intermediate in composition between ordinary carbonates and ortho- or dicarbonates, precisely as the pyrophosphates are intermediate between metaphosphates and normal phosphates. Thus Fritzsche has described a carbonate of magnesium having the formula $Mg^6 C^2 O^7 . 3H^2 O$. According to Mulder and Hochstetter, white lead has the formula $Pb^6 C^2 O^7 . H^2 O$. Mountain blue, a crystalline mineral, has the formula $Cu^6 C^2 O^7 . H^2 O$, and carbonate of bismuth the formula $Bi'''^2 C^2 O^7 . H^2 O$. These last three compounds may also be represented as orthocarbonates, with the respective formulæ $Pb^3 HCO^4$, $Cu^3 HCO^4$, and $Bi''' HCO^4$.

The following Table illustrates the relations of the meta- and ortho-salts above spoken of, to one another and to other similar salts.

| Monobasic, HF . HCl. | Bibasic, $H^2O . H^2S$. | Terbasic, $H^3N . H^3P$. | Tetrabasic, $H^4C . H^4Si$? |
|---|---|---|---|
| $H ClO^3$ Chlorate.<br>{ $H NO^3$ Nitrate.<br>{ $M PO^3$ Metaphosphate. | $M^2SO^3$ Sulphite.<br>{ $M^2CO^3$ Carbonate.<br>{ $M^2SiO^3$ Metasilicate. | $M^3PO^3$ Phosphite. | |
| $M ClO^4$ Perchlorate. | $M^2SO^4$ Sulphate. | { $M^3NO^4$ Orthonitrate.<br>{ $M^3PO^4$ Phosphate. | { $M^4CO^4$ Orthocarbonate.<br>{ $M^4SiO^4$ Silicate. |

Hence it appears that a certain relation which exists between the terbasic and monobasic groups, is paralleled by a similar relation between the tetrabasic and bibasic groups. In phosphates and silicates, the dominant tendency is to form ortho-salts with four atoms of oxygen; in nitrates and carbonates the dominant tendency is to form metasalts with three atoms of oxygen; but each class manifests both tendencies. Metaphosphates and metasilicates on the one hand approximate to anhydro-salts, while orthonitrates and orthocarbonates on the other hand resemble super-basic salts. It is observable that both metaphosphates and nitrates, metasilicates and carbonates, differ from normal ter-oxysalts such as chlorates, sulphites, and phosphites, in their absolute inoxidizability.

## LVIII. *On the Analytical Forms called Trees.*—Part II. *By* A. Cayley, *Esq.*

[Continued from vol. xiii. p. 176.]

THE following class of "trees" presented itself to me in some researches relating to functional symbols; viz., attending only to the terminal knots, the trees with one knot, two knots, three knots, and four knots respectively are shown in the figures 1, 2, 3, and 4:

Fig. 4.

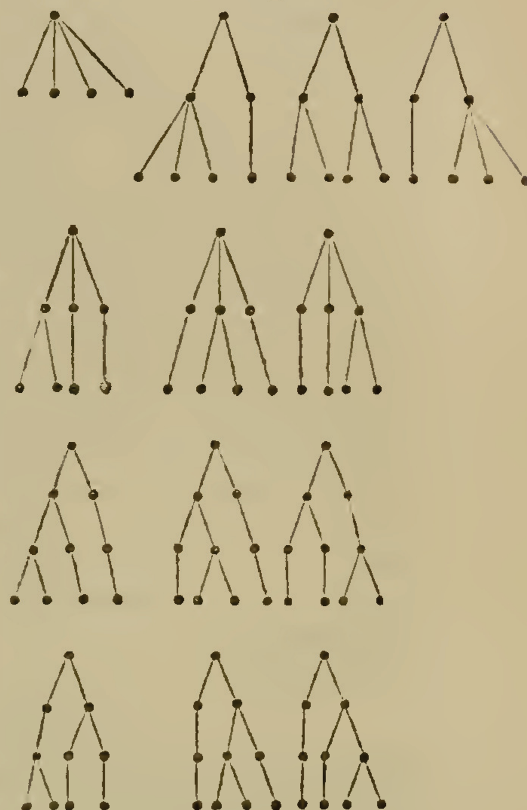

Fig. 1.

