Lectures on Discrete and Polyhedral Geometry

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Introduction

The subject of Discrete Geometry and Convex Polytopes has received much attention in recent decades, with an explosion of the work in the field. This book is an introduction, covering some familiar and popular topics as well as some old, forgotten, sometimes obscure, and at times very recent and exciting results. It is somewhat biased by my personal likes and dislikes, and by no means is a comprehensive or traditional introduction to the field, as we further explain below.

This book began as informal lecture notes of the course I taught at MIT in the Spring of 2005 and again in the Fall of 2006. The richness of the material as well as its relative inaccessibility from other sources led to making a substantial expansion. Also, the presentation is now largely self-contained, at least as much as we could possibly make it so. Let me emphasize that this is neither a research monograph nor a comprehensive survey of results in the field. The exposition is at times completely elementary and at times somewhat informal. Some additional material is included in the appendix and spread out in a number of exercises.

The book is divided into two parts. The first part covers a number of basic results in discrete geometry and with few exceptions the results are easily available elsewhere (to a committed reader). The sections in the first part are only loosely related to each other. In fact, many of these sections are subjects of separate monographs, from which we at times borrow the proof ideas (see reference subsections for the acknowledgements). However, in virtually all cases the exposition has been significantly altered to unify and simplify the presentation. In and by itself the first part can serve as a material for the first course in discrete geometry, with fairly large breadth and relatively little depth (see more on this below).

The second part is more coherent and can be roughly described as the discrete differential geometry of curves and surfaces. This material is much less readily available, often completely absent in research monographs, and, on more than one occasion, in the English language literature. We start with discrete curves and then proceed to discuss several versions of the Cauchy rigidity theorem, the solution of the bellows conjecture and Alexandrov's various theorems on polyhedral surfaces.

Although we do not aim to be comprehensive, the second part is meant to be as an introduction to polyhedral geometry, and can serve as a material for a topics class on the subject. Although the results in the first part are sporadically used in the second part, most results are largely independent. However, the second part requires a certain level of maturity and should work well as the second semester continuation of the first part.

We include a large number of exercises which serve the dual role of possible home assignment and additional material on the subject. For most exercises, we either include a hint or a complete solution, or the references. The appendix is a small collection of standard technical results, which are largely available elsewhere and included here to make the book self-contained. Let us single out a new combinatorial proof of the uniqueness part in the Brunn–Minkowski inequality and an elementary introduction to the theory of places aimed towards the proof of the bellows conjecture.

Organization of the book. The book is organized in a fairly straightforward manner, with two parts, 40 sections, and the increasing level of material between sections and within each section. Many sections, especially in the first part of the book, can be skipped or their order interchanged. The exercises, historical remarks and pointers to the literature are added at the end of each section. Theorems, propositions, lemmas, etc. have a global numbering within each section, while the exercises are numbered separately. Our aversion to formula numbering is also worth noting. Fortunately, due to the nature of the subject we have very few formulas worthy of labeling and those are labeled with AMS-TeX symbols.

The choice of material. Upon inspecting the table of contents the reader would likely assume that the book is organized around "a few of my favorite things" and has no underlying theme. In fact, the book is organized around "our favorite tools", and there are very few of them. These tools are heavily used in the second part, but since their underlying idea is so fundamental, the first part explores them on a more elementary level in an attempt to prepare the reader. Below is our short list, in the order of appearance in the book.

1. *Topological existence arguments.* These basic non-explicit arguments are at the heart of the Alexandrov and Minkowski theorems in Sections 35–37. Sections 4–6 and Subsection 3.5 use (often in a delicate way) the intermediate value theorem, and are aimed to be an introduction to the method.
2. *Morse theory type arguments.* This is the main tool in Section 8 and in the proof of the Fáry–Milnor theorem (Section 24). We also use it in Subsection 1.3.
3. *Variational principle arguments.* This is our most important tool all around, giving alternative proofs of the Alexandrov, Minkowski and Steinitz theorems (Sections 11, 35, and 36). We introduce and explore it in Subsection 2.2, Sections 9 and 10.
4. *Moduli space, the approach from the point of view of algebraic geometry.* The idea of realization spaces of discrete configurations is the key to understand Gluck's rigidity theorem leading to Sabitov's proof of the bellows conjecture (Sections 31 and 34). Two universality type results in Sections 12 and 13 give a basic introduction (as well as a counterpart to the Steinitz theorem).
5. *Geometric and algebraic valuations.* This is a modern and perhaps more technical algebraic approach in the study of polyhedra. We give an introduction in Sections 16 and 17, and use it heavily in the proof of the bellows conjecture (Section 34 and Subsection 41.7).
6. *Local move connectivity arguments.* This basic principle is used frequently in combinatorics and topology to prove global results via local transformations. We introduce it in Section 14 and apply it to scissor congruence in Section 17 and geometry of curves in Section 23.
7. *Spherical geometry.* This is a classical and somewhat underrated tool, despite its wide applicability. We introduce it in Section 20 and use it throughout the second part (Sections 24 and 25, Subsections 27.1, 29.3).

Note that some of these are broader and more involved than others. On the other hand, some closely related material is completely omitted (e.g., we never study the hyperbolic geometry). To quote one modern day warrior, "If you try to please everybody, somebody's not going to like it" [Rum].

Section implications. While most sections are independent, the following list of implications shows which sections are not: $1 \Rightarrow 2, 3, 20$ $5 \Rightarrow 23$ $7 \Rightarrow 28, 36$ $9 \Rightarrow 10 \Rightarrow 25, 40$ $11 \Rightarrow 34$ $12 \Rightarrow 13$ $14 \Rightarrow 17$ $15 \Rightarrow 16 \Rightarrow 17$ $21 \Rightarrow 23$ $22 \Rightarrow 26 \Rightarrow 27 \Rightarrow 37$ $26 \Rightarrow 28, 29, 30$ $25 \Rightarrow 35 \Rightarrow 37 \Rightarrow 38$ $31 \Rightarrow 32 \Rightarrow 33$

Suggested course content. Although our intention is to have a readable (and teachable) textbook, the book is clearly too big for a single course. On a positive side, the volume of book allows one to pick and choose which material to present. Below we present several coherent course suggestions, in order of increasing difficulty.

(**1**) *Introduction to Discrete Geometry* (basic undergraduate course).
§§ 1, 2.1-2, 3, 4, 5.1-2 (+ Prop. 5.9, 5.11), 23.6, 25.1, 19, 20, 8.1-2, 8.4, 9, 10, 12, 13, 14.1-3, 15, 21.1-3, 23.1-2, 23.6, 22.1-5, 26.1-4, 30.2, 30.4, 39, 40.3-4.

(**2**) *Modern Discrete Geometry* (emphasis on geometric rather than combinatorial aspects; advanced undergraduate or first year graduate course).
§§ 4–6, 9, 10, 12–15, 17.5-6, 18, 20–23, 25, 26, 29, 30, 33, 35.5, 36.3-4, 39, 40.

(**3**) *Geometric Combinatorics* (emphasis on combinatorial rather than geometric aspects; advanced undergraduate or first year graduate course).
§§ 25.1, 19, 20, 1–4, 8, 11, 12, 14–17, 23, 22.1-5, 26.1-4, 32, 33, 40.3, 40.4.

(**4**) *Discrete Differential Geometry* (graduate topics course)
§§ 9–11, 21, 22, 24–28, 30–35, 7, 36–38, 40.

(**5**) *Polytopes and Algebra* (intuitive graduate topics course)
Scissor congruence: §§ 15, 16, 14.4, 17, then use [Bolt, Dup, Sah] for further results.
Realization spaces and the bellows conjecture: §§ 11–13, 31, 34, then use [Ric].
Integer points enumeration: use [Barv, §7], [Grub, §19] and [MilS, §12].

On references and the index. Our reference style is a bit idiosyncratic, but, hopefully, is self-explanatory.[1] Despite an apparently large number of references, we made an effort to minimize their number. Given the scope of the field, to avoid an explosion of the references, we often omit important monographs and papers in favor of more recent surveys which contain pointers to these and many other references. Only those sources explicitly mentioned in the remarks and exercises are included. On occasion, we added references to classical texts, but only if we found the exposition in them to be useful in preparing this book. Finally, we gave a certain preference to the important foreign language works that are undeservingly overlooked in the modern English language literature, and to the sources that are freely available on the web, including several US patents. We made a special effort to include the `arXiv` numbers and the shortened clickable web links when available. We sincerely apologize to authors whose important works were unmentioned in favor of recent and more accessible sources.

[1] See also page 432 for the symbol notations in the references.

We use only one index, for both people and terminology. The references have pointers to pages where we use them, so the people in the index are listed only if they are mentioned separately.

On exercises. The exercises are placed at the end of every section. While most exercises are related to the material in the section, the connection is sometimes not obvious and involves the proof ideas. Although some exercises are relatively easy and are meant to be used as home assignments, most others contain results of independent interest. More often than not, we tried to simplify the problems, break them into pieces, or present only their special cases, so that they can potentially be solved by a committed reader. Our intention was to supplement the section material with a number of examples and applications, as well as mention some additional important results.

The exercises range from elementary to very hard. We use the following ranking: exercises labeled [1-], [1] and [1+] are relatively simple and aimed at students, while those labeled [2-], [2] and [2+] are the level of a research paper with the increasing involvement of technical tools and results from other fields. We should emphasize that these rankings are approximate at best, e.g., some of those labeled [1+] might prove to be excessively difficult, less accessible than some of those labeled [2-]. If an exercise has a much easier proof than the ranking suggests, please let me know and I would be happy to downgrade it.

We mark with $\diamondsuit$ the exercises that are either used in the section or are mentioned elsewhere as being important to understanding the material. Some additional, largely assorted and ad hoc exercises are collected in Section 42. These are chosen not for their depth, but rather because we find them appealing enough to be of interest to the reader.

Hints, brief solutions and pointers to the literature are given at the end of the book. While some solutions are as good as proofs in the main part, most are incomplete and meant to give only the first idea of what to do or where to go. Open problems and a few simple looking questions I could not answer are marked with [$*$]. They are likely to vary widely in difficulty.

On figures. There are over 250 color figures in the book, and they are often integral to the proofs.

Acknowledgements. Over the years I benefitted enormously from conversations with a number of people. I would like to thank Arseny Akopyan, Boris Aronov, Imre Bárány, Sasha Barvinok, Bob Connelly, Jesús De Loera, Elizabeth Denne, Nikolai Dolbilin, Maksym Fedorchuk, Alexey Glazyrin, David Jerison, Gil Kalai, Greg Kuperberg, Gilad Lerman, Ezra Miller, Oleg Musin, János Pach, Rom Pinchasi, Yuri Rabinovich, Jean-Marc Schlenker, Richard Stanley, Alexey Tarasov, Csaba Tóth, Uli Wagner and Victor Zalgaller for their helpful comments on geometry and other matters.

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Basic definitions and notations. Let $P \subset \mathbb{R}^d$ be a convex polytope, or a general convex body. We use $S = \partial P$ for the surface of P. To simplify the notation we will always use $\text{area}(S)$, for the surface area of P. Formally, we use $\text{area}(\cdot)$ to denote the $(d-1)$-dimensional volume: $\text{area}(S) = \text{vol}_{d-1}(S)$. We say that a hyperplane H is *supporting P at $x \in S$* if P lies on one side of H, and $x \in H$.

For a set $X \subset \mathbb{R}^d$ we use $\text{cm}(X)$ to denote the *center of mass*[2] and $\text{conv}(X)$ to denote the *convex hull* of X. We use $|xy|$ to denote the distance between points $x, y \in \mathbb{R}^d$. Alternatively, $\|\boldsymbol{e}\|$ denotes the length of a vector $\boldsymbol{e} = \overrightarrow{xy}$. The vectors are always in bold, e.g., O usually denotes the origin, while $\mathbf{0}$ is the zero vector. We let $\langle \boldsymbol{u}, \boldsymbol{w} \rangle$ denote the scalar product of two vectors. The *geodesic distance* between two surface points $x, y \in S$ is denoted by $|xy|_S$. We use $\sphericalangle$ to denote the spherical angles.

All our *graphs* will have *vertices* and *edges*. The edges are sometimes *oriented*, but only when we explicitly say so. In the beginning of Section 21, following a long standing tradition, we study "vertices" *smooth curves*, but to avoid confusion we never mention them in later sections.

In line with tradition, we use the word *polygon* to mean two different things: both a simple closed piecewise linear closed curve and the interior of this curve. This unfortunate lack of distinction disappears in higher dimension, when we consider *space polygons*. When we do need to make a distinction, we use *closed piecewise linear curve* and *polygonal region*, both notions being somewhat unfortunate. A *simple polygon* is always a polygon with no self-intersections. We use $Q = [v_1 \ldots v_n]$ to denote a closed polygon with vertices v_i in this (cyclic) order. We also use (abc) to denote a triangle, and, more generally, $(v_0 v_1 \ldots v_d)$ to denote a $(d+1)$-dimensional simplex. Finally, we use (u, v) for an open interval (straight line segment) or an edge between two vertices, $[xy]$ and $[x, y]$ for a closed interval, either on a line or on a curve, and (xy) for a line through two points.

In most cases, we use *polytope* to mean a convex hull of a finite number of points. Thus the word "polytope" is usually accompanied with an adjective *convex*, except in Sections 15–17, where a polytope is a finite union of convex polytopes. In addition, we assume that it is fully dimensional, i.e., does not lie in an affine hyperplane. In all other cases we use the word *polyhedron* for convex and non-convex surfaces, non-compact intersections of half-spaces, etc. Since we are only concerned with discrete results, we do not specify whether polytopes are open or closed sets in $\mathbb{R}^d$, and use whatever is appropriate in each case.

We make a distinction between *subdivisions* and *decompositions* of a polytope, where the former is required to be a CW complex, while the latter is not. The notions of *triangulation* is so ambiguous in the literature, we use it only for simplicial subdivisions. We call *dissections* the simplicial decompositions, and *full triangulations* the triangulations with a given set of vertices (usually the vertices of a given convex polytope).

Occasionally we use the standard notation for comparing functions: $O(\cdot)$, $o(\cdot)$, $\Omega(\cdot)$ and $\theta(\cdot)$. We use various arrow-type symbols, like $\sim$, $\simeq$, $\leftrightarrow$, $\bowtie$, $\asymp$, $\Leftrightarrow$, etc.,

[2] We consider $\text{cm}(X)$ only for convex of piecewise linear sets X, so it is always well defined.

for different kind of flips, local moves, and equivalence relations. We reserve $\approx$ for numerical estimates.

Finally, throughout the book we employ $[n] = \{1, 2, \ldots, n\}$, $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, $\mathbb{R}_+ = \{x > 0\}$, and $\mathbb{Q}_+ = \{x > 0, x \in \mathbb{Q}\}$. The d-dimensional Euclidean space is always $\mathbb{R}^d$, a d-dimensional sphere is $\mathbb{S}^d$, and a hemisphere is $\mathbb{S}^d_+$, where $d \geq 1$. To simplify the notation, we use $X - a$ and $X + b$ to denote $X \smallsetminus \{a\}$ and $X \cup \{b\}$, respectively.

Part I

Basic Discrete Geometry

1. The Helly theorem

We begin our investigation of discrete geometry with the Helly theorem and its generalizations. That will occupy much of this and the next section. Although these results are relatively elementary, they lie in the heart of discrete geometry and are surprisingly useful (see Sections 3 and 24).

1.1. **Main result in slow motion.**

We begin with the classical *Helly Theorem* in the plane.

Theorem 1.1 (Helly)**.** *Let $X_1, \ldots, X_n \subset \mathbb{R}^2$ be convex sets such that $X_i \cap X_j \cap X_k \neq \varnothing$ for every $1 \le i < j < k \le n$, where $n \ge 3$. Then there exists a point $z \in X_1, \ldots, X_n$.*

In other words, if all triples of convex sets intersect, then *all* sets intersect. The convexity condition in the Helly theorem is necessary, as can be seen in the example in Figure 1.1 below.

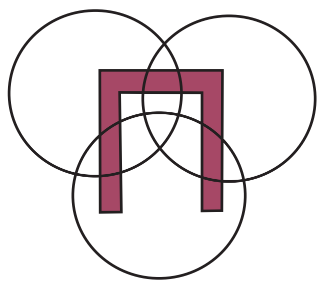

Figure 1.1. The role of convexity in the Helly theorem.

Proof. We prove the result by induction on n. For $n = 3$ there is nothing to prove. For $n = 4$, consider points $v_1 \in X_2 \cap X_3 \cap X_4$, $v_2 \in X_1 \cap X_3 \cap X_4$, $v_3 \in X_1 \cap X_2 \cap X_4$, and $v_4 \in X_1 \cap X_2 \cap X_3$. There are two possibilities: either points v_i are in convex position, or one of them, say v_4, is inside the triangle on remaining the points $(v_1v_2v_3)$. In the second case, $z = v_4$ clearly works. In the first case, take z to be the intersection of two diagonals inside the 4-gon $(v_1v_2v_3v_4)$. For points labeled as in Figure 1.2 we have:

$$z \in \operatorname{conv}\{v_1, v_3\} \cap \operatorname{conv}\{v_2, v_4\} \subset (X_2 \cap X_4) \cap (X_1 \cap X_3),$$

which implies the claim for $n = 4$.

Let us now make a notation which we will use throughout the section. Denote by $X_I = \cap_{i \in I} X_i$, where $I \subset [n] = \{1, \ldots, n\}$. For the induction step, assume that $n > 4$ and for every $(n-1)$-element subset $I \subset [n]$ we have $X_I \neq \varnothing$. Denote by v_i any point in $X_{[n]-i}$, and consider a configuration of 4 points: v_1, v_2, v_3 and v_4. As in the case $n = 4$, there are two possibilities: these points are either in convex position or not. Arguing in the same manner as above we conclude that in each case there exists a point $z \in X_{[n]}$, as desired. $\square$

Here is a natural generalization of the Helly theorem in higher dimensions:

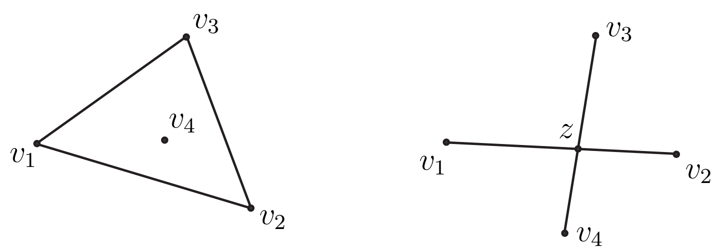

FIGURE 1.2. The Helly theorem for $n = 4$.

Theorem 1.2 (d-dimensional Helly theorem)**.** *Let $X_1, \ldots, X_n \subset \mathbb{R}^d$ be $n \geq d+1$ convex sets such that $X_I \neq \varnothing$ for every subset $I \subset [n]$, $|I| = d+1$. Then there exists a point $z \in X_1, \ldots, X_n$.*

Before we present the proof of Theorem 1.2, let us note that this statement is really about convex polytopes rather than some (at times geometrically complicated) convex sets. This fact will prove useful later on.

Proposition 1.3. *In Theorem 1.2, it suffices to prove the result for convex polytopes.*

Proof. Indeed, for every $I \subset [n]$ such that $X_I \neq \varnothing$ fix a point $x_I \in X_I$. Let $Y_i = \operatorname{conv}\{x_I \mid i \in I\}$. Observe that $Y_I \subset X_I$, so if $X_I = \varnothing$ then $Y_I = \varnothing$. On the other hand, if $X_I \neq \varnothing$, then $Y_I \ni x_I$ since each $Y_i \ni x_I$, for all $i \in I$. Therefore, it suffices to prove the theorem only for the convex polytopes $Y_1, \ldots, Y_n$. $\square$

Our proof of the d-dimensional Helly Theorem proceeds by induction as in the plane. However, for $d > 2$ there are more cases to consider. These cases will be handled by the following simple result.

Theorem 1.4 (Radon)**.** *Let $a_1, \ldots, a_m \in \mathbb{R}^d$ be any $m \geq d+2$ points. Then there exists two subsets $I, J \subset [m]$, such that $I \cap J = \varnothing$ and*

$$\operatorname{conv}\{a_i \mid i \in I\} \cap \operatorname{conv}\{a_j \mid j \in J\} \neq \varnothing.$$

Proof. Write the coordinates of each point $a_i = (a_i^1, \ldots, a_i^d) \in \mathbb{R}^d$, and consider a system of $d+1$ equations and m variables $\tau_1, \ldots, \tau_m$:

$$\begin{cases} \displaystyle\sum_{i=1}^{m} \tau_i \;= 0, & \\ \displaystyle\sum_{i=1}^{m} \tau_i\, a_i^r \;= 0, & \text{for all} \quad 1 \leq r \leq d. \end{cases}$$

Since $m \geq d+2$, the system has a nonzero solution $(\tau_1, \ldots, \tau_m)$. From the first equation, some τ_i are positive and some are negative. Set $I = \{i : \tau_i > 0\}$, $J = \{j : \tau_j < 0\}$, and $c = \sum_{i \in I} \tau_i$. Adding the last d equations as above and rearranging the terms we obtain:

$$\sum_{i \in I} \tau_i\, a_i + \sum_{j \in J} \tau_j\, a_j \;=\; O,$$

$$\sum_{i\in I} \frac{\tau_i}{c}\, a_i \, = \sum_{j\in J} \frac{-\tau_j}{c}\, a_j \, .$$

Since both sides are convex combinations of points a_i from disjoint sets I and J, we obtain the result. $\square$

Proof of Theorem 1.2. Use induction. The case $n = d+1$ is clear. Suppose $n \geq d+2$ and every $(n-1)$-element subset of convex sets X_i has a common point $v_i \in X_{[n]-i}$. By Radon's theorem, there exists two disjoint subsets $I, J \subset [n]$ and a point $z \in \mathbb{R}^d$, such that

$$z \in \operatorname{conv}\{v_i \mid i \in I\} \cap \operatorname{conv}\{v_j \mid j \in J\} \subset X_{[n]\smallsetminus I} \cap X_{[n]\smallsetminus J} \subset X_{[n]}\, ,$$

where the second inclusion follows by definition of points v_i. $\square$

1.2. **Softball geometric applications.** The Helly theorem does not look at all powerful, but it has, in fact, a number of nice geometric applications. We present here several such applications, leaving others as exercises.

Corollary 1.5. *Let $P_1, \ldots, P_n \subset \mathbb{R}^2$ be rectangles with sides parallel to the coordinate axes, such that every two rectangles intersect each other. Then all rectangles have a nonempty intersection.*

Proof. By the Helly theorem, it suffice to show that every three rectangles intersect (see Figure 1.3). Indeed, project three rectangles P_i, P_j, P_k onto the x axis. By the Helly theorem for the line, the three intervals in projection intersect at a point x_0. Similarly, the three intervals in projection of the three rectangles onto the y axis intersect at a point y_0. Therefore, all three rectangles contain point (x_0, y_0), which completes the proof. $\square$

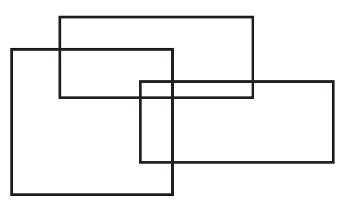

FIGURE 1.3. Three pairwise intersecting rectangles as in Corollary 1.5.

Note that in the proof we used the 1-dimensional version of the Helly theorem, the claim we took for granted when we started the section with $d = 2$ case. Of course, the proof of Theorem 1.2 works fine in this case.

Corollary 1.6. *Let $A \subset \mathbb{R}^2$ be a fixed convex set and let $X_1, \ldots, X_n \subset \mathbb{R}^2$ be any convex sets such that every three of them intersect a translation of A. Then there exists a translation of A that intersects all sets X_i.*

Note that this corollary can be viewed as an extension of the Helly theorem, which is the case when A is a single point (its translations are all points in the plane).

Proof. Denote by $\mathrm{cm}(B)$ the center of mass of the set B. For every i, define Y_i in such a way that $X_i \cap A' \neq \varnothing$ if and only if $\mathrm{cm}(A') \in Y_i$, for every translation A' of A (see Figure 1.4). Now apply the Helly theorem to these convex sets Y_i. For a point $z \in Y_1 \cap \ldots \cap Y_n$ find a translation A'' with $\mathrm{cm}(A'') = z$. By definition of the sets Y_i, the set A'' intersects all sets X_i. □

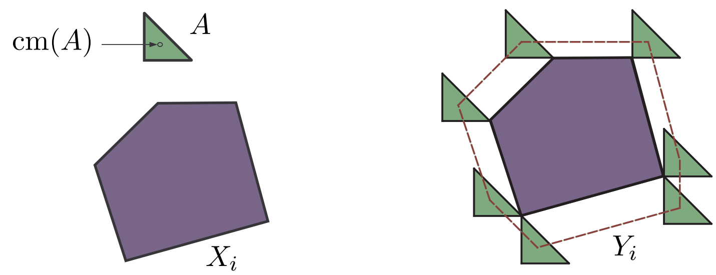

FIGURE 1.4. Convex sets A, X_i, and construction of the set Y_i.

Corollary 1.7. *Let $z_1, \ldots, z_n \in \mathbb{R}^2$ be points in the plane, such that every three of them can be covered by a circle of radius r. Then all points can be covered by a circle of radius r.*

Proof. The result follows immediately from Corollary 1.6 with $x_i = \{z_i\}$, and A is a circle of radius r. □

Note that here we used the other extreme of Corollary 1.6 (setting X_i to be points) than the one which gives the Helly theorem.

Corollary 1.8. *Let $z_1, \ldots, z_n \subset \mathbb{R}^2$ be points in the plane, such that all pairwise distances $|z_i z_j|$ are at most 1. Then all points can be covered by a circle of radius $\frac{1}{\sqrt{3}}$.*

Proof. By Corollary 1.7 it suffices to show that every three points z_i, z_j, and z_k can be covered with a circle of radius $\frac{1}{\sqrt{3}}$. There are three cases: triangle $T = (z_i z_j z_k)$ is either acute, right or obtuse. In the last two case take a circle C centered at the midpoint of the longest edge. Clearly, the radius of C is at most $\frac{1}{2} < \frac{1}{\sqrt{3}}$.

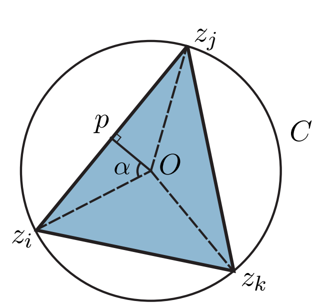

FIGURE 1.5. Computing the radius of the circumscribed circle C.

When T is acute, the proof is apparent from Figure 1.5. Let O be the center of the circumscribed circle C. Suppose $\angle z_i O z_j$ is the largest angle in T and let $\alpha = \angle z_i O p$ be as in the figure. Then $\alpha = \frac{1}{2}(\angle z_i O z_j) \geq \frac{\pi}{3}$. On the other hand, $|z_i p| = \frac{1}{2}|z_i z_j| \leq \frac{1}{2}$, by assumption. Therefore,

$$\text{radius}(C) \,=\, |z_i O| \,=\, \frac{|z_i p|}{\sin\alpha} \,\leq\, \frac{1}{\sqrt{3}},$$

as desired. □

Note that Corollary 1.8 is sharp in a sense that one cannot make the radius smaller (take an equilateral triangle). On the other hand, it is always possible to decrease the shape of the covering set. This approach will be used in the next section.

1.3. **The fractional Helly theorem.** It turns out, one can modify the Helly theorem to require not *all* but a constant proportion of sets X_I to be nonempty. While one cannot guarantee in this case that *all* sets X_i are intersecting (a small fraction of them can even be completely disjoint from the other sets), one can still show that a constant proportion of them do intersect. We will not prove the sharpest version of this result. Instead, we introduce the *Morse function* approach which will be used repeatedly in the study of simple polytopes (see Section 8).

Theorem 1.9 (Fractional Helly)**.** *Let $\alpha > 0$, and $X_1, \ldots, X_n \subset \mathbb{R}^d$ be convex sets such that for at least $\alpha\binom{n}{d+1}$ of $(d+1)$-element sets $I \subset [n]$ we have $X_I \neq \varnothing$. Then there exists a subset $J \subset [n]$, such that $|J| > \frac{\alpha}{d+1}n$, and $X_J \neq \varnothing$.*

Before we prove the theorem, recall Proposition 1.3. Using the same proof verbatim, we conclude that it suffices to prove the result only for convex polytopes.

Now, let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be a generic linear function, i.e., nonconstant on all edges of all polytopes X_I, $I \subset [n]$. Then every polytope X_I will have the *smallest vertex* $\min_\varphi(X_I)$. We will refer to φ as the *Morse function.* We need the following simple result.

Lemma 1.10. *Let $I \subset [n]$ be such that $X_I \neq \varnothing$, and let $v = \min_\varphi(X_I)$ be the minimum of the Morse function. Then there exists a subset $J \subset I$ such that $v = \min_\varphi(X_J)$ and $|J| \leq d$.*

Proof. Consider a set $C := \{w \in \mathbb{R}^d \mid \varphi(w) < \varphi(v)\}$. Clearly, C is convex and $C \cap X_I = \varnothing$. In other words, a family of convex sets $\{C\} \cup \{X_i, i \in I\}$ has an empty intersection, which implies that it contains a $(d+1)$-element subfamily $\{C\} \cup \{X_j : j \in J\}$ for some $J \subset I$, with an empty intersection: $C \cap X_J = \varnothing$. Since all sets $X_J \supset X_I$ contain v, this implies that $\varphi(z) \geq \varphi(v)$ for all $z \in X_J$ (see Figure 1.6). By construction of φ this implies that $v = \min_\varphi(X_J)$. □

Proof of Theorem 1.9. Use a double counting argument. Fix a Morse function φ. Denote by $J = \gamma(I)$ the d-element subset $J \subset [n]$ given by the lemma above.

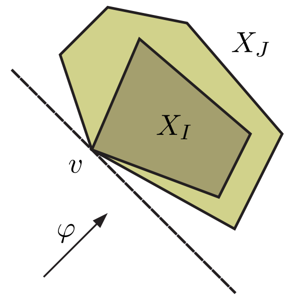

FIGURE 1.6. Morse function in Lemma 1.10.

