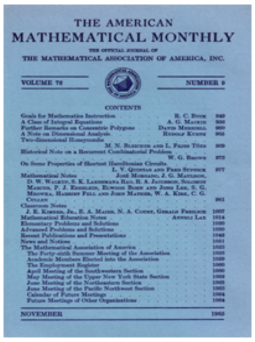

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Historical Note on a Recurrent Combinatorial Problem

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of a "vertical" side equals $\frac{1}{2}w-x$ (Fig. 10), then the total length t of the internal sides is

$$t = w - 2x + \sqrt{(b^2 + 16x)}.$$

The condition of a free minimum is

$$\frac{dt}{dx} = -2 + \frac{16x}{\sqrt{[b^2 + 16x]}} = 0, \quad \text{i.e.,} \quad 2x = \sqrt{\left[\left(\frac{b}{4}\right)^2 + x^2\right]}.$$

This means that the angles of the cell different from a right angle must be 120°.

In view of $\frac{1}{2}w-x>0$, this condition is fulfilled only if $b<\sqrt{12}w$. If $b \geqq \sqrt{12}w$ the best cell is a triangle. This completes the proof of

THEOREM 2. *Among the cells of given area generating a comb of given width the cell having the least perimeter is either an isosceles triangle having an angle* $\geqq 120°$ *at its apex or a pentagon composed of a rectangle and an isosceles triangle having an angle equal to* 120° *at its apex.*

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HISTORICAL NOTE ON A RECURRENT COMBINATORIAL PROBLEM

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1. In two forthcoming papers [45, 46] the author enumerates dissections of the disc into triangles and quadrangles. These results generalize a much studied problem whose history will be described briefly in this paper.

A *dissection* of the disc will be a cell complex whose polyhedron is the closed disc B^2. (The reader who is unfamiliar with this concept can consult, for example, Chapter I of *Riemann Surfaces* by L. Ahlfors and L. Sario, Princeton Univ. Press, 1960.)

It will be required that

a) every edge be incident with two distinct vertices (called its *ends*);
b) no two edges have the same ends;
c) every vertex be incident with at least two edges; and
d) every edge not in the boundary of B^2 be incident with two distinct faces (2-cells).

Alternatively, a dissection of the disc can be thought of as a map on the sphere from which one face whose boundary is a simple polygon is removed; (cf. for example, Chapter 21 of *Introduction to Geometry* by H. S. M. Coxeter, Wiley, 1961; or Chapter 3 of *Generators and Relations for Discrete Groups* by

H. S. M. Coxeter and W. O. J. Moser, Springer, 1957). Conditions a) to d) will be retained.

A dissection is *rooted* if an edge in the boundary of B^2 is oriented and designated as the *root*.

Two rooted dissections will be said to be *isomorphic* if there exists a homeomorphism of B^2 with itself carrying vertices and edges of one dissection respectively onto vertices and edges of the other, and preserving the root (including its orientation).

A dissection will be said to be of type $[n, m]_k$ if

(i) each of its faces is incident with exactly k edges;
(ii) exactly n of its vertices lie in the interior of B^2;
(iii) exactly $m+k$ of its vertices lie in the boundary of B^2.

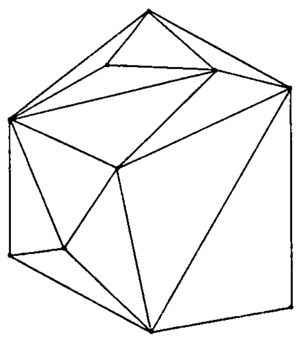

FIG. 1

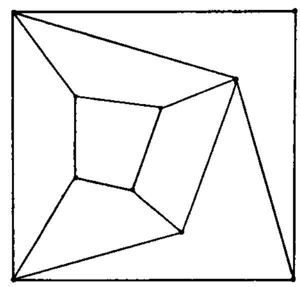

FIG. 2

Examples of dissections of types $[4, 2]_3$ and $[6, 0]_4$ are shown in Figures 1 and 2 respectively. The number, up to isomorphisms, of rooted dissections of type $[n, m]_k$ will be denoted by $D^{(k)}_{n,m}$. In [**45**, **46**] it is shown that

$$(1.1) \qquad D^{(3)}_{n,m} = \frac{2(2m+3)!(4n+2m+1)!}{(m+2)!m!n!(3n+2m+3)!}$$

$$(1.2) \qquad D^{(4)}_{n,m} = \begin{cases} \dfrac{3(3p+4)!(3n+3p+2)!}{(2p+3)!p!n!(2n+3p+4)!} & (m \text{ even}; \; p \underset{\text{def}}{=} m/2) \\ 0 & (m \text{ odd}) \end{cases}$$

for $n \geqq 0$, $m \geqq 0$.

