On the Partitions of a Polygon. By Professor CAYLEY.

[*Read March* 12*th*, 1891.]

1. The partitions are made by non-intersecting diagonals; the problems which have been successively considered are (1) to find the number of partitions of an r-gon into triangles, (2) to find the number of partitions of an r-gon into k parts, and (3) to find the number of partitions of an r-gon into p-gons, r of the form $n(p-2)+2$.

Problem (1) is a particular case of (2); and it is also a particular case of (3); but the problems (2) and (3) are outside each other; for problem (3) a very elegant solution, which will be here reproduced, is given in the paper, H. M. Taylor and R. C. Rowe, "Note on a Geometrical Theorem," *Proc. Lond. Math. Soc.*, t. XIII. (1882), pp. 102–106, and this same paper gives the history of the solution of (1).

The solution of (2) is given in the memoir, Kirkman "On the k-partitions of the r-gon and r-ace," *Phil. Trans.*, t. 147 (for 1857), p. 225; viz., he there gives for the number of partitions of the r-gon into k parts (or, what is the same thing, by means of $k-1$ non-intersecting diagonals) the expression

$$\frac{[r+k-2]^{k-1}\,[r-3]^{k-1}}{[k]^{k-1}\,[k-1]^{k-1}};$$

but there is no complete demonstration of this result.

If $k=r-2$, we have the solution of the problem (1); viz., the number of partitions of the r-gon into triangles is

$$=[2r-4]^{r-3}\div[r-2]^{r-3}.$$

The present paper relates chiefly to the foregoing problem (2), the determination of the number of partitions of the r-gon into k parts, or, what is the same thing, by means of $k-1$ non-intersecting diagonals.

2. Assuming for the moment the foregoing result, then for $k=1$, the number of partitions is $=1$,

for $k=2$ it is $$=\frac{r\,.\,r-3}{2},$$

for $k=3$ it is $$=\frac{r+1\,.\,r\,.\,r-3\,.\,r-4}{12},$$

and so on. As a simple verification, $k=2$, the number of partitions is equal to the number of diagonals, viz., this is number of pairs of summits *less* number of sides, that is

$$\tfrac{1}{2}r\,(r-1)-r,\quad =\tfrac{1}{2}r\,(r-3).$$

For convenience I give the Table on next page, which is a tabulation of the functions

$$U_1=x^3+\ x^4+\ x^5+\ \ x^6+\dots x^r,$$

$$U_2=\qquad 2x^4+5x^5+\ 9x^6+\dots\frac{r.r-3}{2.1}x^r,$$

$$U_3=\qquad 5x^5+21x^6+\dots\frac{r+1.r.r-3.r-4}{3.2.2.1}x^r,$$

$$U_4=\qquad 14x^6+\dots\frac{r+2.r+1.r.r-3.r-4.r-5}{4.3.2.3.2.1}x^r,$$

$$U_5=\qquad \frac{r+3.r+2.r+1.r.r-3.r-4.r-5.r-6}{5.4.3.2.4.3.2.1}x^r.$$

&c.

3. And in connexion herewith I give the Table on page 240, which is a tabulation of the functions

$$V_2=x^6+2x^7+\ 3x^8+\ 4x^9\dots+1\,\frac{r-3}{1}x^{r+2},$$

$$V_3=\qquad 4x^7+14x^8+32x^9\dots+2\,\frac{r+1.r-3.r-4}{3.2.1}x^{r+2},$$

$$V_4=\qquad 14x^8+72x^9\dots+3\,\frac{r+2.r+1.r-3.r-4.r-5}{4.3.3.2.1}x^{r+2},$$

$$V_5=\qquad 48x^9\dots+4\,\frac{r+3.r+2.r+1.r-3.r-4.r-5.r-6}{5.4.3.4.3.2.1}x^{r+2},$$

&c.

$r =$	$k = 1$	2	3	4	5	6	7	8	9	10	11	12	13
3	1												
4	1	2											
5	1	5	5										
6	1	9	21	14									
7	1	14	56	84	42								
8	1	20	120	300	330	132							
9	1	27	225	825	1485	1287	429						
10	1	35	385	1925	5005	7007	5005	1430					
11	1	44	616	4004	14014	28028	32032	19448	4862				
12	1	54	936	7644	34398	91728	148512	143208	75582	16796			
13	1	65	1365	13650	76440	259896	556920	755820	629850	293930	58786		
14	1	77	1925	23100	157080	659736	1790712	3197700	3730650	2735810	1144066	208012	
15	1	90	2640	37400	302940	1534896	5116320	11511720	17587350	17978180	11767536	4457400	742900

$r =$	$k = 2$	3	4	5	6	7	8	9	10	11	12	13
4	1											
5	2	4										
6	3	14	14									
7	4	32	72	48								
8	5	60	225	330	165							
9	6	100	550	1320	1430	572						
10	7	154	1155	4004	7007	6006	2002					
11	8	224	2184	10192	25480	34944	24052	7072				
12	9	312	3822	22932	76440	148512	167076	100776	25194			
13	10	420	6300	47040	199920	514080	813960	775200	396800	90440		
14	11	550	9900	89760	476240	1534996	3197700	4263600	3517470	1634380	326876	
15	12	704	14960	161568	1023264	4093056	10744272	18759840	18573816	15690048	6547520	1188640

4. These functions, U and V, are particular values satisfying the equation

$$(V_2+V_3y+V_4y^2+\ldots) = (U_1+U_2y+U_3y^2+U_4y^3+\ldots)^2;$$

that this is so will appear from the following general investigation.

5. Taking x, y as independent variables, denoting by X an arbitrary function of x, and using accents to denote differentiations in regard to x, we require the following identity :

$$\frac{2}{1.2}X^2+\frac{4y}{1.2.3}(X^3)'+\frac{6y^2}{1.2.3.4}(X^4)''+\ldots$$
$$=\left\{X+\frac{y}{1.2}(X^2)'+\frac{y^2}{1.2.3}(X^3)''+\ldots\right\}^2,$$

which I prove as follows. Writing U to denote the same function of u which X is of x, I start from the equation

$$u=x+yU,$$

which determines u as a function of the independent variables x, y. We have

$$\frac{du}{dy}(1-yU')=U,\quad \frac{du}{dx}(1-yU')=1,$$

where the accent denotes differentiation in regard to u; hence

$$\frac{du}{dy}=U\frac{du}{dx}=\frac{u-x}{y}\frac{du}{dx},\text{ or say }y\frac{du}{dy}=(u-x)\frac{du}{dx}.$$

Writing $$u_1=\int u\,dx,$$

and therefore $$\frac{du_1}{dx}=u,$$

this equation may be written

$$y\frac{d^2u_1}{dx\,dy}-\frac{du_1}{dx}=u\frac{du}{dx}-u-x\frac{du}{dx};$$

or, integrating with respect to x, we have

$$y\frac{du_1}{dy}-u_1=\tfrac{1}{2}u^2-ux,$$

or say $$\frac{2}{y}\,\frac{du_1}{dy}-\frac{2\,(u_1-\frac{1}{2}x^2)}{y^2}=\frac{(u-x)^2}{y^2},$$

that is, $$2\,\frac{d}{dy}\left(\frac{u_1-\frac{1}{2}x^2}{y}\right)=\frac{(u-x)^2}{y^2}.$$