Now consider only $(d+1)$-element subsets $I \subset [n+1]$ Since the number of d-element subsets J is at most $\binom{n}{d}$, there exists a subset J_0 such that $J_0 = \gamma(I)$ for at least

$$\frac{\alpha \binom{n}{d+1}}{\binom{n}{d}} = \alpha \, \frac{n-d}{d+1}$$

different $(d+1)$-element subsets I. Let $v = \min_\varphi(X_{J_0})$. By definition of the map γ, we have $I \supset J_0$ for each I as above. Therefore, there exist at least $\alpha \, \frac{n-d}{d+1}$ different $i = I - J_0$ such that $v \in X_i$. Adding to this all sets X_j with $j \in J_0$, we conclude that v belongs to at least

$$d + \alpha \, \frac{n-d}{d+1} > \alpha \, \frac{n}{d+1}$$

convex subsets X_i. $\square$

1.4. **Exercises.**

Exercise 1.1. (*Infinite Helly theorem*) $\diamondsuit$ [1] Extend the Helly theorem (Theorem 1.2) to infinitely many (closed) convex sets.

Exercise 1.2. *a*) [1-] Let $P \subset \mathbb{R}^3$ be a convex polytope such that of the planes spanned by the faces, every three intersect at a point, but no four intersect. Prove that there exist four such planes which form a tetrahedron containing P.
b) [1-] Generalize the result to polytopes in $\mathbb{R}^d$.

Exercise 1.3. [1+] Prove that every polygon $Q \subset \mathbb{R}^2$ of length L (the sum of lengths of all edges) can be covered by a disk of radius $L/4$.

Exercise 1.4. [1+] Suppose $n \geq 3$ unit cubes are inscribed into a sphere, such that every three of them have a common vertex. Prove that all n cubes have a common vertex.

Exercise 1.5. [1] Suppose every three points of $z_1, \ldots, z_n \in \mathbb{R}^2$ can be covered by a triangle of area 1. Prove that $\text{conv}(z_1, \ldots, z_n)$ can be covered by a triangle of area 4.

Exercise 1.6. [1] Suppose there are n lines in the plane such that every three of them can be intersected with a unit circle. Prove that all of them can be intersected with a unit circle.

Exercise 1.7. (*Generalized Helly theorem*) [2] Let $X_1, \ldots, X_n$ be convex sets in $\mathbb{R}^d$. Suppose the intersections of every $d+1-k$ of them contain an affine k-dimensional subspace. Prove that there exists an affine k-dimensional subspace contained in all subspaces: $H \subset X_i$, for all $1 \le i \le n$.

Exercise 1.8. *a*) [2-] Let $\Delta \subset \mathbb{R}^{d+1}$ be a simplex and let $f : \mathbb{R}^{d+1} \to \mathbb{R}^d$ be a linear map. Denote by $T_1, \ldots, T_{d+2}$ the facets of Δ. Prove that $f(T_1) \cap \ldots \cap f(T_{d+2}) \neq \varnothing$.
b) [2+] Show that part *a*) holds for all continuous maps $f : \partial\Delta \to \mathbb{R}^d$.

Exercise 1.9. *a*) [2-] Let $P \subset \mathbb{R}^{d+1}$ be a convex polytope and let $f : \mathbb{R}^{d+1} \to \mathbb{R}^d$ be a linear map. Prove that there exist two disjoint faces $F, G \subset P$, such that $f(F) \cap f(G) \neq \varnothing$.
b) [1+] Show that part *a*) implies Radon's theorem (Theorem 1.4).
c) [2+] Show that part *a*) holds for all continuous maps $f : \partial P \to \mathbb{R}^d$.
d) [2+] Let $X \subset \mathbb{R}^{d+1}$ be a convex body with nonempty interior and surface $S = \partial X$. Let $f : S \to \mathbb{R}^d$ be a continuous map. Points $x, y \in S$ are called *opposite* if they lie on distinct parallel hyperplanes supporting X. Prove that there exist two opposite points $x, y \in S$ such that $f(x) = f(y)$.

Exercise 1.10. [2] Let $P \subset \mathbb{R}^d$ be a convex polytope different from a simplex. Suppose $X_1, \ldots, X_n \subset \partial P$ are convex sets (thus each lying in a facet of P), such that every d of them intersect. Prove that all of them intersect.

Exercise 1.11. (*Spherical Helly theorem*) *a*) [1] Suppose $X_1, \ldots, X_n \subset \mathbb{S}^2_+$ are convex sets on a hemisphere, such that every three of them intersect. Prove that all X_i intersect.
b) [1+] Suppose $X_1, \ldots, X_n \subset \mathbb{S}^d_+$ are convex sets on a hemisphere, such that every $d+1$ of them intersect. Prove that all X_i intersect.
c) [2-] Suppose $X_1, \ldots, X_n \subset \mathbb{S}^2$ are convex sets on a sphere, such that each of them is inside a hemisphere, every three of them intersect and no four cover the sphere. Prove that all X_i intersect.

Exercise 1.12. *a*) [1] Suppose $z_1, \ldots, z_n \in \mathbb{S}^2$ are points on a unit sphere, such that every three of them can be covered by a spherical disk of radius $r < \pi/3$. Prove that all points can be covered by a spherical circle of radius r.
b) [1] Check that the claim is false if we require only that $r < \pi/2$. Find the optimal constant.

Exercise 1.13. (*Kirchberger*) *a*) [1] Let $X \subset \mathbb{R}^2$ be a set of $n \ge 4$ points in general position. Suppose points in X are colored with two colors such that for every four points there is a line separating points of different color. Prove that there exists a line separating all points in X by color.
b) [1+] Generalize the result to $\mathbb{R}^d$.

Exercise 1.14. Suppose $Q \subset \mathbb{R}^2$ is a simple polygon which contains a point w, such that interval $[w, x]$ lies in Q, for all $x \in Q$. Such Q is called a *star-shaped polygon.* A *kernel* of Q is a set K_Q of points w as above.
a) [1-] Prove that the kernel K_Q is a convex polygon.
b) [1] Prove that the boundary $\partial\mathrm{K}_Q$ lies in the union of ∂Q and the lines spanned by the inflection edges of Q (see Subsection 23.1).
c) [1] Define the *tangent set* L_Q to be the union of all lines supporting polygon Q. Prove that L_Q is the whole plane unless Q is star-shaped. Moreover, the closure of $\mathbb{R}^2 \smallsetminus \mathrm{L}_Q$ is exactly the kernel K_Q.

d) [1+] For a 2-dimensional polyhedral surface $S \subset \mathbb{R}^3$ define the tangent set L_S to be the union of planes tangent to S. Prove that $\mathrm{L}_S = \mathbb{R}^3$ unless S is homeomorphic to a sphere and encloses a star-shaped 3-dimensional polyhedron.

Exercise 1.15. (*Krasnosel'skii*) *a*) [1] Suppose $Q \subset \mathbb{R}^2$ is a simple polygon, such that for every three points $x, y, z \in Q$ there exists a point v such that line segments $[vx], [vy]$, and $[vz]$ lie in Q. Prove that Q is a star-shaped polygon.
b) [1+] Generalize the result to $\mathbb{R}^d$; use $(d+1)$-tuples of points.

Exercise 1.16. *a*) [2-] Let $X_1, \ldots, X_m \subset \mathbb{R}^d$ be convex sets such that their union is also convex. Prove that if every $m-1$ of them intersect, then all of them intersect.
b) [2] Let $X_1, \ldots, X_m \subset \mathbb{R}^d$ be convex sets such that the union of every $d+1$ or fewer of them is a star-shaped region. Then all X_i have a nonempty intersection.
c) [2-] Deduce part *a*) from part *b*).
d) [2+] Find a 'fractional analogue' of part *b*).

Exercise 1.17. *a*) [1] Let $Q \subset \mathbb{R}^2$ be a union of axis parallel rectangles. Suppose for every two points $x, y \in Q$ there exists a point $v \in Q$, such that line segments $[vx]$ and $[vy]$ lie in Q. Prove that Q is a star-shaped polygon.
b) [1+] Generalize the result to $\mathbb{R}^d$; use pairs of points.

Exercise 1.18. *a*) [2-] Suppose $X_1, \ldots, X_n \subset \mathbb{R}^2$ are disks such that every two of them intersect. Prove that there exist four points $z_1, \ldots, z_4$ such that every X_i contains at least one z_j.
b) [2-] In condition of *a*), suppose X_i are unit disks. Prove that three points z_j suffice.

Exercise 1.19. [1+] Prove that for every finite set of n points $X \subset \mathbb{R}^2$ there exist at most n disks which cover X, such that the distance between any two disks is ≥ 1, and the total diameter is $\leq n$.

Exercise 1.20. *a*) [2-] Suppose $X_1, \ldots, X_n \subset \mathbb{R}^2$ are convex sets such that through every four of them there exists a line intersecting them. Prove that there exist two lines ℓ_1 and ℓ_2 such that every X_i intersects ℓ_1 or ℓ_2.
b) [2+] Let $X_1, \ldots, X_n \subset \mathbb{R}^d$ be axis parallel bricks. Suppose for any $(d+1)2^{d-1}$ of these there exists a hyperplane intersecting them. Prove that there exists a hyperplane intersecting all X_i, $1 \leq i \leq n$.

Exercise 1.21. (*Topological Helly theorem*) *a*) [1] Let $X_1, \ldots, X_n \subset \mathbb{R}^2$ be simple polygons[3] in the plane, such that all double and triple intersections of X_i are also (nonempty) simple polygons. Prove that the intersection of all X_i is also a simple polygon.
b) [1+] Generalize part *a*) to higher dimensions.

Exercise 1.22. *a*) [1] Let $X_1, \ldots, X_n \subset \mathbb{R}^2$ be simple polygons in the plane, by which we mean here polygonal regions with no holes. Suppose that all unions $X_i \cup X_j$ are also simple polygons, and that all intersections $X_i \cap X_j$ are nonempty. Prove that the intersection of all X_i is also nonempty.
b) [1+] Let $X_1, \ldots, X_n \subset \mathbb{R}^2$ be simple polygons in the plane, such that all unions $X_i \cup X_j \cup X_k$ are also simple, for all $1 \leq i \leq j \leq k \leq n$. Prove that the intersection of all X_i is nonempty.

Exercise 1.23. (*Weak converse Helly theorem*) *a*) [1] Let $X \subset \mathbb{R}^2$ be a simple polygon such that for every collection of triangles $T_1, \ldots, T_n \subset \mathbb{R}^2$ either $X \cap T_1 \cap \cdots \cap T_n \neq \varnothing$, or $X \cap T_i \cap T_j = \varnothing$ for some $1 \leq i < j < n$. Prove that X is convex.

[3]Here a *simple polygon* is a connected, simply connected finite union of convex polygons.

b) [1+] Generalize part *a*) to higher dimensions.

Exercise 1.24. (*Converse Helly theorem*) *a*) [2-] Let $\mathcal{A}$ be an infinite family of simple polygons in the plane which is closed under non-degenerate affine transformations. Suppose $\mathcal{A}$ satisfies the following property: for every four elements in $\mathcal{A}$, if every three of them intersect, then so does the fourth. Prove that all polygons in $\mathcal{A}$ are convex.
b) [2] Extend this to general compact sets in the plane.
c) [2+] Generalize this to higher dimensions.

Exercise 1.25. (*Convexity criterion*) $\Diamond$ *a*) [2-] Let $X \subset \mathbb{R}^3$ be a compact set and suppose every intersection of X by a plane is contractible. Prove that X is convex.[4]
b) [2+] Generalize this to higher dimensions.

Exercise 1.26. *a*) [2-] Let $\mathcal{A}$ be an infinite family of simple polygons in the plane which is closed under rigid motions and intersections. Prove that all polygons in $\mathcal{A}$ are convex.
b) [2] Extend this to general connected compact sets in the plane.
c) [2] Generalize this to higher dimensions.

1.5. **Final remarks.** For a classical survey on the Helly theorem and its applications we refer to [DGK]. See also [Eck] for recent results and further references. The corollaries we present in this section are selected from [HDK] where numerous other applications of Helly theorem are presented as exercises.

Our proof of the fractional Helly theorem (Theorem 1.9) with minor modifications follows [Mat1, §8.1]. The constant $\alpha/(d+1)$ in the theorem can be replaced with the optimal constant

$$1 - (1-\alpha)^{1/(d+1)},$$

due to Gil Kalai (see [Eck] for references and discussions of this and related results).

[4]Already the case when X is polyhedral (a finite union of convex polytopes) is non-trivial and is a good starting point.

2. Carathéodory and Bárány theorems

This short section is a followup of the previous section. The main result, the Bárány theorem, is a stand-alone result simply too beautiful to be missed. Along the way we prove the classical Sylvester–Gallai theorem, the first original result in the whole subfield of point and line configurations (see Section 12).

2.1. **Triangulations are fun.** We start by stating the following classical theorem in the case of a finite set of points.

Theorem 2.1 (Carathéodory). *Let $X \subset \mathbb{R}^d$ be a finite set of points, and let $z \in \operatorname{conv}(X)$. Then there exist $x_1, \ldots, x_{d+1} \in X$ such that $z \in \operatorname{conv}\{x_1, \ldots, x_{d+1}\}$.*

Of the many easy proofs of the theorem, the one most relevant to the subject is the 'proof by triangulation': given a simplicial triangulation of a convex polytope $P = \operatorname{conv}(X)$, take any simplex containing z (there may be more than one). To obtain a triangulation of a convex polytope, choose a vertex $v \in P$ and triangulate all facets of P (use induction on the dimension). Then consider all cones from v to the simplices in the facet triangulations.[5]

One can also ask how many simplices with vertices in X can contain the same point. While this number may vary depending on the location of the point, it turns out there always exists a point z which is contained in a constant proportion of all simplices.

Theorem 2.2 (Bárány). *For every $d \geq 1$ there exists a constant $\alpha_d > 0$, such that for every set of n points $X \subset \mathbb{R}^d$ in general position, there exists a point $z \in \operatorname{conv}(X)$ contained in at least $\alpha_d \binom{n}{d+1}$ simplices $(x_1 \ldots x_{d+1})$, $x_i \in X$.*

Here the points are in *general position* if no three points lie on a line. This result is nontrivial even for convex polygons in the plane, where the triangulations are well understood. We suggest the reader try to prove the result in this special case before going through the proof below.

2.2. **Infinite descent as a mathematical journey.** The method of infinite descent is a basic tool in mathematics, with a number of applications in geometry. To illustrate the method and prepare for the proof of the Bárány theorem we start with the following classical result of independent interest:

Theorem 2.3 (Sylvester–Gallai). *Let $X \in \mathbb{R}^2$ be a finite set of points, not all on the same line. Then there exists a line containing exactly two points in X.*

Proof. Suppose every line which goes through two points in X, goes also through a third such point. Since the total number of such lines is finite, consider the shortest distance between points and lines. Suppose the minimum is achieved at a pair: point $y \in X$ and line ℓ. Suppose also that ℓ goes through points x_1, x_2, x_3 (in that

[5] Formally speaking, in dimensions at least 4, this argument produces a *dissection*, and not necessarily a (face-to-face) *triangulation* (see Exercise 2.1). Of course, a dissection suffices for the purposes of the Carathéodory theorem. We formalize and extend this approach in Subsection 14.5.

order). Since sum of the angles $\angle x_1yx_2 + \angle x_2yx_3 < \pi$, at least one of them, say $\angle x_1yx_2 < \pi/2$. Now observe that the distance from x_2 to line (x_1, y) is smaller that the distance from y to ℓ, a contradiction (see Figure 2.1). $\square$

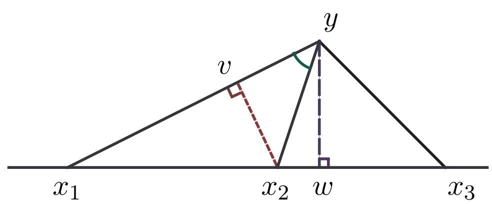

FIGURE 2.1. Descent step $[yw] \to [x_2v]$ in the proof of the Sylvester–Gallai theorem.

One can think of the proof of the Sylvester–Gallai theorem as an algorithm to find a line with exactly two points, by "descending" through the pairs (point, line) with shorter and shorter distances. Of course, for infinite configurations, where every line contains ≥ 3 points (e.g., a square grid), this "descent" is infinite indeed.

The following result is a "colorful" generalization of the Carathéodory theorem, and has a 'proof from the book', again by an infinite descent.

Theorem 2.4 (Colorful Carathéodory)**.** *Let $X_1, \ldots, X_{d+1} \subset \mathbb{R}^d$ be finite sets of points whose convex hulls contain the origin, i.e., $O \in \operatorname{conv}(X_i)$, $1 \leq i \leq d+1$. Then there exist points $x_1 \in X_1, \ldots, x_{d+1} \in X_{d+1}$, such that $O \in \operatorname{conv}\{x_1, \ldots, x_{d+1}\}$.*

Here the name come from the idea that points in the same set X_i are colored with the same color, these colors are different for different X_i, and the theorem shows existence of a simplex with vertices of different color.

Proof. We call $(x_1x_2 \ldots x_{d+1})$ with $x_i \in X_i$, the *rainbow simplex*. Suppose none of the rainbow simplices contain the origin. Let $\Delta = (x_1x_2 \ldots x_{d+1})$ be the closest rainbow simplex to O, and let $z \in \Delta$ be the closets point of Δ to O. Denote by H the hyperplane containing z and orthogonal to (Oz). Clearly, some vertex of Δ, say, x_1, does not lie on H. Since $\operatorname{conv}(X_1) \ni O$ and O, x_1 lie on different sides of H, there exists a point $y \in X_1$ which lies on the same side of H as O. Now observe that $\Delta' := (yx_2 \ldots x_{d+1})$ is a rainbow simplex which contains an interval (yz) lying on the same side of H as O (see Figure 2.2 for the case when z lies in the relative interior of a facet). We conclude that Δ' is closer to O, a contradiction. $\square$

2.3. **Generalized Radon theorem.** Recall the Radon theorem (Theorem 1.4), which states that every set of size $\geq d+2$ can be split into two subsets whose convex hulls intersect. It is natural to ask for the smallest number $t = t(d, r)$ of points needed so they can be split into r sets whose convex hulls intersect. A well known theorem of Tverberg gives $t(d, r) = (r-1)(d+1) + 1$. We present here an easy proof of a weaker version of this result.

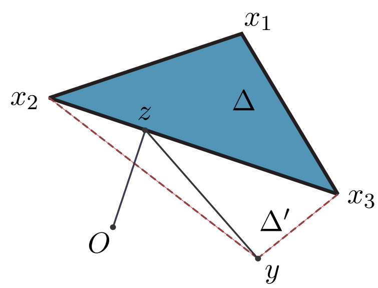

FIGURE 2.2. Descent step $(x_1x_2x_3) \to (yx_2x_3)$ on rainbow simplices.

Theorem 2.5 (Tverberg). *Let $n \geq (r-1)(d+1)^2+1$ for $d, r \in \mathbb{N}$. Then for every n points $x_1, \ldots, x_n \in \mathbb{R}^d$ there exist r subsets $I_1, \ldots, I_r \subset [n]$, such that $I_i \cap I_j = \varnothing$, $i \neq j$, and*

$$\bigcap_{i=1}^{r} \operatorname{conv}\bigl(X_{I_i}\bigr) \neq \varnothing.$$

Proof. Let $k = (r-1)(d+1)$, and $s = n-k$. Observe that every $d+1$ subsets Y_i, $1 \leq i \leq d+1$, of $X = \{x_1, \ldots, x_n\}$ of size s have a common point. Indeed, a simple counting gives: $|Y_1| = n-k$, $|Y_1 \cap Y_2| \geq n-2k$, ..., $|Y_1 \cap Y_2 \cap \ldots \cap Y_{d+1}| \geq n-(d+1)k \geq 1$. In other words, for every $Y_1, \ldots, Y_{d+1}$ as above, their convex hulls $B_i = \operatorname{conv}(Y_i)$ have a point in common (which must lie in X). Therefore, by the Helly theorem (Theorem 1.2), all convex hulls of s-tuples (and, therefore, m-tuples, $m \geq s$) of points in X have a point $z \in \operatorname{conv}(X)$ in common (not necessarily in X).

By the Carathéodory theorem (Theorem 2.1), since $|X| \geq d+2$, we can choose a $(d+1)$-element subset $X_1 \subset X$ such that $z \in \operatorname{conv}(X_1)$. Since $|X \smallsetminus X_1| = n-(d+1) \geq s$, from above $z \in \operatorname{conv}(X \smallsetminus X_1)$. Again, by the Carathéodory theorem, there exists $X_2 \subset (X \smallsetminus X_1)$ such that $z \in \operatorname{conv}(X_2)$. Repeating this r times we find r disjoint $(d+1)$-element subsets $X_1, \ldots, X_r \subset X$ whose convex hulls contain z. $\square$

2.4. **Proof of Theorem 2.2.** Let $r = \lfloor n/(d+1)^2 \rfloor$. By Theorem 2.5, there exist disjoint subsets $X_1, \ldots, X_r \subset X$ whose convex hulls intersect, i.e., contain some point $z \in \operatorname{conv}(X)$. Now, by the colorful Carathéodory theorem (Theorem 2.4), for every $(d+1)$-element subset $J = \{j_1, \ldots, j_{d+1}\} \subset [r]$ there exists a rainbow simplex $\Delta_J = (y_1, \ldots, y_{d+1})$, $y_i \in X_{J_i}$, containing point z. In other words, if we think of $[r]$ as the set of different colors, then there exists a J-colored simplex containing z, for every subset J of $d+1$ colors. Therefore, the total number of such simplices is

$$\binom{r}{d+1} = \binom{\lfloor n/(d+1)^2 \rfloor}{d+1} > \alpha_d \binom{n}{d+1},$$

as desired. $\square$

2.5. **Exercises.**

Exercise 2.1. $\Diamond$ [1-] Check that the argument below Theorem 2.1 produces a (face-to-face) triangulation in $\mathbb{R}^3$. Explain what can go wrong in $\mathbb{R}^4$.

Exercise 2.2. (*Dual Sylvester–Gallai theorem*) [1] Let L be a finite set of lines in the plane, not all going through the same point. Prove that there exists a point $x \in \mathbb{R}^2$ contained in exactly two lines.

Exercise 2.3. $\Diamond$ *a*) [1] Let L be a finite set of lines in the plane, not all going through the same point. Denote by p_i the number of points which lie in exactly i lines, and by q_i the number of regions in the plane (separated by L) with i sides. Prove:

$$\sum_{i\geq 2}(3-i)p_i \,+\, \sum_{i\geq 3}(3-i)q_i \,=\, 3$$

Conclude from here that $p_2 \geq 3$, implying the dual Sylvester–Gallai theorem.
b) [1] Use combinatorial duality "points $\leftrightarrow$ lines" to similarly show that for a finite set of points, not all on the same line, there exists at least three lines, each containing exactly two points.

Exercise 2.4. (*Graham–Newman problem*) [2-] Let $X \in \mathbb{R}^2$ be a finite set of points, not all on the same line. Suppose the points are colored with two colors. A line is called *monochromatic* if all its points in X have the same color. Prove that there exists a monochromatic line containing at least two points in X.

Exercise 2.5. [1] Let $X = \{x_1, \ldots, x_n\} \subset \mathbb{R}^2$ be a finite set of points, not all on the same line. Suppose real numbers $a_1, \ldots, a_n$ are associated with the points such that the sum along every line is zero. Prove that all numbers are zero.

Exercise 2.6. [1] Let $X_1, \ldots, X_n \subset \mathbb{R}^d$ be any convex sets, and $z \in \mathrm{conv}(X_1 \cup \ldots \cup X_n)$. Then there exist a subset $I \subset [n]$, $|I| = d+1$, such that $z \in \mathrm{conv}\,(\cup_{i\in I} X_i)$.

Exercise 2.7. [1+] Let $P \subset \mathbb{R}^d$ be a simple polytope with m facets. Fix $n = m - d$ vertices $v_1, \ldots, v_n$. Then there exists a vertex w and facets $F_1, \ldots, F_n \subset P$, such that $w, v_i \notin F_i$, for all $1 \leq i \leq n$.

Exercise 2.8. (*Steinitz*) *a*) [1+] Let $X \subset \mathbb{R}^d$ be a finite set of points, let $P = \mathrm{conv}(X)$ and let $z \in P \smallsetminus \partial P$ be a point in the interior of P. Then there exists a subset $Y \subset X$, $|Y| \leq 2d$, such that $z \in Q \smallsetminus \partial Q$ is the point in the interior of $Q = \mathrm{conv}(Y)$.[6]
b) [1-] Show that the upper bound $|Y| \leq 2d$ is tight.

Exercise 2.9. *a*) [2-] Let $X_1, \ldots, X_n$ be convex sets in $\mathbb{R}^d$ such that the intersection of every $2d$ of them has volume ≥ 1. Then the intersection of all X_i has volume $\geq c$, where $c = c(d) > 0$ is a universal constant which depends only on d.
b) [1-] Check that the number $2d$ cannot be lowered.

Exercise 2.10. *a*) [1-] Suppose points x_1, x_2 and x_3 are chosen uniformly and independently at random from the unit circle centered at the origin O. Show that the probability that $O \in (x_1x_2x_3)$ is equal to $1/4$.
b) [1+] Generalize part *a*) to higher dimensions.

[6]Think of this result as a variation on the Carathéodory theorem.

Exercise 2.11. [1] Let $X = [x_1 \dots x_n] \subset \mathbb{R}^2$ be a convex polygon. Prove that there exist $n-2$ points $y_1, \dots, y_{n-2} \subset \mathbb{R}^2$ such that every triangle on X contains at least one point y_i.[7] Show that this is impossible to do with fewer than $n-2$ points.

Exercise 2.12. Let $P \subset \mathbb{R}^3$ be a convex polytope of volume 1. Suppose $X \subset P$ is a set of n points.
a) [1-] Prove that there exists a convex polytope $Q \subset P \smallsetminus X$ such that $\mathrm{vol}(Q) \geq \frac{1}{n}$.
b) [1+] For every $n = 3(2^k - 1)$, prove that there exists a convex polytope $Q \subset P \smallsetminus X$ such that $\mathrm{vol}(Q) \geq 2^{-k}$.

Exercise 2.13. (*Cone triangulation*) [1] Let $C \subset \mathbb{R}^d$ be a *convex cone*, defined as the intersection of finitely many halfspaces containing the origin O. A cone in R^d is called *simple* if it has d faces. Prove that C can be subdivided into convex cones.

Exercise 2.14. *a*) [1+] Let $Q \subset \mathbb{R}^3$ be a space polygon, and let $P \subset \mathbb{R}^3$ be the convex hull of Q. Prove that every point $v \in P$ belongs to a triangle with vertices in Q.
b) [2-] Extend this result to $\mathbb{R}^d$ and general connected sets Q.

Exercise 2.15. [2+] We say that a polygon $Q \subset \mathbb{R}^{2d}$ is *convex* if every hyperplane intersects it at most $2d$ times. Denote by P the convex hull of Q. Prove that every point $v \in P$ belongs to a d-simplex with vertices in Q.

Exercise 2.16. Let $P \subset \mathbb{R}^d$ be a convex polytope with n facets, and let $X = \{x_1, \dots, x_n\}$ be a fixed subset of interior points in P. For a facet F of P and a vertex x_i, define a *pyramid* $\Phi_i(F) = \mathrm{conv}(F, x_i)$.
a) [2] Prove that one can label the facets $F_1, \dots, F_d$ in such a way that pyramids $\Phi_i(F_i)$ do not intersect except at the boundaries.
b) [2] Prove that one can label the facets $F_1, \dots, F_d$ in such way that the pyramids $\Phi_i(F_i)$ cover the whole P.

Exercise 2.17. (*Colorful Helly theorem*) [2-] Let $\mathcal{F}_1, \dots, \mathcal{F}_{d+1}$ be finite families of convex sets in $\mathbb{R}^d$. Suppose for every $X_1 \in \mathcal{F}_1, \dots, X_{d+1} \in \mathcal{F}_{d+1}$ we have $X_1 \cap \ldots \cap X_{d+1} \neq \varnothing$. Prove that $\cap_{X \in \mathcal{F}_i} X \neq \varnothing$ for some $1 \leq i \leq d+1$.

Exercise 2.18. [2] Let $X_1, \dots, X_{d+1} \subset \mathbb{R}^d$ be finite sets of points such that $O \in \mathrm{conv}(X_i \cup X_j)$, for all $1 \leq i < j \leq d+1$. Then there exist points $x_1 \in X_1, \dots, x_{d+1} \in X_{d+1}$, such that $O \in \mathrm{conv}\{x_1, \dots, x_{d+1}\}$.

Exercise 2.19. *a*) [1+] Let $P, Q \subset \mathbb{R}^d$ be two convex polytopes. Prove that $P \cup Q$ is convex if and only if every interval $[vw] \subset P \cup Q$ for all vertices v of P and w of Q.
b) [2] Generalize part *a*) to a union of n polytopes.

Exercise 2.20. *a*) [1] A triangle Δ is contained in a convex, centrally symmetric polygon Q. Let Δ' be the triangle symmetric to Δ with respect to a point $z \in \Delta$. Prove that at least one of the vertices of Δ' lies in Q or on its boundary.
b) [1-] Extend part *a*) from triangles Δ to general convex polygons.

[7]This immediately implies that at least one of y_i is covered with at least $\frac{1}{n-2}\binom{n}{3}$ triangles. Obviously, this is much weaker than the constant proportion of triangles given by the Bárány theorem.

2.6. **Final remarks.** The method of infinite descent is often attributed to Pythagoras and his proof of irrationality of $\sqrt{2}$. In the modern era it was reintroduced by Euler in his proof of Fermat's last theorem for the powers 3 and 4. For the history of the Sylvester–Gallai theorem (Theorem 2.3), let us quote Paul Erdős [Erd]:

> In 1933 while reading the beautiful book "Anschauliche Geometrie" of Hilbert and Cohn–Vossen [HilC], the following pretty conjecture occurred to me: Let $x_1, \ldots, x_n$, be a finite set of points in the plane not all on a line. Then there always is a line which goes through exactly two of the points. I expected this to be easy but to my great surprise and disappointment I could not find a proof. I told this problem to Gallai who very soon found an ingenious proof. L. M. Kelly noticed about 10 years later that the conjecture was not new. It was first stated by Sylvester in the Educational Times in 1893. The first proof though is due to Gallai.

