. On the other hand, there is a direct way to compute these numbers from the previous number. To be precise, if the number of triangulations of an n-gon is P, then the number of triangulations of an $(n+1)$-gon is $\frac{4n-6}{n}P$. In fact, instead of using this relation, one can compute the number of triangulations explicitly, as the following product of fractions:

$$\frac{2}{2} \cdot \frac{6}{3} \cdot \frac{10}{4} \cdot \frac{14}{5} \cdot \frac{18}{6} \cdot \frac{22}{7} \cdots \frac{4n-10}{n-1}$$

where the numerators grow linearly as $4n$, while the denominators grows as n. Following this formula, we carefully compute a new table, hopefully avoiding any mistakes.

$$T_3 = 1,\ T_4 = 2,\ T_5 = 5,\ T_6 = 14,\ T_7 = 42,\ T_8 = 132,\ T_9 = 429,\ T_{10} = 1430,$$
$$T_{11} = 4862,\ T_{12} = 16,796,\ T_{13} = 58,786,\ T_{14} = 208012,\ T_{15} = 742,900,$$
$$T_{16} = 2,674,440,\ T_{17} = 9,694,845,\ T_{18} = 35,357,670,\ T_{19} = 129,644,790$$
$$T_{20} = 477,638,700,\ T_{21} = 1,767,263,190,\ T_{22} = 6,564,120,420,$$
$$T_{23} = 24,466,267,020,\ T_{24} = 91,482,563,640,\ T_{25} = 343,059,613,650.$$

Source: This article appeared in the "Summary" section of *Novi Commentarii Academiae Scientiarum Imperialis Petropolitanae*, vol. 7, pp. 13–15 (dated 1758/59, but published only in 1761). The Latin original and Johann Segner's full article can be downloaded from translator's *Catalan Numbers Page* at `http://tinyurl.com/afrd34j`

Authorship: The article is unsigned, but the authorship by Euler is both evident and reported by numerous sources. The further evidence is given by comparison with a letter by Euler to Christian Goldbach written on 4 September 1751, published by a mathematician, protégé and grandson-in-law Nicolas Fuss. It contains the same product formula (in addition to other results), and is very similar is both the language and notation. The letter is available from the Dartmouth University's Euler website: `http://tinyurl.com/awu3pk3`

Remark: This is a very loose translation from Latin, made solely for my own investigation into the history of *Catalan numbers*, and with no other purpose. The writing in the article is very old fashioned, with long-winded convoluted sentences, imprecise meanings and no paragraph breaks. Rather than adhere to the original wording and style, I tried to preserve the mathematical spirit, while stating the results using modern terminology and notation. On a side note, all numbers Euler computed are correct and well organized.

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