6. But, from the equation

$$u=x+yU,$$

we have $$u=x+yX+\frac{y^2}{1.2}\,(X^2)'+\frac{y^3}{1.2.3}\,(X^3)''+\ldots,$$

and thence $$u_1=\tfrac{1}{2}x^2+yX_1+\frac{y^2}{1.2}\,X^2+\frac{y^3}{1.2.3}\,(X^3)'+\ldots,$$

if for a moment X_1 is written for $\int X dx$. And hence, from the relation obtained above, we have the required identity

$$\frac{2}{1.2}\,X^2+\frac{4y}{1.2.3}\,(X^3)'+\frac{6y^2}{1.2.3.4}\,(X^4)''+\ldots$$
$$=\left\{X+\frac{y}{1.2}\,(X^2)'+\frac{y^2}{1.2.3}\,(X^3)''+\ldots\right\}^2.$$

This of course gives the series of identities

$$\frac{2}{1.2}\,X^2\;=X^2,$$

$$\frac{4}{1.2.3}\,(X^3)'=\frac{2}{1.2}\,X\,(X^2)',$$

$$\frac{6}{1.2.3.4}\,(X^4)''=\frac{2}{1.2.3}\,X\,(X^3)''+\left\{\frac{1}{1.2}\,(X^2)'\right\}^2;$$

or say $$X^2\;=X^2,$$
$$(X^3)'=\tfrac{3}{2}X\,(X^2)',$$
$$(X^4)''=\tfrac{4}{3}X\,(X^3)''+\left\{(X^2)'\right\}^2,$$

any of which may be easily verified.

7. I multiply each side of the identity by x^2, and write

$$U_1 = x.X, \qquad V_2 = x^2\frac{2}{1.2}X^2,$$

$$U_2 = x\frac{1}{1.2}(X^2)', \qquad V_3 = x^2\frac{4}{1.2.3}(X^3)',$$

$$U_3 = x\frac{1}{1.2.3}(X^3)'', \qquad V_4 = x^2\frac{6}{1.2.3.4}(X^4)'';$$

$$U_4 = x\frac{1}{1.2.3.4}(X^4)''', \qquad V_5 = x^2\frac{8}{1.2.3.4.5}(X^5)'''.$$

$$\vdots \qquad\qquad \vdots$$

We thus obtain two sets of functions U, V, satisfying the before-mentioned equation. We have

$$(V_2+yV_3+y^2V_4+\dots) = (U_1+yU_2+y^2U_3+\dots)^2;$$

and it will be observed that we have, moreover, the relations

$$U_2 = \tfrac{1}{2}x\,(x^{-2}V_2)', \quad U_3 = \tfrac{1}{4}x\,(x^{-2}V_3)', \quad U_4 = \tfrac{1}{6}x\,(x^{-2}V_4)', \dots.$$

8. In particular, if $\qquad X = \dfrac{x^2}{1-x},$

then the general term

in X is x^{r-1}, the first term occurring when $r = 3, \dots$

in X^2 is $(r-3)\,x^r$, " " $r = 4, \dots$

in X^3 is $\dfrac{r-3.r-4}{1.2}\,x^{r+1}$, " " $r = 5, \dots$

in X^4 is $\dfrac{r-3.r-4.r-5}{1.2.3}\,x^{r+2}$, " " $r = 6, \dots$

from which it appears that, for this value of X, U_1, U_2, U_3, U_4, &c. have the before-mentioned values (No. 2), and further that V_2, V_3, V_4, V_5, &c. have also the before-mentioned values (No. 3).

9. We do not absolutely require, but it is interesting to obtain, the finite expressions of these functions. We have

$$
\begin{aligned}
(1-x)\ U_1 &= x^3\,(1),\\
(1-x)^3\ U_2 &= x^4\,(2-x),\\
(1-x)^5\ U_3 &= x^5\,(5-4x+x^2),\\
(1-x)^7\ U_4 &= x^6\,(14-14x+6x^2-x^3),\\
(1-x)^9\ U_5 &= x^7\,(42-48x+27x^2-8x^3+x^4),\\
(1-x)^{11}\ U_6 &= x^8\,(132-165x+110x^2-44x^3+10x^4-x^5)\ ;\\
&\ \vdots\\
(1-x)^2\ V_2 &= x^6\,(1),\\
(1-x)^4\ V_3 &= x^7\,(4-2x),\\
(1-x)^6\ V_4 &= x^8\,(14-12x+3x^2),\\
(1-x)^8\ V_5 &= x^9\,(48-54x+24x^2-4x^3),\\
(1-x)^{10}\ V_6 &= x^{10}\,(165-220x+132x^2-40x^3+5x^4),\\
&\ \vdots
\end{aligned}
$$

and here the factors in ( ) satisfy the series of relations

$$
\begin{aligned}
1 &= 1^2,\\
4-2x &= 2\,(2-x),\\
14-12x+3x^2 &= 2.1\,(5-4x+x^2)+(2-x)^2,\\
48-54x+24x^2-4x^3 &= 2.1\,(14-14x+6x^2-x^3)+2\,(2-x)(5-4x+x^2),\\
&\ \vdots
\end{aligned}
$$

corresponding to $\qquad V_2 = U_1^2,\quad V_3 = 2U_1U_2,\ \&c.,$

given by the before-mentioned equation (No. 7), between the functions V and U.

10. It is to be shown that, taking $U_1, U_2, U_3, \ldots$ for the functions which belong to the partitions of the r-gon (assumed to be unknown functions of r and the suffixes), and connecting them with a set of

functions V_2, V_3, V_4, ... by the relations

$$U_2 = \tfrac{1}{2}x\,(x^{-2}V_2)',\quad U_3 = \tfrac{1}{4}x\,(x^{-2}V_3)',\quad U_4 = \tfrac{1}{6}x\,(x^{-2}V_4)',\ \&c.,$$

then we have the foregoing identical equation

$$(V_2+V_3y+V_4y^2+\dots) = (U_1+U_2y+U_3y^2+U_4y^3+\dots)^2.$$

This implies the relations

$$\begin{aligned} V_2 &= U_1^2, \\ V_3 &= 2U_1U_2, \\ V_4 &= 2U_1U_3+U_2^2, \\ V_5 &= 2U_1U_4+2U_2U_3, \\ &\&c. \end{aligned}$$

Thus, if U_1 is known, the equation

$$V_2 = U_1^2$$

determines V_2, and then

$$U_2 = \tfrac{1}{2}x\,(x^{-2}V_2)'$$

determines U_2, so that U_1, U_2 are known; and we thence in the same way find successively U_3 and V_3, U_4 and V_4, and so on; that is, assuming only that U_1 has the before-mentioned value,

$$U_1 = x^3+x^4+x^5+\dots+x^r\dots,$$

it follows that all the remaining functions U and V must have their before-mentioned values. But the function

$$U_1 = x^3+x^4+x^5+\dots,$$

where each coefficient is $=1$, is evidently the proper expression for the generating function of the number of partitions of the r-gon into a single part; and we thus arrive at the proof that the remaining functions U, which are the generating functions for the number of partitions of the r-gon into 2, 3, 4, ... k, parts, have their before-mentioned values.

