$$S_4 = s_1 s_2 s_3 (2k^2 - 1 - k^4 \Sigma s_1^2 + k^4 \Sigma s_1^2 s_2^2 - k^4 s_1^2 s_2^2 s_3^2) \quad + d_1 d_2 d_3 \Sigma s_1 c_2 c_3 d_1,$$
$$C_4 = c_1 c_2 c_3 (1 - k^2 \Sigma s_1^2 + k^4 \Sigma s_1^2 s_2^2 - k^4 s_1^2 s_2^2 s_3^2) - \quad (1-k^2) \quad \Sigma s_2 s_3 c_1 d_2 d_3,$$
$$D_4 = d_1 d_2 d_3 (1 - k^2 \Sigma s_1^2 + k^2 \Sigma s_1 s_2^2 - k^2 s_1^2 s_2^2 s_3^2) + k^2 (1-k^2) s_1 s_2 s_3 \Sigma s_1 c_1 c_3 d_1,$$
$$N_4 = 1 - k^2 \Sigma s_1^2 + k^4 \Sigma s_1^2 s_2^2 + k^2 (1 - 2k^2) s_1^2 s_2^2 s_3^2 \quad + k^2 c_1 c_2 c_3 \Sigma s_2 s_3 c_1 d_2 d_3.$$

10. If we put $w = 0$, the second set reduce to the corresponding expressions for $u+v$ multiplied by $c_1 c_2$; and the third set reduce to the same quantities multiplied by $d_1 d_2$. When $u=v=w$, the formulæ reduce directly to the known expressions for $3u$.

It is noteworthy that the expressions for the functions of $u+v+w$, recently communicated to the Society by Mr. M. M. U. Wilkinson,* reduce, when $w=0$, to the forms S_4, C_4, D_4, and N_4; and that, when rendered homogeneous by the introduction of the letter n, they are symmetrical with respect to the four letters s, c, d, and n, and therefore constitute of themselves a complete system.

Note on a Geometrical Theorem. By H. M. Taylor and R. C. Rowe.

[*Read March 9th*, 1882.]

1. This note contains the solution of a generalized form of a problem interesting, if only from the names of the mathematicians whose attention it attracted half a century ago. The problem is,—"To find the number of independent ways in which a polygon of m sides can be divided into triangles by means of non-intersecting diagonals."

The solution seems to have been first discovered by Euler, who gives, but without proof, the number required for polygons up to the enneagon. The proof is supplied by J. A. de Segner, in a paper in the Petersburg Transactions [Nov. Comm., t. vii., pp. 203–210].

The question was next taken up in a series of four papers in Liouville's Journal for 1838–9. Proposed by Terquem, it was answered first by Lamé in a letter to Liouville (t. iii., pp. 505–7), and further discussed by O. Rodrigues (t. iii., pp. 547–8), M. J. Binet (t. iv., pp. 79–90), and E. Catalan (t. iv., pp. 91–94).

2. The result is most conveniently expressed in terms, not of the number (m) of angles of the polygon, but of the number (n) of triangles into which it can be divided. This number will be called the 'order' of the polygon; and it is clear that $m = n+2$. We write P_n for the required number of partitions in the case of a polygon of the n^{th} order.

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* [See infra, pp. 106—109, and p. 70.]

3. Lamé, in the paper referred to above, arrives by geometrical reasoning at the two functional equations

$$(\alpha) \dots\dots P_{n+1} = P_n + P_1 P_{n-1} + P_2 P_{n-2} + \dots + P_{n-1} P_1 + P_n,$$

$$(\beta) \dots\dots \quad P_n = \frac{n+2}{2(n-1)} (P_1 P_{n-1} + P_2 P_{n-2} + \dots + P_{n-1} P_1);$$

whence, immediately, it follows that

$$(\gamma) \dots\dots P_{n+1} = \frac{4n+2}{n+2} P_n.$$

Rodrigues gives an elegant direct demonstration of (γ). Binet's paper contains a solution by the method of generating functions with various 'reflections,' while Catalan, using a somewhat longer process, adds to the above equations a fourth,

$$(\delta) \dots\dots P_{n-1} - n\, P_{n-2} + \frac{n-1\,.\,n-2}{1\,.\,2} P_{n-3} - \dots = 0.$$

From (γ), together with the obvious relation $P_1 = 1$, we obtain at once

$$P_n = \frac{2n\,!}{n\,!\; \overline{n+1}\,!}.$$

4. It does not seem to have occurred to any of these writers that the problem admits of extension to the case of division into polygons other than triangles, and none of their methods is alone sufficient for this purpose, though the solution for the general case is as easy as for the simple case of the triangle.