Fig. 2.

Fig. 3.

and similarly for any number of knots. The trees with four knots are formed first from those of one knot by attaching thereto in every possible way (one way only) four knotted branches; secondly, from those with two knots by attaching thereto in every possible way (three different ways) four knotted branches; and thirdly, from those with three knots by attaching thereto in every possible way (three different ways) four knotted branches,—the original knots of the trees of one knot and two and three knots, being no longer terminal knots, are disregarded. The total numbers of trees with one knot and with two and three knots being respectively 1, 1, 3; the total number of trees with four knots is $1.1+3.1+3.3=13$. And in general, if the number of trees with $m$ knots is $\phi m$, then it is easy to see that we have

$$\phi m=\phi 1+\frac{m-1}{1}\phi 2+\frac{m-1.m-2}{1.2}\phi 3\ldots+\frac{m-1}{1}\phi(m-1);$$

or what is the same thing,

$$2\phi m=\phi 1+\frac{m-1}{1}\phi 2+\frac{m-1.m-2}{1.2}\phi 3\ldots+\frac{m-1}{1}\phi(m-1)+\phi m.$$

Whence if

$$u=\phi 1+\frac{x}{1}\phi 2+\frac{x^2}{1\,.\,2}\phi 2+\ldots,$$

we obtain

$$\begin{aligned} e^x\,.\,u= & \quad \phi 1 \\ & + x\,.\,(\phi 1+\phi 2) \\ & + \frac{x^2}{1\,.\,2}(\phi 1+2\phi 2+\phi 3) \\ & + \&c. \\ = & \quad 2\phi 1-1 \\ & + x\,.\,2\phi 2 \\ & + \frac{x^2}{1\,.\,2}\,.\,2\phi 3 \\ & + \&c.; \end{aligned}$$

that is,

$$e^x u=2u-1,$$

and thence

$$u=\frac{1}{2-e^x},$$

which gives for $\phi m$ the expression

$$\phi m=1\,.\,2\,.\,3\ldots(m-1)\text{ coeff. }x^{m-1}\text{ in }\frac{1}{2-e^x};$$

and the value of $\phi m$ might easily be obtained in an explicit form in terms of the differences of the powers of zero. The values of $\phi m$ are, for

$$\begin{array}{llllllllll} m=1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & \&c. \\ \phi m=1, & 1, & 3, & 13, & 75, & 541, & 4683, & 47293. & \end{array}$$

In the foregoing problem, the number of branches descending from a non-terminal knot is one, two, or more. But assume that the number of branches descending from a non-terminal knot is always two; so that attending, as before, only to the terminal knots, the trees with two knots, three knots, four knots respectively are shown in the figures, 5, 6, and 7.

Fig. 5.

Fig. 6.

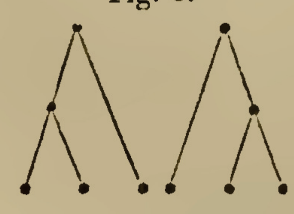

Fig. 7.

This corresponds to the following problem in the theory of symbols; viz. if A, B, C, D, &c. are symbols capable of successive binary combinations, but do not satisfy the associative law, what is the number of the different significations of the ambiguous expressions ABC, ABCD, ABCDE, &c. respectively? For instance, AB has only one meaning; ABC may mean either A . BC or AB . C. In like manner ABCD may mean A(B.CD), or AB . CD, or (AB . C)D, or (A . BC)D, or A(BC . D); the numbers 1, 2, 5 being those of the trees in the last three figures respectively; and similarly for any greater number of symbols.