For more on the history, variations and quantitative extensions of the Sylvester–Gallai theorem see [ErdP, BorM], [PacA, §12] and [Mat1, §4]. We return to this result in Section 12. The method of infinite descent makes another appearance in Section 9, where it has a physical motivation.

The Bárány theorem (Theorem 2.2) and the colorful Carathéodory theorem (Theorem 2.4) are due to I. Bárány [Bar3]. For extensions and generalizations of results in this section see [Mat1]. Let us mention that the exact asymptotic constant for α_2 in the Bárány theorem (Theorem 2.2) is equal to 2/9. This result is due to Boros and Füredi [BorF] (see Subsection 4.2 for the lower bound and Exercise 4.10 for the upper bound).

3. The Borsuk conjecture

3.1. **The story in brief.** Let $X \subset \mathbb{R}^d$ be a compact set, and let $\operatorname{diam}(X)$ denotes the largest distance between two points in X. The celebrated *Borsuk conjecture* claims that every convex set[8] with $\operatorname{diam}(X) = 1$ can be subdivided into $d+1$ parts, each of diameter < 1. This is known to be true for $d = 2, 3$. However for large d the breakthrough result of Kahn and Kalai showed that this is spectacularly false [KahK]. While we do not present the (relatively elementary) disproof of the conjecture, we will prove the classical (now out of fashion) Borsuk theorem (the $d = 2$ case). We will also establish the Borsuk conjecture for smooth convex bodies. Finally, we use a basic topological argument to prove that three parts are not enough to subdivide the sphere $\mathbb{S}^2$ into parts of smaller diameter. We continue with the topological arguments in the next three sections.

3.2. **First steps.** Before we show that every plane convex set can be subdivided into three parts of smaller diameter, let us prove a weaker result, that four parts suffice for all polygons.

Proposition 3.1. *Let $X \subset \mathbb{R}^2$ be a convex polygon with* $\operatorname{diam}(X) = 1$*. Then X can be subdivided into 4 disjoint convex bodies with diameter* $\leq \sqrt{\frac{2}{3}}$.

Proof. Let X be a convex polygon with vertices $z_1, \ldots, z_n$ and $\operatorname{diam}(X) = 1$. From Corollary 1.8 we know that the points z_i, and hence the whole polygon, can be covered by a circle C of radius $\frac{1}{\sqrt{3}}$. Subdividing the circle by two orthogonal diameters we obtain a subdivision of X into four parts (see Figure 3.1). Since the diameter of each part is at most $\sqrt{\frac{2}{3}}$, we obtain the result. $\square$

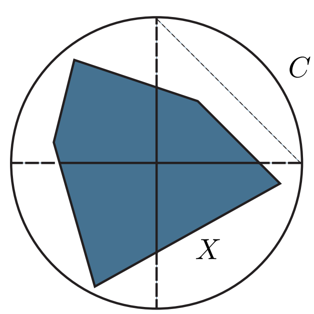

Figure 3.1. Dividing polygons into four parts of smaller diameter.

In the next subsection we show how this construction can be modified to reduce the number of regions of smaller diameter from four to three.

[8] Of course, convexity here is entirely irrelevant since taking a convex hull does not increase the diameter. Still, we would like to keep this condition to help visualize the problem.

3.3. **Borsuk theorem by extended meditation.** We are ready to prove the Borsuk conjecture for $d = 2$.

Theorem 3.2 (Borsuk)**.** *Let $X \subset \mathbb{R}^2$ be a convex body with* $\operatorname{diam}(X) = 1$*. Then X can be subdivided into 3 disjoint convex bodies with diameter $\le (1-\varepsilon)$, for some fixed $\varepsilon > 0$ independent of X.*

Let us try to modify the idea in the proof of Proposition 3.1 above. To obtain a legitimate proof one has to deal with two problems:

(I) we need to extend Corollary 1.8 to infinite sets of points;

(II) we need to subdivide our disk into three parts of smaller diameter.

There are several simple ways to resolve the problem (I). First, we can go to the proof of the corollary, starting with the Helly theorem, and prove that it holds for infinitely many convex sets. We will leave this check as an exercise to the reader (see Exercise 1.1).

Alternatively, one can consider a sequence of unit disks obtained by adding rational points in the body one by one. Since they lie in a compact set, their centers have to lie at a distance ≤ 1 from the first point, and there exists a converging subsequence. The limiting disk of that subsequence is the desired covering unit disk (see Exercise 7.1).

In a different direction, one can also consider polygon approximations of the convex body X (see Figure 3.2) to obtain a covering disk of radius $(1+\varepsilon)\operatorname{diam}(X)$. If the proof of the second part is robust enough, as we present below, this weaker bound suffices.

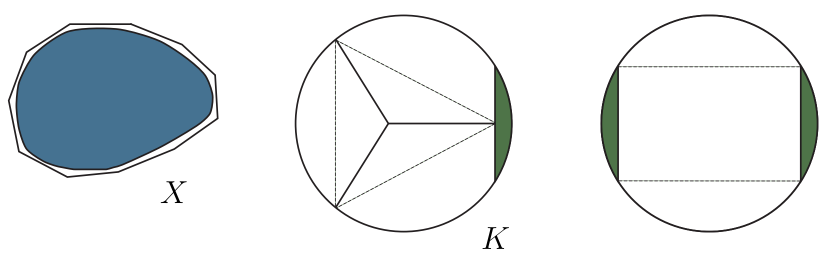

FIGURE 3.2. Resolving problems (I) and (II).

Now, getting around problem (II) may seem impossible since the equilateral triangle inscribed into a circle C of radius $\frac{1}{\sqrt{3}}$ has side 1. On the other hand, such a circle C is "almost enough". Indeed, if X is covered with a *truncated disk* K as shown in Figure 3.2, no matter how small the shaded segment is, the inscribed equilateral triangle has side < 1, and the subdivision as in the figure immediately proves the Borsuk theorem.

Finally, making two opposite sectors in disk C small enough we can make sure that the distance between them is $> (1+\varepsilon)$, and thus the set X does not intersect at least one of two sectors. In other words, the set X is covered with a truncated disk, which completes the proof of the Borsuk theorem. $\square$

3.4. **Making it smooth helps.** As it turns out, for smooth convex bodies the Borsuk conjecture is much easier, the reason being the simple structure of points at distance 1. Let us present a simple and natural argument proving this.

Theorem 3.3 (Hadwiger)**.** *Let $X \subset \mathbb{R}^d$ be a smooth convex body of* $\operatorname{diam}(X) = 1$*. Then X can be subdivided into $d+1$ disjoint bodies of diameter < 1.*

Proof. Let $U = \partial X$ be the surface of X, and let S be the surface of a unit ball. We say that two points $u, v \in U$ are *opposite* if the tangent hyperplanes to u and v are orthogonal to the interval (u, v).

Observe that it suffices to subdivide X in such a way that no part contains a pair of the opposite points. Clearly, the distance between any two points, at least one of which is in the interior in X, is strictly smaller than 1. On the other hand, if the tangent plane T_u is not orthogonal to (u, v), then the distance can be increased locally as in Figure 3.3. Thus the distance $|uv| < 1$ unless u and v are opposite.

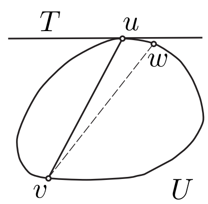

FIGURE 3.3. If the tangent line T is not orthogonal to (uv), then the distance $|uv|$ can be increased locally: $|vw| > |vu|$.

Now consider a continuous map $f : U \to \mathbb{S}^{d-1}$ defined by the condition that tangent surfaces to $u \in U$ and $s = f(u) \in \mathbb{S}^{d-1}$ have the same normals (see Figure 3.4). Subdivide $\mathbb{S}^{d-1}$ into $d+1$ parts such that no part contains two opposite points. One possible way to do this is take a small cap around the 'North Pole' and cutting along 'meridians' to create equal size segments.[9] Use f^{-1} to pull the subdivision back onto U (see Figure 3.4).

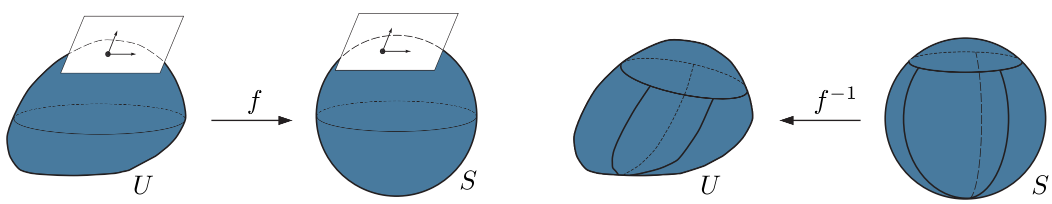

FIGURE 3.4. Map $f : U \to \mathbb{S}^{d-1}$, the subdivision of S and U.

By construction, the opposite points of U are mapped into the opposite points of $\mathbb{S}^{d-1}$. Therefore, no part of the subdivision of U contains opposite points. Now

[9]Alternatively, inscribe a regular d-dimensional simplex into $\mathbb{S}^{d-1}$ and take central projections of its facets.

subdivide X by taking a cone from any interior point $O \in X$ onto each part of the subdivision of U. From the observation above, only the opposite points can be at distance 1, so the diameter of each part in X is < 1. $\square$

3.5. **Do we really need so many parts?** It is easy to see that many convex polytopes in $\mathbb{R}^d$, such as hypercubes and cross-polytopes (generalized octahedra), can be subdivided into just two parts of smaller diameter. In fact, this holds for every centrally symmetric convex polytope (see Exercise 3.2). On the other hand, we clearly need at least $d+1$ parts to subdivide a regular simplex Δ in $\mathbb{R}^d$ since no part can contain two of the vertices of Δ. Now, what about a unit ball? From the proof of Hadwiger's Theorem 3.3, any improvement in the $d+1$ bound would imply the same bound for all smooth bodies. The classical Borsuk-Ulam theorem implies that this is impossible for any d. For $d = 2$ this is easy: take a point on the boundary between two parts on a circle and observe that the opposite point cannot belong to either of the parts then. We prove this result only for $d = 3$.

Proposition 3.4 (3-dim case)**.** *A unit ball $B \subset \mathbb{R}^3$ cannot be subdivided into three disjoint bodies of smaller diameter.*

Proof. From the contrary, consider a subdivision $B = X \sqcup Y \sqcup Z$. Consider the sphere $\mathbb{S}^2 = \partial B$ and the relative boundary $\Gamma = \partial(X \cap \mathbb{S}^2)$. Observe that Γ is a union of m closed non-intersecting curves, for some $m \geq 1$. Let Γ' be the set of points opposite to Γ. Since $\Gamma \cap \Gamma' = \varnothing$, the set $\Gamma \cup \Gamma'$ consists of $2m$ closed non-intersecting lines. Note (or prove by induction) that $Q = \mathbb{S}^2 \smallsetminus (\Gamma \cup \Gamma')$ consists of $(2m+1)$ regions of connectivity, some of which are centrally symmetric to each other and thus come in pairs (see Figure 3.5). Since Q is itself centrally symmetric, by the parity it contains a centrally symmetric region $U \subset Q$. Connect a point $u \subset U$ by a path γ in U to its opposite u', and consider a closed path ζ obtained as a union of γ with the opposite path γ', i.e., let $\zeta = \gamma \cup \gamma'$. Map ζ onto a circle $\mathbb{S}^1$ such that the opposite points in ζ are mapped onto the opposite points in $\mathbb{S}^1$. Since $\zeta \subset Y \sqcup Z$, the problem is reduced to the $d = 2$ case. $\square$

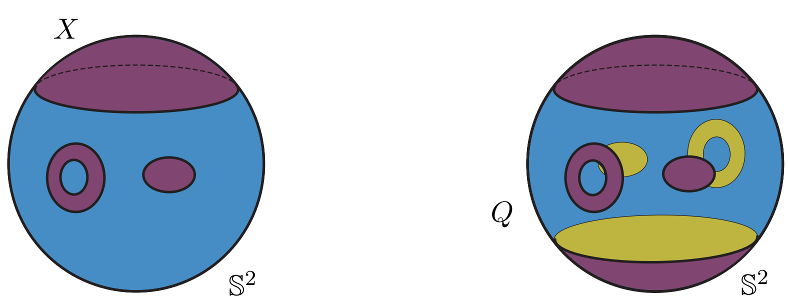

FIGURE 3.5. Boundary $\Gamma = X \cap \mathbb{S}^2$ and a region $Q = \mathbb{S}^2 \smallsetminus (\Gamma \cup \Gamma')$.

3.6. **Exercises.**

Exercise 3.1. [1-] Prove that every convex polygon X of area 1 can be covered by a rectangle of area 2.

Exercise 3.2. *a*) [1] Prove that centrally symmetric polytopes in $\mathbb{R}^d$ can be subdivided into two subsets of smaller diameter.
b) [1] Prove the Borsuk conjecture for centrally symmetric convex sets $X \subset \mathbb{R}^d$.

Exercise 3.3. [1-] Prove that every set of n points in $\mathbb{R}^3$ with diameter ℓ, can be covered by a cube with side length ℓ.

Exercise 3.4. (*Pál*) *a*) [1+] Prove that every convex set of unit diameter can be covered by a hexagon with side 1.
b) [1-] Prove that every convex set of unit diameter can be subdivided into three convex sets of diameter at most $\frac{\sqrt{3}}{2}$.
c) [2-] Prove that every planar set of n points of unit diameter can be partitioned into three subsets of diameter less than $\frac{\sqrt{3}}{2}\cos\frac{2\pi}{3n(n-1)}$.

Exercise 3.5. (*Borsuk conjecture in* $\mathbb{R}^3$) [2] Prove the Borsuk conjecture for convex sets $X \subset \mathbb{R}^3$.

Exercise 3.6. (*Convex sets of constant width*) $\Diamond$ Let $X \subset \mathbb{R}^d$ be a convex set. Clearly, for every vector $\boldsymbol{u}$ there exists two supporting hyperplanes H_1, H_2 with $\boldsymbol{u}$ as normals. The distance between H_1 and H_2 is called the *width in direction* $\boldsymbol{u}$. We say that X has *constant width* width(X) if it has the same width in every direction. An example of a convex set with constant width r is a ball of radius $r/2$. In the plane, another important example is the *Reuleaux triangle* R obtain by adding three circular segments to an equilateral triangle (see Figure 3.6).

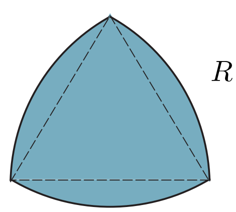

FIGURE 3.6. Reuleaux triangle R.

a) [1] Check that width$(X) = \operatorname{diam}(X)$ for every set $X \subset \mathbb{R}^d$ with constant width. Conclude that for every set X there exists a convex set $Y \supset X$ of constant width with the same diameter: $\operatorname{diam}(Y) = \operatorname{diam}(X)$.
b) [1+] Prove that the perimeter of every region $X \subset \mathbb{R}^2$ with constant width 1 is equal to π. Conclude from here that in the plane the circle has the largest area among convex sets of given constant width.
c) [2+] Prove that in the plane the Reuleaux triangle R has the least area among convex sets of given constant width.
d) [1-] Find two convex bodies in $\mathbb{R}^3$ with constant width 1 and different surface area.

Exercise 3.7. [1] Let $X \subset \mathbb{R}^2$ be a body of constant width. Observe that every rectangle circumscribed around X is a square (see Subsection 5.1). Prove that the converse result is false even if one assumes that X is smooth.

Exercise 3.8. (*Covering disks with disks*) Denote by R_k the maximum radius of a disk which can be covered with k unit disks.
a) [1] Prove that $R_2 = 1$, $R_3 = \frac{2}{\sqrt{3}}$ and $R_4 = \sqrt{2}$.
b) [1] Prove that $R_5 > 1.64$.
c) [2-] Find the exact value for R_5.
d) [1+] Prove that $R_7 = 2$.
e) [1+] Prove that $R_9 = 1 + 2\cos\frac{2\pi}{8}$.

Exercise 3.9. A finite set of points $A \subset \mathbb{R}^2$ is said to be *coverable* if there is a finite collection of disjoint unit disks which covers A. Denote by N the size of the smallest set A which is not coverable.
a) [1] Prove that $N < 100$.
b) [1] Prove that $N > 10$.

3.7. **Final remarks.** Recall that our proof of the Borsuk conjecture in dimension 2 is *robust*, i.e., produces a subdivision into three parts, such that each of them have diam $\le (1-\varepsilon)$, for some universal constant $\varepsilon > 0$. Although not phrased that way by Borsuk, we think this relatively small extension makes an important distinction; we emphasize the robustness in the statement of Theorem 3.2. Dimension 3 is the only other dimension in which the Borsuk conjecture has been proved, by Eggleston in 1955, and that proof is also robust (a simplified proof was published by Grünbaum [BolG]). However, our proof of the Hadwiger theorem (Theorem 3.3) is inherently different and in fact cannot be converted into a robust proof. We refer to [BMS, §31] for an overview of the subject, various positive and negative results and references.

The 'extended meditation' proof of the Borsuk theorem is due to the author, and is similar to other known proofs. The standard proof uses Pál's theorem (see Exercise 3.4) that every such X can be covered by a regular hexagon with side 1, which gives an optimal bound on ε (see [BolG, HDK]). The proof in the smooth case is a reworked proof given in [BolG, §7]. The proof in Proposition 3.4 may seem simple enough, but in essence it coincides with the early induction step of a general theorem. For this and other applications of the topological approach see [Mat2].

The high-dimensional counterexamples of Kahn–Kalai [KahK] (to the Borsuk conjecture) come from a family of polytopes spanned by subsets of vertices of a d-dimensional cube, whose vertices are carefully chosen to have many equal distances. We refer to [AigZ, §15] for an elegant exposition. In addition, we recommend a videotaped lecture by Gil Kalai [Kal2] which gives a nice survey of the current state of art with some interesting conjectures and promising research venues.

4. Fair division

In this section we begin our study of topological arguments, which is further continued in the next two sections. Although the results in this section are largely elementary, they have beautiful applications.

4.1. Dividing polygons.

Let $Q \subset \mathbb{R}^2$ be a convex polygon in the plane. An *equipartition* is a subdivision of Q by lines into parts of equal area. Here is the first basic equipartition result.

Proposition 4.1 (Equipartition with two lines)**.** *For every convex polygon $Q \subset \mathbb{R}^2$ there exist two orthogonal lines which divide Q into four parts of equal area.*

Proof. Fix a line $\ell \in \mathbb{R}^2$ and observe that by continuity there exists a unique line $\ell_1 \| \ell$ which divides Q into two parts of equal area. Similarly, there exists a unique line $\ell_2 \perp \ell$ which divides Q into two parts of equal area. Now lines ℓ_1 and ℓ_2 divide Q into four polygons, which we denote by A_1, A_2, A_3 and A_4 (see Figure 4.1). By construction,

$$\begin{aligned} \text{area}(A_1) + \text{area}(A_2) \ &= \ \text{area}(A_2) + \text{area}(A_3) \ = \ \text{area}(A_3) + \text{area}(A_4) \\ &= \ \text{area}(A_4) + \text{area}(A_1) \ = \ \frac{1}{2}\,\text{area}(Q), \end{aligned}$$

which implies that $\text{area}(A_1) = \text{area}(A_3)$ and $\text{area}(A_2) = \text{area}(A_4)$. Now rotate line ℓ continuously, by an angle of $\pi/2$. By the uniqueness, we obtain the same division of Q with two lines, with labels A_i shifted cyclically. Thus, the function $\text{area}(A_1) - \text{area}(A_2)$ changes sign and is equal to zero for some direction of ℓ. Then, the corresponding lines ℓ_1 and ℓ_2 give the desired equipartition. $\square$

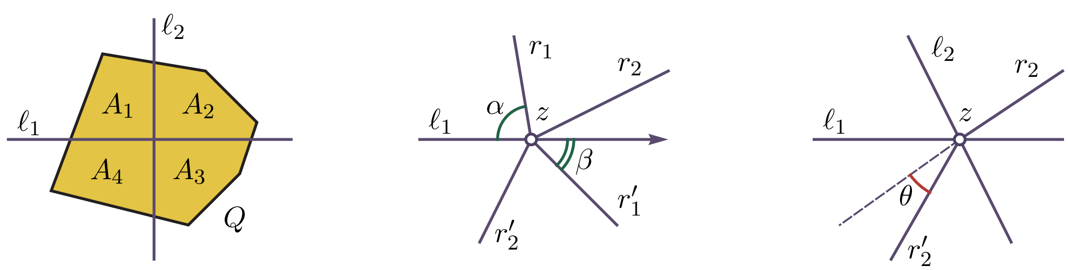

FIGURE 4.1. Equipartitions with two and three lines.

The next result is even more impressive as it shows existence of an equipartition of the polygon into six parts with three *concurrent lines* (lines that are distinct and have a common point). Let us note that it is impossible to make an equipartition of a convex polygon with three lines into seven parts (see Exercise 4.7).

Theorem 4.2 (Equipartition with three lines)**.** *For every convex polygon $Q \subset \mathbb{R}^2$ there exist three concurrent lines which divide Q into six parts of equal area.*

Proof. Fix a line $\ell \subset \mathbb{R}^2$ and take $\ell_1 \| \ell$ which divides Q into two parts of equal area. For every point $z \in \ell_1$ there there is a unique collections of rays r_2, r_3, r_2' and r_3' which start at z and together with ℓ_1 divide Q into six parts of equal area (see Figure 4.1). Move z continuously along ℓ_1. Observe that the angle α defined as in the figure, decreases from π to 0, while angle β increases from 0 to π. By continuity, there exists a unique point z such that r_2 and r_2' form a line. Denote this line by ℓ_2. As in the previous proof, rotate ℓ continuously, by an angle of π. At the end, we obtain the same partition, with the roles of r_3 and r_3' interchanged. Thus, the angle θ between lines spanned by r_3 and r_3' changes sign and at some point is equal to zero. Rays r_3 and r_3' then form a line which we denote by ℓ_3. Then, lines ℓ_1, ℓ_2 and ℓ_3 gives the desired equipartition. □

4.2. **Back to points, lines and triangles.** The following result is a special case of the Bárány theorem (Theorem 2.2), with an explicit constant. We obtain it as easy application of the equipartition with three lines (Theorem 4.2).

Theorem 4.3 (Boros–Füredi)**.** *Let $X \subset \mathbb{R}^2$ be a set of $n = 6k$ points in general position. Then there exists a point z contained in at least $8k^3$ triangles with vertices in X.*

Note that $8k^3 = \frac{2}{9}\binom{n}{3} + O(n^2)$, which gives an asymptotic constant $\alpha_2 = \frac{2}{9}$ in the Bárány theorem. In fact, this constant cannot be improved (see Exercise 4.7). The proof is based on the following variation on Theorem 4.2.

Lemma 4.4. *Let $X \subset \mathbb{R}^2$ be a set of $n = 6k$ points in general position. Then there exist three intersecting lines which separate X into six groups with k points each.*

The proof of the lemma is similar to the proof of Theorem 4.2, and takes into account that the points are in general position (see Exercise 4.9).

Proof of Theorem 4.3. Denote by $A_1, \ldots, A_6$ the regions in the plane divided by three lines as in Lemma 4.4 (see Figure 4.2). Let z be the intersection point. Consider what type of triangles must contain z. First, all triangles with one vertex in each of A_1, A_3 and A_4, must contain z. The same is true for regions A_2, A_4 and A_6. This gives $2k^3$ triangles containing z.

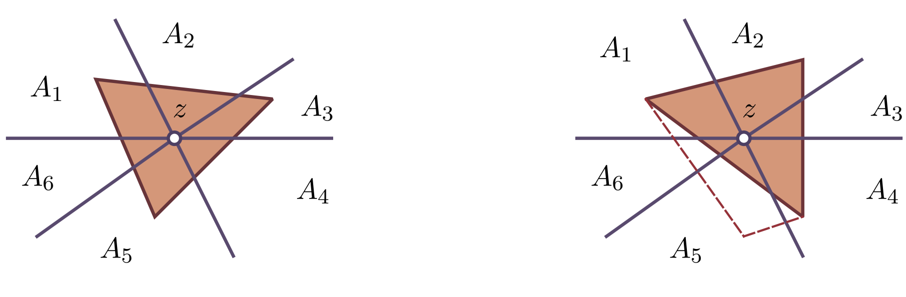

FIGURE 4.2. Two types of triangles containing z.

Similarly, for every two vertices in the opposite regions A_1 and A_4 there exist at least $2k$ ways to form a triangle which contains z, with either all vertices in $A_2 \cup A_3$ or with all vertices in $A_4 \cup A_5$ (see Figure 4.2). This gives at least $2k^3$ triangles containing z. Since the same argument holds for the other two pairs of opposite regions, we get at least $6k^3$ triangles of this type, and $8k^3$ in total. □

4.3. **Inscribed chords.** Let $f : [0,1] \to \mathbb{R}$ be a continuous function with $f(0) = f(1) = 0$. Consider two points $x, y \in [0,1]$ such that $y - x = \ell$ and $f(x) = f(y)$. We call the chord $[x, y]$ the *inscribed chord of length* ℓ. Now, is it true that for every $0 < \ell < 1$ there exists an inscribed chord of length ℓ? While the answer to this question is easily negative (see Figure 4.3), there is a surprising connection to topological arguments earlier in this section.

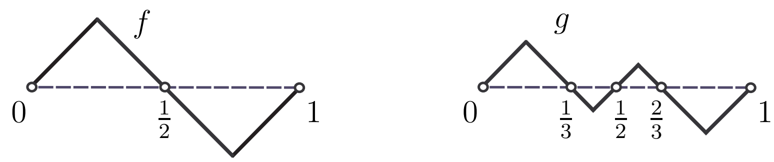

FIGURE 4.3. Functions f and g with $\mathcal{D}(f) = \big[0, \frac{1}{2}\big]$ and $\mathcal{D}(g) = \big[0, \frac{1}{3}\big] \cup \big[\frac{1}{2}, \frac{2}{3}\big]$.

Let us first show that there is always an inscribed chord of length $\frac{1}{2}$. Indeed, if $f(\frac{1}{2}) = 0$, we found our inscribed chord. Suppose now that $f(\frac{1}{2}) > 0$, and consider $g(x) = f(x + \frac{1}{2}) - f(x)$, where $x \in [0, \frac{1}{2}]$. Since $g(x)$ is continuous, $g(0) > 0$, and $g(\frac{1}{2}) < 0$, we conclude that $g(z) = 0$ for some $z \in [0, \frac{1}{2}]$. Thus, $[z, z + \frac{1}{2}]$ is the desired inscribed chord of length $\frac{1}{2}$.

Now that we know that some distances always occur as lengths of inscribed chords, let us restate the problem again. Denote by $\mathcal{D}(f)$ the set of distances between points with equal values:

$$\mathcal{D}(f) \, = \, \big\{ y - x \mid f(x) = f(y), 0 \le x \le y \le 1 \big\}.$$

Observe that for a given f, the set $\mathcal{D}(f)$ contains all distances small enough. However, even for very small $\epsilon > 0$, there exists a distance $\ell < \epsilon$ and a function f, such that $\ell \notin \mathcal{D}(f)$ (see Exercise 4.2). In fact the only distances guaranteed to be in $\mathcal{D}(f)$ are given by the following theorem.

Theorem 4.5 (Inscribed chord theorem)**.** *For every continuous function $f : [0,1] \to \mathbb{R}$ with $f(0) = f(1) = 0$, we have $\frac{1}{n} \in \mathcal{D}(f)$, for all $n \in \mathbb{N}$.*

In other words, there always exist inscribed chords of length $1/n$. The theorem follows easily from the following attractive lemma.

Lemma 4.6. *For every $a, b \notin \mathcal{D}(f)$ and f as in the theorem, we have $a + b \notin \mathcal{D}(f)$.*

Proof of Theorem 4.5. Assume $\frac{1}{n} \notin \mathcal{D}(f)$. By the lemma, we have $\frac{2}{n} \notin \mathcal{D}(f)$. Similarly, $\frac{3}{n} \notin \mathcal{D}(f)$, etc. Thus, $\frac{n}{n} = 1 \notin \mathcal{D}(f)$, which is impossible since $f(0) = f(1)$. □

Proof of Lemma 4.6. Denote by G the graph of function f, and by G_ℓ the graph G shifted by ℓ, where $\ell \in \mathbb{R}$. Attach to G two vertical rays: at the first (global) maximum of f pointing up and at the first (global) minimum pointing down (see Figure 4.4). Do the same with G_ℓ. Now observe that $\ell \in \mathcal{D}(f)$ is equivalent to having G and G_ℓ intersect. Since $a, b \notin \mathcal{D}(f)$, then graph G does not intersect graphs G_{-a} and G_b. By construction, graph G divides the plane into two parts. We conclude that graphs G_{-a} and G_b lie in different parts, and thus do not intersect. By the symmetry, this implies that G and G_{a+b} do not intersect, and therefore $a + b \notin \mathcal{D}(f)$. □

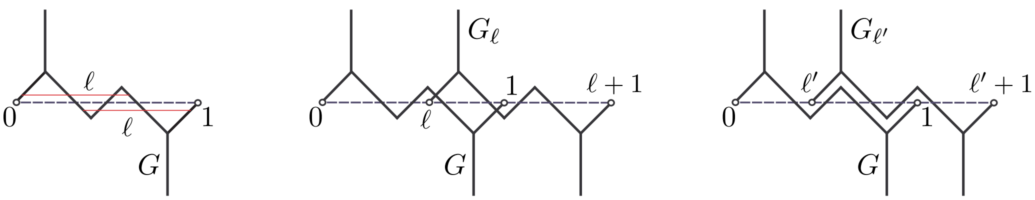

FIGURE 4.4. Graphs G, G_ℓ and $G_{\ell'}$, where $\ell \in \mathcal{D}(f)$ and $G \cap G_\ell \neq \varnothing$, while $\ell' \notin \mathcal{D}(f)$ and $G \cap G_{\ell'} = \varnothing$.