5. We notice first that there is a necessary relation between the number of sides of the polygon to be resolved and that of the polygons into which it is to be resolved. If the latter number be p, we find the number (m) of sides of the polygon which can be divided by diagonals into n p-gons (*i.e.*, the polygon which is "of the n^{th} order for division into p-gons") thus. There are clearly $n-1$ diagonals, and each of them is a side of two component polygons. All the sides of the m-gon, together with $n-1$ diagonals each counted twice, are therefore all the sides of n p-gons, so that

$$m + 2(n-1) = np,$$

or

$$m = (p-2)\, n + 2.$$

6. To find the number, P_n, of ways in which the m-gon, whose vertices we will call $A_1, A_2 \dots A_m$, can be divided into n p-gons, we fix our attention on one side, for instance $A_1 A_2$. This must be a side of one of the component polygons: suppose that, in a partition chosen at random, the other vertices of the p-gon which has $A_1 A_2$ for a side are A_a, A_b, &c., the order being $A_1\, A_a\, A_b \dots A_2$. It is obviously a necessary

condition that the polygon which has the angular points from A_1 to A_a inclusive for its vertices should be capable of division into p-gons; and let it be of the order a. Similarly, the polygon with the angular points from A_a to A_b for its vertices must be divisible into p-gons,—let its order be b, and so on. Then the first of these polygons is divisible into a p-gons in P_a ways; the second into b p-gons in P_b ways, and so on. It follows that the number of independent ways in which the m-gon can be divided into p-gons, *with the condition that the p-gon which has* A_1A_2 *for a side is* $A_1A_aA_b \dots A_2$, is $P_a P_b P_c \dots$; for the central polygon $A_1A_aA_b \dots A_2$ serves as a neutral zone to prevent mutual interference among the partitions of the polygons $A_1 \dots A_a$, $A_a \dots A_b$, &c.

Both the number and the magnitude of the quantities $a, b, c, \dots$ are subject to a condition; for, since there is one of these quantities corresponding to a polygon cut off by each side of the central polygon save A_1A_2, their number is $p-1$; also, the total number of resulting p-gons is obtained by adding unity for the central polygon to a, b, &c., for the polygons $A_1 \dots A_a$, $A_a \dots A_b$, &c.; we have, therefore, the further condition
$$a+b+c+\dots = n-1.$$

It is to be observed that, if any of the vertices $A_1, A_a, A_b \dots$ are consecutive, there will be a diminution in the number of the quantities a, b, c, &c.; for instance, if A_a were consecutive to A_1, we should have $a = 0$. We can, however, allow for this by the introduction of a symbol P_0 equal in fact to unity. We have, then, the final result,

$$P_n = \Sigma P_a P_b P_c \dots, \quad \text{where} \quad a+b+c+\dots = n-1,$$

the number of the quantities $a, b, c \dots$ being $p-1$, and zero being admissible as a value for any of them.

7. This result may also be obtained by a method which will assign a meaning to the symbol

$$\Sigma P_a P_b P_c \dots (a+b+c+\dots = n-1),$$

where the number of factors in each term is less than $p-1$.

For the sake of clearness we will consider the case of division into pentagons, so that $p = 5$, although the method is quite general. Let the polygon to be divided be $ABCD \dots XYZA$ and be of the order n (so that it has $3n+2$ sides).

Let S_n be the number of ways of division, subject to the condition that no diagonal shall pass through either of three assigned consecutive angular points, say A, Z, Y. Then, since B must be joined to X to form the pentagon $ABXYZ$, while the polygon cut off by this diagonal (which is of the order $n-1$) can then be divided in all possible independent ways, we have

$$S_n = P_{n-1}.$$

Next, let R_n be the number of ways of division subject to the condition that no diagonal shall pass through either of two assigned consecutive angular points, say Z, Y. Then this number = the number of partitions in which, as a further condition, no diagonal passes through A + the number in which A is joined to no vertex further round the polygon (in the direction ABC...) than E + the number in which it is joined to no vertex further round than H + ...

$$= S_n+P_1 S_{n-1}+P_2 S_{n-2}+\ldots+P_{n-1} S_1$$

$$= P_0 P_{n-1}+P_1 P_{n-2}+P_2 P_{n-3}+\ldots+P_{n-1} P_0$$

$$= \Sigma P_a P_b \; (a+b=n-1).$$

Again, let Q_n be the number of ways of division subject to the condition that no diagonal passes through an assigned angular point, say Y. Then this number = the number of partitions in which no diagonal passes through Z + the number in which Z is joined to no vertex further round than D + the number in which it is joined to no vertex further round than G + ...

$$= R_n+P_1 R_{n-1}+\ldots+P_{n-1} R_1$$

$$= \Sigma P_a P_b P_c \; (a+b+c=n-1).$$

And, in the same way, we shall obtain

$$P_n = Q_n+P_1 Q_{n-1}+\ldots+P_{n-1} Q_1$$

$$= \Sigma P_a P_b P_c P_d \; (a+b+c+d=n-1).$$

Hence, in general, if P_n be the number of ways of dividing a polygon of order n into p-gons, the expression

$$\Sigma P_a P_b P_c \ldots (a+b+c+\ldots=n-1),$$

there being r factors in each term, is the number of ways of division under the condition that no diagonal is to pass through either of $p-r-1$ assigned consecutive angular points.

8. It only remains to solve the functional equation at the end of §6; which amounts to the statement that, if

$$f(x) = 1+P_1 x+P_2 x^2+ \ldots \text{ ad. inf.},$$

then f satisfies the equation

$$f^{p-1} = \frac{f-1}{x}$$

or

$$f = 1+xf^{p-1};$$

whence, by Lagrange's Theorem,

$$f = 1+x\,[f^{p-1}]+\frac{x^2}{1.2}\left[\frac{d}{df}f^{2(p-1)}\right]+\&c.,$$

where, in the square brackets, f is to be put equal to unity after differentiation.

Hence
$$P_n = \frac{1}{n!}\left[\frac{d^{n-1}}{df^{n-1}} f^{n(p-1)}\right]$$
$$= (np-n)(np-n-1)\dots(np-2n+2) \div n!$$
$$= \frac{\{(p-1)n\}!}{n!\,\{(p-2)n+1\}!},$$

which is therefore the number of independent ways in which a polygon of the order n (*i.e.*, a polygon of $(p-2)n+2$ sides) can be divided into p-gons.