11. Considering, then, the partition problem from the point of view just referred to, I write A_r, B_r, C_r, ... for the number of partitions of

an r-gon into 1 part, 2 parts, 3 parts, &c., and form therewith the generating functions

$$\begin{aligned} U_1 &= A_3x^3+A_4x^4 \quad \ldots + \quad A_rx^r \ \ldots, \\ U_2 &= \qquad\quad B_4x^4 \quad \ldots + \quad B_rx^r \ \ldots, \\ U_3 &= \qquad\qquad\quad C_5x^5 \ldots + \quad C_rx^r \ \ldots, \\ &\ \vdots \end{aligned}$$

and also the functions

$$\begin{aligned} V_2 &= \quad \frac{2}{4}B_4x^6 \quad \ldots + \frac{2}{r}B_rx^{r+2}\ldots, \\ V_3 &= \qquad\quad \frac{4}{5}C_5x^7\ldots + \frac{4}{r}C_rx^{r+2}\ldots, \\ &\ \vdots \end{aligned}$$

where observe that the functions U, V are such that

$$U_2 = \tfrac{1}{2}x\,(x^{-2}V_2)', \quad U_3 = \tfrac{1}{4}x\,(x^{-2}V_3)', \quad U_4 = \tfrac{1}{6}x\,(x^{-2}V_4)', \ \ \&c.$$

To fix the ideas, consider an r-gon which is to be divided into six parts. Choosing any particular summit, and from this summit drawing a diagonal successively to each of the non-adjacent $r-3$ summits, we divide the r-gon into two parts in $r-3$ different ways; viz., the two parts are

$$\begin{array}{lcl} \text{a } \ 3\text{-gon} & \text{and} & (r-1)\text{-gon}, \\ \quad\ 4\text{-gon} & \text{and} & (r-2)\text{-gon}, \\ \qquad\vdots & & \\ (r-1)\text{-gon} & \text{and} & 3\text{-gon}; \end{array}$$

say any one of these ways is

$$\text{an } \ \alpha\text{-gon} \quad \text{and} \quad \beta\text{-gon}, \quad \alpha+\beta = r-2.$$

Next, writing $$a+b = 6,$$

that is,

$$\begin{aligned} a,\ b = {} & 1,\ 5, \\ & 2,\ 4, \\ & 3,\ 3, \\ & 4,\ 2, \\ & 5,\ 1, \end{aligned}$$

we divide in every possible way the α-gon into a parts, and the β-gon into b parts (so dividing the r-gon into six parts). Observing that A, B, C, D, E, F are the letters belonging to the numbers 1, 2, 3, 4, 5, 6, respectively, the number of parts which we thus obtain (corresponding to the different values of a, b) are

$$A_\alpha E_\beta + B_\alpha D_\beta + C_\alpha C_\beta + D_\alpha B_\beta + E_\alpha A_\beta,$$

and summing for the different values of α, β $(\alpha+\beta = r+2)$, the whole number of parts is

$$= \text{coeff. } x^{r+2} \text{ in } (U_1U_5 + U_2U_4 + U_3U_3 + U_4U_2 + U_5U_1),$$

that is, $\quad$ in $(2U_1U_5 + 2U_2U_4 + U_3^2)$.

12. To obtain the whole number of the partitions of the r-gon into six parts, we must perform the foregoing process successively with each summit of the r-gon as the summit from which is drawn the diagonal which divides the r-gon into two parts; that is, the number found as above is to be multiplied by r. We thus obtain all the partitions repeated a certain number of times, viz., each partition into six parts is a partition by means of five diagonals, and is thus obtainable by the foregoing process, taking any one of the ten extremities of these diagonals as the point from which is drawn the diagonal which divides the r-gon into two parts; that is, we have to divide the foregoing product by 10. The final result thus is

$$\frac{10}{r} F_r = \text{coeff. } x^{r+2} \text{ in } (2U_1U_5 + 2U_2U_4 + U_3^2),$$

where $\quad \dfrac{10}{r} F_r$ is $=$ coeff. x^{r+2} in V_6;

we thus have $\quad V_6 = 2U_1U_5 + 2U_2U_4 + U_3^2.$

13. The reasoning is perfectly general; and applying it successively to the partitions into two parts, three parts, &c., we have

$$V_2 = U_1^2,$$

$$V_3 = 2U_1U_2,$$

$$V_4 = 2U_1U_3 + U_2^2,$$

$$V_5 = 2U_1U_4 + 2U_2U_3,$$

where any function V is related to the corresponding function U as above. The value of U_1 is obviously

$$U_1 = x^3+x^4+x^5+\ldots = \frac{x^3}{1-x};$$

and hence the several functions U and V have the values above written down; the general term of U_k is

$$\frac{[r+k-2]^{k-1}\,[r-3]^{k-1}}{[k]^{k-1}\,[k-1]^{k-1}}\,x^r;$$

and the number of partitions of the r-gon into k parts is equal to the coefficient of x^r in this general term.

14. In the investigations which next follow, I consider, without using the method of generating functions, the problem of the partition of the r-gon into 2, 3, 4 or 5 parts; it will be convenient to state the results as follows:

Number of Partitions.

2 parts $\quad \frac{r}{2}\;A,$

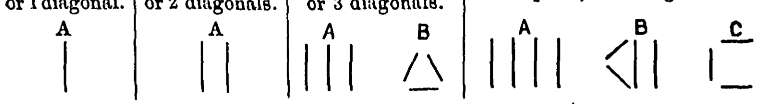
3 parts $\quad \frac{r}{4}\;2A,$

4 parts $\quad \frac{r}{6}\,(3A+2B),$

5 parts $\quad \frac{r}{8}\,(4A+8B+2C);$

where the capital letters refer to different "diagonal-types," thus:

2 parts, or 1 diagonal.	3 parts, or 2 diagonals.	4 parts, or 3 diagonals.	5 parts, or 4 diagonals.



viz.: if, in a polygon divided into k parts by means of $k-1$ diagonals, we delete all the sides of the polygon, leaving only the diagonals, then these will present themselves under distinct forms, which are what I call "diagonal-types"; for instance,

$$k = 4,$$

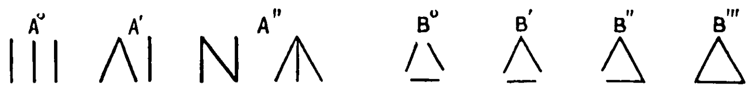
there are the two types A and B shown in the above diagram for four parts.

15. It is to be observed that we have sub-types corresponding to the coalescence of the terminal points of different diagonals; thus,

$$k = 4.$$

Writing now A° and B° to denote the forms without coalescences, we have the sub-types A°, A', A'' and B°, B, B', B'', B''', as follows:

4 parts, or 3 diagonals.



where observe that under A'' are included two distinct forms, which, nevertheless, by reason that there is in each of them the same number $(=2)$ of coalescences, are reckoned as belonging to the same sub-type.