2. The problem of computing the number of ways of dissecting a convex polygonal region into triangular regions by means of its diagonals appears to have been first posed by Euler to Segner, who developed a recurrence for the numbers which we have denoted by $D^{(3)}_{0,m}$. In his memoir on the subject [2] Segner included a table of values of $D^{(3)}_{0,m}$ for $m < 18$; because of an unfortunate computational error, however, the values for $m > 11$ were incorrect. This latter fact was pointed out by Euler, who published the correct values for $m < 23$; Euler also stated without proof the equivalent of our formula (1.1) for $n = 0$, viz.

$$D_{0,m}^{(3)} = \frac{(2m+2)!}{(m+1)!(m+2)!}$$

The more general problem of computing the number of dissections of a convex n-gonal region into m-gonal regions by means of its diagonals was posed by Pfaff to N. von Fuss, who generalized Segner's recurrence [3].

The Euler-Segner problem reappeared in the years 1838–1839 in a series of papers in Liouville's Journal [4–10 inclusive] in which it was solved in various ways, notably by Binet, who used generating functions, and Rodrigues, who provided a very elegant direct solution. The Pfaff-Fuss problem was again considered in [11, 12, 13].

Numerous authors have observed [5, 7, 21, etc.] the equivalence of the Euler-Segner problem with the following algebraic problem: In how many ways can a product of n factors be interpreted in a non-commutative non-associative algebra? This latter problem and its generalizations are posed periodically in the Problems section of this MONTHLY [23, 24, 27, 32, 35, 43] and have also been considered in [18, 22, 25, 28, 29, 30, 36, 38, 39, 40].

The equivalence of the Euler-Segner problem to yet other algebraic and combinatorial problems has been demonstrated in [21, 26, 30, 31, 34, 41, etc.].

Generalizations of the Pfaff-Fuss problem (still involving dissections of a polygonal region by its diagonals) have been considered in [19, 20, 31]. Further generalizations (allowing internal vertices) were considered at great length by Kirkman [14, 15, 16, etc.] who developed numerous recurrences in an attempt to enumerate polyedra (sic).

The accompanying bibliography (undoubtedly incomplete) traces the appearances of these problems in print in chronological order of publication. The author will be indebted to any reader who can supply missing references.

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ON SOME PROPERTIES OF SHORTEST HAMILTONIAN CIRCUITS

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1. Introduction. In this paper proofs are given of the following "folk theorems."

By the *covertex class* of a polygon is meant the class of all polygons having the same vertex set as the given polygon.

THEOREM 1. *A spherical or planar polygon which is shortest of its covertex class and whose vertices are noncogeodesic cannot intersect itself.*

THEOREM 2. *A spherical or planar polygon which is shortest of its covertex class and whose vertices are noncogeodesic must contain the vertices on the boundary of its convex hull in their cyclic order.*

When on a 2-sphere we shall assume that the vertex set of a polygon does not contain a pair of antipodal points and that all edges are minor geodesic arcs.

The symbols for polygons $[V_1 \cdots V_n]$ are to be considered cyclic and symmetric. The symbol

$$[V_1 \cdots V_{i-1}(V_i \cdots V_j)V_{j+1} \cdots V_n]$$

will denote the *arc-inversion* that operates on the polygon

$$[V_1 \cdots V_{i-1}V_i \cdots V_{j-1}V_jV_{j+1} \cdots V_n]$$

to produce

$$[V_1 \cdots V_{i-1}V_jV_{j-1} \cdots V_iV_{j+1} \cdots V_n].$$

2. Proof of Theorem 1. Let h denote a polygon which intersects itself. Then there is a closed edge of h which intersects an (open) edge of h. Let P_aP_b be the open edge and denote the closed edge by $\mathrm{Cl}(P_cP_d)$, where

$$h = [P_aP_b \cdots P_cP_d \cdots].$$

We have the following cases:

1. The intersection of P_aP_b and $\mathrm{Cl}(P_cP_d)$ is a point, i.e., the intersection point is P_c, P_d, or a point of P_cP_d.

2. The intersection of P_aP_b and P_cP_d is a geodesic arc and when P_aP_b and