Some Elliptic Function Formulæ. *By the* Rev. M. M. U. Wilkinson.

[*Read March 9th*, 1882.]

Since
$$k^2 \operatorname{sn}\alpha \operatorname{sn}\beta \operatorname{sn}(\alpha-\beta) = Z(\alpha-\beta) - Z\alpha + Z\beta,$$
$$k^2 \operatorname{sn}\beta \operatorname{sn}\gamma \operatorname{sn}(\beta-\gamma) = Z(\beta-\gamma) - Z\beta + Z\gamma,$$
$$k^2 \operatorname{sn}\gamma \operatorname{sn}\alpha \operatorname{sn}(\gamma-\alpha) = Z(\gamma-\alpha) - Z\gamma + Z\alpha,$$
$$k^2 \operatorname{sn}(\alpha-\beta)\operatorname{sn}(\beta-\gamma)\operatorname{sn}(\gamma-\alpha) = -Z(\alpha-\beta) - Z(\beta-\gamma) - Z(\gamma-\alpha);$$

we have
$$\operatorname{sn}\alpha \operatorname{sn}\beta \operatorname{sn}(\alpha-\beta) + \operatorname{sn}\beta \operatorname{sn}\gamma \operatorname{sn}(\beta-\gamma) + \operatorname{sn}\gamma \operatorname{sn}\alpha \operatorname{sn}(\gamma-\alpha)$$
$$= -\operatorname{sn}(\alpha-\beta)\operatorname{sn}(\beta-\gamma)\operatorname{sn}(\gamma-\alpha) \quad \dots\dots\dots\dots (1);$$

a known formula. Substituting $\alpha + iK'$ for α, &c., we have
$$\operatorname{sn}\alpha \operatorname{sn}(\beta-\gamma) + \operatorname{sn}\beta \operatorname{sn}(\gamma-\alpha) + \operatorname{sn}\gamma \operatorname{sn}(\alpha-\beta)$$
$$= -k^2 \operatorname{sn}\alpha \operatorname{sn}\beta \operatorname{sn}\gamma \operatorname{sn}(\beta-\gamma)\operatorname{sn}(\gamma-\alpha)\operatorname{sn}(\alpha-\beta) \dots (2),$$

since
$$k \operatorname{sn}\alpha \operatorname{sn}(\alpha+iK') = 1.$$

If we examine the way in which (1) can be obtained by means of the Addition Formulæ, it will appear that, *mutatis mutandis*, sn, cn, dn can be permuted, and corresponding results obtained as follows:—

$$\operatorname{dn}\alpha \operatorname{dn}\beta(\operatorname{dn}^2\alpha - \operatorname{dn}^2\beta)(\operatorname{sn}\gamma\operatorname{cn}\gamma\operatorname{dn}\beta + \operatorname{sn}\beta\operatorname{cn}\beta\operatorname{dn}\gamma)(\operatorname{sn}\alpha\operatorname{cn}\alpha\operatorname{dn}\gamma + \operatorname{sn}\gamma\operatorname{cn}\gamma\operatorname{dn}\alpha)$$
$$+\operatorname{dn}\beta \operatorname{dn}\gamma(\operatorname{dn}^2\beta - \operatorname{dn}^2\gamma)(\operatorname{sn}\alpha\operatorname{cn}\alpha\operatorname{dn}\gamma + \operatorname{sn}\gamma\operatorname{cn}\gamma\operatorname{dn}\alpha)(\operatorname{sn}\beta\operatorname{cn}\beta\operatorname{dn}\alpha + \operatorname{sn}\alpha\operatorname{cn}\alpha\operatorname{dn}\beta)$$
$$+\operatorname{dn}\gamma \operatorname{dn}\alpha(\operatorname{dn}^2\gamma - \operatorname{dn}^2\alpha)(\operatorname{sn}\beta\operatorname{cn}\beta\operatorname{dn}\alpha + \operatorname{sn}\alpha\operatorname{cn}\alpha\operatorname{dn}\beta)(\operatorname{sn}\gamma\operatorname{cn}\gamma\operatorname{dn}\beta + \operatorname{sn}\beta\operatorname{cn}\beta\operatorname{dn}\gamma)$$
$$= \operatorname{sn}\alpha\operatorname{cn}\alpha\operatorname{sn}\beta\operatorname{cn}\beta\operatorname{dn}\alpha\operatorname{dn}\beta\operatorname{dn}^2\gamma(\operatorname{dn}^2\alpha - \operatorname{dn}^2\beta + \operatorname{dn}^2\beta - \operatorname{dn}^2\gamma + \operatorname{dn}^2\gamma - \operatorname{dn}^2\alpha) + \dots$$
$$\dots + \operatorname{dn}^2\alpha\operatorname{dn}^2\beta(\operatorname{dn}^2\alpha - \operatorname{dn}^2\beta)\operatorname{sn}^2\gamma\operatorname{cn}^2\gamma + \dots$$
$$= \operatorname{dn}^2\alpha\operatorname{dn}^2\beta(\operatorname{dn}^2\alpha - \operatorname{dn}^2\beta)\operatorname{sn}^2\gamma\operatorname{cn}^2\gamma + \operatorname{dn}^2\beta\operatorname{dn}^2\gamma(\operatorname{dn}^2\beta - \operatorname{dn}^2\gamma)\operatorname{sn}^2\alpha\operatorname{cn}^2\alpha$$
$$+\operatorname{dn}^2\gamma\operatorname{dn}^2\alpha(\operatorname{dn}^2\gamma - \operatorname{dn}^2\alpha)\operatorname{sn}^2\beta\operatorname{cn}^2\beta.$$