16. The numbers called A, B, C, &c. have values which may be directly determined. I write down as follows:

1 diagonal $A = \dfrac{r-3}{1}$,

2 diagonals $A = \dfrac{r-3.r-2.r-1}{6} - \dfrac{r-3}{1} = \dfrac{r-3.r-4}{6}(r+1)$.

$$\begin{array}{rrr} \text{Calculation is} & r-2.r-1 = & r^2-3r+2 \\ & -6 & -6 \\ \hline & & r^2-3r-4 \\ & & = r-4.r+1. \end{array}$$

3 diagonals

$$A = \frac{r-3.r-2.r-1.r.r+1}{120} - 2\left(\frac{r-3.r-2.r-1}{6} - \frac{r-3}{1}\right) - \frac{r-3}{1}$$

$$= \frac{r-3.r-2.r-1.r.r+1}{120} - 2\frac{r-3.r-2.r-1}{6} + 1\frac{r-3}{1}$$

$$= \frac{r-3.r-4.r-5}{120}(r^2+7r+2).$$

Calculation is

$$
\begin{array}{ll}
r-2.r-1.r.r+1 = & r^4-2r^3-\ \ r^2+\ \ 2r \\
\quad -40.r-2.r-1 & \qquad -40r^2+120r-\ 80 \\
\qquad\qquad +120 & \qquad\qquad\qquad\quad +120 \\
\hline
& r^4-2r^3-41r^2+122r+\ 40 \\
& = r-4.r-5.r^2+7r+2.
\end{array}
$$

$$B = \frac{r-5.r-4.r-3.r-2.r-1}{120}$$

$$= \frac{r-3.r-4.r-5}{120}(r-1.r-2)$$

4 diagonals $A = \dfrac{r-3.r-2.r-1.r.r+1.r+2.r+3}{5040}$

$$-3\frac{r-3.r-2.r-1.r.r+1}{120}+3\frac{r-3.r-2.r-1}{6}-1\frac{r-3}{1}$$

$$= \frac{r-3.r-4.r-5.r-6}{5040}(r^3+18r^2+65r).$$

Calculation is

$$
\begin{array}{rl}
r-2.r-1.r.r+1.r+2.r+3 = & r^6+3r^5-\ \ 5r^4-\ 15r^3+\ \ \ 4r^2+\ \ 12r \\
-126.r-2.r-1.r.r+1 & \qquad\ -126r^4+252r^3+\ 126r^2-\ 252r \\
+2520.r-2.r-1 & \qquad\qquad\qquad\qquad +2520r^2-7560r+5040 \\
-5040 & \qquad\qquad\qquad\qquad\qquad\qquad\qquad -5040 \\
\hline
& r^6+3r^5-131r^4+237r^3+2650r^2-7800r \\
& = r-4.r-5.r-6.r^3+18r^2+65r.
\end{array}
$$

$$B = \frac{r-5.r-4.r-3.r-2.r-1.r.r+1}{5040} - \frac{r-5.r-4.r-3.r-2.r-1}{120}$$

$$= \frac{r-3.r-4.r-5.r-6}{5040}(r-1.r-2.r+7),$$

$$C = \frac{r-7.r-6.r-5.r-4.r-3.r-2.r-1}{5040}$$

$$= \frac{r-3.r-4.r-5.r-6}{5040}(r-1.r-2.r-7).$$

Calculation is

$$
\begin{array}{rl}
r.r+1 & = r^2+r \\
-42 & \quad\ \ -42 \\
\hline
& r^2+r-42 \\
& = r-6.r+7.
\end{array}
$$

17. To explain the formation of these expressions, observe that:

One diagonal.—There must be on each side of the diagonal, or say in each of the two "intervals" formed by the diagonal, two sides; there remain $r-4$ sides which may be distributed at pleasure between the two intervals, and the number of ways in which this can be done is

$$=\frac{r-3}{1}.$$

Two diagonals.—There must be on each side of the two diagonals, or say in two of the four intervals formed by the diagonals, two sides; there remain $r-4$ sides to be distributed between the same four intervals, and the number of ways in which this can be done is

$$=\frac{r-3.r-2.r-1}{6}.$$

But we must exclude the distributions where there is 0 side in the one interval and 0 side in the other interval between the two diagonals; the number of these is that for the case of the coalescence of the two diagonals into a single diagonal, viz., it is

$$=\frac{r-3}{1};$$

and thus the number required is

$$\frac{r-3.r-2.r-1}{6}-\frac{r-3}{1}.$$

18. Three diagonals, A.—There must be on each side of the three diagonals, that is, in two of the six intervals formed by the diagonals, two sides; there remain $r-4$ sides to be distributed between the same six intervals, and the number of ways in which this can be done is

$$=\frac{r-3.r-2.r-1.r.r+1}{120}.$$

But we must exclude distributions which would permit the coalescence of the first and second, or of the second and third, or of all three of the diagonals. For the coalescence of the first and second diagonals (the third diagonal not coalescing) the term to be subtracted is

$$\frac{r-3.r-2.r-1}{6}-\frac{r-3}{1};$$

and the same number for the coalescence of the second and third diagonals (the first diagonal not coalescing); that is, the last-mentioned number is to be multiplied by 2; and for the coalescence of all three diagonals the number to be subtracted is

$$=\frac{r-3}{1};$$

we have thus the foregoing value

$$\frac{r-3.r-2.r-1.r.r+1}{120}-2.\frac{r-3.r-2.r-1}{6}+1\frac{r-3}{1},$$

where it will be observed that we have the binomial coefficients 1, 2, 1 with the signs $+$, $-$, $+$.

Three diagonals, *B*.—There must be outside each of the three diagonals, that is, in three of the six intervals formed by the diagonals, two sides; and there remain $r-6$ sides to be distributed between the six intervals; the number of ways in which this can be done is

$$=\frac{r-5.r-4.r-3.r-2.r-1}{120};$$

and there is here no coalescence of diagonals, so that this is the number required.

19. Four diagonals, *A*.—There must be on each side of the four diagonals, that is, in two of the eight intervals formed by the diagonals, two sides; there remain $r-4$ sides to be distributed between the eight intervals, and the number of ways in which this can be done is

$$\frac{r-3.r-2.r-1.r.r+1.r+2.r+3}{5040}.$$

But this number requires to be corrected for coalescences, as in the case Three diagonals, *A*, and the required number is thus found to be

$$\frac{r-3.r-2.r-1.r.r+1.r+2.r+3}{5040}-3\frac{r-3.r-2.r-1.r.r+1}{120}$$
$$+3\frac{r-3.r-2.r-1}{6}+1\frac{r-3}{1}.$$

Four diagonals, *B*.—There must be outside of three of the diagonals, that is, in each of three of the eight intervals formed by the diagonals, two sides; there remain $r-6$ sides to be distributed between the eight intervals, and the number of ways in which this can be done is

$$\frac{r-5.r-4.r-3.r-2.r-1.r.r+1}{5040}.$$

There is a correction for the coalescence of two of the diagonals, giving rise to a form such as Three diagonals, B, and consequently a term

$$-\frac{r-5.r-4.r-3.r-2.r-1}{120},$$

which, with the first-mentioned term, gives the required number.