Let $\phi m$ be the required value corresponding to the number $m$; then we may in any manner whatever separate the number $m$ into two parts $m'$, $m''$, and then combining *inter se* the $m'$ knots (or symbols) and the $m''$ knots (or symbols) respectively, ultimately combine the two combinations; hence a part of $\phi m$ is $\phi m' . \phi m''$. The assumed definition of $\phi m$ does not apply to the case $m=1$; but if we write $\phi 1=1$, then the foregoing consideration shows that we have

$$\begin{aligned}\phi m = & \ \phi 1 \phi(m-1) \\ & + \phi 2 \phi(m-2) \\ & \ \vdots \\ & + \phi(m-1)\phi 1);\end{aligned}$$

from which it is easy to calculate

$$\phi 1=1,\ \phi 2=1,\ \phi 3=2,\ \phi 4=5,\ \phi 5=14,\ \phi 6=42,\ \phi 7=132,\ \text{\&c.}$$

But to obtain the law, consider the generating function

$$u=\phi 1+x\phi 2+x^2\phi 3+\text{\&c.};$$

we have

$$u^2=\phi 1\ \phi 1+x(\phi 1\ \phi 2+\phi 2\ \phi 1)+x^2(\phi 1\ \phi 3+\phi 2\ \phi 2+\phi 3\ \phi 1)+\text{\&c.},$$

which is

$$=\phi 2+x\phi 3+x^2\phi 4+\text{\&c.};$$

and we have therefore

$$xu^2=u-1,$$

and consequently

$$u=\frac{1-\sqrt{1-4x}}{2x}.$$

But

$$\sqrt{1-4x}=1-\frac{1}{2}4x+\frac{\frac{1}{2}\cdot-\frac{1}{2}}{1.2}(4x)^2-\frac{\frac{1}{2}\cdot-\frac{1}{2}\cdot-\frac{3}{2}}{1.2.3}(4x)^3+\text{\&c.}$$

$$=1-2x-2x^2-4x^3-10x^4+\text{\&c.},$$

and therefore

$$u=\frac{1-\sqrt{1-4x}}{2x}=1+1x+2x^2+5x^3+\text{\&c.},$$

the series of coefficients 1, 1, 2, 5, &c. agreeing with the values already found. The expression for the general term is at once seen to be

$$\phi m = \frac{1 . 3 . 5 \ldots 2m-3}{1 . 2 . 3 \ldots m} 2^{m-1},$$

which is a remarkably simple form.

2 Stone Buildings, W.C.,
June 9, 1859.

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## LIX. *Notices respecting New Books.*

*An Essay on the Theory of Equations. By* G. B. Jerrard. London: Taylor and Francis.

THE main object of this pamphlet is to show that the *general* equation of the fifth degree admits of solution. Has Mr. Jerrard succeeded in establishing this proposition? We shall not take upon ourselves to affirm it. Lagrange, Vandermonde, Euler, Galois, and many other distinguished mathematicians who devoted their powerful intellects to the question, failed in obtaining a direct answer—although the investigations into which they were thus led have made us acquainted with many propositions of great beauty and generality, and have considerably extended the domains of algebra.

But Mr. Jerrard's result is in direct opposition to a proposition given by Abel, the proof of which, afterwards simplified by Wantzel, has been received by eminent analysts.

Now it is impossible for both conclusions to be correct; and having looked very carefully into Wantzel's proof as given by Serret (*Algèbre Supérieure*), we acknowledge we can find no flaw in it. Mr. Jerrard's solution runs to such a length, and is besides so intricate from the constant introduction of new symbols, that we fairly confess we had not courage to go through the whole of it. Mr. Jerrard is a veteran in this subject, and has done good service. To him we owe a remarkable proposition, which enables us to transform any equation into another which shall want the second, third, and fourth terms, or else the second, third, and fifth, by the solution of a cubic equation in the former case, and a biquadratic in the latter. Any investigation of his must therefore not be rejected hastily, even when in contradiction to others. In the present case, however, instead of limiting his conclusion to the statement, "whence I infer the possibility of solving any proposed equation of the fifth degree," it would certainly be worth his trouble to obtain the solution and apply it to one of his own simplified trinomials.