Again,

$$(\mathrm{dn}^2\beta-\mathrm{dn}^2\alpha)\,(\mathrm{dn}^2\alpha-\mathrm{dn}^2\gamma)\,(\mathrm{dn}^2\gamma-\mathrm{dn}^2\beta)$$

$$=-\frac{k^4}{k'^2}\left[\mathrm{dn}^2\alpha\,\mathrm{dn}^2\beta(\mathrm{dn}^2\alpha-\mathrm{dn}^2\beta)\frac{1-\mathrm{dn}^2\gamma}{k^2}\cdot\frac{\mathrm{dn}^2\gamma-k'^2}{k^2}+\ldots\right]$$

$$=-\frac{k^4}{k'^2}\left[\mathrm{dn}^2\alpha\,\mathrm{dn}^2\beta\,(\mathrm{dn}^2\alpha-\mathrm{dn}^2\beta)\,\mathrm{sn}^2\gamma\,\mathrm{cn}^2\gamma+\ldots\right];$$

so that we have

$$\frac{k'^2}{k^4}\cdot\frac{\mathrm{dn}^2\beta-\mathrm{dn}^2\gamma}{\mathrm{sn}\,\gamma\,\mathrm{cn}\,\gamma\,\mathrm{dn}\,\beta+\mathrm{sn}\,\beta\,\mathrm{cn}\,\beta\,\mathrm{dn}\,\gamma}\cdot\frac{\mathrm{dn}^2\gamma-\mathrm{dn}^2\alpha}{\mathrm{sn}\,\alpha\,\mathrm{cn}\,\alpha\,\mathrm{dn}\,\gamma+\mathrm{sn}\,\gamma\,\mathrm{cn}\,\gamma\,\mathrm{dn}\,\alpha}$$

$$\times\frac{\mathrm{dn}^2\alpha-\mathrm{dn}^2\beta}{\mathrm{sn}\,\beta\,\mathrm{cn}\,\beta\,\mathrm{dn}\,\alpha+\mathrm{sn}\,\alpha\,\mathrm{cn}\,\alpha\,\mathrm{dn}\,\beta}$$

$$=\frac{\mathrm{dn}\,\beta\,\mathrm{dn}\,\gamma\,(\mathrm{dn}^2\beta-\mathrm{dn}^2\gamma)}{\mathrm{sn}\,\gamma\,\mathrm{cn}\,\gamma\,\mathrm{dn}\,\beta+\mathrm{sn}\,\beta\,\mathrm{cn}\,\beta\,\mathrm{dn}\,\gamma}+\ldots;$$

and, since $\dfrac{\mathrm{sn}\,(\alpha-\beta)}{\mathrm{cn}\,(\alpha-\beta)}=\dfrac{\mathrm{sn}^2\alpha-\mathrm{sn}^2\beta}{\mathrm{sn}\,\beta\,\mathrm{cn}\,\beta\,\mathrm{dn}\,\alpha+\mathrm{sn}\,\alpha\,\mathrm{cn}\,\alpha\,\mathrm{dn}\,\beta}$,

we have

$$\mathrm{dn}\,\beta\,\mathrm{dn}\,\gamma\,\frac{\mathrm{sn}\,(\beta-\gamma)}{\mathrm{cn}\,(\beta-\gamma)}+\ldots=\frac{k'^2\,\mathrm{sn}\,(\beta-\gamma)\,\mathrm{sn}\,(\gamma-\alpha)\,\mathrm{sn}\,(\alpha-\beta)}{\mathrm{cn}\,(\beta-\gamma)\,\mathrm{cn}\,(\gamma-\alpha)\,\mathrm{cn}\,(\alpha-\beta)}\ \ldots(3);$$

and, in like manner, or from (3) by changing

$$\mathrm{sn},\ \mathrm{cn},\ \mathrm{dn},\ k^2,\ k'^2,\ \text{ into }\ k\,\mathrm{sn},\ \mathrm{dn},\ \mathrm{cn},\ \frac{1}{k^2},\ -\frac{k'^2}{k^2},$$

we have

$$\mathrm{cn}\,\beta\,\mathrm{cn}\,\gamma\,\frac{\mathrm{sn}\,(\beta-\gamma)}{\mathrm{cn}\,(\beta-\gamma)}+\ldots=-\frac{k'^2\,\mathrm{sn}\,(\beta-\gamma)\,\mathrm{sn}\,(\gamma-\alpha)\,\mathrm{sn}\,(\alpha-\beta)}{\mathrm{cn}\,(\beta-\gamma)\,\mathrm{cn}\,(\gamma-\alpha)\,\mathrm{cn}\,(\alpha-\beta)}\ldots(4);$$

and, increasing in (3) α, &c., by K, we get

$$\mathrm{dn}\,\alpha\,\frac{\mathrm{sn}\,(\beta-\gamma)}{\mathrm{cn}\,(\beta-\gamma)}+\mathrm{dn}\,\beta\,\frac{\mathrm{sn}\,(\gamma-\alpha)}{\mathrm{cn}\,(\gamma-\alpha)}+\mathrm{dn}\,\gamma\,\frac{\mathrm{sn}\,(\alpha-\beta)}{\mathrm{cn}\,(\alpha-\beta)}$$