Four diagonals, C.—There must be outside of each of the diagonals, that is, in each of four of the eight intervals formed by the diagonals, two sides; there remain $r-8$ sides to be distributed between the eight intervals, and the number of ways in which this can be done is

$$\frac{r-7.r-6.r-5.r-4.r-3.r-2.r-1}{5040},$$

which is the required number.

20. In the expressions of No. 14, A, $2A$, $3A+2B$, $4A+8B+2C$, if we regard the terminals of the diagonals as given points, then (1) we have two summits, which can be joined in one way only, giving rise to the diagonal-type A; (2) we have four summits, which can be joined in two ways only, so as to give rise to the diagonal-type A; (3) we have six summits, which can be joined in three ways so as to give rise to a diagonal-type A, and in two ways so as to give rise to a diagonal-type B; and (4) we have eight summits, which can be joined in four ways so as to give rise to a diagonal-type A, in eight ways so as to give rise to a diagonal-type B, and in two ways so as to give rise to a diagonal-type C; we have thus the linear forms in question. To obtain the number of partitions, we have in each case to multiply by r. To explain this, after the polygon is drawn, imagine the summits to be numbered 1, 2, 3, ... r in succession (the numbering may begin at any one of the r summits), and, regarding each of these numberings as giving a different partition, we should have the factor r. But in fact the partitions so obtained are not all of them distinct, but we have in each case a system of partitions repeated as many times as there are summits of the diagonals, that is, a number of times equal to twice the number of the diagonals; and we have thus, after the multiplication by r, to divide by the numbers 2, 4, 6, 8, in the four cases respectively.

21. We hence have immediately

Two parts, number of partitions

$$=\frac{r}{2}A=\frac{r.r-3}{2.1};$$

Three parts, number of partitions

$$=\frac{r}{2}A=\frac{r+1.r.r-3.r-4}{3.2.2.1};$$

Four parts, number of partitions

$$=\frac{r}{6}(3A+2B)=\frac{r+2.r+1.r.r-3.r-4.r-5}{4.3.2.3.2.1},$$

the calculation being

$$\begin{array}{rl} 3(r^2+7r+2)= & 3r^2+21r+\ 6 \\ +2.r-1.r-2 & +2r^2-\ 6r+\ 4 \\ \hline & 5r^2+15r+10 \\ = & 5.r+1.r+2; \end{array}$$

Five parts, number of partitions

$$=\frac{r}{8}(4A+8B+2C)=\frac{r+3.r+2.r+1.r.r-3.r-4.r-5.r-6}{5.4.3.2.4.3.2.1},$$

the calculation being

$$\begin{array}{rl} 4(r^3+18r^2+65r)= & 4r^3+72r^2+260r \\ +8.r-1.r-2.r+7 & +8r^3+32r^2-152r+112 \\ +2.r-1.r-2.r-7 & +2r^3-20r^2+\ 46r-\ 28 \\ \hline = & 14r^3+84r^2+154r+\ 84 \\ = & 14(r^3+6r^2+\ 11r+6) \\ = & 14.r+1.r+2.r+3. \end{array}$$

To complete the theory, it would be in the first instance necessary to find for any given number of diagonals, $k-1$, whatever, the number and form of the diagonal-types, A, B, C, &c.; this is itself an interesting question in the Theory of Partitions, but I have not considered it.

22. Although the foregoing process (which, it will be observed, deals with the diagonal-types, without any consideration of the sub-types) is the most simple for the determination of the numbers A, B, C, &c., yet it is interesting to give a second process. Considering the several cases in order:

One diagonal, A.—The diagonal has two summits; we must have on each side of it one summit, and there remain $r-4$ summits which may be distributed between the two intervals formed by the diagonals. This can be done in $\frac{r-3}{1}$ ways, or we have, as before,

$$A = \frac{r-3}{1}.$$

Two diagonals, A.—The diagonals have four summits; we must have outside each diagonal one summit, and there remain $r-6$ summits to be distributed between the four intervals formed by the diagonals; this can be done in $\frac{r-5.r-4.r-3}{6}$ ways, or we have this value for A°. But the two top summits of the diagonals, or the two bottom summits, may coalesce; in either case the diagonals have three summits. We must have outside each diagonal one summit, and there remain $r-5$ summits to be distributed between the three intervals formed by the diagonals; the number of ways in which this can be done is

$$= \frac{r-4.r-3}{2},$$

say this is the value of A'. And we then have $A = A^\circ + 2A'$,

$$= \frac{r+1.r-3.r-4}{6}, \text{ as before.}$$

The calculation is $\qquad r-5+6 = r+1.$

23. Three diagonals, A.—See No. 15 for the figures of the sub-types. We have

$$A = A^\circ + 4A' + 4A'',$$

where the coefficients, 4 and 4, are the number of ways in which A' and A'' respectively can be derived from A° by coalescences of summits. For A° the diagonals have six summits, and there must be outside of two diagonals one summit; there remain $r-8$ summits to be distributed between the six intervals formed by the diagonals, and we have

$$A^\circ = \frac{r-7.r-6.r-5.r-4.r-3}{120}.$$

For A' the diagonals have five summits, and we must have outside of each of two diagonals, one summit; there remain $r-7$ summits to be distributed between the five intervals formed by the diagonals; we thus have

$$A' = \frac{r-6.r-5.r-4.r-3}{24}.$$

For A'' the diagonals have four summits; there must be outside of each of two diagonals one summit, and there remain $r-6$ summits to be distributed between the four intervals formed by the diagonals; we thus have

$$A'' = \frac{r-5.r-4.r-3}{6}.$$

The foregoing values give

$$A = \frac{r-3.r-4.r-5}{120}(r^2+7r+2),$$

as before. The calculation is

$$\begin{array}{rr} r-6.r-7 = & r^2-13r+\ 42 \\ +20.r-6 & +20r-120 \\ +80 & +\ 80 \\ \hline & r^2+\ 7r+\ \ 2 \end{array}$$

Three diagonals, B.—See No. 15 for the figures of the sub-types. We have

$$B = B^\circ+3B'+3B''+B'''.$$

For B° the diagonals have six summits, and there must be outside each of the three diagonals one summit; there remain $r-9$ summits to be distributed between the six intervals formed by the diagonals. We thus have

$$B^\circ = \frac{r-8.r-7.r-6.r-5.r-4}{120}.$$

Similarly,

$$B' = \frac{r-7.r-6.r-5.r-4}{24},\quad B''=\frac{r-6.r-5.r-4}{6},\quad B'''=\frac{r-5.r-4}{2}.$$