4.4. **Helping pirates divide the loot fairly.** Imagine two pirates came into possession of a pearl necklace with white and black pearls. We assume that the number of pearls of each color is even, so each pirate wants exactly half the white pearls and half the black pearls. Can they cut the necklace in just two places (between pearls) so that the resulting two pieces satisfy both pirates? Surprisingly, this is always possible. An example of such fair division is shown in Figure 4.5. The following result extends this to k pirates.

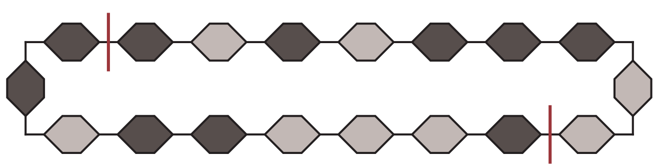

FIGURE 4.5. Fair division of a necklace with two cuts.

Theorem 4.7 (Splitting necklaces)**.** *Suppose a pearl necklace has kp white pearls and kq black pearls, for some integers k, p and q. Then one can cut the necklace in at most $2(k-1)$ places so that the remaining pieces can be rearranged into k groups, with p white and q black pearls in each group.*

In fact, we show that each of the k pirates gets either one or two continuous pieces of the necklace. Note also that the bound on the number of cuts in the theorem is tight: if black pearls are separated from white pearls, at least $k - 1$ cuts are needed to divide pearls of each color.

Proof. Denote by $n = p + q$ the number of pearls each pirate must receive. Fix a starting point 0 and an orientation of the necklace. Let $a(x)$ and $b(x)$ denote the number of white and black pearls among the first x pearls after 0. Consider a discrete function $f : \{0, 1, \ldots, kn\} \to \mathbb{Z}$, defined as $f(x) = a(x)/p - b(x)/q$. Extend f linearly to the whole interval $[0, kn]$. Observe that $f(0) = f(kn) = 0$. Apply the inscribed chord theorem (Theorem 4.5) to obtain $x, y \in [0, kn]$ such that $y - x = n$ and $f(y) - f(x) = 0$. Since the number of pearls between x and y must be integral, we can round x, y down and obtain a fair $\frac{1}{k}$ portion of the necklace for the first pirate. Repeat the procedure by cutting out a fair $\frac{1}{k-1}$ portion of the remaining necklace, etc. At the end, we have $k - 1$ pirates who make two cuts each, giving the total of $2(k - 1)$ cuts. $\square$

4.5. **Exercises.**

Exercise 4.1. [1-] Let $f : \mathbb{R} \to \mathbb{R}$ be a periodic continuous piecewise linear function. Prove that f has inscribed chords of any length.

Exercise 4.2. $\Diamond$ [1+] For every $\alpha \notin \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ find a continuous function $f : [0, 1] \to \mathbb{R}$ with $f(0) = f(1) = 0$, and such that $\alpha \notin \mathcal{D}(f)$. In other words, prove that Theorem 4.5 cannot be extended to other values.

Exercise 4.3. [1+] In the conditions of Theorem 4.5, prove that for every integer n there are at least n inscribed chords whose lengths are multiples of $1/n$.

Exercise 4.4. *a*) [1-] Consider two convex polygons in the plane. Prove that there exists a line which divides both of them into halves of equal area.
b) [1-] Same with two halves of equal perimeter.
c) [1] Consider three convex polytopes in $\mathbb{R}^3$. Prove that there always exist a plane which divides each of the three polytopes into halves of equal volume.
d) [1] Prove or disprove: there always exist a plane which divides each of the three polytopes into halves with equal surface area.

Exercise 4.5. *a*) [1+] Let $Q_1, \ldots, Q_m \subset \mathbb{R}^2$ be convex polygons in the plane with weights $w_1, \ldots, w_m \in \mathbb{R}$ (note that the weights can be negative). For a region $B \subset \mathbb{R}^2$ define the *weighted area* as $w_1 \,\text{area}(B \cap Q_1) + \ldots + w_m \text{area}(B \cap Q_m)$. Prove that there exist two orthogonal lines which divides the plane into four parts of equal weighted area.
b) [1+] Generalize Theorem 4.2 to weighted areas.

Exercise 4.6. [2-] Let Q be a convex polygon in the plane. A line is called a *bisector* if it divides Q into two parts of equal area. Suppose there exists a unique point $z \in Q$ which lies on at least three bisectors (from above, there is at least one such point). Prove that z is the center of symmetry.

Exercise 4.7. $\Diamond$ [1] Prove that for every convex polygon $Q \subset \mathbb{R}^2$, no three lines can divide it into seven parts of equal area.

Exercise 4.8. [1] Let $Q \subset \mathbb{R}^2$ be a convex polygon. A *penta-partition* of Q is a point $z \in Q$ and five rays starting at z, which divide $\mathbb{R}^2$ into five equal cones, and divide Q into five polygons of equal area. Prove or disprove: every Q as above has a penta-partition.

Exercise 4.9. [1] Prove Lemma 4.4.

Exercise 4.10. ♢ [1+] Give an example showing that the $\alpha_2 = \frac{2}{9}$ in Bárány's theorem is optimal (see Theorem 2.2 and Theorem 4.3).

Exercise 4.11. [2] Let $Q \subset \mathbb{R}^2$ be a convex polygon in the plane. Prove that there exists a convex quadrilateral $X = [x_1x_2x_3x_4]$ such that X together with lines (x_1x_3) and (x_2x_4) divide Q into eight regions of equal area (see Figure 4.6).[10] Generalize this to polygons Q_i and weighted area (see Exercise 4.5).

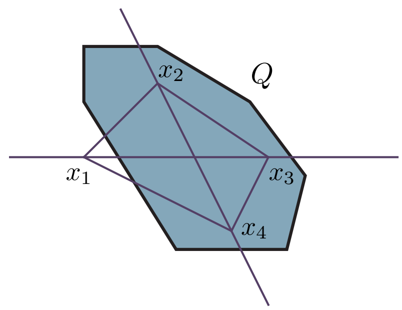

FIGURE 4.6. Cutting a polygon into eight regions of equal area.

Exercise 4.12. [2-] Let $P \subset \mathbb{R}^3$ be a convex polytope. Prove that there exist three orthogonal planes, such that every two of them divide P into four parts of equal volume.

Exercise 4.13. *a*) [1+] Suppose a necklace has $2p$ white, $2q$ black, and $2r$ red pearls. Prove that three cuts suffice to divide the necklace fairly between two pirates.
b) [1-] Check that two cuts may not be enough.
c) [2] Prove that for k pirates and $s \geq 2$ types of pearls, $s(k-1)$ cuts always suffice.

Exercise 4.14. [1] Two pirates found a necklace with pearls of n different sizes, and each size comes in two colors: black and white. They observed that the sizes of white pearls increase clockwise, while the size of black pearls increase counterclockwise. Pirates want to divide the necklace so that each gets a pearl of every size. Prove that two cuts always suffice.

Exercise 4.15. (*Two integrals theorem*) Let $f, g : [0,1] \to [0,1]$ be two continuous functions such that $\int_0^1 f = \int_0^1 g = 1$.
a) [1] Prove that there exist $0 \leq \alpha < \beta \leq 1$, such that $\int_\alpha^\beta f = \int_\alpha^\beta g = 1/2$.
b) [1+] Suppose for some $0 < a < 1$ and every $0 \leq \alpha < \beta \leq 1$, either $\int_\alpha^\beta f \neq a$ or $\int_\alpha^\beta g \neq a$. Prove that there exist $0 \leq \alpha < \beta \leq 1$, such that $\int_\alpha^\beta f = \int_\alpha^\beta g = 1 - a$.
c) [1] Prove that for every $n \in \mathbb{N}$, there exist $0 \leq \alpha < \beta \leq 1$, such that $\int_\alpha^\beta f = \int_\alpha^\beta g = 1/n$.

4.6. **Final remarks.** The equipartition with two lines goes back to [CouR, §6.6]. Theorem 4.2 is proved in [Buck] (see also Exercises 4.5 and 4.7). For the background, generalizations and references on classical equipartition results see [Grü2, §4.2].

[10]This is called *cobweb equipartition*.

The Boros–Füredi theorem is proved in [BorF]. Our proof follows a recent paper [Bukh]. For a matching lower bound, the original proof has been shown to be false, and a complete proof is given in [BMN] (see Exercise 4.10).

The inscribed chord theorem (Theorem 4.5) is usually attributed to Lévy (1934). It was pointed out in [Fle] that the result was first discovered by Ampère in 1806. The proof we present is due to Hopf [Hop1]. Our presentation follows [Lyu, §34].

Theorem 4.7 is due to Goldberg and West (1985), and was further generalized a number of times. Notably, Alon showed in [Alon] that for k pirates and $s \geq 2$ types of pearls, $s(k-1)$ cuts suffice (see Exercise 4.13). We refer to [Mat2, §3.2, §6.6] for further results, proofs and references. Our presentation is a variation on several known proofs and was partly influenced by [Tot].

5. Inscribed and circumscribed polygons

In this section we study the inscribed figures in planar piecewise linear curves. Our main goal is the resolution of the *square peg problem*, but along the way we prove a number of variations and special cases. One notable tool is the *mountain climbing lemma*, an elementary result of independent interest and wide applicability. In the next section, we explore space polygons inscribed into surfaces in $\mathbb{R}^3$. The results of this section make a reappearance in Sections 21, 23 and 24, when we start a serious study of the geometry of plane and space curves.

5.1. **Where it all begins.**

We start with the following easy result which gives a flavor of results both in this and in the next section.

Let $X \subset \mathbb{R}^2$ be a convex set in the plane. We say that X has a *circumscribed square* W if X lies inside W and every edge of W contains at least one point $x \in X$ (see Figure 5.1).

Proposition 5.1. *Every convex set $X \subset \mathbb{R}^2$ has a circumscribed square.*

Proof. For every unit vector $\boldsymbol{u} \in \mathbb{R}^2$ denote by $f(\boldsymbol{u})$ the distance between two lines orthogonal to $\boldsymbol{u}$ and supporting X (see Figure 5.1). Let $g(\boldsymbol{u}) = f(\boldsymbol{u}) - f(\boldsymbol{u}')$, where $\boldsymbol{u}'$ is orthogonal to $\boldsymbol{u}$. Observe that $g(\boldsymbol{u})$ changes continuously as $\boldsymbol{u}$ changes from $\boldsymbol{v}$ to $\boldsymbol{v}'$, for any $\boldsymbol{v} \perp \boldsymbol{v}'$. Since $g(\boldsymbol{v}) = -g(\boldsymbol{v}')$, for some $\boldsymbol{u}$ we have $g(\boldsymbol{u}) = 0$. Now X is inscribed into a square bounded by the four lines orthogonal to $\boldsymbol{u}$ and $\boldsymbol{u}'$. $\square$

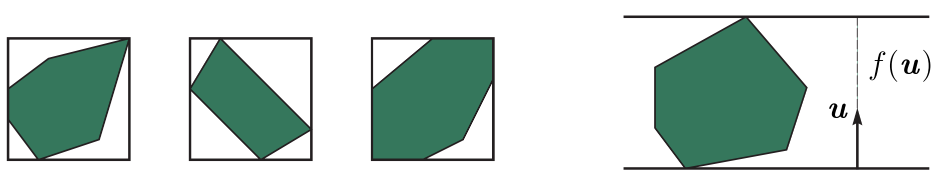

Figure 5.1. Polygons with circumscribed squares; width $f(\boldsymbol{u})$ in direction $\boldsymbol{u}$.

Consider now a superficially similar problem of squares inscribed into simple polygons, often called the *square peg problem*. We say that a closed curve $C \subset \mathbb{R}^2$ has an *inscribed square* if there exists four distinct points on the curve which form a square (see Figure 5.2). Does the inscribed square always exist? If C is a general Jordan curve, this is a classical problem, open for nearly a century. However, if the curve is "nice enough" in a sense that it is either piecewise linear or smooth, or has a certain degree of regularity, then it does have an inscribed square (see Subsection 5.8 for the references).

Many results in this section were obtained as an attempt to resolve the inscribed square problem and are stated for simple polygons.[11] Eventually we prove that every simple polygon has an inscribed square (see Subsection 5.6). Let us note that this

[11]While some of these results easily extend to general Jordan curves, others do not, often for delicate reasons. We hope the reader enjoys finding them.

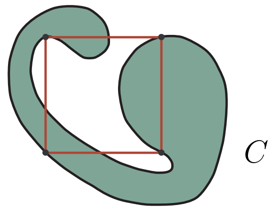

FIGURE 5.2. Jordan curve C and an inscribed square.

implies the claim for self-intersecting closed polygons as well, since taking any simple cycle in it suffices.

In a different direction, one can ask if Proposition 5.1 generalizes to higher dimensions. The answer is yes, but the proof is more delicate. In three dimensions this is called the *Kakutani theorem*; we prove it in the next section. In fact, much of the next section is based on various modifications and generalizations of the Kakutani theorem.

5.2. **Inscribing triangles is easy.** Let $X \subset \mathbb{R}^2$ be a simple polygon in the plane, and let $A \subset \mathbb{R}^2$ denote the region enclosed by X, i.e., $X = \partial A$. We say that an equilateral triangle is *inscribed* into X if there exist three distinct points $y_1, y_2, y_3 \in X$ such that $|y_1y_2| = |y_1y_3| = |y_2y_3|$.

Proposition 5.2. *For every simple polygon $X = [x_1 \ldots x_n] \subset \mathbb{R}^2$ and a point $z \in X$ in the interior of an edge in X, there exists an equilateral triangle (y_1y_2z) inscribed into X. The same holds for every vertex $z = x_i$ with $\angle x_{i-1}x_ix_{i+1} \geq \pi/3$.*

First proof. Denote by X' the clockwise rotation of X around z by angle $\pi/3$. Clearly, $\text{area}(X) = \text{area}(X')$. Thus for every z in the interior of an edge in X, polygons X and X' intersect at z and at least one other point v. Denote by u the counterclockwise rotation of v around z by $\pi/3$. Then (z, u, v) is the desired inscribed triangle.

The same argument works when z is a vertex as in the proposition. Indeed, if $z = x_i$ and $\angle x_{i-1}x_ix_{i+1} \geq \pi/3$, then the interiors of the corresponding regions X and X' intersect. $\square$

Second proof. Suppose polygon X is oriented clockwise. For every point $u \in X$, consider an equilateral triangle (zuv) oriented clockwise. Observe that for u placed further on the same edge as z we can make $v \in A$, while for u at maximal distance from z we have either $v \in X$ (in which case we are done), or $v \notin A$. The latter case follows by continuity of v, and completes the proof of the first part. The same argument works when z is a vertex with $\angle x_{i-1}x_ix_{i+1} \geq \pi/3$. $\square$

Note that the first proof is more natural and gives an explicit construction of an inscribed triangle. However, the second proof is more amenable to generalizations, as can be seen in the following 3-dimensional generalizations of Proposition 5.2.

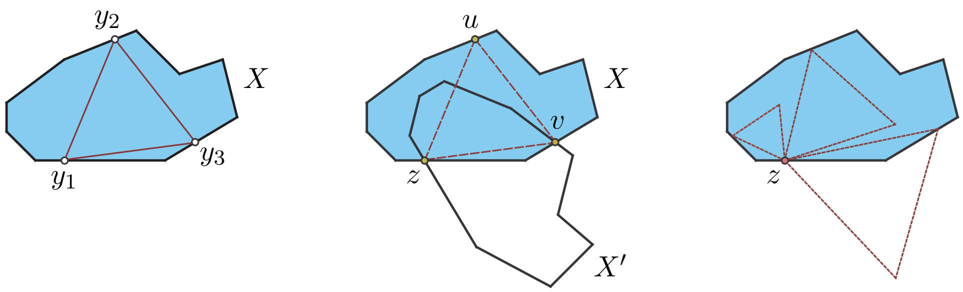

FIGURE 5.3. Inscribed equilateral triangle $(y_1y_2y_2)$ into a polygon X and its two constructions.

Theorem 5.3. *For every simple space polygon $X \subset \mathbb{R}^3$ and a point $z \in X$ in the interior of an edge in X, there exists an equilateral triangle (y_1y_2z) inscribed into X. The same holds for every vertex $z = x_i$ with $\angle x_{i-1}x_ix_{i+1} \geq \pi/3$.*

Proof. For every $u \in X$, denote by $C(u)$ the circle of all points $v \in \mathbb{R}^3$, so that (zuv) is an equilateral triangle (see Figure 5.4). Observe that when u is close to z the circle $C(u)$ is linked with X, and when u is at maximal distance from z, $C(u)$ is not linked with X. Therefore, $C(u_0)$ intersects with X, for some $u_0 \in X$. The intersection point v_0 together with u_0 and z form the desired equilateral triangle. The case of a vertex $z = x_i$ is similar. □

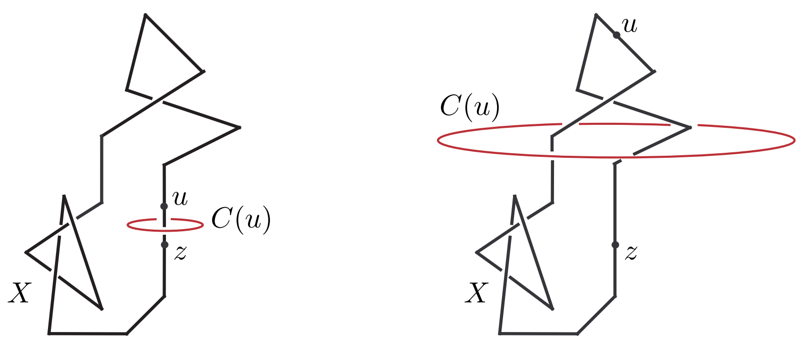

FIGURE 5.4. Space polygon X and two circles $C(u)$ for different u.

5.3. **Need to inscribe rectangles? Topology to the rescue.** We say that X has an *inscribed rectangle* if there exist four distinct points $x_1, \ldots, x_4 \in X$ which form a rectangle.

Proposition 5.4. *Every simple polygon $X \subset \mathbb{R}^2$ has an inscribed rectangle.*

First proof. Think of $X = [x_1 \ldots x_n] \subset \mathbb{R}^2$ as lying on a horizontal plane in $\mathbb{R}^3$. For every two points $u, v \in X$ let $h(u, v)$ be a point at height $|uv|$ which projects onto the midpoint of (u, v):

$$h(u, v) = \left(\frac{\overrightarrow{Ou} + \overrightarrow{Ov}}{2}, \frac{|uv|}{2} \right) \in \mathbb{R}^3.$$

By construction, $h(u, v)$ forms a piecewise linear surface H which lies above X and has X as a boundary (see Figure 5.5).

Assume first that H is self-intersecting. This means that there exist points u, v, u' and v' in X, such that $|uv| = |u'v'|$ and the midpoints of (u, v) and (u', v') coincide. Then $[uvu'v']$ is the desired inscribed rectangle.

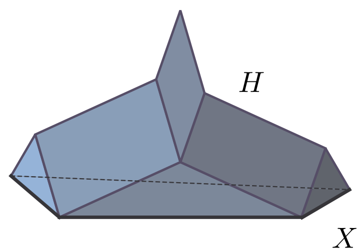

FIGURE 5.5. Triangles and parallelograms in the surface H.

Suppose now that H is embedded, i.e., not self-intersecting. Attach the interior of X to H and denote by S the resulting polyhedral surface. Observe that S has one n-gonal face, n triangles corresponding to edges in X, and $n(n-1)/2$ parallelograms corresponding to unordered pairs of these edges (see Figure 5.5). Therefore, the surface S has the total of $k = 1 + n + \binom{n}{2}$ faces, and

$$m = \frac{1}{2} \left(n + 3 \cdot n + 4 \cdot \frac{n(n-1)}{2} \right) = n^2 + n \quad \text{edges}.$$

On the other hand, the vertices of S are x_i and points $h(x_i, x_j)$, which gives the total of $N = n + \binom{n}{2}$ vertices. Therefore, the Euler characteristic χ of the surface S is equal to: $\chi = N - m + k = 1$, which implies that S is non-orientable. Thus, S cannot be embedded into $\mathbb{R}^3$, a contradiction. $\square$

While the counting argument in the second part of the proof is elementary, it can be substituted by the following even more straightforward topological argument.

Second proof. Let H and S be as in the first proof. Observe that H is homeomorphic to the space of unordered pairs of points $u, v \in X$. Since the space of ordered pairs is a torus, its quotient space H is homeomorphic to the Möbius strip with the polygon X (corresponding to pairs with $u = v$) as its boundary (see Figure 5.6). Therefore, the surface S is homeomorphic to $\mathbb{RP}^2$ and cannot be embedded into $\mathbb{R}^3$. This shows that H is self-intersecting, which implies the result. $\square$

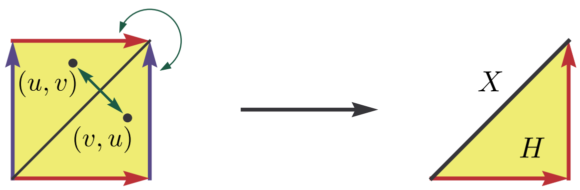

Figure 5.6. Surface H as a quotient space of a torus; H is homeomorphic to a Möbius strip with X as a boundary.

5.4. **Climbing mountains together.** Suppose two climbers stand on different sides at the foot of a two-dimensional (piecewise linear) mountain. As they move toward the top of the mountain, they can move up and down; they are also allowed to move forward or backtrack. The question is whether they can coordinate their movements so they always remain at the same height and together reach the top of the mountain.

We will show that the answer is yes and give two elementary proofs. However, let us mention here that the problem is not as simple as it might seem at first. For example, for the mountain as in Figure 5.7 the first climber needs to move along intervals

$$(1, 2, 3, 4, 5, 6, 6, 5, 4, 3, 2, 2, 3, 4, 5, 6, 7)\,,$$

while the second climbers moves along intervals

$$(1', 2', 3', 3', 3', 4', 5', 6', 6', 6', 7', 8', 9', 9', 9', 10', 11')\,.$$

In other words, the first climber needs to first go almost all the way up, then almost all the way down, and finally all the way up just to be on the same level with the second climber at all times.

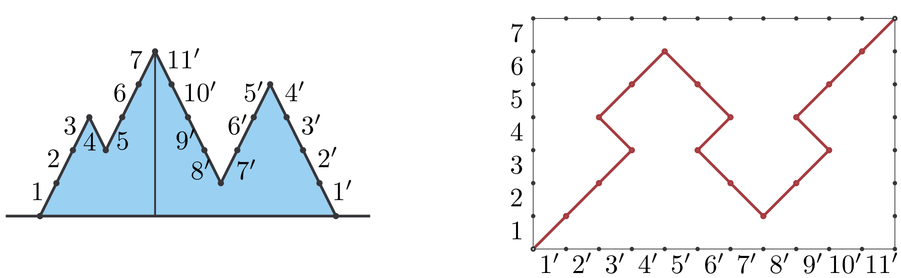

Figure 5.7. Two mountain climbers.

Theorem 5.5 (Mountain climbing lemma)**.** *Let $f_1, f_2 : [0,1] \to [0,1]$ be two continuous piecewise linear functions with $f_1(0) = f_2(0) = 0$ and $f_1(1) = f_2(1) = 1$. Then there exist two continuous piecewise linear functions $g_1, g_2 : [0,1] \to [0,1]$, such that $g_1(0) = g_2(0) = 0$, $g_1(1) = g_2(1) = 1$, and*

$$f_1(g_1(t)) \;=\; f_2(g_2(t)) \quad \textit{for every} \ \ t \in [0,1].$$

The mountain climbing lemma is a simple and at the same time a powerful tool. We will use it repeatedly in the next subsection to obtain various results on inscribed polygons.

First proof. Consider a subset $A \subset [0,1]^2$ of pairs of points at the same level:

$$A = \{(t_1, t_2) \mid f_1(t_1) = f_2(t_2)\}.$$

By the assumptions, $(0,0), (1,1) \in A$. We need prove that $(0,0)$ and $(1,1)$ lie in the same connected component of A. This follows from the fact that $(0,1)$ and $(1,0)$ lie in different connected components of $[0,1]^2 \smallsetminus A$. Indeed, if one climber is going up the mountain and the other is going down, there is a time t when they are at the same level. □

Second proof. Let us first assume that f_1 and f_2 are *generic*, i.e., all peaks and valleys of the mountain are at different levels. Define the set A as in the first proof. Let us show that A is a continuous piecewise linear curve from $(0,0)$ to $(1,1)$. Indeed, check the transitions at peaks and valleys, where exactly four possibilities for a change in the direction of A can occur (see Figure 5.8). We conclude that A is a continuous curve with $(0,0)$ and $(1,1)$ its only possible endpoints. which proves the theorem in the generic case. In a non-generic case, perturb two functions and use the limit argument. Namely, consider a sequence of pairs of functions $(f_1^{(i)}, f_2^{(i)})$, $i = 1, 2, \ldots$ □

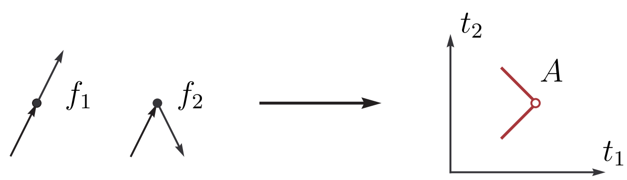

FIGURE 5.8. One of the four possible local changes of A.

Third proof. Consider the subset $A \subset [0,1]^2$ defined in the first proof. Think of A as a finite graph with straight edges. Observe that except for $(0,0)$ and $(1,1)$, every vertex has degree 2 or 4, where the vertices of degree 4 appear when two corresponding peaks of two valleys are at the same level. This immediately implies that $(0,0)$ and $(1,1)$ lie in the same connected component of A. □

Remark 5.6. Let us mention that the first proof is misleadingly simple and suggests that the mountain climbing lemma holds for *all* continuous functions. This is not true (see Exercise 5.23), as we are implicitly using the fact that A is "sufficiently nice." Formalizing this observation is a good exercise which we leave to the reader.

For more than two climbers, climbing (two-dimensional) mountains of the same height, one can still coordinate them to always remain at the same level. While the first proof does not work in this case, the second and third proof extend verbatim (see Exercise 5.24).

Consider the distance between two climbers in the mountain climbing lemma. Clearly, this distance changes continuously from the maximal in the beginning down to zero. As in

Subsection 4.3, one can think of these as of inscribed chords. In other words, the lemma implies that in contrast with the general mountains, where negative heights are allowed, there exist inscribed intervals of all length, not just the integer fractions as in Theorem 4.5.

5.5. **Rhombi in polygons.** Let $X \subset \mathbb{R}^2$, be a simple polygon in the plane. An *inscribed rhombus* is a rhombus with four distinct vertices in X.

Theorem 5.7. *Every point of a simple polygon in $\mathbb{R}^2$ is a vertex of an inscribed rhombus.*

Proof. Let $X \subset \mathbb{R}^2$ be a simple polygon, and let v be a point outside of X. Denote by $y, z \in X$ the points of minimal and maximal distance from v. Consider the distance functions f_1 and f_2 from v to points $x \in X$ on each of the $[yz]$ portions of X. By the mountain climbing lemma (Theorem 5.5), one can continuously move points x_1 and x_2 from y to z so they remain at equal distance from v.[12] Denote by w the fourth point of the rhombus with three other vertices v, x_1 and x_2 (see Figure 5.9). Since z is at maximal distance from v, as x_1 and x_2 approach z, the point w is outside of X. Similarly, assuming y is not a vertex of X and points v, x_1, x_2 are close enough to y, then w is inside X. Therefore, by continuity, for some $x_1, x_2 \in X$ there is a point $w \in X$, so that $R_v = [vx_1wx_2]$ is a rhombus.

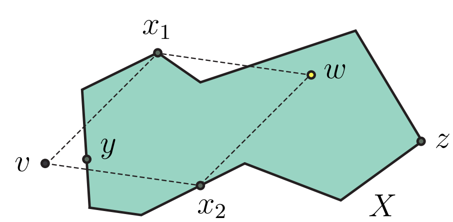

FIGURE 5.9. A construction of rhombi $[vx_1wx_2]$.

Now, let y be a fixed point in the interior of an edge e in X. Consider a sequence of generic points v outside of X which converge to y. Let us show that the sequence of rhombi R_v converges to an inscribed rhombus R_y, i.e., the rhombi do not degenerate in the limit. Denote by X' a polygon symmetric to X with respect to y. We assume that X has no parallel edges. Then $X \cap X'$ consists of an interval on the edge e and finitely many points u_k. One can choose points v to approach y in the direction that is neither orthogonal to intervals (y, u_k), nor to the edge e. When the points v approach y, we have the points x_1 and x_2 approach points u_k, and by construction this implies that R_v does not degenerate.

It remains to show that the inscribed rhombus exists when X has parallel edges, or when y is a vertex. In both cases, use can now use the *limit argument*, by letting v

[12]Here we implicitly use the fact that one can parameterize each path from y to z so that the distance functions f_1 and f_2 are piecewise linear. This can be done by first setting values for the vertices of X (say, by assigning the edges of each path an equal "length") and then by extending the parametrization to the edges.

approach y. Indeed, since X is simple, the only way the rhombi do not converge to a rhombus in the limit is when all four vertices converge to a point. Clearly, the latter is impossible for the limit of polygons. This completes the proof. $\square$

Theorem 5.8. *Every simple polygon in the plane has an inscribed rhombus with two sides parallel to a given line.*

Proof. Let $X = [x_1 \ldots x_n] \subset \mathbb{R}^2$ be a simple polygon and let ℓ be a given line. We assume that ℓ is not parallel to any edge of X. Denote by $\varphi : \mathbb{R}^2 \to \mathbb{R}$ a linear function constant on ℓ. Let y and z be two vertices of X with the minimum and maximum values of φ. We can assume for now that these vertices are uniquely defined, and use the limit argument otherwise.

Use the mountain climbing lemma (Theorem 5.5) to continuously move points $u_1 = u_1(t)$ and $u_2 = u_2(t)$ from y to z, so that $\varphi(u_1(t)) = \varphi(u_2(t))$, for all $t \in [0, 1]$. Reverse the time to continuously move points $w_1 = w_1(t) = u_1(1-t)$ and $w_2 = w_2(t) = u_2(1-t)$ from z to y. Clearly, $\varphi(w_1(t)) = \varphi(w_2(t))$ for all $t \in [0, 1]$. Denote by $f_1(t) = |u_1(t)u_2(t)|$ and $f_2(t) = |w_1(t)w_2(t)|$ the distances between these points.