$$=\mathrm{dn}\,\alpha\,\mathrm{dn}\,\beta\,\mathrm{dn}\,\gamma\,\frac{\mathrm{sn}\,(\beta-\gamma)\,\mathrm{sn}\,(\gamma-\alpha)\,\mathrm{sn}\,(\alpha-\beta)}{\mathrm{cn}\,(\beta-\gamma)\,\mathrm{cn}\,(\gamma-\alpha)\,\mathrm{cn}\,(\alpha-\beta)}\ \ldots(5);$$

increasing in (4) α, &c., by $K+iK'$, we have

$$\mathrm{cn}\,\alpha\,\frac{\mathrm{sn}\,(\beta-\gamma)}{\mathrm{cn}\,(\beta-\gamma)}+\mathrm{cn}\,\beta\,\frac{\mathrm{sn}\,(\gamma-\alpha)}{\mathrm{cn}\,(\gamma-\alpha)}+\mathrm{cn}\,\gamma\,\frac{\mathrm{sn}\,(\alpha-\beta)}{\mathrm{cn}\,(\alpha-\beta)}$$

$$=k^2\,\mathrm{cn}\,\alpha\,\mathrm{cn}\,\beta\,\mathrm{cn}\,\gamma\,\frac{\mathrm{sn}\,(\beta-\gamma)\,\mathrm{sn}\,(\gamma-\alpha)\,\mathrm{sn}\,(\alpha-\beta)}{\mathrm{dn}(\beta-\gamma)\,\mathrm{dn}\,(\gamma-\alpha)\,\mathrm{dn}\,(\alpha-\beta)}\ \ldots(6);$$

It can be e seen from (1) that

$$\mathrm{sn}(\alpha+\beta+\gamma) = \frac{\{\mathrm{sn}(\alpha+\beta)\,\mathrm{sn}\,\beta-\mathrm{sn}\,\gamma\,\mathrm{sn}(\alpha+\gamma)\}}{\mathrm{n}(\alpha+\gamma)\,\mathrm{sn}\,\beta-\mathrm{sn}(\beta+\alpha)\,\mathrm{sn}\,\gamma}$$

$$= \frac{\{\mathrm{sn}\,\beta\,(\mathrm{sn}^2\alpha-\mathrm{sn}^2\beta)(\mathrm{sn}\,\alpha\,\mathrm{cn}\,\gamma\,\mathrm{dn}\,\gamma-\mathrm{sn}\,\gamma\,\mathrm{cn}\,\alpha\,\mathrm{dn}\,\alpha)-\mathrm{sn}\,\gamma\,(\mathrm{sn}^2\alpha-\mathrm{sn}^2\gamma)(\mathrm{sn}\,\alpha\,\mathrm{cn}\,\beta\,\mathrm{dn}\,\beta-\mathrm{sn} \quad \alpha\,\mathrm{dn}\,\alpha\}}{\mathrm{n}^2\alpha-\mathrm{sn}^2\gamma)\,\mathrm{sn}\,\beta\,(\mathrm{sn}\,\alpha\,\mathrm{cn}\,\beta\,\mathrm{dn}\,\beta-\mathrm{sn}\,\beta\,\mathrm{cn}\,\alpha\,\mathrm{dn}\,\alpha)-\mathrm{sn}\,\gamma\,(\mathrm{sn}^2\alpha-\mathrm{sn}^2\beta)\,(\mathrm{sn}\,\alpha\,\mathrm{cn}\,\gamma\,\mathrm{dn}\,\gamma-\mathrm{sn}\,\gamma \quad \mathrm{ln}\,\alpha)}$$

$$= \frac{\alpha\,\mathrm{dn}\,\alpha\,\mathrm{sn}\,\beta\,\mathrm{sn}\,\gamma\,(\mathrm{sn}^2\beta-\mathrm{sn}^2\gamma)+\mathrm{cn}\,\beta\,\mathrm{dn}\,\beta\,\mathrm{sn}\,\gamma\,\mathrm{sn}\,\alpha\,(\mathrm{sn}^2\gamma-\mathrm{sn}^2\alpha)+\mathrm{cn}\,\gamma\,\mathrm{dn}\,\gamma\,\mathrm{sn}\,\alpha\,\mathrm{sn}\,\beta\,(\mathrm{sn}^2 \quad {}^2\beta)}{\mathrm{cn}\,\alpha\,\mathrm{dn}\,\alpha\,\mathrm{sn}\,\alpha\,(\mathrm{sn}^2\beta-\mathrm{sn}^2\gamma)+\mathrm{cn}\,\beta\,\mathrm{dn}\,\beta\,\mathrm{sn}\,\beta\,(\mathrm{sn}^2\gamma-\mathrm{sn}^2\alpha)+\mathrm{cn}\,\gamma\,\mathrm{dn}\,\gamma\,\mathrm{sn}\,\gamma\,(\mathrm{sn}^2\alpha-\mathrm{sn}^2}\quad\ldots\ldots(7);$$