Hence
$$B=\frac{r-5.r-4.r-3.r-2.r-1}{120},$$

as before. The calculation is

$$\begin{array}{rr} r-6.r-7.r-8 = & r^3-21r^2+146r-336 \\ +15.r-6.r-7 & +15r^2-195r+630 \\ +60.r-6 & +\ 60r-360 \\ +60 & +\ 60 \\ \hline & r^3-\ 6r^2+\ 11r-\ \ 6 \\ & = r-1.r-2.r-3. \end{array}$$

24. Four diagonals, A.—The figures of the sub-types of A, B, C can be supplied without difficulty. We have

$$A = A^\circ + 6A' + 12A'' + 8A''',$$

where I remark that the numerical coefficients 1, 6, 12, 8 are the terms of $(1, 2)^3$. We have

$$A^\circ = \frac{r-9.r-8.r-7.r-6.r-5.r-4.r-3}{5040},$$

$$A' = \frac{r-8.r-7.r-6.r-5.r-4.r-3}{720},$$

$$A'' = \frac{r-7.r-6.r-5.r-4.r-3}{120},$$

$$A''' = \frac{r-6.r-5.r-4.r-3}{24},$$

and thence $A = \dfrac{r-3.r-4.r-5.r-6}{5040}(r^3+18r^2+65r)$,

as before. The calculation is

$$\begin{array}{rr} r-9.r-8.r-7 = & r^3-24r^2+191r-\ \ 504 \\ +42.r-8.r-7\quad & +42r^2-630r+2352 \\ +504.r-7\quad & +504r-3528 \\ +1680\quad & +1680 \\ \hline & r^3+18r^2+\ \ 65r \end{array}$$

Four diagonals, B.—We have

$$B = B^\circ + 5B' + 9B'' + 7B''' + 2B^{\text{iv}},$$

where the coefficients, 1, 5, 9, 7, 2, are the terms of $(1, 1)^3(1, 2)$. We have

$$B^\circ = \frac{r-10.r-9.r-8.r-7.r-6.r-5.r-4}{5040},$$

$$B' = \frac{r-9.r-8.r-7.r-6.r-5.r-4}{720},$$

$$B'' = \frac{r-8.r-7.r-6.r-5.r-4}{120},$$

$$B''' = \frac{r-7.r-6.r-5.r-4}{24},$$

$$B^{\text{iv}} = \frac{r-6.r-5.r-4}{6},$$

and thence $B = \frac{r-3.r-4.r-5.r-6}{5040}r-1.r-2.r+7$

as before. The calculation is

$$\begin{array}{rr} r-10.r-9.r-8.r-7 = & r^4-34r^3+431r^2-2414r+\ 5040 \\ +35.r-9.r-8.r-7 \quad & +35r^3-840r^2+6685r-17640 \\ +378.r-8.r-7 \quad & +378r^2-5670r+21168 \\ +1470.r-7 \quad & +1470r-10290 \\ +1680 \quad & +\ 1680 \\ \hline & r^4+r^3-31r^2+\ 71r-\ \ 42 \\ & =r-3.r-2.r-1.r+7. \end{array}$$

Four diagonals, C.—We have

$$C = C^{\circ}+4C'+6C''+4C'''+1C^{\text{iv}},$$

where the coefficients are the terms of $(1, 1)^4$. We have

$$C^{\circ} = \frac{r-11.r-10.r-9.r-8.r-7.r-6.r-5}{5040},$$

$$C' = \frac{r-10.r-9.r-8.r-7.r-6.r-5}{720},$$

$$C'' = \frac{r-9.r-8.r-7.r-6.r-5}{120},$$

$$C''' = \frac{r-8.r-7.r-6.r-5}{24},$$

$$C^{\text{iv}} = \frac{r-7.r-6.r-5}{6},$$

and thence $C = \frac{r-7.r-6.r-5.r-4.r-3.r-2.r-1}{5040}$.

I omit the calculation, as the equation is at once seen to be a particular case of a known factorial formula.

25. We may analyse the partitions of an r-gon into a given number of parts, according to the nature of the parts, that is the numbers of the sides of the several component polygons. It is for this purpose convenient to introduce the notion of "weight"; say a triangle has the weight 1, then a quadrangle, as divisible into two triangles, has

the weight 2, a pentagon, as divisible into three triangles, has the weight 3, ... and generally an r-gon, as divisible into $r-2$ triangles, has the weight $r-2$. It at once follows that, if

$$W = w+w', \quad \text{or} \quad = w+w'+w'', \text{ \&c.},$$

then a polygon of weight W is divisible into two polygons of the weights w, w', or into three polygons of the weights w, w', w'' respectively; and so on. Thus the 2-partitions of an 8-gon (weight $= 6$) are 15, 24, and 33; the 3-partitions are 114, 123, 222, and so on. Of course the number of the partitions 15, 24, 33, is equal to the whole number of the 2-partitions of the 8-gon, that is $= 20$; the number of the partitions 114, 123, 222, is equal to the whole number of the 3-partitions of the 8-gon, that is it is $= 120$; and so in other cases. It is easy to derive in order one from the other the numbers of the partitions of each several kind of the polygons of the several weights 2, 3, 4, 5, 6, &c.; and I write down the accompanying Table (No. 26), facing page 260, the process for the construction being as follows:

27. The first column (2 parts) is at once obtained. For a polygon of an odd number of sides, for instance the 9-gon (weight $= 7$), imagining the summits numbered in order 1, 2, ... 9, we divide this into a triangle and octagon, or obtain the partitions 16, by drawing the diagonals 13, 24, ... 81, 92: viz., the number is $= 9$. In the Table this is written, $16 = 9$; and so in other cases. Similarly we divide it into a quadrangle and heptagon, or obtain the partitions 25, by drawing the diagonals 14, 25, ... 82, 93: viz., the number is again $= 9$; and we divide it into a pentagon and a hexagon, or obtain the partitions 34, by drawing the diagonals 15, 26, ... 83, 94: viz., the number is $= 9$,

and here $$9+9+9 = 27,$$

the whole number of 2-partitions of the 9-gon. For a polygon of an even number of sides, for instance the 10-gon (weight $= 8$), the process is a similar one, the only difference being that for the division into two hexagons (that is, for the partitions 44) each partition is thus obtained twice, or the number of such partitions is $\tfrac{1}{2}10, = 5$; the numbers for the partitions 17, 26, 35, 44, thus are 10, 10, 10, 5; and we have

$$10+10+10+5 = 35,$$

the whole number of the 2-partitions of the 10-gon.

28. To obtain the second column (3 parts)—suppose, for instance, the 3-partitions of the 9-gon; these are 115, 124, 133, 233. We obtain

the number of the partitions 115 from the terms

$$16 = 9 \quad \text{and} \quad 25 = 9$$

of the first column: viz., in 16, changing the 6 into 15, that is, dividing the polygon of weight 6 into two parts of weights 1 and 5 respectively; this can be done in eight ways (see, higher up, 15 = 8 in the first column), and we thus obtain the number of partitions

$$9 \times 8 = 72;$$

and again, in 25, changing the 2 into 11, that is, dividing the polygon of weight 2 into two parts each of weight 1; this can be done in two ways (see, higher up, 11 = 2 in the first column), and we thus obtain the number of partitions

$$9 \times 2 = 18;$$

we should thus have, for the number of partitions 115, the sum

$$72 + 18 = 90,$$

only, as it is easy to see, each partition is obtained twice, and the number of the partitions 115 is the half of this, = 45. And by the like process it is found that the numbers of the partitions 124, 133, 233 are equal to 90, 45, 45 respectively; and then, as a verification we have

$$45 + 90 + 45 + 45 = 225,$$

the whole number of the 3-partitions of the 9-gon.