Suppose f_1 maximizes at time t_0. Consider points $v_1 = u_1(t_0)$ and $v_2 = u_2(t_0)$. By construction, $w_1(1-t_0) = v_1$ and $w_2(1-t_0) = v_2$. Rescale the time parameter t so that points u_1, w_1 leave y and z at $t = 0$, and reach v_1 at $t = 1$. We have $f_1(0) = f_2(0) = 0$, and $f_1(1) = f_2(1) = |v_1v_2|$. Use the mountain climbing lemma (Theorem 5.5) to continuously move points u_1, w_1 until they reach v_1, so that the corresponding distances are equal: $f_1(t) = f_2(t)$ for all $t \in [0, 1]$. We obtain four points moving continuously toward v_1 and v_2, while forming a parallelogram $[u_1u_2w_2w_1]$ with two sides parallel to ℓ. Since the difference between side lengths $g(t) = |u_1u_2| - |u_1w_1|$ is initially negative and positive when they reach v_1 and v_2, one of these parallelograms is the desired rhombus. $\square$

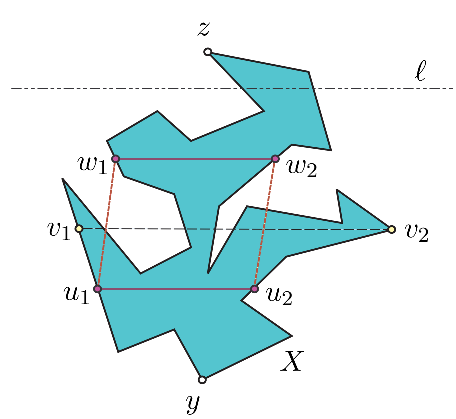

FIGURE 5.10. Inscribed parallelogram with two sides parallel to a given line.

Now let us consider the inscribed rhombi with one diagonal parallel to a given line. We start with a special case of convex polygons.

Proposition 5.9. *Every convex polygon in the plane has an inscribed rhombus with a diagonal parallel to a given line.*

This result is a special case of a result for general simple polygons which again uses the mountain climbing lemma (Theorem 5.5). We present two simple proofs, both related to far-reaching generalizations (see e.g., Exercise 5.19).

First proof. Let $X = \partial A$ be a convex polygon in the plane and let ℓ be a given line. For a point $x \in A \smallsetminus X$, denote by a_1, a_2 the intersections of a line though x and parallel to ℓ with X. Similarly, denote by b_1, b_2 the intersections of a line through x and orthogonal to ℓ with X (see Figure 5.11). Let $f : A \to A$ be a function defined by

$$f(x) = \mathrm{cm}\{a_1, a_2, b_1, b_2\},$$

where $\mathrm{cm}\{\cdot\}$ denotes the *center of mass*. Clearly, function f is continuous and can be extended by continuity to X. By the Brouwer fixed point theorem, there exists a point $z \in A$ such that $z = f(z)$. Then z is a midpoint of (a_1, a_2) and of (b_1, b_2), so $[a_1 b_1 a_2 b_2]$ is the desired inscribed rhombus. □

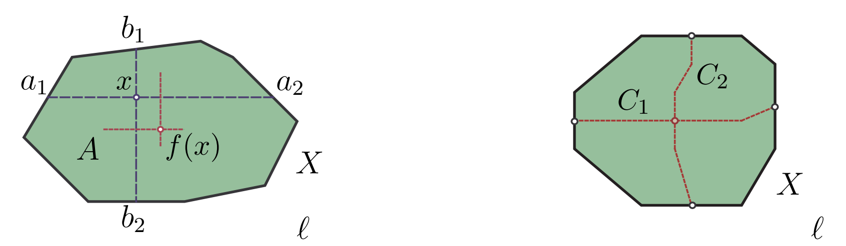

FIGURE 5.11. Function $f : A \to A$ in the first proof of Proposition 5.9. Average curves C_1 and C_2 in the second proof.

Second proof. Think of ℓ as a horizontal line. Denote by C_1 the locus of midpoints of vertical lines intersecting a given polygon X. Similarly, denote by C_2 the locus of midpoints of horizontal lines (see Figure 5.11). The endpoints of C_1 are the leftmost and rightmost points of X (or the midpoints of the vertical edges). Therefore, curve C_2 separates them, and thus intersects C_1. The intersection point is the center of the desired rhombus. □

Perhaps surprisingly, the extension of Proposition 5.9 to all simple polygons is more delicate. As in the proof of Theorem 5.8, we need to use the mountain climbing lemma twice.

Theorem 5.10. *Every simple polygon in the plane has an inscribed rhombus with a diagonal parallel to a given generic line.*

Proof. Let $X \in \mathbb{R}^2$ be a simple polygon, and let ℓ be a given generic line. Think of ℓ being horizontal. Let $\varphi(x)$ denote the *height* of a point $x \in X$. As in the proof of Theorem 5.8, let y and z be the points at minimum and maximum height. Since ℓ is generic, these points are unique. Without loss of generality, we assume that $\varphi(y) = 0$ and $\varphi(z) = 1$.

Use the mountain climbing lemma (Theorem 5.5) to continuously and piecewise linearly (in time t) move points $u_1 = u_1(t)$ and $u_2 = u_2(t)$ from y to z, so that they remain at the same height, for all $t \in [0, 1]$ (see Figure 5.12). We will show that $(u_1(t), u_2(t))$ is a diagonal of a rhombus, for some $t \in [0, 1]$. Let us remark here that not all pairs of points at the same height may necessarily appear as (u_1, u_2).

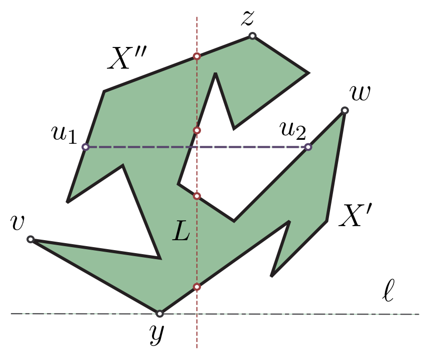

FIGURE 5.12. Line $L \perp \ell$ through midpoints of an interval $(u_1, u_2) \parallel \ell$.

For every $t \in [0, 1]$, consider a vertical line L through the midpoints of $(u_1(t), u_2(t))$ and plot the height $h = \varphi(x)$ of the intersection points $x \in L \cap X$ in a graph

$$\Gamma = \{(t, h), 0 \leq t \leq 1\}.$$

By construction, the resulting graph Γ is a union of non-intersecting piecewise linear curves. There are at least two curve endpoints at $t = 0$ and at least two curve endpoints at $t = 1$. We will prove that there are at least two continuous curves in Γ with one endpoint at $t = 0$ and one at $t = 1$ (see Figure 5.13).

Denote by v and w the leftmost and the rightmost points in X (see Figure 5.11). They separate the polygon into two curves X' and X''. Since the intersection points always lie between these, there is always an odd number of intersection points with X'. Therefore, there exists a continuous curve in Γ with one endpoint in X' at $t = 0$ and one endpoint in X' at $t = 1$. The same is true for X'', which proves the claim. Denote these curves by C' and C'', respectively. In general, curves C' and C'' are not functions of t. Use the mountain climbing lemma to parameterize the curves

$$C' = \{(h_1(\tau), t_1(\tau)), \tau \in [0, 1]\} \quad \text{and} \quad C'' = \{(h_2(\tau), t_2(\tau)), \tau \in [0, 1]\},$$

so that $t_1(\tau) = t_2(\tau)$. Now define the *average curve* C^* of C' and C'' as

$$C^* = \{(h_1(\tau)/2 + h_2(\tau)/2, t_1(\tau)), \tau \in [0, 1]\}.$$

By construction, curve C^* is continuous, starts at $t = 0$ and ends at $t = 1$.

Consider a curve H given by the function $h(t) = \varphi(u_1(t))$. Plot H and the average curve C^* on the same graph as in Figure 5.13. Since $h(t)$ is a continuous function,

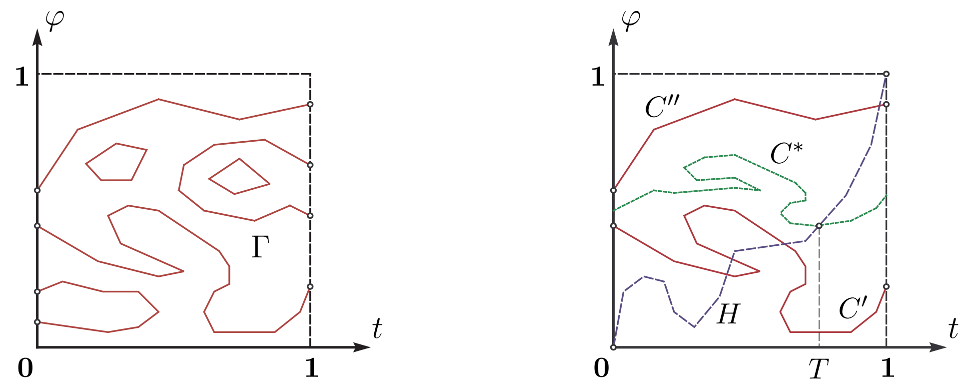

FIGURE 5.13. Graph Γ with curves C', C'', the average curve C^*, and curve H.

$h(0) = 0$ and $h(1) = 1$, we conclude that C^* intersects H at some $t = T$. Then $(u_1(T), u_2(T))$ is the diagonal of a rhombus inscribed into X. $\square$

5.6. **Squares in polygons.** We are ready to prove now the main result of this section: every simple polygon in the plane has an inscribed square. We present an easy proof for convex polygons and a more delicate proof in the general case. Another proof is given in Subsection 23.6, and is based on a different approach.

Proposition 5.11. *Every convex polygon in the plane has an inscribed square.*

Proof. Assume that convex polygon $X \subset \mathbb{R}^2$ has no parallel edges. By Proposition 5.9 and Exercise 5.2, for every direction $\boldsymbol{u}$ there exists a unique rhombus inscribed into X with a diagonal parallel to $\boldsymbol{u}$. Denote by $g(\boldsymbol{u})$ the difference in the diagonal lengths and check that $g(\boldsymbol{u})$ is continuous (see Exercise 5.2). Since $g(\boldsymbol{u})$ changes sign when vector $\boldsymbol{u}$ rotates by $\pi/2$, there exists a vector $\boldsymbol{e}$ such that $g(\boldsymbol{e}) = 0$. This gives the desired inscribed square. $\square$

Theorem 5.12 (Square peg theorem). *Every simple polygon in the plane has an inscribed square.*

The proof of this theorem will be given in Subsection 23.6, based on other ideas.

5.7. **Exercises.**

Exercise 5.1. *a*) [1-] Prove that for every simple polygon $X \subset \mathbb{R}^2$ and an interior point O, there exist points $x, y \in X$ such that O is a midpoint of (x, y).
b) [1] Let $X \subset \mathbb{R}^2$ be a convex polygon with the center of mass $\mathrm{cm}(X) = O$. Prove that X has an inscribed parallelogram with the center of mass at O.
c) [1-] Let $X = [x_1 \ldots x_n] \subset \mathbb{R}^2$ be a non-convex simple polygon. Prove that there exists a vertex x_i and points $y, z \in X$ such that x_i is a midpoint of (y, z). Check that this does not necessarily hold for all vertices x_i in the interior of the convex hull of X.

Exercise 5.2. (*Inscribed rhombi*) $\diamondsuit$ *a*) [1] Prove that every convex polygon in the plane with no parallel edges has at most one inscribed rhombus with a diagonal parallel to a given line.
b) [1] In conditions of *a*), prove that as the line rotates, the rhombi change continuously.
c) [1] Same for inscribed rhombi with an edge parallel to a given line.
d) [1] Find examples of polygons with parallel edges which do have inscribed rhombi as in parts *a*) and *b*). Prove that whenever a polygon has two such rhombi, it has infinitely many of them.
e) [1-] Find a convex polygon $X \subset \mathbb{R}^2$ with non-parallel edges and a vertex v, such that X has exactly two inscribed rhombi with vertices at v.

Exercise 5.3. *a*) [1] Let $X \subset \mathbb{R}^2$ be a simple polygon. Prove that there exist a cyclic quadrilateral (inscribed into a circle) which is inscribed into X, and whose vertices divide the closed curve X into four arcs of equal length.
b) [1] Let $X \subset \mathbb{R}^3$ be a simple space polygon. Prove that there exist a quadrilateral inscribed into X whose vertices lie in a plane and divide X into four arcs of equal length.

Exercise 5.4. [1] Let $R \subset \mathbb{R}^2$ be the Reuleaux triangle defined in Exercise 3.6. Prove that R has no inscribed regular n-gons, for all $n \geq 5$.

Exercise 5.5. [1] Let $Q_1, Q_2, Q_3 \subset \mathbb{R}^2$ be three (non-intersecting) piecewise linear curves with the same endpoint x and other endpoints y_1, y_2, and y_3, respectively. Let $Q = Q_1 \cup Q_2 \cup Q_3$ be the union of these curves. Prove that Q has an inscribed equilateral triangle with a vertex at either y_1, y_2 or y_3.

Exercise 5.6. [1] Let $Q = \partial A$, $A \subset \mathbb{R}^2$ be a simple polygon in the plane and let $x_1, x_2, x_3 \in Q$ be three distinct points on Q. Prove that there exists a circle inside A which contains at least one point from the three closed arcs of Q separated by the points x_i.

Exercise 5.7. [2-] Let $X \subset \mathbb{R}^2$ be a simple polygon. Use Exercise 5.5 to prove that for all but at most two points $z \in X$, there exists an equilateral triangle inscribed into X with a vertex at z.

Exercise 5.8. [1-] Prove or disprove: for every convex polygon $Q \subset \mathbb{R}^2$ containing the origin O in its relative interior, there exists a triangle inscribed into Q, with the center of mass at O.

Exercise 5.9. *a*) [1-] Prove that for every three distinct parallel lines in the plane there exists an equilateral triangle with a vertex on each line.
b) [1] Prove that for every $d+1$ distinct parallel hyperplanes in $\mathbb{R}^d$ there exists a regular simplex with a vertex on each hyperplane.
c) [1] Prove or disprove: such a simplex is uniquely determined up to a rigid motion.

Exercise 5.10. [1+] Prove that for every three non-intersecting lines $\ell_1, \ell_2, \ell_3 \subset \mathbb{R}^3$ there exists a unique triangle $\Delta = (a_1, a_2, a_3)$, such that $a_i \in \ell_i$ and the lines have equal angles with the adjacent edges of Δ. For example, the angle between ℓ_1 and (a_1, a_2) must be equal to the angle between ℓ_1 and (a_1, a_3).

Exercise 5.11. *a*) [1-] Prove or disprove: for every simple convex cone $C \subset \mathbb{R}^3$ there exists a plane L, such that $C \cap L$ is an equilateral triangle.
b) [1-] Prove or disprove: for every simple convex cone $C \subset \mathbb{R}^3$ with equal face angles, there exists a unique plane L, such that $C \cap L$ is an equilateral triangle.

c) [1] Let C be a simple cone with three right face angles. Prove that for every triangle T there exists a plane L, such that $C \cap L$ is congruent to T.
d) [1] Prove that every convex cone $C \subset \mathbb{R}^3$ with four faces there exists a plane L, such that $C \cap L$ is a parallelogram.

Exercise 5.12. *a*) [1-] Prove that for every tetrahedron $\Delta \subset \mathbb{R}^3$ there exists a plane L, such that $\Delta \cap L$ is a rhombus. Prove that there are exactly three such planes.
b) [1] Suppose the resulting three rhombi are homothetic. Prove that Δ is equihedral (see Exercise 25.12).
c) [1] Suppose the resulting three rhombi are squares. Prove that Δ is regular.

Exercise 5.13. [1+] Prove or disprove: every space polygon $Q \subset \mathbb{R}^3$ has an inscribed rectangle.

Exercise 5.14. [1] Let $X \subset \mathbb{R}^2$ be a simple polygon in the plane. Prove that there exists a rectangle Q with exactly three vertices of Q lying in X.

Exercise 5.15. Let $Q \subset \mathbb{R}^3$ be a simple space polygon.
a) [2+] Prove that for every $k \geq 3$ there exists an equilateral k-gon inscribed into Q. In other words, prove that there exist distinct points $y_1, \ldots, y_k \in Q$ such that

$$|y_1y_2| \, = \, \ldots \, = \, |y_{n-1}y_n| \, = \, |y_ny_1| \, .$$

b) [$*$] Prove that for every $k \geq 3$ there exists an *equiangular* k-gon inscribed into Q. In other words, prove that there exist distinct points $y_1, \ldots, y_k \in Q$ such that

$$\angle \, y_ny_1y_2 \, = \, \angle \, y_1y_2y_3 \, = \, \ldots = \, \angle \, y_{n-1}y_ny_1 \, .$$

Exercise 5.16. *a*) [1+] Prove that for every convex plane polygon and every $\rho \neq 1$ there exists at least two inscribed rectangles with *aspect ratio* (ratio of its sides) equal to ρ.
b) [2] Prove that for every simple plane polygon there exists an inscribed rectangle with a given *aspect ratio* (ratio of its sides).
c) [$*$] Let $Q \subset \mathbb{R}^2$ be an isosceles trapezoid. Prove that for every simple plane polygon there exists an inscribed polygon *similar* to Q (equal up to homothety).
d) [1] Prove that part *c*) does not extend to any other quadrilateral Q.

Exercise 5.17. $\diamondsuit$ Let ℓ_1, ℓ_2, ℓ_3 and ℓ_4 be generic lines in the plane.
a) [1] Prove that there are exactly three squares with all vertices on different lines ℓ_i. Formalize explicitly what it means to be generic.
b) [1] Prove that there are 12 more squares with vertices on lines ℓ_i.
c) [1] Compute the number of rectangles with given aspect ratio (ratio of its sides), such that all vertices lie on different lines.
d) [1] Let $Q \subset \mathbb{R}^2$ be a simple quadrilateral with different edge lengths. Compute the number of quadrilaterals similar to Q, with all vertices on lines ℓ_i.

Exercise 5.18. *a*) [2+] Prove that every simple space polygon in $\mathbb{R}^3$ has an inscribed flat (i.e., coplanar) rhombus.
b) [1-] Prove that every flat rhombus inscribed into a sphere is a square. Use part *a*) to conclude that every simple spherical polygon has an inscribed square.
c) [2-] Use part *b*) and a limit argument to give another proof that every simple polygon in the plane has an inscribed square.

Exercise 5.19. An *equihedral octahedron* Q is defined as a convex hull of three orthogonal intervals (diagonals of Q) intersecting at midpoints. Clearly all eight triangular faces in Q are congruent.

a) [1+] Let $P \subset \mathbb{R}^3$ be a convex polytope which is in general position with respect to the orthogonal axes. Prove that P has an inscribed equihedral octahedron with its diagonals parallel to the axes.
b) [1] Show that the general position condition on P is necessary.
c) [1] Find a convex polytope $P \subset \mathbb{R}^3$ which has at least two inscribed equihedral octahedra with parallel diagonals.

Exercise 5.20. (*Inscribed octahedra*) *a*) [2+] Prove that every convex polytope $P \subset \mathbb{R}^3$ has an inscribed regular octahedron.
b) [2+] Generalize the result to (non-convex) surfaces embedded in $\mathbb{R}^3$ and homeomorphic to a sphere.
c) [2+] Generalize the result to higher dimensions.

Exercise 5.21. [2-] Find a convex polytope in $\mathbb{R}^3$ which has no inscribed bricks (rectangular parallelepipeds).

Exercise 5.22. (*The table and the chair theorems*) Let $Q = \partial A$, $A \subset \mathbb{R}^2$ be a convex polygon, and let $f : \mathbb{R}^2 \to \mathbb{R}_+$ be a continuous piecewise linear function which is zero outside of Q.
a) [2+] Prove that for every $c > 0$ there exist a square of size c, such that $\operatorname{cm}[x_1x_2x_3x_4] \subset A$ and $f(x_1) = f(x_2) = f(x_3) = f(x_4)$.[13]
b) [2+] Prove that for every triangle $\Delta \subset \mathbb{R}^2$ there exist a translation $\Delta' = [x_1x_2x_3]$ such that $\operatorname{cm}(\Delta') \in A$ and $f(x_1) = f(x_2) = f(x_3)$.[14]
c) [2-] Prove that part *a*) does not hold for non-convex A.

Exercise 5.23. (*Mountain climbing lemma for general functions*) $\Diamond$ *a*) [1-] Show that the mountain climbing lemma (Theorem 5.5) does not extend to all continuous functions with the same boundary conditions.
b) [1+] Extend the mountain climbing lemma to all piecewise algebraic functions.

Exercise 5.24. (*Generalized mountain climbing lemma*) $\Diamond$ [1-] Let $f_i : [0,1] \to [0,1]$ be k continuous piecewise linear functions with $f_i(0) = 0$ and $f_i(1) = 1$, for all $1 \le i \le k$. Prove that there exist k continuous piecewise linear functions $g_i : [0,1] \to [0,1]$, such that $g_i(0) = 0$, $g_i(1) = 1$ for all $1 \le i \le k$, and

$$f_1(g_1(t)) \,=\, \ldots \,=\, f_k(g_k(t)) \quad \text{for every} \quad t \in [0,1].$$

Exercise 5.25. (*Ladder problem*) [2-] Let C be a piecewise linear curve in the plane which begins and ends with a straight unit segment. Prove that there is a continuous family Υ of unit intervals which are inscribed into C containing both end segments.[15]

Exercise 5.26. (*Ring width problem*) [2] Let $X \subset \mathbb{R}^2$ be a simple polygon with two rays attached as in Figure 5.14. Think of X as if it was made out of metal.
a) [2-] A *metallic ring of diameter* ℓ is an interval of length ℓ whose endpoints lie on different sides of X. Suppose a metallic ring of diameter ℓ can slide from one ray into another. Prove that so can a metallic ring of diameter ℓ', for all $\ell' > \ell$.
b) [2-] An *elastic ring of diameter* ℓ is an interval of length ℓ whose endpoints lie on different sides of X, and whose length is allowed to decrease to any length smaller than ℓ. Suppose an

[13] In other words, there is a way to place a table of every size on a hill; thus the "table theorem".
[14] This is the "chair theorem".
[15] Thus, two people can coordinate their movements to move along a path while carrying a ladder.

elastic ring of diameter ℓ can slide from one ray into another. Prove that so can a metallic ring of diameter ℓ.[16]
c) [2] Show that the ray condition can be removed in part a).

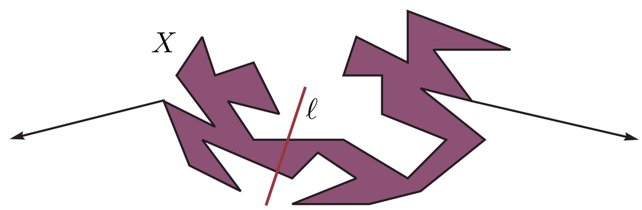

FIGURE 5.14. Polygon X and a ring of length ℓ.

5.8. **Final remarks.** The piecewise linear version of the mountain climbing lemma (Theorem 5.5) is considerably simpler than the general case [Kel]. Both our proofs follow the proof idea in [GPY], stated there in a more general context.

Our proof of Theorem 5.3 and the second (topological) proof of Proposition 5.4 follows [Nie]. Proposition 5.9 was first proved in [Emch] and generalized in [HLM]. Our proof of the proposition follows the proof in [Kra], where the most general result on inscribed parallelepipeds is established (see Exercise 5.19).

The problem of inscribed squares has a long history. It was first proved by Emch for convex curves (see [Emch]), by Shnirelman [Shn] (see also [Gug2]) for sufficiently smooth curves, and further extended in [Gri, Jer, Stro] and other papers. The proof of the square peg theorem (Theorem 5.12) given in Subsection 5.6. We refer to surveys [KleW, Problem 11] and [Nie] for more on the subject. See also [Mey3] for the references and connections to the table and chair theorems (Exercise 5.22).

[16] Warning: it takes an effort to realize that both parts of the problem are in fact not obvious.

6. Dyson and Kakutani theorems

In this short section we continue the study of inscribed figures, concentrating on *inscribed tripods*. We first prove Dyson's and Kakutani's theorems and then their common generalization, which is itself a special case of the *Knaster problem* (see Section 6.7). These results will not be used later on, but the underlying continuity arguments will be repeatedly used throughout the book.

6.1. **Centrally symmetric squares in space polygons.** Now we can move into $\mathbb{R}^3$ and prove the following attractive result. We say that a polygon $Q \subset \mathbb{R}^3$ is *simple* if it is not self-intersecting.

Theorem 6.1. *Let $Q \subset \mathbb{R}^3$ be a simple polygon centrally symmetric with respect to the origin O. Then there exists a square inscribed into Q and centrally symmetric with respect to O.*

Proof. For simplicity, we will refer to the opposite points v and $v' = -v$ in Q as the same point, hoping this would not lead to a confusion.

For every point $v \in Q$ consider a plane H_v containing O and perpendicular to (O, v). We say that Q is *generic* if the intersection $T_v = Q \cap H_v$ contains at most one vertex (in fact, an identified opposite pair of vertices) of Q. Then T_v is always finite, since otherwise it must contain the whole edge e in Q, and thus at least two vertices.

Assume for now that Q is *generic*. Consider a graph $\Gamma = \{(v, w) \in Q^2 \mid v \in Q, w \in T_v\}$ which can be viewed as a subset of a square with the opposite sides identified. We claim that since Q is generic, Γ is a union of a finite number of disjoint curves. This follows from the fact that locally, around every non-vertex point $v \in Q$, the graph Γ is a union of disjoint curves.

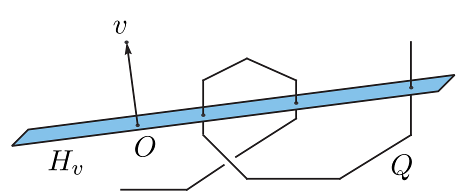

Figure 6.1. Portion of a polygon Q, plane H_v and three points of intersection in T_v.

Observe that for a generic point $v \in Q$ the number of points in T_v is odd. This follows from assumption that Q is simple and from the fact that a plane H_v separates two opposite points of a polygon Q. Therefore, at least one of the curves γ in Γ is defined on the whole Q.

Finally, for every point $v \in Q$ let $f(v) = |vO|$. In this notation, we need to find two points $v, w \in Q$ such that $(v, w) \in \Gamma$ and $f(v) = f(w)$. Suppose there is no such pair. Then all curves in Γ split into symmetric pairs of curves C_i and C_i' with $f(v) > f(w)$ and $f(v) < f(w)$. Since Γ does not intersect the diagonal $\{(v, v), v \in Q\}$, every two corresponding curves $C_i \cup C_i'$ together have an even number of points in the intersection T_v, for a generic v. This gives a contradiction and completes the proof when the polygon Q is generic.

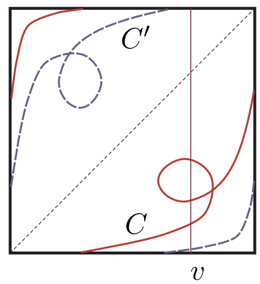

FIGURE 6.2. Curve $C \subset \Gamma$, the corresponding symmetric curve C', and four points in the intersection $T_v \subset \Gamma$.

To remove the assumption that Q is generic, perturb all the vertices. For vertices v_1 and v_2 there is a unique central hyperplane H containing them, and the normal to H will not intersect Q when v_1, v_2 are in general position. Consider now a family of generic polygons converging to Q and take the limit of the squares inscribed into them. Since Q is embedded, it does not contain the origin O. Therefore, the squares do not degenerate in the limit, since otherwise they would contract to O. Thus, in the limit we obtain the desired inscribed square. $\square$

6.2. **Centrally symmetric squares in polytopes.** The following result is the first of the two main results in this section.

Theorem 6.2 (Dyson). *Let $P \subset \mathbb{R}^3$ be a convex polytope containing the origin O in its relative interior. Then there exists a centrally symmetric square inscribed into P.*

Proof. Let us prove that the surface $S = \partial P$ of the polytope P contains a centrally symmetric polygon Q. The result then follows from Theorem 6.1.

Let P' be a reflection of P in O, and let $S' = \partial P'$. Since $O \in P$, then $P \cap P' \neq \varnothing$, and thus the intersection $W = S \cap S'$ is nonempty. Indeed, if $W = \varnothing$, then one of the polytopes P, P' would be contained in another and thus has a greater volume. Assume for now that W is one-dimensional, and moreover a union of non-intersecting polygons $Q_1, \ldots, Q_n$. These polygons Q_i lie on the surface S homeomorphic to a sphere $\mathbb{S}^2$. Note that for all i, the reflection Q_i' of a polygon Q_i is also one of the polygons in W.

Let us prove that one of the polygons Q_i is centrally symmetric. Consider a radial projection $\pi : W \to \mathbb{S}^2$ of W onto a sphere centered at O. Then $\pi(Q_i)$ are non-intersecting Jordan curves on $\mathbb{S}^2$. By contradiction, suppose none of the $\pi(Q_i)$ is centrally symmetric. By analogy with the proof of Proposition 3.4, one of the connected components of $\mathbb{S}^2 - \pi(W)$ is a centrally symmetric region $U \subset \mathbb{S}^2$. For $x \in \mathbb{S}^2$, denote by $f(x) = |\pi^{-1}(x)O|$ the distance from the origin to the point which projects onto x, and let $g(x) = f(x) - f(x')$, where $x' \in S$ is a reflection of x. By construction, the function g is continuous and $g(x) \neq 0$ on U. On the other hand, $g(x) = -g(x')$ for all $x, x' \in U$, a contradiction.

Suppose now that the above assumption is false, i.e., that $W \subset S$ is not a union of non-intersecting polygons. Perturb the vertices of P so that no two faces are parallel and no vertex is opposite to a point on a face. The resulting polytope satisfies the assumption, and the theorem follows by the limit argument. $\square$

6.3. **Circumscribed cubes.** It is natural to assume that Proposition 5.1 can be extended to higher dimensions. This is indeed possible, but the proof is no longer straightforward. We present only the 3-dimensional case here (see also Exercise 6.12).