Again, sinc

$$\mathrm{cn} \quad \iota+\beta)-\mathrm{sn}\,\alpha\,\mathrm{dn}\,\beta\,\mathrm{cn}(\alpha+\beta) = \frac{\mathrm{cn}\,\alpha\,(\mathrm{sn}^2\alpha-\mathrm{sn}^2\beta)-\mathrm{sn}\,\alpha\,\mathrm{dn}\,\beta\,(\mathrm{sn}\,\alpha\,\mathrm{cn}\,\alpha\,\mathrm{dn}\,\beta-\mathrm{sn}\,\beta\,\mathrm{cn}\,\beta\,\mathrm{dn}}{\mathrm{sn}\,\alpha\,\mathrm{cn}\,\beta\,\mathrm{dn}\,\beta-\mathrm{sn}\,\beta\,\mathrm{cn}\,\alpha\,\mathrm{dn}\,\alpha}$$

$$= \frac{\mathrm{cn}\,\alpha\,\mathrm{sn}^2\alpha\,(1-\mathrm{dn}^2\beta)-\mathrm{sn}\,\beta\,(\mathrm{cn}\,\alpha\,\mathrm{sn}\,\beta-\mathrm{sn}\,\alpha\,\mathrm{dn}\,\beta\,\mathrm{cn}\,\beta\,\mathrm{dn}\,\alpha)}{\mathrm{sn}\,\alpha\,\mathrm{cn}\,\beta\,\mathrm{dn}\,\beta-\mathrm{sn}\,\beta\,\mathrm{cn}\,\alpha\,\mathrm{dn}\,\alpha}$$

$$= \frac{\mathrm{sn}\,\beta\,(k^2\,\mathrm{cn}\,\alpha\,\mathrm{sn}\,\beta\,\mathrm{sn}^2\alpha-\mathrm{cn}\,\alpha\,\mathrm{sn}\,\beta+\mathrm{sn}\,\alpha\,\mathrm{dn}\,\beta\,\mathrm{cn}\,\beta\,\mathrm{dn}\,\alpha)}{\mathrm{sn}\,\alpha\,\mathrm{cn}\,\beta\,\mathrm{dn}\,\beta-\mathrm{sn}\,\beta\,\mathrm{cn}\,\alpha\,\mathrm{dn}\,\alpha} = \mathrm{si} \quad \alpha;$$

we have, then,

$$\mathrm{sn}\,\beta\,\mathrm{dn}(\alpha+\gamma)\,\mathrm{cn}(\alpha+\beta+\gamma) = \mathrm{cn}\,\beta\,\mathrm{sn}(\alpha+\beta+\gamma)-\mathrm{sn}(\alpha+\gamma)\,\mathrm{dn}\,\beta;$$

and

$$\mathrm{sn}\,\gamma\,\mathrm{dn}(\alpha+\beta)\,\mathrm{cn}(\alpha+\beta+\gamma) = \mathrm{cn}\,\gamma\,\mathrm{sn}(\alpha+\beta+\gamma)-\mathrm{sn}(\alpha+\beta)\,\mathrm{dn}\,\gamma;$$

$$\mathrm{cn}(\alpha+\beta+\gamma) = \frac{\mathrm{sn}(\alpha+\beta)\,\mathrm{cn}\,\beta\,\mathrm{dn}\,\gamma-\mathrm{sn}(\alpha+\gamma)\,\mathrm{cn}\,\gamma\,\mathrm{dn}\,\beta}{\mathrm{sn}\,\beta\,\mathrm{cn}\,\gamma\,\mathrm{dn}(\alpha+\gamma)-\mathrm{sn}\,\gamma\,\mathrm{cn}\,\beta\,\mathrm{dn}(\alpha+\beta)}$$

$$= \frac{(\mathrm{sn}^2\alpha-\mathrm{sn}^2\beta) \quad \mathrm{dn}\,\gamma\,(\mathrm{sn}\,\alpha\,\mathrm{cn}\,\gamma\,\mathrm{dn}\,\gamma-\mathrm{sn}\,\gamma\,\mathrm{cn}\,\alpha\,\mathrm{dn}\,\alpha)-(\mathrm{sn}^2\alpha-\mathrm{sn}^2\gamma)\,\mathrm{cn}\,\gamma\,\mathrm{dn}\,\beta\,(\mathrm{sn}\,\alpha\,\mathrm{cn}\,\beta\,\mathrm{dn}\,\beta-\mathrm{sn}\,\beta\,\mathrm{cn}\,\alpha\,\mathrm{d}}{\mathrm{sn}\,\beta\,\mathrm{cn}\,\gamma\,(\mathrm{sn} \quad \iota\,\mathrm{cn}\,\gamma-\mathrm{sn}\,\gamma\,\mathrm{dn}\,\gamma\,\mathrm{cn}\,\alpha)\,(\mathrm{sn}\,\alpha\,\mathrm{cn}\,\beta\,\mathrm{dn}\,\beta-\mathrm{sn}\,\beta\,\mathrm{cn}\,\alpha\,\mathrm{dn}\,\alpha) \\ -\mathrm{sn}\,\gamma\,\mathrm{cn}\,\beta\,(\mathrm{sn}\,\alpha\,\mathrm{dn}\,\alpha\,\mathrm{cn}\,\beta-\mathrm{sn}\,\beta\,\mathrm{dn}\,\beta\,\mathrm{cn}\,\alpha)\,(\mathrm{sn}\,\alpha\,\mathrm{cn}\,\gamma\,\mathrm{dn}\,\gamma- \quad \mathrm{n}\,\alpha\,\mathrm{dn}\,\alpha)}$$