29. The third column (4 parts) is derived in like manner from the second column by aid of the first column; and so in general each column is derived in like manner from the column which immediately precedes it, by aid of the first column. And we have for the numbers in each compartment of any column the verification that the sum of these numbers is equal to the whole number (for the proper values of k and r) of the k-partitions of the r-gon.

It might be possible, by an application of the method of generating functions, to find a law for the numbers in any compartment of a column of the table; but I have not attempted to make this investigation.

30. In the table in No. 2, the numbers 1, 2, 5, 14, 42, &c., of the diagonal line show the number of partitions of the triangle, the quadrangle, the 5-gon, ... r-gon into triangles: viz., these numbers show the number of partitions of the r-gon into $r-2$ parts, that is, into triangles; and, for the r-gon, writing

$$k = r - 2,$$

26. THE TABLE REFERRED TO.

[To face page 260

<table>
<tr><th>r</th><th>W</th><th>2 PARTS</th><th>3 PARTS</th><th>4 PARTS</th><th>5 PARTS</th><th>6 PARTS</th><th>7 PARTS</th><th>8 PARTS</th></tr>
<tr><td>3</td><td>1</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>4</td><td>2</td><td>11 = 2
2</td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>5</td><td>3</td><td>12 = 5
5</td><td>111 5 × 2 = 10
2) 10 = 5</td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>6</td><td>4</td><td>13 = 6
22 = 3
9</td><td>112 6 × 5 = 30
3 × 4 = 12
2) 42 = 21</td><td>1111 21 × 2 = 42
3) 42 = 14</td><td></td><td></td><td></td><td></td></tr>
<tr><td>7</td><td>5</td><td>14 = 7
23 = 7
14</td><td>113 7 × 6 = 42
7 × 2 = 14
2) 56 = 28

122 7 × 3 = 21
7 × 5 = 35
2) 56 = 28
56</td><td>1112 28 × 5 = 140
28 × 4 = 112
3) 252 = 84</td><td>11111 84 × 2 = 168
4) 168 = 42</td><td></td><td></td><td></td></tr>
<tr><td>8</td><td>6</td><td>15 = 8
24 = 8
33 = 4
20</td><td>114 8 × 7 = 56
8 × 2 = 16
2) 72 = 36

123 8 × 7 = 56
8 × 6 = 48
4 × 10 = 40
2) 144 = 72

222 8 × 3 = 24
2) 24 = 12
120</td><td>1113 36 × 6 = 216
72 × 2 = 144
3) 360 = 120

1122 36 × 3 = 108
72 × 5 = 360
12 × 6 = 72
3) 540 = 180
300</td><td>11112 120 × 5 = 600
180 × 4 = 720
4) 1320 = 330</td><td>111111 330 × 2 = 660
5) 660 = 132</td><td></td><td></td></tr>
</table>

26. THE TABLE REFERRED TO.—*Continued.*

r	W	2 PARTS	3 PARTS	4 PARTS	5 PARTS	6 PARTS	7 PARTS	8 PARTS
9	7	16 = 9 25 = 9 34 = 9 27	115 9 × 8 = 72 9 × 2 = 18 2) 90 = 45 124 9 × 8 = 72 9 × 7 = 63 9 × 5 = 45 2) 180 = 90 133 9 × 4 = 36 9 × 6 = 54 2) 90 = 45 223 9 × 7 = 63 9 × 3 = 27 2) 90 = 45 225	1114 45 × 7 = 315 90 × 2 = 180 3) 495 = 165 1123 45 × 7 = 315 90 × 6 = 540 45 × 10 = 450 45 × 4 = 180 3) 1485 = 495 1222 90 × 3 = 270 45 × 5 = 225 3) 495 = 165 825	11113 165 × 6 = 990 495 × 2 = 990 4) 1980 = 495 11122 165 × 3 = 495 495 × 5 = 2475 165 × 6 = 990 4) 3960 = 990 1485	111112 495 × 5 = 2475 990 × 4 = 3960 5) 6435 = 1287	1111111 1287 × 2 = 2574 6) 2574 = 429	
10	8	17 = 10 26 = 10 35 = 10 44 = 5 35	116 10 × 9 = 90 10 × 2 = 20 2) 110 = 55 125 10 × 9 = 90 10 × 8 = 80 10 × 5 = 50 2) 220 = 110 134 10 × 9 = 90 10 × 7 = 70 5 × 12 = 60 2) 220 = 110 224 10 × 8 = 80 5 × 6 = 30 2) 110 = 55 233 10 × 4 = 40 10 × 7 = 70 2) 110 = 55 385	1115 55 × 8 = 440 110 × 2 = 220 3) 660 = 220 1124 55 × 8 = 440 110 × 7 = 770 110 × 5 = 550 55 × 4 = 220 3) 1980 = 660 1133 55 × 4 = 220 110 × 6 = 660 55 × 2 = 110 3) 990 = 330 1223 110 × 7 = 770 110 × 3 = 330 55 × 6 = 330 55 × 10 = 550 3) 1980 = 660 2222 55 × 3 = 165 3) 165 = 55 1925	11114 220 × 7 = 1540 660 × 2 = 1320 4) 2860 = 715 11123 220 × 7 = 1540 660 × 6 = 3960 330 × 10 = 3300 660 × 4 = 2640 4) 11440 = 2860 11222 660 × 3 = 1980 660 × 5 = 3300 55 × 8 = 440 4) 5720 = 1430 5005	111113 715 × 6 = 4290 2860 × 2 = 5720 5) 10010 = 2002 111122 715 × 3 = 2145 2860 × 5 = 14300 1430 × 6 = 8580 5) 25025 = 5005 7007	1111112 2002 × 5 = 10010 5005 × 4 = 20020 6) 30030 = 5005	11111111 5005 × 2 = 10010 7) 10010 = 1430

the number is $$= \frac{[2r-4]^{r-3}}{[r-2]^{r-3}}.$$

If, as above, taking the weight of the triangle to be 1, we write

$$r-2 = w,$$

then the number is $$= \frac{[2w]^{w-1}}{[w]^{w-1}},$$

viz., this is the expression for the number of partitions of the polygon of weight w, or $(w+2)$-gon, into triangles.