We say that a convex set $X \subset \mathbb{R}^3$ has a *circumscribed cube* if there exists a cube $C \subset \mathbb{R}^3$, such that $X \subseteq C$ and every (2-dimensional) face of C contains a point of X.

Theorem 6.3 (Kakutani)**.** *Every convex set $X \subset \mathbb{R}^3$ has a circumscribed cube.*

Proof. Let $O \in X$ be the origin, and let $\mathbb{S}^2$ be the unit sphere centered at O. For a point $x \in \mathbb{S}^2$, denote by $f(x)$ the distance between planes orthogonal to (Ox) and supporting X on both sides. Think of f as a (symmetric) function on a sphere $f : \mathbb{S}^2 \to \mathbb{R}$. It suffices to show that there exists three points $x, y, z \in \mathbb{S}^2$ such that $(Ox), (Oy)$ and (Oz) are orthogonal to each other and such that $f(x) = f(y) = f(z)$. We prove this claim by contradiction.

Think of (xyz) as a right equilateral triangle on $\mathbb{S}^2$, i.e., a spherical triangle with edge lengths $\pi/2$. For every right equilateral triangle $(xyz) \subset \mathbb{S}^3$, denote by $\tau(x, y, z)$ the orthogonally projection of the point $(f(x), f(y), f(z)) \in \mathbb{R}^3$ onto the plane $L = \{(a, b, c), a+b+c = 0\} \subset \mathbb{R}^3$ containing the origin O. Clearly, the map τ is continuous and if $f(x) = f(y) = f(z)$, then $\tau(x, y, z) = O$. We assume that $\tau(x, y, z) \neq O$ for all right equilateral triangles.

Fix the three points $x_0 = (1, 0, 0)$, $y_0 = (0, 1, 0)$ and $z_0 = (0, 0, 1)$ in the sphere. Observe that $\tau(x_0, y_0, z_0), \tau(y_0, z_0, x_0)$ and $\tau(z_0, x_0, y_0)$ are equilateral triangles in L. Denote by $\mathbb{G} = \mathrm{SO}(3, \mathbb{R})$ the group of orthogonal rotations, whose elements can be identified with the right equilateral triangles (x, y, z) of the same orientation. Consider a curve $\Gamma_0 \subset \mathbb{G}$ which connects (x_0, y_0, z_0) to (y_0, z_0, x_0). Use the symmetry to connect (y_0, z_0, x_0) to (z_0, x_0, y_0) and (z_0, x_0, y_0) to (x_0, y_0, z_0) with the rotations of Γ_0. Now these three copies of Γ_0 form a closed curve $\Gamma_1 \subset \mathbb{G}$. Suppose $\tau(x, y, z) \neq O$ for every point $(x, y, z) \in \Gamma_1$. Denote by $\gamma_0 = \tau(\Gamma_0)$ and $\gamma_1 = \tau(\Gamma_1)$ the curves in $L \smallsetminus O$. By construction, γ_1 consists of three rotated copies of γ_0 (see Figure 6.3). Therefore, the *argument* of γ_1, defined as 2π times the winding number around O, is nonzero and is of the form $(3m + 1)2\pi$.

Finally, recall that $\mathbb{G} = \mathrm{SO}(3, \mathbb{R})$ is homeomorphic to $\mathbb{RP}^3$, and hence the fundamental group $\pi(\mathbb{G}) = \mathbb{Z}_2$. Therefore the curve $\Gamma_2 = 2\Gamma_1$ is contractible in $\mathbb{G}$. On the other hand, its image $\gamma_2 = \tau(\Gamma_2) = 2\gamma_1$ is not contractible in $L \smallsetminus O$, a contradiction. $\square$

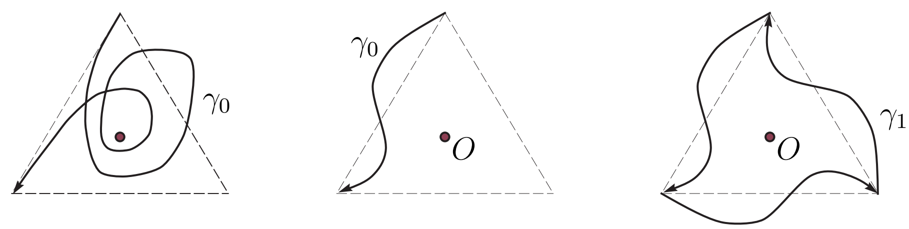

FIGURE 6.3. Two examples of curves γ_0 and an example of curve γ_1 with winding number 1.

6.4. **Tripods standing on surfaces.** Let $S \subset \mathbb{R}^3$ be an embedded compact orientable surface (without boundary), and denote by P its interior: $S = \partial P$. Let $O \in P \smallsetminus S$ be a fixed point in the interior of S. We say that a tetrahedron $(Ouvw)$, where $u, v, w \in S$ is a (α, β)-*tripod* if $|Ou| = |Ov| = |Ow|$, $\angle uOv = \alpha$, $\angle uOw = \alpha$, and $\angle vOw = \beta$.

Theorem 6.4 (The tripod theorem). *Let $S \subset \mathbb{R}^3$ be an embedded compact surface without boundary, and let O be a point inside S. Suppose angles α and β satisfy $0 < \alpha, \beta \le \pi$ and $\beta \le 2\alpha$. Then there exist points $u, v, w \in S$, such that the tetrahedron $(Ouvw)$ is a (α, β)-tripod.*

There are several ways to think of this result. Visually, it is saying that on every surface one can place a *tripod* (a three-legged stand with given angles α, β between its legs) such that the top is placed at any given point O (see Figure 6.4).

On the other hand, the tripod theorem can be viewed as a generalization of the Kakutani theorem (Theorem 6.3). Indeed, let $f : \mathbb{S}^2 \to \mathbb{R}_+$ be a function as in the proof of the Kakutani theorem. Consider a surface $S \subset \mathbb{R}^3$ of points at distance $f(x)$ in direction (Ox), $x \in \mathbb{S}^2$. Clearly, the surface S is homeomorphic to a sphere. Now observe that the $(\pi/2, \pi/2)$-tripod is the desired right equilateral triangle corresponding to directions of the circumscribed cube.

Similarly, the tripod theorem can be viewed as a variation on the Dyson theorem (Theorem 6.2). Indeed, let $f : \mathbb{S}^2 \to \mathbb{R}_+$ be the distance function to a point on the surface ∂P. Now the $(\pi/2, \pi)$-tripod corresponds exactly to three of the four vertices of an inscribed square.

Proof. Let x_0 and x_1 be points on the surface S with the largest and the smallest distance from O, respectively. We can always assume that $x_0 \ne x_1$, since otherwise $S = \mathbb{S}^2$ and the result is trivial. Let $\{x_t, t \in [0,1]\}$ be a path on the surface S between x_0 and x_1. Let C_t be the set of points $y \in \mathbb{R}^3$ such that $|Oy| = |Ox_t|$ and $\angle yOx_t = \alpha$, for all $t \in [0,1]$. Clearly, C_t is a circle which changes continuously with t.

Observe that C_0 has no points inside S, while C_1 has no points outside S. Let τ be the supremum of $t \in [0,1]$, such that C_t contains two outside points $a_t, b_t \notin P$ at angle $\beta = \angle a_t O b_t$. If $\tau = 0$, then tetrahedron $(Ox_0a_0b_0)$ is the desired tripod. Now assume that $\tau > 0$. Denote by (a_τ, b_τ) the limit of a sequence of intervals (a_t, b_t) as $t \to \tau$. Note that $a_\tau, b_\tau \in S$ since otherwise τ is not a supremum defined above. We conclude that $(Ox_\tau a_\tau b_\tau)$ is the tripod as in the theorem. $\square$

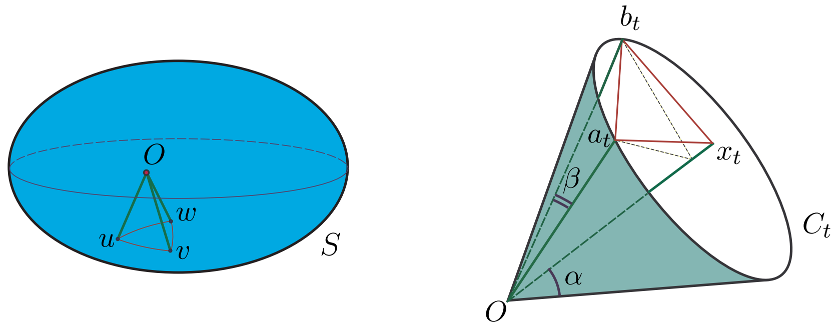

FIGURE 6.4. A (α,β)-tripod $(Oxyz)$ standing on the surface S and an element in its construction.

6.5. **General tripods.** The problem of inscribed tripods can be extended to include tripods with three possible angles. The following result shows that this is possible for the *star-shaped surfaces* $S \subset \mathbb{R}^3$ defined as surfaces of regions which contain the origin O and every interval (Ox) for all $x \in S$. We present it in the form amenable to generalizations to higher dimensions, where it is called *the Knaster problem.*

Theorem 6.5 (Knaster's problem in $\mathbb{R}^3$)**.** *Let $f : \mathbb{S}^2 \to \mathbb{R}_+$ be a continuous function on a unit sphere, and let $x_1, x_2, x_3 \in \mathbb{S}$ be three fixed points. Then there exists a rotation $\rho \in \mathrm{SO}(3,\mathbb{R})$, such that*

$$f(\rho(x_1)) \, = \, f(\rho(x_2)) \, = \, f(\rho(x_3))\,.$$

In other words, the theorem says that there exists a spherical triangle $(y_1y_2y_3)$ congruent to $(x_1x_2x_3)$, similarly oriented, and such that $f(y_1) = f(y_2) = f(y_3)$. When

$$|x_1x_2|_{\mathbb{S}^2} \, = \, \alpha, \quad |x_1x_3|_{\mathbb{S}^2} \, = \, \alpha, \quad |x_2x_3|_{\mathbb{S}^2} \, = \, \beta,$$

we obtain the tripod theorem (Theorem 6.4) for star surfaces.

6.6. **Exercises.**

Exercise 6.1. a) [1-] Prove that for every tetrahedron $\Delta \subset \mathbb{R}^3$ there exists a unique circumscribed parallelepiped which touches all edges of Δ.
b) [1-] Prove that the circumscribed parallelepiped as above is a brick (has right face angles) if and only if Δ is equihedral, i.e., has congruent faces (see Exercise 25.12).

Exercise 6.2. $\lozenge$ [1-] Extend Theorem 6.1 to self-intersecting centrally symmetric polygons.

Exercise 6.3. [1] The *aspect ratio* ρ of a rectangle is the ratio of its sides. Prove that every centrally symmetric polygon $Q \subset \mathbb{R}^3$ contains an inscribed centrally symmetric rectangle with a given aspect ratio $\rho > 0$. When $\rho = 1$ this is the claim of Theorem 6.1.

Exercise 6.4. [2] A quadrilateral A in $\mathbb{R}^3$ is called *regular* if all four sides are equal and all four angles are equal. Prove that every space polygon in $\mathbb{R}^3$ has an inscribed regular quadrilateral. Generalize this to $\mathbb{R}^d$, for all $d \geq 3$.

Exercise 6.5. $\Diamond$ [1] Let A be a quadrilateral inscribed into a space polygon $Q \subset \mathbb{R}^3$. Note that for A to be a square it has to satisfy five equations, while there are only four degrees of freedom to choose A. Formalize this observation. Conclude that a generic space polygon does not have an inscribed square.

Exercise 6.6. [2+] Let $\Delta \subset \mathbb{R}^d$ be a fixed simplex. Prove that every convex polytope $P \subset \mathbb{R}^d$ has an inscribed simplex Δ' similar to Δ. Generalize this to polyhedral surfaces homeomorphic to a sphere.

Exercise 6.7. [1] Let $Q(z) = [ABCD] \subset \mathbb{R}^3$ be a quadrilateral with $A = (0,0,0)$, $B = (0,1,0)$, $C = (0,1,z)$ and $D = (1,0,0)$. Check that $Q(1)$ does not have an inscribed square. Use convergence argument to show that $Q(z)$ does not have inscribed squares, for all sufficiently small $z > 0$.

Exercise 6.8. $\Diamond$ [1] Check that Dyson's theorem (Theorem 6.2) holds for all (non-convex) polyhedra $P \subset \mathbb{R}^3$ whose surface $S = \partial P$ is a star surface with respect to O. What about general 2-dimensional orientable polyhedral surfaces?

Exercise 6.9. [2-] Use Exercise 6.3 to prove that every convex polyhedron $P \subset \mathbb{R}^3$ has an inscribed rectangle with a given aspect ratio and given center inside P.

Exercise 6.10. [1-] Let $P \subset \mathbb{R}^3$ be a convex polytope centrally symmetric at the origin O. Prove that there exists a regular octahedron inscribed into P and centrally symmetric at O.

Exercise 6.11. [1+] Let $P \subset \mathbb{R}^3$ be a convex polytope centrally symmetric at the origin O. Prove that there exists a regular hexagon inscribed into P and centrally symmetric at O.

Exercise 6.12. (*Generalized Kakutani's theorem in higher dimensions*) [2] Prove that every convex set $X \subset \mathbb{R}^d$ can has a circumscribed hypercube.

Exercise 6.13. [2-] Prove that every centrally symmetric convex polytope in $\mathbb{R}^3$ has an inscribed cube.

Exercise 6.14. (*Knaster problem in* $\mathbb{R}^3$) $\Diamond$ Recall that $\mathrm{SO}(3,\mathbb{R})$ is homeomorphic to $\mathbb{RP}^3$.
a) [1+] A manifold Z is called *unicoherent* if for every closed, connected subsets $A, B \subset X$ with $X = A \cup B$, the intersection $A \cap B$ is connected. Prove that $\mathbb{RP}^3$ is unicoherent.

b) [1+] Let X be a unicoherent manifold, and let σ be a fixed point free involution on X. Suppose a subset $A \subset X$ satisfies the following conditions:
(*i*) A is a closed subset of X,
(*ii*) A is invariant under σ,
(*iii*) A separates x from $\sigma(x)$, for all $x \in X$.
Then there exists a connected subset $B \subset A$ which satisfies (*i*)–(*iii*).

c) [1] Let $X = \mathrm{SO}(3,\mathbb{R})$ and f as in Theorem 6.5. Define functions $g, g' : X \to \mathbb{R}$ by

$$g(\rho) \,=\, f(\rho(x_1)) - f(\rho(x_3)), \quad g'(\rho) \,=\, f(\rho(x_2)) - f(\rho(x_3)), \quad \text{where} \ \ \rho \in X.$$

Define A and A' the subsets of rotations $\rho \in X$ with $g(\rho) = 0$ and $g'(\rho) = 0$, respectively. Denote by $\xi, \xi' \in \mathrm{SO}(3,\mathbb{R}^3)$ the involutions which satisfy $\xi(x_1) = x_3$, $\xi'(x_2) = x_3$, and define the involutions σ and σ' on X as multiplications by ξ and ξ' from the right:

$$\sigma(\rho) \,=\, \rho \cdot \xi, \quad \sigma'(\rho) \,=\, \rho \cdot \xi'.$$

Check that $g \circ \sigma = -g$, $g' \circ \sigma' = -g'$, and that the pairs (σ, A) and (σ', A') satisfy conditions of part *b*).

d) [1] Denote by $B \subset A$ and $B' \subset A'$ the connected subsets obtained by applying part *b*) to sets A and A' in part *c*). Assume that $B \cap B' = \varnothing$. Use the fact that σ, σ' preserve the Haar measure on $\mathrm{SO}(3, \mathbb{R})$ to obtain a contradiction. Conclude from here Theorem 6.5.

Exercise 6.15. (*General tripods on surfaces*) $\diamondsuit$ [$*$] Let $S \subset \mathbb{R}^3$ be an embedded orientable surface and let O be a point inside S. For a simple cone C with face angles α, β and γ, define a (α, β, γ)-*tripod* on S to be a triple of points $u, v, w \in S$, such that the tetrahedron $(Ouvw)$ has $\angle uOv = \alpha$, $\angle vOw = \beta$, $\angle wOv = \gamma$, and $|Ou| = |Ov| = |Ow|$. Prove that for every C as above, there exists a (α, β, γ)-tripod on S.

6.7. **Final remarks.** Kakutani's theorem (Theorem 6.3) was proved in [Kaku] and became a source of inspiration for a number of results. It was extended in [YamY] to higher dimensions (Exercise 6.12). Our version of Dyson's theorem is different from the original [Dys], and can be viewed as a variation on Kakutani's theorem. Dyson noticed that in the proof Kakutani never used the fact that f is centrally symmetric, so he showed that general functions on spheres always have two orthogonal diameters such that the function is constant on all four endpoints. This result, in turn, was also extended in a number of ways (see e.g. Exercise 6.8 and 6.9).

Our proof of the tripod theorem (Theorem 6.4) is based on [Gor], which in turn follows the proof in [YamY]. Theorem 6.5 is due to Floyd who used a topological argument to prove it [Flo]. The proof is outlined in Exercise 6.14. The generalization to all embedded 2-dimensional surfaces is stated as Exercise 6.15.

Knaster's problem was formulated by B. Knaster in 1947 in an attempt to further generalize the Kakutani theorem. Namely, he conjectured that for every $f : \mathbb{S}^d \to \mathbb{R}^m$ and $x_1, \ldots, x_n \in \mathbb{S}^d$, where $n = d - m + 2$, there exists a rotation $\rho \in \mathrm{SO}(m, \mathbb{R})$ such that $f(\rho(x_1)) = \ldots = f(\rho(x_n))$. This conjecture was refuted in a number of cases, for example for all $d > m > 2$ [Chen], but remained open for $m = 1$ until recently, when it was disproved for $d > 10^{12}$ [KasS].

7. Geometric inequalities

The isoperimetric problems are some of the oldest problems in geometry. There are numerous monographs written on the subject, so we see little need to expand on this (see Subsection 7.9). We do need, however, the important *Brunn–Minkowski inequality*, including the equality part of it, to prove the Minkowski theorem in Section 36. Thus, we present an elementary introduction to the subject, trying to make it as painless as possible.

7.1. **Isoperimetry in the plane.** It is well known that among all regions in the plane with the same area, the disk has the smallest perimeter. Proving this result is actually more delicate than it seems at first sight. The following proof due to Steiner is beautiful, but also (somewhat) incorrect. While we explain the problem in the next subsection, it is a nice exercise to find the error.

Throughout this section we always assume that our sets are convex. This allows us to avoid certain unpleasantness in dealing with general compact sets.[17]

Theorem 7.1 (Isoperimetric inequality in the plane)**.** *Among all convex sets in the plane with a given area, the disk has the smallest perimeter.*

Incorrect proof of Theorem 7.1. The proof can be split into several easy steps. Let $X \subset \mathbb{R}^2$ be a compact set in the plane. We say that the set is *optimal* if it minimizes the perimeter given the area. We will try to characterize the optimal sets and eventually show that only a disk can be optimal.

1) We say that two points x and y on the boundary $P = \partial X$ are *opposite* if they divide P into two equal halves. The interval (xy) is called a *diameter* in this case. We claim that in an optimal set X the diameter divides the area into two equal parts. If not, attach the bigger part and its reflection to obtain a set Y of equal perimeter and bigger area (see Figure 7.1).

2) Let (xy) be a diameter in an optimal set X, and let $P = \partial X$. For every point $z \in P$ we then have $\angle xzy = \pi/2$. Indeed, from 1) the diameter splits the area into equal parts. If $\angle xzy \neq \pi/2$, take chords $[xy]$, $[yz]$ and denote by A, B the regions they separate as in the figure. Now attach A and B to a triangle $(x'y'z')$ with sides $|x'z'| = |xz|$, $|z'y'| = |zy|$, and $\angle x'y'z' = \pi/2$. Copy the construction symmetrically on the other side of the diameter (see Figure 7.1). Since $\text{area}(xyz) < \text{area}(x'y'z')$ and areas of segments remain the same, we obtain a region X' with the same perimeter and bigger area.

In summary, we showed that in an optimal set the angle at all points of the boundary to a fixed diameter must be $\pi/2$. Thus X is a circle. This completes the proof. □

[17] Otherwise, we have to specify that our set X is measurable, has a measurable boundary, etc. For all purposes in this section the reader can assume that X has a piecewise analytic boundary. Then it is easy to show that the convex hull $\text{conv}(X)$ has a smaller perimeter but a bigger area, thus reducing the problem to the convex case.

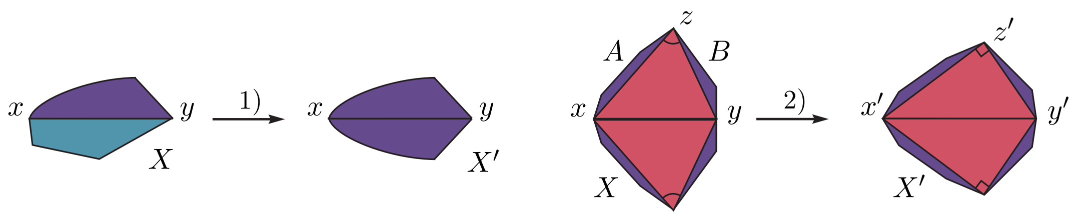

FIGURE 7.1. Increasing the area of convex sets while preserving diameter.

7.2. **Reaching the optimum.** The problem with the argument above is that at no point do we show that the optimum *exists*. Naturally, if we set upon finding a convex subset with *maximum* perimeter given the area, then the optimum does not exist. Hence, showing that all sets except for the circles are *not* optimal does not *prove* that the circles are optimal, and thus does not (yet) resolve the problem (in either case). There are several ways to get around the problem. Let us describe two most standard approaches.

(I) An easy patch is by an abstract *compactness argument*. Recall the classical Blaschke selection theorem (see Exercise 7.1), which states that a set $\mathcal{C}(\Lambda)$ of convex subsets of a compact set $\Lambda \subset \mathbb{R}^2$ is a compact. While this result is inapplicable to a non-compact plane $\Lambda = \mathbb{R}^2$, we can restrict Λ as follows.

Consider all convex sets with area π and ask which of them has the smallest perimeter. Without loss of generality we can assume that our convex sets contain the origin $O \in \mathbb{R}^2$. Let Λ be a circle with radius 4. If a set X contains both the origin O and point $x \notin \Lambda$, then perimeter of X is at least $2|Ox| \geq 8 > 2\pi$, i.e., bigger than perimeter of a unit circle. Therefore, the optimum, if exists, must be achieved on a convex subset of Λ.

Now, the arguments 1) and 2) in the proof above show that the perimeter is not minimal among all sets with area π except for unit circles. On the other hand, by compactness of $\mathcal{C}(\Lambda)$, at least one minimum must exist. Therefore, the unit circle is the desired minimum.

(II) Another way to prove the result is to exhibit an *explicit convergence* to a circle. That is, start with a set X, apply rule 1) and then repeatedly apply rules 2) to points $z \in P$ chosen to split the perimeter into 2^r pieces of equal length (occasionally one must also take convex hulls). We already know that under these transformations the perimeter stays the same while the area increases. Eventually the resulting region converges to a circle. This implies that the area of all sets with given perimeter was smaller than that of circles, as desired. We will skip the (easy) details.

Remark 7.2. Let us emphasize the similarities and differences between the two approaches, and the incorrect proof above. In all cases, we use the same variational principle, by exhibiting simple rules of how to improve the desired parameter (area/perimeter^2). In (I) we then used a general compactness argument to show that the optimum must exist. Alternatively, in (II) we showed that these rules when applied infinitely often, in the limit will transform any initial object to an optimum.

While the first approach is often easier, more concise and elegant as a mathematical argument, the second is more algorithmic and often leads to a better understanding of the problem. While, of course, there is no need to use the variational principle to construct a circle, making a distinction between two different approaches will prove useful in the future.

7.3. **How do you add sets?** Let us define the *Minkowski sum* of any two sets $A, B \subset \mathbb{R}^d$ as follows:

$$A + B := \{a + b \mid a \in A, b \in B\}.$$

For example, if A and B are two intervals starting at the origin O, then $A + B$ is a parallelogram spanned by the intervals. Similarly, if $A \subset \mathbb{R}^d$ is a convex set and B is a closed ball of radius $\varepsilon > 0$ centered at the origin, then $A+B$ consists of all points z at distance at most ε from A (see Figure 7.2).

Let us make several observations on the Minkowski sum. First, the sum is symmetric and respects translation, i.e., if A' is a translate of A, and B' is a translate of B, then $A' + B'$ is a translate of $A + B$. On the other hand, the Minkowski sum does not respect rotations: if A' and B' are (possibly different) rotations of A and B, then $A' + B'$ does not have to be a rotation of $A + B$, as the example with two intervals shows.

One more definition. For every $\lambda \in \mathbb{R}$, let $\lambda A = \{\lambda \cdot a \mid a \in A\}$. We call set A an *expansion* of B if there exists $\lambda \neq 0$, such that $A = \lambda B$. Clearly, if $A = \lambda B$, and $\lambda \neq 0$, then $B = (1/\lambda)A$. In other words, if A is an expansion of B, then B is an expansion of A.

Now, suppose H_1 and H_2 are two parallel hyperplanes. Then $H = \lambda_1 H_1 + \lambda_2 H_2$ is yet another parallel hyperplane. For $A_1 \subset H_1$ and $A_2 \subset H_2$ the set $B := \frac{1}{2}(A_1 + A_2)$ is the *average* of sets A_1 and A_2, consisting of midpoints of intervals between the sets. The set B lies in the hyperplane $H_3 = \frac{1}{2}(H_1 + H_2)$, in the middle between hyperplanes H_1 and H_3.

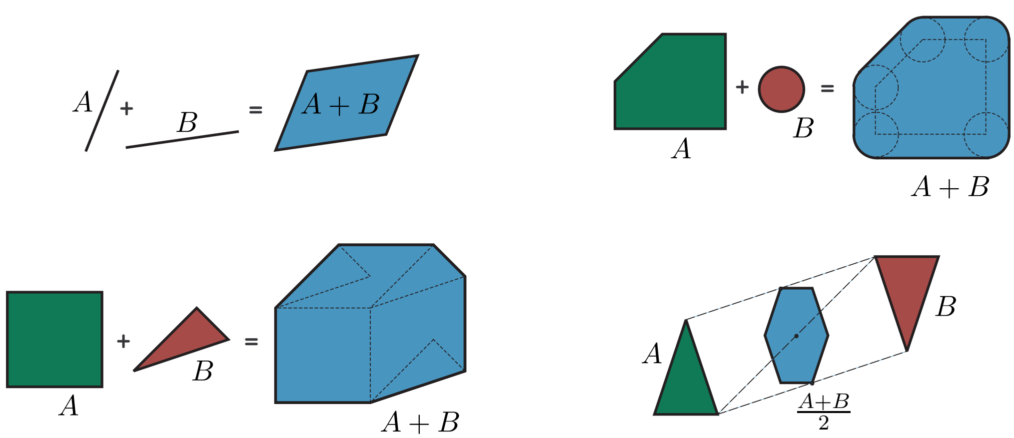

FIGURE 7.2. Four examples of the Minkowski sum of convex sets.

Recall that by $\text{area}(S) = \text{vol}_{d-1}(S)$ we denote the $(d-1)$-dimensional volume of the surface $S \subset \mathbb{R}^d$. The following result expresses the area of S via the volume of the Minkowski sums.

Proposition 7.3. *Let $X \subset \mathbb{R}^d$ be a convex set with the surface $S = \partial X$, and let B be a unit ball. Then:*

$$\text{area}(S) \, = \, \frac{d}{dt} \, \text{vol}(X + tB) \, \big|_{t=0} \, .$$

The proof is clear for polytopes and requires a straightforward convergence argument for general sets. In fact, everywhere below the reader can use this proposition as a definition of the surface area of general convex sets.

7.4. **Slicing polytopes with hyperplanes.** The following theorem is the key result in the field, marking the beginning of modern studies of geometric inequalities. We will need it to solve the isoperimetric problem in $\mathbb{R}^d$, and later on to prove the Minkowski uniqueness theorem (see Section 36).

Theorem 7.4 (Brunn–Minkowski inequality)**.** *Let $A, B \subset \mathbb{R}^d$ be two convex bodies. Then*

$$\text{vol}(A+B)^{1/d} \, \geq \, \text{vol}(A)^{1/d} \, + \, \text{vol}(B)^{1/d}.$$

Moreover, the inequality becomes an equality only if A is an expansion of B.

We prove in this section only the first part of the theorem. The second part is proved in the Appendix (see Subsection 41.5).

The following interesting result is actually a corollary from the Brunn–Minkowski inequality. It was, in fact, the original motivation for Brunn's studies (the work of Minkowski came later). We deduce this theorem from Theorem 7.4 in the next subsection.

Theorem 7.5 (Brunn)**.** *Let P be a convex body and let H_1, H_2 and H_3 be three parallel hyperplanes in $\mathbb{R}^d$ intersecting P in that order. Denote by $A_i = P \cap H_i$, $1 \leq i \leq 3$. Then:* $\text{area}(A_2) \geq \min\{\text{area}(A_1); \text{area}(A_3)\}$.

Example 7.6. Suppose $d = 2$ and line H_2 is a midway between H_1 and H_3. Then a stronger statement is true: $\text{area}(A_2) \geq \big(\text{area}(A_1) + \text{area}(A_3)\big)/2$. On the other hand, taking averages is not useful in higher dimensions. Take a simplex $P = \{0 \leq x_1 \leq x_2 \leq \ldots \leq x_d \leq 1\} \subset \mathbb{R}^d$, and let H_i be the hyperplanes defined by linear equations $x_1 = a_i$, where $a_1 = 0$, $a_2 = \frac{1}{2}$, and $a_3 = 1$. Clearly, the hyperplanes are parallel and at equal distance from each other. On the other hand, the areas of the intersections are given by

$$\text{area}(A_1) = 0 \, , \quad \text{area}(A_2) = \frac{1}{2^d n!} \, , \quad \text{and} \quad \text{area}(A_3) = \frac{1}{n!} \, .$$

Thus, the best we can hope for is $\text{area}(A_2) \geq \big(\text{area}(A_1) + \text{area}(A_3)\big)/2^d$, an unattractive inequality formally not implying Brunn's theorem. On the other hand, this does suggest that taking d-th root is a good idea, since in this case we have $\text{area}(A_2)^{1/d} = \big(\text{area}(A_1)^{1/d} + \text{area}(A_3)^{1/d}\big)/2$.