$$= \frac{\mathrm{sn}\,\alpha\,\mathrm{cn}\,\beta\,\mathrm{cn}\,\gamma[\quad \mathrm{sn}^2\alpha-\mathrm{sn}^2\beta)-\mathrm{dn}^2\beta(\mathrm{sn}^2\alpha-\mathrm{sn}^2\gamma)]+\mathrm{cn}\,\gamma\,\mathrm{cn}\,\alpha\,\mathrm{dn}\,\alpha\,\mathrm{dn}\,\beta\,\mathrm{sn}\,\beta(\mathrm{sn}^2\alpha-\mathrm{sn}^2\gamma)-\mathrm{cn}\,\alpha\,\mathrm{cn}\,\beta\,\mathrm{sn}\,\gamma\,\mathrm{dn}\,\gamma\,\mathrm{dn} \quad {}^2\alpha-\mathrm{sn}^2\beta)}{\mathrm{sn}\,\beta\,\mathrm{cn}\,\beta\,\mathrm{dn}\,\beta \quad \mathrm{dn}\,\alpha\,\mathrm{cn}^2\gamma-\mathrm{sn}^2\gamma\,\mathrm{cn}^2\alpha\,\mathrm{dn}\,\alpha)+\mathrm{sn}\,\alpha\,\mathrm{cn}\,\alpha\,\mathrm{dn}^2\alpha\,(\mathrm{sn}^2\gamma\,\mathrm{cn}^2\beta-\mathrm{sn}^2\beta\,\mathrm{cn}^2\gamma)+\mathrm{sn}\,\gamma\,\mathrm{cn}\,\gamma\,\mathrm{dn}\,\gamma\,\mathrm{dn}\,\alpha\,(\mathrm{sn}^2\beta\,\mathrm{cn} \quad \mathrm{n}^2\alpha\,\mathrm{cn}^2\beta)}$$

$$= \frac{\mathrm{sn}\,\alpha\,\mathrm{dn}\,\alpha\,\mathrm{cn} \quad (\mathrm{sn}^2\beta-\mathrm{sn}^2\gamma)+\mathrm{sn}\,\beta\,\mathrm{dn}\,\beta\,\mathrm{cn}\,\gamma\,\mathrm{cn}\,\alpha\,(\mathrm{sn}^2\gamma-\mathrm{sn}^2\alpha)+\mathrm{sn}\,\gamma\,\mathrm{dn}\,\gamma\,\mathrm{cn}\,\alpha\,\mathrm{cn}\,\beta\,(\mathrm{sn}^2\alpha-\mathrm{sn}^2\beta)}{\mathrm{sn}\,\alpha\,\mathrm{c} \quad \alpha\,(\mathrm{sn}^2\beta-\mathrm{sn}^2\gamma)+\mathrm{sn}\,\beta\,\mathrm{cn}\,\beta\,\mathrm{dn}\,\beta\,(\mathrm{sn}^2\gamma-\mathrm{sn}^2\alpha)+\mathrm{sn}\,\gamma\,\mathrm{cn}\,\gamma\,\mathrm{dn}\,\gamma\,(\mathrm{sn}^2\alpha-\mathrm{sn}^2\beta)}\quad\ldots\ldots\ldots\ldots(8),$$

and, in like mai or by permuting cn, dn, &c., we have

$$\mathrm{dn}(\alpha+\beta+\gamma) \quad \frac{\alpha\,\mathrm{cn}\,\alpha\,\mathrm{dn}\,\beta\,\mathrm{dn}\,\gamma\,(\mathrm{sn}^2\beta-\mathrm{sn}^2\gamma)+\mathrm{sn}\,\beta\,\mathrm{cn}\,\beta\,\mathrm{dn}\,\gamma\,\mathrm{dn}\,\alpha\,(\mathrm{sn}^2\gamma-\mathrm{sn}^2\alpha)+\mathrm{sn}\,\gamma\,\mathrm{cn}\,\gamma\,\mathrm{dn}\,\alpha\,\mathrm{dn}\,\beta\,(\mathrm{sn}^2\alpha \quad \beta)}{\mathrm{sn}\,\alpha\,\mathrm{cn}\,\alpha\,\mathrm{dn}\,\alpha\,(\mathrm{sn}^2\beta-\mathrm{sn}^2\gamma)+\mathrm{sn}\,\beta\,\mathrm{cn}\,\beta\,\mathrm{dn}\,\beta\,(\mathrm{sn}^2\gamma-\mathrm{sn}^2\alpha)+\mathrm{sn}\,\gamma\,\mathrm{cn}\,\gamma\,\mathrm{dn}\,\gamma\,(\mathrm{sn}^2\alpha-\mathrm{sn}^2}\ldots(9).$$

We may remark that we consider the formulæ

$$\operatorname{sn}(\alpha+\beta) = \operatorname{sn}^2\alpha - \operatorname{sn}^2\beta, \qquad (\div)$$

$$\operatorname{cn}(\alpha+\beta) = \operatorname{sn}\alpha\operatorname{cn}\alpha\operatorname{dn}\beta - \operatorname{sn}\beta\operatorname{cn}\beta\operatorname{dn}\alpha, \qquad (\div)$$

$$\operatorname{dn}(\alpha+\beta) = \operatorname{sn}\alpha\operatorname{dn}\alpha\operatorname{cn}\beta - \operatorname{sn}\beta\operatorname{dn}\beta\operatorname{cn}\alpha, \qquad (\div)$$

where denominator $= \operatorname{sn}\alpha\operatorname{cn}\beta\operatorname{dn}\beta - \operatorname{sn}\beta\operatorname{cn}\alpha\operatorname{dn}\alpha$,

simpler than the elementary formulæ usually given. These do not involve k explicitly, and have numerators and denominators one degree lower than those have.