31. The question considered by Taylor and Rowe, in the paper referred to in No. 1, is that of the partition of the r-gon into p-gons, for p, a given number > 3; this implies a restriction on the form of r, viz., we must have $r-2$ divisible by $p-2$. In fact (generalizing the definition of w), if we attribute to a p-gon the weight 1, and accordingly to a polygon divisible into w p-gons the weight w, then, r being the number of summits, we must have

$$r = (p-2)\,w+2.$$

In particular, if $p = 4$, or the r-gon is to be divided into quadrangles, then r is necessarily even, and for the values

$$w = 1, 2, 3, \ldots,$$

we have $$r = 4, 6, 8, \ldots.$$

32. To fix the ideas, I assume $p = 4$, or consider the problem of the division of the $(2w+2)$-gon into quadrangles. Writing

$$w-1 = a+b+c,$$

we take at pleasure any one side of the $(2w+2)$-gon, making this the first side of a quadrangle, the second, third, and fourth sides being diagonals of the polygon such that outside the second side we have a polygon of weight a, outside the third side a polygon of weight b, and outside the fourth side a polygon of weight c. Any one or more of the numbers a, b, c may be $= 0$ (they cannot be each of them $= 0$ except in the case $w = 1$). The meaning is that the corresponding side of the quadrangle, instead of being a diagonal, is a side of the $(2w+2)$-gon, viz., there is no polygon outside such side. Suppose in general that P_w is the number of ways in which a polygon of weight w can be divided in quadrangles, and let each of the polygons of weights a, b, c respectively, be divided into quadrangles, the number of ways in which this can be done is $P_aP_bP_c$; and it is to be noticed that, if for instance $a=0$, then the number is $= P_b P_c$, viz., the formula remains true if only we assume $P_0=1$. The number of partitions thus obtained is $\Sigma P_a P_b P_c$, where the summation extends to all the partitions of $w-1$ into the parts a, b, c (zeros admissible and the order of the parts being attended to). And we thus obtain *all* the partitions of the $(2w+2)$-gon into w parts; for first the partitions so obtained are all distinct from each other, and next every partition of the $(2w+2)$-gon into w parts is a partition in which the selected side of the $(2w+2)$-gon is a side of some one of the quadrangles. That is,

we have $$P_w = \Sigma P_a P_b P_c \text{ } (a, b, c \text{ as above});$$

and it hence appears that, considering the generating function

$$f = 1+P_1x+P_2x^2+P_3x^3+\dots,$$

we have $$f = 1+xf^3.$$

The reasoning is precisely the same if, instead of a division into quadrangles, we have a division into p-gons; the only difference is that instead of the three parts a, b, c, we have the $p-1$ parts $a, b, c \dots$, and the equation for f thus is

$$f = 1+xf^{p-1}.$$

33. Writing for a moment

$$f = u+xf^{p-1},$$

and expanding by Lagrange's theorem, we have

$$f = u+\frac{x}{1}(u^{p-1})+\frac{x^2}{1.2}(u^{2(p-1)})'\dots+\frac{x^w}{1.2\dots w}(u^{w(p-1)})'^{\dots(w-1)}+\dots,$$

viz., after the differentiation, writing $u=1$, we have

$$P_w = \frac{[(p-1)\,w]^{w-1}}{[w]^{w-1}},$$

where it will be recollected that for the number of sides of the polygon,

we have $$r = (p-2)\,w+2.$$

In the case of the partition into triangles, $p=3$, and we have the before mentioned value

$$[2w]^{w-1} \div [w]^{w-1}, \quad w = r-2.$$

Thursday, April 9th, 1891.

Major P. A. MACMAHON, R.A., F.R.S., Vice-President, in the Chair.

Mr. J. Rose-Innes, B.Sc. London, B.A. Camb., was elected a member.

The following communication was made :—

The Analytical Forms called Trees, with Applications to the Combinations of certain Electrical Quantities, and to the Compositions of Multipartite Numbers: Major Macmahon. (Mr. Walker in the Chair.)

A discussion followed, in which Messrs. Kempe, Hammond, and S. Roberts took part.

Mr. Kempe spoke on the flaw in his *proof* "On the Map-colour Theorem," which had recently been detected by Mr. P. J. Heawood, and showed that a statement by the latter at the close of his paper failed.* He further stated that the subject had long engaged his attention, and that he was still unable to solve the question to his satisfaction.

Mr. Tucker communicated a paper by Mr. Culverwell, "Compounded Solutions in the Calculus of Variations."

The following presents were received :—

"Educational Times," for April.

"Transactions of the Cambridge Philosophical Society," Vol. xv., Part I.

"Proceedings of the Cambridge Philosophical Society," Vol. VII., Part III.

"Œuvres Complètes de Christiaan Huygens," Vol. III., 4to; La Haye, 1890.

"Bulletin des Sciences Mathématiques," Tome xv., Feb., 1891.

"Bulletin de la Société Mathématique de France," Tome XIX., Nos. 1 and 2.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. VII., Fasc. 5 and 6, Roma, 1891.

"Beiblätter zu den Annalen der Physik und Chemie," Band xv., Stück 3; Leipzig, 1891.

"Rendiconti del Circolo Matematico di Palermo," Tomo v., Fasc. I. e II.

"Bollettino delle Pubblicazioni Italiane, ricevute per Diritto di Stampa," Nos. 125 and 126, and two parts of Index.

* *Cf. Proc. Lond. Math. Soc.*, Appendix to Vol. XXI.

"Berichte über die Verhandlungen der Königlich-Sächsischen Gesellschaft der Wissenschaften zu Leipzig," 1890, III. and IV.; Leipzig, 1891.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. x., No. 1.

"Jahrbuch über die Fortschritte der Mathematik," Band xx., Heft 2; Jahrgang 1888, Heft 2.

A number of pamphlets by Dr. L. Kronecker, viz.—

Offprints from the "Sitzungsberichte der Königlich-Preussischen Akademie der Wissenschaften zu Berlin," 1888, Art. xvi., xviii.; 1889, Art. vi., x., xiv., xviii., xix., xxx., xxxi., xlii.; 1890, Art. vi., vii., xiv., xvi., xxvi., xxviii., xxx., xxxvi., xl., xlviii., liii.; 1891, Art. ii., iii. From "Crelle's Journal," Band c., Heft 4; ci., 4; cii., 3; civ., 4; cv., 2. And "Festschrift . . . . Mathematischen Gesellschaft in Hamburg 1890" (Uber die Dirichletsche Methode der Werthestimmung der Gaussschen Reihen); Leipzig, 1890.

Also a number by M. A. Mannheim, viz.—

Offprints from "Rendiconti del Circolo Matematico," Tomo III. (1889, 10 Marzo).

"Comptes Rendus," 10 Fév., 24 Fév., 3 Nov., 1 Dec., 1890; 2 Fév., 2 Mars, 1891.

"Journal de l'École Polytechnique," LX^e cahier, 1890.

Stability of Orbits. *By* Mr. A. G. Greenhill.

[*Read May* 10*th*, 1888.]

1. The discussion of the stability of a circular orbit described under a central force is given in Section IX., Book I., of Newton's *Principia;* and it is there shown that if the force is proportional to a single power of the distance, represented by μr^{n-3}, the apsidal angle in the circular orbit, when slightly disturbed, is $\pi/n^{\frac{1}{2}}$; so that n must be positive for the apsidal angle to be real, and for the orbit to be stable.

Representing as usual the reciprocal of r by u, and the central force P by μu^m, then for the circular orbit to be stable, m must not be greater than 3.