7.5. **The functional version.** Let $A, B \subset \mathbb{R}^d$ be two convex bodies and let $X_t = (1-t)A + tB$, where $t \in [0,1]$. As we mentioned in the example above, the first part of the following proposition immediately implies Theorem 7.5.

Proposition 7.7. *The function $\varphi(t) = \mathrm{vol}(X_t)^{1/d}$ is convex on $[0,1]$.*[18] *Moreover, $\varphi'(0) \geq \mathrm{vol}(B) - \mathrm{vol}(A)$, and the inequality becomes an equality if and only if A is an expansion of B.*

Proof of Theorem 7.5 modulo Proposition 7.7. Let $D = \lambda A_1 + (1-\lambda)A_3$, where λ is given by the ratio of the distance between hyperplanes: $\lambda = \mathrm{dist}(H_1, H_2)/\mathrm{dist}(H_1, H_3)$. By construction, set D is a convex body which lies in hyperplane H_2. On the other hand, by definition of the Minkowski sum, we have $D \subset \mathrm{conv}\{A_1, A_3\} \subset P$. Thus, $D \subset A_2$, and by Proposition 7.7 we conclude:

$$\mathrm{vol}(A_2) \,\geq\, \mathrm{vol}(D) \,\geq\, \min\{\mathrm{vol}(A_1), \mathrm{vol}(A_3)\}\,,$$

as desired. $\square$

Proof of Proposition 7.7 modulo Theorem 7.4. For every three values $0 \leq t' < t < t'' \leq 1$, consider $A = X_{t'}$, $B = X_{t''}$ and $C = X_t$. We have $C = \lambda A + (1-\lambda)B$, where $\lambda = (t - t'')/(t' - t'')$. By the Brunn–Minkowski inequality we have:

$$\begin{aligned}\mathrm{vol}(C)^{1/d} &\geq \mathrm{vol}\big(\lambda A\big)^{1/d} + \mathrm{vol}\big((1-\lambda)B\big)^{1/d} = \big[\lambda^d \mathrm{vol}(A)\big]^{1/d} + \big[(1-\lambda)^d \mathrm{vol}(B)\big]^{1/d} \\ &\geq \lambda \mathrm{vol}(A)^{1/d} + (1-\lambda)\,\mathrm{vol}(B)^{1/d}\,,\end{aligned}$$

which proves the first part.

Observe that $\varphi(0) = \mathrm{vol}(A)$ and $\varphi(1) = \mathrm{vol}(B)$. By convexity, the right derivative $\varphi'(0)$ is well defined (see Figure 7.3). From above and by the Brunn–Minkowski inequality, $\varphi(\frac{1}{2}) = \frac{1}{2}(\varphi(A) + \varphi(B))$ if and only if A is an expansion of B. Therefore, by convexity we have $\varphi'(0) \geq (\varphi(\frac{1}{2}) - \varphi(0))/\frac{1}{2} \geq \mathrm{vol}(B) - \mathrm{vol}(A)$, and the equality holds if and only if A is an expansion of B. $\square$

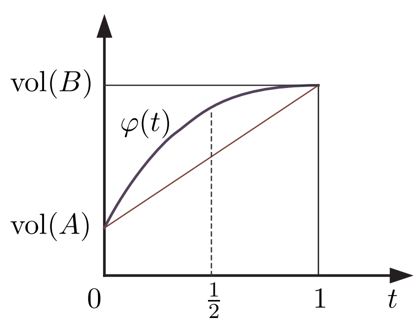

FIGURE 7.3. Convexity of $\varphi(t) = \mathrm{vol}(X_t)^{1/d}$, where $X_t = (1-t)A + tB$.

[18]Sometimes these functions are called *concave downwards*, *upper convex*, etc. All we can do here to convey the meaning is to point to Figure 7.3 and hope the reader is not too confused.

7.6. **The general isoperimetric inequality.** Here is the main result in this section:

Theorem 7.8 (Isoperimetric inequality in $\mathbb{R}^d$)**.** *Among all convex sets in $\mathbb{R}^d$ with a given volume, the ball has the smallest surface area.*

Proof. Let A be a convex set, and let B be a unit ball in $\mathbb{R}^d$. By Proposition 7.7, the function $\varphi(t) = \mathrm{vol}\big((1-t)A + tB\big)$ is convex on $[0,1]$. Therefore, by Proposition 7.3 we obtain:

$$\begin{aligned}\mathrm{area}(A) \ &= \ \frac{d}{dt}\,\mathrm{vol}(A + tB)\,\big|_{t=0} = \ \frac{d}{dt}\left[(1+t)^d\,\mathrm{vol}\left(\frac{A}{1+t} + \frac{tB}{1+t}\right)\right]\,\Big|_{t=0} \\ &= \ \frac{d}{dt}\left[(1+t)\,\varphi\left(\frac{t}{1+t}\right)\right]^d\,\Big|_{t=0} = d\,\varphi(0)^{d-1}\left[\varphi(0)+\varphi'(0)\right].\end{aligned}$$

By Proposition 7.7, we have $\varphi'(0) \geq \mathrm{vol}(B) - \mathrm{vol}(A)$, and the inequality becomes an equality if and only if A is an expansion of B. Since $\mathrm{vol}(A) = \varphi(0)$, the area$(A)$ minimizes only when A is an expansion of B, i.e., when A is a ball. □

Remark 7.9. In the spirit of Remark 7.2, it is instructive to decide what type of argument is used in this proof: did we use a *compactness* or an *explicit convergence* argument? From above, the rule for decreasing $\psi(X) = \mathrm{area}(X)/\mathrm{vol}(X)^{(d-1)/d}$ turns out to be quite simple: take $X_t = (1-t)X + tB$, where B is a unit ball. Letting $t \to 1$ makes X_t approach the unit ball. Thus the proof can be viewed as an explicit convergence argument, perhaps an extremely simple case of it.

7.7. **Brick-by-brick proof of the Brunn–Minkowski inequality.** Perhaps surprisingly, the proof of Theorem 7.4 will proceed by induction. Formally, let us prove the inequality for *brick regions* defined as disjoint unions of bricks (parallelepipeds with edges parallel to the coordinate axes). Note that we drop the convexity condition. A union of bricks does not have to be connected or the bricks be of the same size: any disjoint union of bricks will do (see Figure 7.4). Denote by $\mathcal{B}_d$ the set of all brick regions in $\mathbb{R}^d$.

Lemma 7.10. *If the Brunn–Minkowski inequality holds for brick regions $A, B \in \mathcal{B}_d$, it also holds for general convex regions in $\mathbb{R}^d$.*

We continue with the proof of the inequality. The lemma will be proved later.

Proof of the Brunn–Minkowski inequality for brick regions. Let A and B be two brick regions. We use induction on the total number k of bricks in A and B. For the base of induction, suppose $k = 2$, so both sets consist of single bricks: $A = [x_1 \times x_2 \times \ldots \times x_d]$ and $B = [y_1 \times y_2 \times \ldots \times y_d]$. The Brunn–Minkowski inequality then becomes the Minkowski inequality[19] (Theorem 41.4).

For the induction step, suppose the result holds for brick regions with k or fewer bricks (in total), $k \geq 2$. Now, take two brick regions $A, B \in \mathcal{B}_d$ with $k+1$ bricks. Suppose A contains at least two bricks (otherwise we can relabel the regions). Fix

[19]While we move the proof of the Minkowski inequality to the appendix as a standard result, we do suggest the reader go over the proof so as to acquaint yourself with the proof style of inequalities.

these two bricks, say $P, Q \subset A$. Observe that there always exists a hyperplane $H \subset \mathbb{R}^d$ with normal to one of the axes, which separates bricks P and Q. Denote by A_1 and A_2 portions of A lying on the two sides of H. Since H either avoids or divides each brick into two parts, we have $A_1, A_2 \in \mathcal{B}_d$. Note also that A_1 and A_2, each have strictly fewer bricks than A.

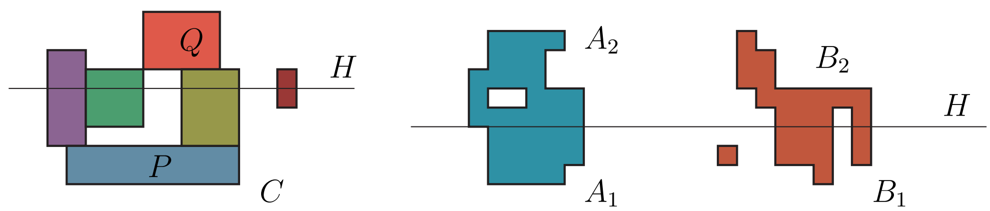

FIGURE 7.4. Brick region C on plane presented as a disjoint union of rectangles with different colors. Two regions $A, B \in \mathcal{B}_2$ divided by a line H with the same area ratios: $\theta = 2/5$.

Let $\theta = \mathrm{vol}(A_1)/\mathrm{vol}(A)$. Recall that the volume of the Minkowski sum is independent of translation. Thus, we can assume that the hyperplane H goes through the origin $O \in \mathbb{R}^d$. Now translate set B so that H divides B into two sets B_1 and B_2 with the same volume ratio: $\theta = \mathrm{vol}(B_1)/\mathrm{vol}(B)$. We have

$$\mathrm{vol}(A+B) \geq \mathrm{vol}(A_1+B_1) + \mathrm{vol}(A_2+B_2),$$

since the Minkowski sums A_1+B_1 and A_2+B_2 lie on different sides of H and thus do not intersect (except at the boundary). Since the total number of bricks in (A_1, B_1) and in (A_2, B_2) is less than k we can apply to them the inductive assumption:

$$\begin{aligned}
\mathrm{vol}(A+B) &\geq \mathrm{vol}(A_1+B_1) + \mathrm{vol}(A_2+B_2) \\
&\geq \left[\mathrm{vol}(A_1)^{1/d} + \mathrm{vol}(B_1)^{1/d}\right]^d + \left[\mathrm{vol}(A_2)^{1/d} + \mathrm{vol}(B_2)^{1/d}\right]^d \\
&\geq \left[\big(\theta\,\mathrm{vol}(A)\big)^{1/d} + \big(\theta\,\mathrm{vol}(B)\big)^{1/d}\right]^d \\
&\qquad + \left[\big((1-\theta)\,\mathrm{vol}(A)\big)^{1/d} + \big((1-\theta)\,\mathrm{vol}(B)\big)^{1/d}\right]^d \\
&\geq \left[\theta + (1-\theta)\right] \cdot \left[\mathrm{vol}(A)^{1/d} + \mathrm{vol}(B)^{1/d}\right]^d \\
&\geq \left[\mathrm{vol}(A)^{1/d} + \mathrm{vol}(B)^{1/d}\right]^d .
\end{aligned}$$

This completes the induction step and proves the theorem. □

Proof of Lemma 7.10. For general convex regions $A, B \subset \mathbb{R}^d$ the following is a standard convergence argument. First, consider parallel hyperplanes $x_i = m/2^n$, $m \in \mathbb{Z}$, $i \in [d]$. They subdivide the regions into small interior cubes and partly-filled cubes near the boundary. Denote by A_n and B_n the union of all cubes intersecting A and B, respectively. Observe that $A_1 \supseteq A_2 \supseteq \ldots$, and the same holds for $\{B_n\}$ and $\{A_n + B_n\}$. On the other hand, $\bigcap_n A_n = A$ since for every point $a \notin A$ there exists an $\varepsilon > 0$, such that a cube with side ε containing a does not intersect A. This gives $\mathrm{vol}(A_n) \to \mathrm{vol}(A)$ and $\mathrm{vol}(B_n) \to \mathrm{vol}(B)$, as $n \to \infty$. In the topology defined by the distance $d(A, B)$ (see Exercise 7.1), this implies that $A_n \to A$ and $B_n \to B$, as $n \to \infty$.

Finally, $A+B \supseteq \bigcap_n A_n+B_n$ since for every $c_n = a_n+b_n \to c \in \mathbb{R}^d$, by compactness of A, B there exist converging subsequences $a_n \to a$ and $b_n \to b$, and thus $c = a+b \in A+B$. On the other hand, $A_n + B_n \supseteq A + B$ for all $n \in \mathbb{N}$, which implies that $A_n + B_n \to A + B$ as $n \to \infty$. Now we can apply the inequality to the brick regions A_n and B_n:

$$\begin{aligned}\mathrm{vol}(A+B)^{1/d} &= \lim_{n\to\infty} \mathrm{vol}(A_n+B_n)^{1/d} \ge \lim_{n\to\infty} \mathrm{vol}(A_n)^{1/d} + \lim_{n\to\infty} \mathrm{vol}(B_n)^{1/d} \\ &\ge \mathrm{vol}(A)^{1/d} + \mathrm{vol}(B)^{1/d},\end{aligned}$$

which completes the proof of the lemma. □

7.8. **Exercises.**

Exercise 7.1. (*Blaschke selection theorem*) ♢ [1+] For every subset $\Lambda \subset \mathbb{R}^d$, let $\mathcal{C}(\Lambda)$ be the set of convex subsets of Λ. Define a distance $d(X, Y)$ between two convex sets $X, Y \subset \mathcal{C}(\mathbb{R}^d)$ as follows:

$$d(X, Y) := \mathrm{vol}(X \smallsetminus Y) + \mathrm{vol}(Y \smallsetminus X).$$

Check that $d(X, Y)$ is a metric on $\mathcal{C}(\mathbb{R}^d)$. Prove that $\mathcal{C}(\Lambda) \subset \mathcal{C}(\mathbb{R}^d)$ is compact if and only if set $\Lambda \subset \mathbb{R}^d$ is compact. Conclude that an infinite family of convex subsets inside a unit cube has a converging subsequence.

Exercise 7.2. Let $X \subset \mathbb{R}^3$ be a convex polytope. Define $\rho(X)$ to be the set of points obtained by reflections of one point in X across another: $\rho(X) = \{2x - y \mid x, y \in X\}$.
a) [1-] Show that $\rho(X)$ is also a convex polytope.
b) [1-] Consider a sequence $X_1, X_2, \ldots$, where $X_{i+1} = \rho(X_i)$ and X_1 is a cube. Compute polytope X_n for all $n > 1$.
c) [1] Same question when X_1 is a regular tetrahedron, octahedron and icosahedron.
d) [1] Same question when X_1 is a standard tetrahedron and a Hill tetrahedron (see Subsection 31.6).
e) [1] Prove or disprove: for every fixed X_1, the number of faces of X_n is bounded, as $n \to \infty$.
f) [1+] Scale polytopes Y_n to have the same unit volume: $Y_n = X_n/\mathrm{vol}(X_n)$. Prove that the sequence Y_n converges to a convex body.

Exercise 7.3. (*Steiner symmetrization*) Another one of several Steiner's ingenious proofs of the isoperimetry in the plane is based on the following construction. Take a convex set $X \subset \mathbb{R}^2$ and any line ℓ. Divide X into infinitely many intervals $[a, b]$, $a, b \in \partial X$, orthogonal to ℓ. Move each interval $[a, b]$ along the line (a, b) so that it is symmetric with respect to ℓ (See Figure 7.5). Denote by $Y = S(X, \ell)$ the resulting set. The resulting map $\varsigma : X \to Y$ is called the *Steiner symmetrization*.
a) [1] Prove that $\mathrm{diam}(Y) \le \mathrm{diam}(X)$, $\mathrm{area}(Y) = \mathrm{area}(X)$ and $\mathrm{perimeter}(Y) \le \mathrm{perimeter}(X)$.
b) [1+] Prove that $C = \varsigma(A) + \varsigma(B)$ fits inside $D = \varsigma(A + B)$, for all convex $A, B \subset \mathbb{R}^2$.
c) [1] Prove that C is a translate of D if and only if A and B are homothetic.

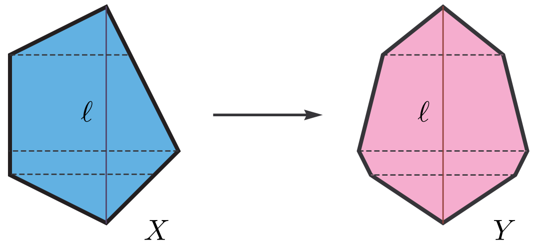

FIGURE 7.5. The Steiner symmetrization $\varsigma : X \to Y$.

$d)$ [1+] Use the Steiner symmetrization to obtain two new proofs of Theorem 7.1, by employing both types of arguments, (I) and (II), discussed above.

Exercise 7.4. (*Hammer's X-ray problem*) [2-] Let $X, X' \subset \mathbb{R}^2$ be two convex polygons in the plane. Fix three lines $\ell_1, \ell_2, \ell_3 \subset \mathbb{R}^2$ whose normals are not rationally dependent. Suppose $S(X, \ell_i) \sim S(X, \ell_i)$ for $i = 1, 2, 3$, where $X \sim Y$ if Y is a translate of X. Prove that $X \sim X'$.

Exercise 7.5. [1] Here is yet another attempt at symmetrization. Let $Q \subset \mathbb{R}^3$ be a convex polygon containing the origin O. For every line ℓ through O, shift the interval $\ell \cap Q$ to make it centrally symmetric at O. Prove or disprove: the resulting body is convex.

Exercise 7.6. [1] Prove that of all n-gons in the plane with given perimeter the inscribed regular n-gon has the largest area. Deduce from here the isoperimetry of a circle in the plane.

Exercise 7.7. (*Bonnesen's inequality*) [1+] Let $Q \subset \mathbb{R}^2$ be a convex polygon, $a = \text{area}(Q)$, and $\ell = \text{perimeter}(Q)$. Let r and R denote the radius of the maximal circle inside Q and of the minimal circle outside Q. Prove that $\ell^2 - 4\pi a \geq \pi^2(R - r)^2$.

Exercise 7.8. (*Spherical isoperimetric inequality*) [2-] Let X be a simple spherical polygon of length ℓ on a unit sphere $\mathbb{S}^2$. Denote by $a = a(X)$ the area of one of the sides. Prove the isoperimetric inequality $\ell^2 \leq a(4\pi - a)$.

Exercise 7.9. [1] Let $v_1, \ldots, v_n \in \mathbb{R}^2$ be a set of vectors such that $|v_1| + \ldots + |v_n| = 1$. Prove that there exists a subset $I \subseteq \{1, \ldots, n\}$ such that

$$\Big|\sum_{i \in I} v_i\Big| \geq \frac{1}{\pi}\,.$$

Check that the $1/\pi$ constant is optimal.

Exercise 7.10. (*Urysohn's inequality*) a [1] Let X be a convex body. Prove that $\text{area}(X) \leq \frac{\pi}{4}\, \text{diam}(X)^2$, where the equality holds if and only if X is a disk.
$b)$ [1+] Prove that $\text{area}(X) \leq \frac{\pi}{4} w(X)^2$, where $w(X)$ is the *average width* of X (the width is defined as $f(\boldsymbol{u})$ in the proof of Proposition 5.1).
$c)$ [2-] Generalize parts $a)$ and $b)$ to higher dimensions.

Exercise 7.11. (*Monotonicity of the area*) $a)$ [1-] Let $Q_1, Q_2 \subset \mathbb{R}^2$ be convex polygons, such that Q_1 is inside Q_2. Prove that the perimeter of Q_1 is smaller than the perimeter of Q_2.

b) [1-] Let $P_1, P_2 \subset \mathbb{R}^3$ be a convex polytopes such that P_1 is inside P_2. Prove that $\operatorname{area}(P_1) \le \operatorname{area}(P_2)$.

Exercise 7.12. Let $P_1 \subset P_2$ be two convex polytopes in $\mathbb{R}^3$, and let L_1, L_2 denote their total edge lengths.
a) [1] Prove that $L_1 \le L_2$ when the polytopes are bricks.
b) [1] Prove that $L_1 \le L_2$ when the polytopes are parallelepipeds.
c) [1-] Prove or disprove that $L_1 \le L_2$ when the polytopes are tetrahedra.
d) [1] Prove or disprove that $L_1 \le L_2$ when the polytopes are combinatorially equivalent polytopes with parallel corresponding faces.

Exercise 7.13. Let $P \subset \mathbb{R}^3$ be a convex polytope which lies inside a unit sphere $\mathbb{S}^2$. Denote by L the sum of edge lengths of P, and by A the surface area.
a) [1] Prove that $A \le L$. Check that this inequality is tight.
b) [1] Let O be the center of $\mathbb{S}^2$, and suppose O lies in the interior of P. For every edge $e = (v, w)$ of P and every face F containing e, denote by $\gamma(e, F)$ the interior angle between F and triangle (Ovw). Prove that $A \le \xi L$, where $\xi = \max_{e,F} \cos \gamma(e, F)$.

Exercise 7.14. Let $P \subset \mathbb{R}^3$ be a convex polytope such that a circle of radius r fits inside each face.
a) [2] Denote by S the sum of squares of edge lengths of P. Prove that $S \ge 48\, r^2$. Show that the equality holds only for a cube.
b) [2] Denote by L the sum of edge lengths of P. Prove that $L \ge 12\sqrt{3}\, r$. Show that the equality holds only for a tetrahedron.

Exercise 7.15. *a*) [2] Suppose a convex polytope $P \subset \mathbb{R}^3$ contains a unit sphere. Prove that the sum of edge lengths $L \ge 24$. Show that the equality holds only for the cube.
b) [2-] Suppose P is *midscribed* around a unit sphere, i.e., all edges of P touch the sphere. Compute L for all regular polytopes and for all n-prisms. Show that there exist polyhedra with a smaller L.

Exercise 7.16. (*Zonotopes*) $\Diamond$ Let $P \subset \mathbb{R}^3$ be a convex polytope whose faces are centrally symmetric polygons. Such polytopes are called *zonotopes*.
a) [1-] Subdivide each face of P into parallelograms.
b) [1] Prove that P can be subdivided into *parallelepipeds*, defined as the Minkowski sum of three independent vectors.
c) [1] Prove that P is centrally symmetric.
d) [1+] Prove that P is the Minkowski sum of the intervals.
e) [1] Suppose each face of P is a parallelogram. Prove that the number of faces of P is two times a triangular number.
f) [1] Use Exercise 2.3 to show that every polytope $P \subset \mathbb{R}^3$ has at least 6 parallelograms.

7.9. **Final remarks.** The "proof" by Steiner is one of his five proofs of the isoperimetry of a disk (Theorem 7.1), published in 1841. The mistake remained unnoticed until 1882. Another one of his 'proofs' is based on the Steiner symmetrization (Exercise 7.3). In all fairness to Steiner, the rigor of the work by contemporaries is questionable as well. For more on Steiner's work and other early results on isoperimetric problems see [Bla3, Kry].

The proof of the general isoperimetric inequality (Theorem 7.8) via the Brunn–Minkowski inequality goes back to Minkowski [Min]. Both results can be found in a number of textbooks (see [BonF, Grub, Had1, Hör, Schn2]). We follow [Mat1, §12.2] in our presentation of the first part of Theorem 7.4. The second part is given in Subsection 41.5, and with all its tediousness it is our own. The advantage of this proof is that it can be extended to a variety of non-convex regions, e.g., connected regions with smooth boundary.

Let us also mention the *Alexandrov–Fenchel inequality* generalizing the Brunn–Minkowski inequality to mixed volumes of many different convex bodies [BZ3, §4.2], and the *Prékopa-Leindler inequality*, which is a dimension-free extension [Grub, §8.5]. We refer to [Gar, BZ3] for further generalizations, geometric inequalities, their applications and numerous references.

Different proofs of the Blaschke selection theorem (Exercise 7.1) can be found in [BolY, §4.3], [Egg1, §4.2] and [Fej2, Ch. 2, §1]. A completely different approach to the isoperimetric inequalities via averaging is given in Exercise 24.7. The spherical analogue of the isoperimetric inequality is given in Subsection 28.5.

8. Combinatorics of convex polytopes

In this section we study graphs and f-vectors of convex polytopes. Most proofs use *Morse functions*, a powerful technical tool useful throughout the field and preludes the variational principle approach in the next three sections. Among other things, we prove the *Dehn-Sommerville equations* generalizing *Euler's formula*. This is just about the only place in the book where we do exact counting. We also show that the graphs of convex polytopes in $\mathbb{R}^3$ are 3-connected (see *Balinski's theorem*), an important result leading to the Steinitz theorem in Section 11. We also introduce the *permutohedron* and the *associahedron* (see Examples 8.4 and 8.5), which are used later in Sections 16 and 17.

8.1. **Counting faces is surprisingly easy.** Define *simple polytopes* in $\mathbb{R}^d$ to be polytopes where all vertices are adjacent to exactly d edges. These polytopes are dual to *simplicial polytope*, whose faces are simplices. It is important to note that the combinatorics of simplicial polytopes (the poset of faces of all dimensions) does not change when we slightly perturb the vertices. Similarly, the combinatorics of simple polytopes does not change when we slightly perturb the normals to facet hyperplanes.

Let $P \subset \mathbb{R}^d$ be a simple polytope. Denote by $f_i = f_i(P)$ the number of i-dimensional faces, $0 \le i \le d$. For example, $f_d = 1$ and f_0 is the number of vertices in P. The sequence $(f_0, f_1, \ldots, f_d)$ is called the *f-vector* of P.

The generalized Euler's formula for all convex polytopes is the following equation:

$$(\heartsuit) \quad f_0 - f_1 + f_2 - \ldots + (-1)^d f_d = 1\,.$$

We will show that this is not the only linear relation on f_i, but the first in a series.

Theorem 8.1 (Dehn–Sommerville equations)**.** *The f-vector of a simple polytope $P \subset \mathbb{R}^d$ satisfies the following linear equations:*

$$\sum_{i=k}^{d} (-1)^i \binom{i}{k} f_i \,=\, \sum_{i=d-k}^{d} (-1)^{d-i} \binom{i}{d-k} f_i\,, \quad \textit{for all} \ \ 0 \le k \le d.$$

Now observe that the case $k = 0$ corresponds to Euler's formula $(\heartsuit)$.

Proof. Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be a *Morse function*, defined as a linear function on $\mathbb{R}^d$ that is nonconstant on edges of the polytope P. Let $\Gamma = (V, E)$ be the graph of P, where V and E are the sets of vertices and edges, respectively. Since P is simple, the graph Γ is *d-regular*, i.e., the degree of every vertex is d.

Consider an acyclic orientation $\mathcal{O}_\varphi$ of edges of Γ according to φ (edges are oriented in the direction of increase of φ). For a vertex $v \in V$ let the *index* of v, denoted $\operatorname{ind}_\varphi(v)$, be the number of outgoing edges in $\mathcal{O}$. Let h_i be the number of vertices $v \in V$ with $\operatorname{ind}_\varphi(v) = i$. The sequence $(h_0, h_1 \ldots, h_d)$ is called the *h-vector* of P. We will show that the numbers h_i depend only on Γ and are in fact independent of the Morse function φ.

For every face F of P denote by $\min_\varphi(F)$ the unique vertex $v \in F$ with the smallest value of φ. Looking at the acyclic orientation $\mathcal{O}_\varphi$ this is the unique source point in an induced subgraph $\Gamma|_F$. Let us count the number f_k of k-dimensional faces in P by looking at their minimum vertices. Observe that for all $v \in V$, every subset (even the empty set) of $\mathrm{ind}_\varphi(v)$ increasing edges spans a face. Summing over all vertices $v \in V$, for the number of such k-element subsets we obtain:

$$(*) \qquad f_k \,=\, \sum_{v\in V,\, \mathrm{ind}_\varphi(v)\ge k} \binom{\mathrm{ind}_\varphi v}{k} \,=\, \sum_{i=k}^{d} \binom{i}{k} h_i.$$

Writing

$$\mathcal{F}(t) \,=\, \sum_{i=0}^{d} f_i\, t^i, \qquad \mathcal{H}(t) \,=\, \sum_{i=0}^{d} h_i\, t^i,$$

we can rewrite $(*)$ as $\mathcal{F}(t) = \mathcal{H}(t+1)$. Indeed,

$$\begin{aligned}\mathcal{H}(t+1) \,&=\, \sum_{i=0}^{d} h_i\,(t+1)^i \,=\, \sum_{i=0}^{d} h_i \left[\sum_{k=0}^{i} \binom{i}{k} t^k\right] \,=\, \sum_{k=0}^{d} t^k \left[\sum_{i=k}^{d} \binom{i}{k} h_i\right] \\ &=\, \sum_{k=0}^{d} t^k\, f_k \,=\, \mathcal{F}(t).\end{aligned}$$

This implies that the h-vector of P as defined above is independent of the Morse function φ. Now consider the Morse function $\psi = -\varphi$ and observe that $\mathrm{ind}_\psi(v) = d - \mathrm{ind}_\varphi(v)$, for all $v \in V$. Thus, $h_k = h_{d-k}$. Expanding $\mathcal{H}(t) = \mathcal{F}(t-1)$, we obtain:

$$h_k \,=\, (-1)^k \sum_{i=k}^{d} (-1)^i \binom{i}{k} f_i \quad \text{and} \quad h_{d-k} = (-1)^{d-k} \sum_{i=d-k}^{d} (-1)^i \binom{i}{d-k} f_i\,,$$

which implies the result. □

Remark 8.2. The combinatorial interpretations of the integers h_i in the proof above imply the inequalities $h_i \ge 0$ for all $0 \le i \le d$. For example, we have $h_0 = 1$, $h_1 = f_{d-1} - d \ge 0$, and $h_2 = f_{d-2} - (d-1)f_{d-1} + \binom{d}{2} \ge 0$. In fact, much stronger inequalities are known:

$$h_0 \,\le\, h_1 \,\le\, h_2 \,\le\, \ldots \,\le h_r\,, \quad \text{where } r = \lfloor d/2 \rfloor.$$

For example, $h_1 \ge h_0$ gives an obvious inequality for the number of facets in P: $f_{d-1} \ge d+1$. On the other hand, already $h_2 \ge h_1$ gives an interesting inequality:

$$f_{d-2} \,-\, d f_{d-1} \,\ge\, \binom{d+1}{2}.$$

For more on this inequality see Remark 27.4.