They become vanishing fractions when

$$\alpha-\beta = 2miK'+2nK.$$

Thursday, April 6th, 1882.

S. ROBERTS, Esq., F.R.S., President, in the Chair.

Messrs. Buchheim, Muir, and Charles Smith were admitted into the Society.

The following communications were made:—

"The Algebraic Solution of the Modular Equation for the Septic Transformation," G. S. Ely.

"Note on the Condensation of Skew Determinants which are partially zero-axial, and on a Symmetric Determinant connected with Lagrange's Interpolation Problem:" T. Muir.

"On the Addition Equations for the Elliptic and Theta Functions of the Sum of n Arguments:" Rev. M. M. U. Wilkinson.

"On the General Equation of the Second Degree referred to Tetrahedral Coordinates:" Rev. A. J. C. Allen.

"On certain Loci and Envelopes belonging to triangles of given form inscribed and circumscribed to a given triangle:" Prof. Wolstenholme.

"On Binomial Biordinals:" Sir J. Cockle.

"On the Coordinates of a Plane Curve in Space:" H. W. Lloyd Tanner.

"On Polygons circumscribed about a confocal Cubic:" R. A. Roberts.

The following presents were received:—

"Carte-de-Visite Likeness," from F. Scott Haydon, B.A.

"Educational Times" for April, 1882.

"American Journal of Mathematics," Vol iv., No. 2.

"The Mathematical Visitor," Vol. ii., No. 1: from the Editor, Artemas Martin, M.A.

"Bulletin des Sciences Mathématiques et Astronomiques," Tome v, Août, Sept., 1881.

"Beiblätter zu den Annalen der Physik und Chemie," Band vi., Stück 3, No. 3, 1882.

"Bulletin de la Société Mathématique de France," Tome x., No. 1; Paris, 1882.

"Crelle," 92 Band, 1° and 2° Heften; Berlin, 1882.

"Journal de L'École Polytechnique," Cinquantième Cahier, Tome xxxi.; Paris, 1881.

"Atti della R. Accademia dei Lincei," Vol. vi., Fasc. 7°, 8°; Roma, 1882.

"Mittheilungen der Mathematischen Gesellschaft in Hamburg," No. 2 (ausgegeben im März 1882).

(i.) "Lösung der auf die trilineare Verwandtschaft ausgedehnten Projectivitätsproblem."

(ii.) "Elementares Beweis des Feuerbach'schen Satzes," von Dr. Hermann Schubert; Hamburg, 1882.

"Note on Mr. Russell's Paper 'On certain Definite Integrals, No. 10.'" By W. Spottiswoode, P.R.S., from "Proceedings of Royal Society," No. 218, 1882.

"Mathematical Fragments," by the late William Kingdon Clifford, F.R.S.

On the Addition Equations for the Elliptic and Theta Functions of the Sum of n Arguments. *By* Rev. M. M. U. Wilkinson.

[*Read April 6th*, 1882.]

We define as follows:—

$$V(\alpha, \beta) = \frac{\Theta^2 0\, \Theta_1(\alpha-\beta)}{\Theta^2\alpha\, \Theta^2\beta};$$

$$V(\alpha, \beta, \gamma) = \frac{\Theta^6 0\, \Theta_1(\alpha-\beta)\, \Theta_1(\alpha-\gamma)\, \Theta_1(\beta-\gamma)}{\Theta^4\alpha\, \Theta^3\beta\, \Theta^3\gamma};$$

or $$V(\alpha, \beta, \gamma) = \frac{\Theta^2 0\, \Theta_1(\alpha-\beta)\, \Theta_1(\alpha-\gamma)}{\Theta^3\alpha\, \Theta\beta\, \Theta\gamma} V(\beta, \gamma);$$

$$\ldots \quad \ldots \quad = \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$V(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n) = \frac{\Theta^n 0\, \Theta_1(\alpha_1-\alpha_2)\, \Theta_1(\alpha_1-\alpha_3) \ldots \Theta_1(\alpha_1-\alpha_n)}{\Theta^n\alpha_1\, \Theta\alpha_2\, \Theta\alpha_3 \ldots \Theta\alpha_n} V(\alpha_2, \alpha_3, \ldots \alpha_n);$$

The definitions which we shall now give are by determinants. One line of the determinant only will be written down. The other lines are in every case got by substituting the letters in order.

$$S(\alpha, \beta) = |\, \mathrm{sn}^2\alpha, 1 \,|;$$

$$C(\alpha, \beta) = |\, \mathrm{sn}\,\alpha\, \mathrm{cn}\,\alpha,\ \mathrm{dn}\,\alpha \,|;$$

$$D(\alpha, \beta) = |\, \mathrm{sn}\,\alpha\, \mathrm{dn}\,\alpha,\ \mathrm{cn}\,\alpha \,|;$$

$$N(\alpha, \beta) = -\,|\, \mathrm{cn}\,\alpha\, \mathrm{dn}\,\alpha,\ \mathrm{sn}\,\alpha \,|;$$