8.2. **A few good examples would not hurt.** The Dehn-Sommerville equations above may seem rather pointless, but in fact the Morse function approach can be used to compute the f-vectors. Below we present three such examples for some 'combinatorial polytopes'.

Example 8.3. (*Hypercube*) Let $C_d \in \mathbb{R}^d$ be a standard d-dimensional cube $[0,1]^d$. Consider the Morse function $\varphi(x_1,\ldots,x_d) = x_1 + \ldots + x_d$. Clearly, $\mathrm{ind}_\varphi(x_1,\ldots,x_d) = d - \varphi(x_1,\ldots,x_d)$, for all $x_i \in \{0,1\}$, and we obtain $h_i = \binom{d}{i}$. Therefore,

$$\mathcal{H}(t) \, = \, \sum_{i=1}^{d} \binom{d}{i} t^i \, = \, (1+t)^d,$$

and

$$\mathcal{F}(t) \, = \, \mathcal{H}(t+1) \, = \, (2+t)^d \, = \, \sum_{i=1}^{d} 2^{d-i} \binom{d}{i} t^i \, .$$

The resulting formula $f_i = 2^{d-i}\binom{d}{i}$ can, of course, be obtained directly, but it is always nice to see something you know obtained in an unexpected albeit complicated way.

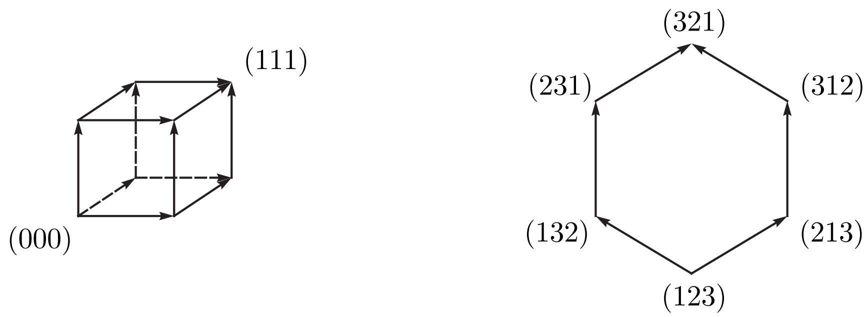

FIGURE 8.1. Acyclic orientations $\mathcal{O}_\varphi$ of a cube C_3 and a permutohedron P_3.

Example 8.4. (*Permutohedron*) Let $P_n \subset \mathbb{R}^n$ be a *permutohedron*, defined as the convex hull of all $n!$ permutation vectors $(\sigma(1),\ldots,\sigma(n))$. Note that P_n is simple and $(n-1)$-dimensional. The graph Γ_n of P_n is the Cayley graph of the symmetric group with the adjacent transpositions as the generating set:

$$\Gamma_n \, = \, \mathrm{Cayley}\big(S_n, \{(1,2),(2,3),\ldots,(n-1,n)\}\big).$$

Consider the Morse function

$$\varphi(x_1, x_2 \ldots, x_n) \, = \, x_1 \, + \, \epsilon \, x_2 \, + \, \ldots + \, \epsilon^{n-1} \, x_n \, .$$

If $\epsilon = \epsilon(n) > 0$ is sufficiently small, the resulting acyclic orientation $\mathcal{O}_\varphi$ of Γ_n makes a partial order on permutations that is a suborder of the lexicographic order. In particular, the identity permutation $(1,2,\ldots,n)$ is the smallest and $(n,n-1,\ldots,1)$ is the largest permutation. Now observe that for every permutation $\sigma \in S_n$ the number of outgoing edges $\mathrm{ind}_\varphi(\sigma)$ is equal to the number of *ascents* in σ, i.e., the number of $1 \le i < n$ such that $\sigma(i) < \sigma(i+1)$. Therefore, h_i is equal to the *Eulerian number* $A(n,i)$, defined as the number of permutations $\sigma \in S_n$ with i ascents. We refer to Exercise 8.17 for further properties of Eulerian numbers.

Example 8.5. (*Associahedron*) Let $Q \subset \mathbb{R}^2$ be a fixed convex n-gon (not necessarily regular), and let T_n be the set of triangulations of Q. We denote the vertices of Q by integers $i \in [n]$. Euler proved that $|T_n| = C_{n-2}$, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ are the *Catalan numbers*. For every vertex $v \in Q$ in a triangulation $\tau \in T_n$, denote by $\xi_\tau(v)$ the sum of areas of triangles in τ that contain v. Let $R_n \subset \mathbb{R}^n$ be a convex hull of all C_{n-2} functions $\xi_\tau : [n] \to \mathbb{R}^n$, defined as $v \to \xi_\tau(v)$, for all $\xi_\tau \in \mathbb{R}^n$. The polytope R_n is called an *associahedron*.

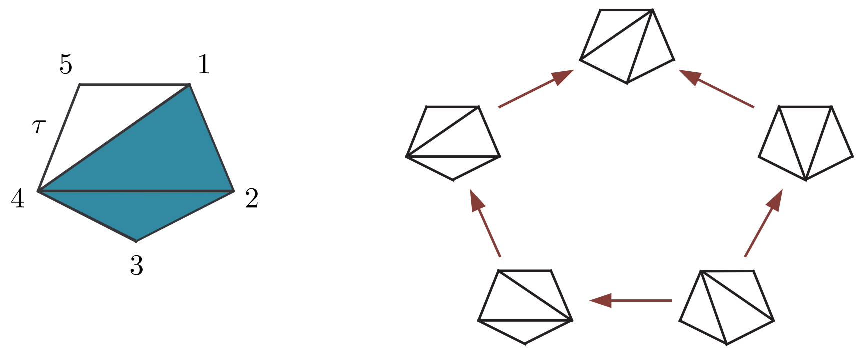

FIGURE 8.2. Triangulation $\tau \in T_5$ with $\xi_\tau(2) = \text{area}(124) + \text{area}(234)$, and an acyclic orientation $\mathcal{O}_\varphi$ of the associahedron R_5.

One can check that R_n is in fact $(n-3)$-dimensional; for example R_4 is an interval and R_5 is a pentagon (see Exercise 8.2). The edges of R_n correspond to pairs of triangulations which differ in exactly one edge, so R_n is a simple polytope. More generally, the k-dimensional faces of R_n correspond to subdivisions of Q with $(n-3-k)$ noncrossing edges, and the containment of faces is by subdivision. As in Example 8.4, consider the Morse function $\varphi : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\varphi(x_1, x_2, \ldots, x_n) \,=\, x_1 \,+\, \epsilon\, x_2 \,+\, \ldots + \,\epsilon^{n-1} x_n\,,$$

where $\epsilon > 0$ is small enough. This defines an acyclic orientation $\mathcal{O}_\varphi$ of the graph Γ_n of the associahedron (see Figure 8.2). For example, it is easy to see that the triangulation with edges $\{(1,3), \ldots, (1, n-1)\}$ is maximal with respect to φ.

Recall the standard bijection $\beta : \tau \to t$ from polygon triangulations to binary trees, where the triangles correspond to vertices, the triangle adjacent to $(1,n)$ corresponds to the root, and the adjacent triangles correspond to the left and right edges to the tree (see Figure 8.3). Observe that the number of outgoing edges $\text{ind}_\varphi(\tau)$ is equal to the number of left edges in $\beta(\tau)$. Therefore, h_i is equal to the number $B_{n-2,i}$ of binary trees with $n-2$ vertices and i left edges (see Exercise 8.18 for a closed formula).

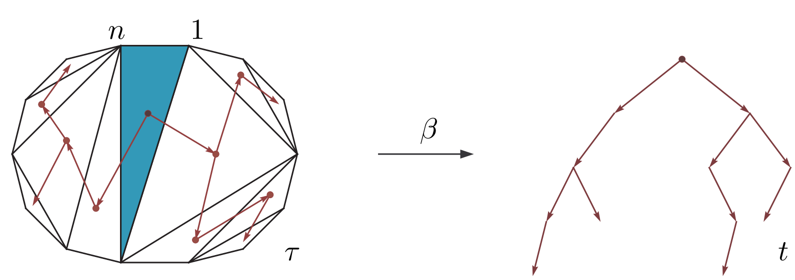

FIGURE 8.3. Map $\beta : \tau \to t$. Here $n = 14$, $\text{ind}_\varphi(\tau) = 7$.

8.3. **Kalai's simple way to tell a simple polytope from its graph.** The proof of the Dehn-Sommerville equations shows that given an acyclic orientation $\mathcal{O}_\varphi$ coming from a Morse function φ we can compute the f-vector of a simple polytope. It turns out that we can obtain the whole poset of faces given just the graph Γ of the polytope.

Theorem 8.6 (Blind–Mani)**.** *Let $P \in \mathbb{R}^d$ be a simple polytope and let Γ be its graph. Then Γ determines the entire combinatorial structure of P.*

In other words, given a graph Γ one can decide whether a given subset of vertices forms a k-dimensional face. The proof we present below is the celebrated proof by Kalai which uses insight of Morse functions to resolve the problem.

Proof. For a subset of vertices $B \subset V$ of the graph Γ to form a k-dimensional face the induced subgraph $\Gamma|_B$ must be connected and k-regular. However, this is clearly not enough. For example, a graph of 3-dimensional cube contains a cycle of length 6 which is not a two-dimensional face. We will use certain acyclic orientations of Γ to give a complete characterization.

Consider an acyclic orientation $\mathcal{O}$ of Γ. We say that $\mathcal{O}$ is *good* if for every face $F \subset P$, the corresponding graph Γ_F has exactly one source. Otherwise, $\mathcal{O}$ is called *bad.* Note that in every face F, the acyclic orientation of Γ_F must have at least one source.

Denote by $h_i^{\mathcal{O}}$ the number of vertices of index i, i.e., the number of vertices with exactly i outgoing edges and let

$$\alpha(\mathcal{O}) \, = \, h_0^{\mathcal{O}} \, + \, 2\, h_1^{\mathcal{O}} \, + \, 4\, h_2^{\mathcal{O}} \, + \, \ldots \, + \, 2^d h_d^{\mathcal{O}} \, .$$

Finally, let $\beta = \beta(P) = f_0 + f_1 + \ldots + f_d$ be the total number of faces of all dimensions. Arguing as in the proof of the Dehn-Sommerville equations, we conclude:

- $\alpha(\mathcal{O}) \geq \beta$ for all $\mathcal{O}$,
- $\alpha(\mathcal{O}) = \beta$ if and only if $\mathcal{O}$ is good.

To show this, count every face of every dimension according to the source vertices. Clearly, for a vertex v with index i, the number of such faces is at most 2^i, the number of all possible subsets of edges coming out of v. Now observe that good orientations are exactly those where every face is counted exactly once.

This gives us a (really slow) algorithm for computing $\beta(P)$: compute $\alpha(\mathcal{O})$ for all acyclic orientations $\mathcal{O}$ of Γ and take the smallest value. Of course, the total number of faces is only a tiny fraction of the information we want. What is more important is that this approach gives a way to decide whether a given orientation is good. Having found one good orientation $\mathcal{O}$, we can take the numbers $h_i^{\mathcal{O}}$ to compute the f-vector by the formula $f_i = \sum_{k=i}^d h_k^{\mathcal{O}} \binom{k}{i}$. Still, this approach does not show how to start with a good orientation of Γ, choose i edges coming out of some vertex v and compute the whole face. Morse functions again come to the rescue.

Let $B \subset V$ be a subset of vertices, a candidate for a face. We say that B is *final* with respect to an acyclic orientation $\mathcal{O}$ if no edge is coming out of B. We are now ready to state the main claim:

Face criterion. *An induced connected k-regular subgraph $H = \Gamma_B$ on a set of vertices B is a graph of a face of P if and only if there exists a good orientation $\mathcal{O}$ of G, such that B is final with respect to $\mathcal{O}$.*

The 'only if' part can be proved as follows. Take a simple polytope P and a face F with the set of vertices $B \subset V$. Consider any linear function φ which maximizes on F. Now perturb φ to make it a Morse function but keep B final with respect to φ. This is clearly possible and gives the desired construction.

The 'if' part is more delicate. Let v be any source point of the graph H with respect to $\mathcal{O}$ (for now we do not know if such a point is unique). Take all k edges in H that are coming out of v. Since $\mathcal{O}$ is good, there exists a k-dimensional face F with v as its minimum. Since B is final with respect to $\mathcal{O}$, we have $F \subseteq B$ and the graph $Y = \Gamma_F$ is a subgraph of H. Finally, since P is simple and F is k-dimensional, the graph Y of Γ is also k-regular.

Now compare graphs H and Y. Both are connected, k-regular, and $Y \subseteq H$. This implies that $Y = H$, which proves the face criterion and the theorem.[20] $\square$

Remark 8.7. A couple of things are worth noting while we are on the subject. First, the above criteria gives a 'really slow' algorithm for deciding if a subset is a facet. A polynomial time algorithm was recently found in [Fri]. Second, one cannot easily use this idea to decide whether a given graph is a graph of a simple polytope; this problem is believed to be very difficult (see [Kaib]).

8.4. **Graph connectivity via Morse functions.** It is well known and easy to prove that graphs of 3-dimensional polytopes are 3-connected, but the corresponding result for d-dimensional polytopes is less standard. The proof again is based on Morse functions.

A graph is called *d-connected* if the removal of any $d-1$ vertices leaves a connected graph.

Theorem 8.8 (Balinski)**.** *The graph Γ of a d-dimensional polytope P is d-connected.*

Proof. Let V be a set of vertices in Γ and let $X = \{x_1, \ldots, x_{d-1}\}$ be any subset of $d-1$ vertices. We need to prove that $\Gamma|_{V \smallsetminus X}$ is connected. Fix a vertex $z \in V \smallsetminus X$ and let $\psi : \mathbb{R}^d \to \mathbb{R}$ be a linear function which is constant on $X + z$:

$$\psi(x_1) = \ldots = \psi(x_{d-1}) = \psi(z) = c,$$

for some $c \in \mathbb{R}$. The set of vertices $v \notin (X + z)$ is now split into two subsets: $V_+ = \{v \in V \mid \psi(v) > c\}$ and $V_- = \{v \in V \mid \psi(v) < c\}$. There are two cases:

1) One of the sets V_- or V_+ is empty. Say, $V_- = \varnothing$. Perturb ψ to obtain a Morse function $\varphi : \mathbb{R}^d \to \mathbb{R}$ (i.e., a function which is nonconstant on the edges of P). Orient all edges in Γ, from a vertex with smaller to a vertex with larger value of φ, and denote by w_+ the maximal vertex. Now observe that for every vertex $v \in V_+ - w_+$ there is an edge coming out of v. Continue walking along the path of increasing edges until we eventually reach the unique maximum w_+. Similarly, since $|X| < d$, vertex z also has an edge connecting it to V_+. Therefore, all vertices in $V \smallsetminus X$ are connected to w_+, which finishes the proof of this case.

[20] While the implication is obvious in this case, it can be viewed as a special case of a general 'local $\Rightarrow$ global' principle. In a very similar context this idea also appears in the proof of Lemmas 14.7 and 35.1.

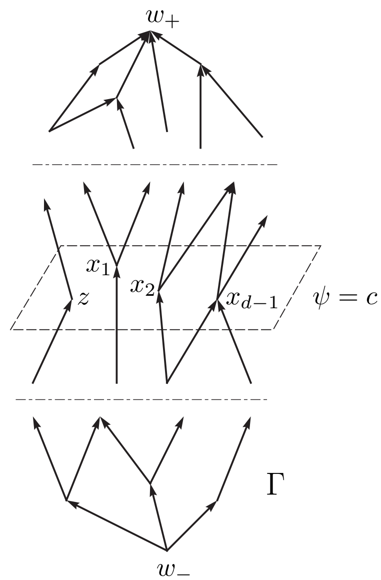

FIGURE 8.4. An acyclic orientation in the proof of Balinski's theorem.

2) Suppose $V_+, V_- \neq \varnothing$. Again, perturb ψ to obtain a Morse function $\varphi : \mathbb{R}^d \to \mathbb{R}$. By the argument above, all vertices in V_+ are connected to the maximum w_+ and all vertices in V_- are connected to the minimum w_-. It suffices to show that z is connected to both V_+ and V_-, since that would make all vertices connected (see Figure 8.4).

Suppose that z does not have an edge connecting it to V_-. Making a small perturbation φ' of ψ, we can make z the minimum in $X + z$. By the assumption, all edges adjacent to z either connect it with X or V_+. Therefore, all edges adjacent to z are increasing, and z is the minimum in V. On the other hand, we still have $\varphi'(v) < \varphi'(z)$ for all vertices $v \in V_-$, a contradiction. This completes the proof. $\square$

8.5. **Exercises.**

Exercise 8.1. $\diamondsuit$ [1-] Denote by $\mathcal{F}_d \subset \mathbb{R}^{d+1}$ the subspace spanned by all f-vectors of convex polytopes in $\mathbb{R}^d$. Prove that the Dehn–Sommerville equations form a basis in the dual to the quotient space $\mathbb{R}^{d+1}/\mathcal{F}_d$. In other words, prove that the equations are linearly independent and imply all linear equations which hold for all f-vectors.

Exercise 8.2. (*Associahedron*) $\diamondsuit$ *a*) [1] Prove that the associahedron R_n defined in Example 8.5, is $(n-3)$-dimensional.
b) [1] Prove that the edges of R_n correspond to diagonal flips.
c) [1] Give a complete combinatorial description of the facets of R_n.

Exercise 8.3. $\diamondsuit$ [1-] Let $P \subset \mathbb{R}^d$ be a convex polytope and let Γ be its graph. Suppose x, y are vertices of P lying on the same side of a hyperplane L. Then there exists a path from x to y in Γ that does not intersect L.

Exercise 8.4. $\lozenge$ [1+] Let $P \subset \mathbb{R}^d$ be a convex polytope with n vertices. For a vertex $v \in P$ denote by C_v the infinite cone with a vertex at v, which is spanned by the edges of P containing v. Prove that P is the intersection of at most $n-d+1$ cones C_v. Deduce from here the Balinski's theorem (Theorem 8.8).

Exercise 8.5. Let Γ be a graph of a convex polytope $P \subset \mathbb{R}^d$.
a) [1-] Prove that if $P \subset \mathbb{R}^3$ is simplicial, then the edges of P can be colored with two colors so that between every two vertices there are monochromatic paths of both colors.
b) [1-] Find all simple polytopes $P \subset \mathbb{R}^3$ for which such a 2-coloring exists.
c) [1] Generalize part *a*) to simplicial polytopes in $\mathbb{R}^d$ and $d-1$ colors.

Exercise 8.6. [1] Let $P \subset \mathbb{R}^d$ be a convex polytope and let Γ be its graph. Prove that there is an *embedding* of K_{d+1} into Γ, i.e., there exist $d+1$ different vertices in P every two of which are connected by pairwise disjoint paths in Γ.

Exercise 8.7. [1] Let $P \subset \mathbb{R}^3$ be a convex polytope with even-sided faces. Prove or disprove: the edges of P can be 2-colored so that every face has an equal number of edges of each color.

Exercise 8.8. *a*) $\lozenge$ [1-] Prove that every convex polytope $P \subset \mathbb{R}^3$ has a vertex of degree at most 5.
b) [2-] Prove that every convex polytope $P \subset \mathbb{R}^3$ has two adjacent vertices whose total degree is at most 13.
c) [2-] Prove that every convex polytope $P \subset \mathbb{R}^3$ with even-sided faces has two adjacent vertices whose total degree is at most 8.
d) [1] Show that the upper bounds in part *b*) and *c*) are sharp.

Exercise 8.9. *a*) [1] Let $P \subset \mathbb{R}^3$ be a convex polytope and let L be the sum of edge lengths of P. Prove that $L \geq 3\,\mathrm{diam}(P)$.
b) [1+] Let Γ be the graph of a convex polytope $P \subset \mathbb{R}^3$ with n vertices. Prove that the diameter of Γ is at most $(n-1)/3$. Show that this inequality is sharp.

Exercise 8.10. Let Γ be the graph of a simple convex polytope $P \subset \mathbb{R}^3$ with triangular and hexagonal faces.
a) [1-] Prove that P has exactly four triangular faces.
b) [2-] Prove that the number of hexagonal faces P is even.
c) [2] Let $P \subset \mathbb{R}^3$ be a simple polytope, whose faces have $3, 6, 9, \ldots$ sides. Prove that the number of faces of P is even.

Exercise 8.11. [1-] Let $Q \subset \mathbb{R}^2$ be a closed (possibly self-intersecting) polygonal curve, such that all angles are $< \pi$. Fix a Morse function, i.e., a linear function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ which is non-constant on the edges of Q. Prove that Q is a convex polygon if and only if φ has a unique local maximum.

Exercise 8.12. (*Neighborly polytopes*) [1] Construct explicitly a convex polytope $P \subset \mathbb{R}^4$ with 6 vertices and $\binom{6}{2} = 15$ edges. In general, we say that a polytope is *neighborly* if its graph is a complete graph. For all $d \geq 4$ construct a neighborly polytope in $\mathbb{R}^d$ with $d+2$ vertices.

Exercise 8.13. [1] Let $P \subset \mathbb{R}^d$ be a convex polytope with $n \geq d+2$ vertices. Prove that there exists a vertex $v \in P$ and two facets, such that $v \notin F_1, F_2$. Moreover, we can make F_1 and F_2 *adjacent*, which means that their intersection must be a $(d-2)$-dimensional face.

Exercise 8.14. [1] Let $P \subset \mathbb{R}^d$ be a convex polytope with $n \geq d+2$ vertices. Prove that for every k, $1 \leq k \leq d-1$, there exist a k-face and a $(d-k)$-dimensional face of P which are disjoint.

Exercise 8.15. Consider all cross sections of a convex polytope $P \subset \mathbb{R}^3$ which do not contain any vertices.
a) [1-] Can all these cross sections be triangles?
b) [1-] Can all these cross sections be quadrilaterals?
c) [1-] Can all these cross sections be odd-sided polygons?
d) [1] Suppose all vertices of P have even degree. Prove that every cross section of P is an even-sided polygon.

Exercise 8.16. Denote by $\alpha(\Gamma)$ the length of the longest cycle in the graph Γ (cycles do not have repeated vertices).
a) [1-] Find a convex polytope P with n vertices, such that the graph $\Gamma = \Gamma(P)$ has $\alpha(\Gamma) < n$.
b) [1+] Construct a sequence of convex polytopes $\{P_k\}$ with graphs $\Gamma_k = \Gamma(P_k)$ on n_k vertices, such that $n_i \to \infty$, and $\alpha(\Gamma_k) = O(n_k^{1-\varepsilon})$ as $k \to \infty$, for some $\varepsilon > 0$.

Exercise 8.17. (*Eulerian numbers*) $\Diamond$ In the notation of Example 8.4, let $A(n,k)$ be the number of permutations $\sigma \in S_n$ with k ascents. The integers $(A_0, A_1, \ldots, A_{n-1})$ are called *Eulerian numbers.*
a) [1-] Prove that Eulerian numbers satisfy:

$$A(n,k) = (n-k)A(n-1,k-1) + (k+1)A(n-1,k).$$

b) [1] Prove that the Eulerian numbers satisfy

$$A_k = n!\,\mathrm{vol}(Q_{n,k}),$$

where $Q_{n,k} \subset \mathbb{R}^n$ is a convex polytope defined by the inequalities

$$k \leq x_1 + \ldots + x_n \leq k+1, \qquad 0 \leq x_i \leq 1,\ 1 \leq i \leq n.$$

c) [1] Observe that that the Eulerian numbers satisfy

$$A(n,k) = n!\,\mathrm{vol}(P_{n,k}),$$

where $P_{n,k} \subset \mathbb{R}^n$ is a union of simplices $0 \leq x_{\sigma(1)} \leq \ldots \leq x_{\sigma(n)} \leq 1$, over all $\sigma \in S_n$ with exactly k ascents. Find an explicit piecewise-linear volume-preserving map $\varphi : Q_{n,k} \to P_{n,k}$. This gives an alternative proof of *b*).

Exercise 8.18. (*Narayana numbers*) $\Diamond$ In the notation of Example 8.5, let $B_{n,k}$ be the number of binary trees with n vertices and k left edges. Numbers $B_{n,k}$ are called *Narayana numbers.*
a) [1-] Use induction to prove a closed formula:

$$B_{n,k} = \frac{1}{n}\binom{n}{k}\binom{n}{k+1}.$$

b) [1] Use binomial identities to obtain a closed formula for the f-vector of associahedron R_n:

$$f_i = \frac{1}{n-d-2}\binom{n-3}{n-d-3}\binom{2n-d-4}{n-d-3}.$$

c) [1-] Use induction to give an alternative proof of this formula.

Exercise 8.19. $\Diamond$ The *Birkhoff polytope* B_n is defined by the following equations and inequalities:

$$\sum_{i=1}^{n} a_{i,j} = \sum_{j=1}^{n} a_{i,j} = 1, \quad \text{and } a_{i,j} \geq 0, \quad \text{for all } 1 \leq i, j \leq n.$$

a) [1-] Prove that B_n has dimension $(n-1)^2$.
b) [1] Prove that B_n has $n!$ vertices which correspond to permutation matrices.
c) [1] Prove that the edges of B_n correspond to multiplication of permutations by single cycles. Conclude the graph of B_n has diameter two, and the vertices have degree $\theta\big((n-1)!\big)$.
d) [1-] Check that B_3 is both neighborly (see Exercise 8.12) and simplicial, and that B_n, $n \geq 4$, are neither.
e) [1-] Compute $\operatorname{vol}(B_3)$.
f) [2-] Let G be a subgraph of $K_{n,n}$. We say that G is *elementary* if every edge of G belongs to some perfect matching in G. Prove that the poset of elementary subgraphs of $K_{n,n}$ ordered by inclusion is isomorphic to the face lattice of B_n.
g) [2-] Describe the f-vector of B_n in terms of elementary graphs.

Exercise 8.20. $\Diamond$ Define the *transportation polytope* $T_{m,n}$ by the following equations and inequalities:

$$\sum_{i=1}^{m} a_{i,j} = m, \quad \sum_{j=1}^{n} a_{i,j} = n, \quad \text{and } a_{i,j} \geq 0, \quad \text{for all } 1 \leq i \leq m, \ 1 \leq j \leq n.$$

a) [1-] Prove that $T_{m,n}$ has dimension $(m-1)(n-1)$.
b) [2-] Prove that $T_{m,n}$ is simple for all $(m,n) = 1$.
c) [1] Show that $T_{n,n+1}$ has $(n+1)^{n-1} n!$ vertices. To do that, give a bijective proof between these vertices of $T_{n,n+1}$ and labeled trees on $n+1$ vertices, where both vertices and edges are labeled, and use Cayley's formula.
d) [1] Find a complete description of the edges of $T_{n,n+1}$ in terms of these labeled trees.
e) [1+] Use the Morse function approach to compute h_1 and h_2 of the simple polytope $T_{n,n+1}$. From here, obtain $f_2\big(T_{n,n+1}\big)$.

Exercise 8.21. (*Klyachko lemma*) [1+] Let $P \subset \mathbb{R}^3$ be a convex polytope. Fix $\varepsilon > 0$. Suppose on every facet F in P, there is a particle which moves clockwise along the edges of F with the speed at least ε. Prove that at some point some two particles will collide.

Exercise 8.22. (3^d-*conjecture*) Let $P \subset \mathbb{R}^d$ be a centrally symmetric convex polytope with f-vector $(f_0, f_1, \ldots, f_d)$.
a) [1+] For $d = 3$, prove that $f_0 + f_1 + f_2 + f_3 \geq 27$.
b) [2+] For $d = 4$, prove that $f_0 + f_1 + f_2 + f_3 + f_4 \geq 81$.
c) [$*$] Prove that $f_0 + f_1 + f_2 + \ldots + f_d \geq 3^d$, for all $d \geq 1$.

8.6. **Final remarks.** The generalization ($\heartsuit$) of Euler's formula is due to Poincaré (1897). The Dehn–Sommerville equations (Theorem 8.1) in dimensions 4 and 5 were discovered by Dehn (1905) and generalized to all dimensions by Sommerville (1927). Note that in the original form, the equations gave a different basis in the affine space spanned by the f-vectors of all simple polytopes in $\mathbb{R}^d$ (cf. Exercise 8.1). Our proof is standard and can be found in [Brø] (see also [Barv, § Vi.6]), while a traditional proof can be found in [Grü4, §9.2]. See also [Zie1, §8.3] for the dual treatment (called *shelling*) of simplicial (rather than simple) polytopes, as this approach is connected to further combinatorial questions and ideas.

The proof of Theorem 8.6 is essentially the same as Kalai's original argument [Kal1] (see also [Zie1, §3.4]). Our proof of Balinski's theorem (Theorem 8.8) is a variation on the original proof [Bali] and the argument in [Bar3] (see also [Zie1, §3.5] and Exercise 8.4), but the use of Morse functions allows us to avoid some unnecessary technicalities.

Eulerian numbers appear in numerous instances in enumerative combinatorics; it is one of the two major *statistics* on permutations.[21] We refer to [Sta3] for other appearances of Eulerian numbers, Catalan numbers and polygon subdivisions. The associahedron is also called the *Stasheff polytope*; in our definition we follow [GKZ, Chapter II.7]. The calculations in the examples are partly ours and partly folklore. For more on permutohedron and associahedron see [EKK, Zie1] and references therein. See also Section 14 for more on triangulations.

Finally, let us mention that Morse functions are basic tools in *Morse theory*, a classical branch of topology. We refer to [For] for a friendly treatment of the discrete case and further references.

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[21]This statistic is also called *Eulerian*. The other major statistic on S_n, by the number of inversions in permutations, is called *Mahonian*, after British combinatorialist Percy MacMahon.

9. Center of mass, billiards and the variational principle

This is our first introduction to the variational principle approach. After several elementary results on the center of mass of convex polytopes, we explore closed billiards trajectories in smooth convex bodies and in convex polygons. These results lead the way to the analysis of closed geodesics on convex polytopes in the next section. The variational principle approach will also be used throughout the book.

9.1. How to roll a polytope.

Here is a basic test whether your intuition is combinatorial/algebraic or physical/geometric: try proving the following result.

Theorem 9.1. *Let $P \in \mathbb{R}^d$ be a convex polytope, and let $z \in P$ be any point. Prove that there exists a facet F such that an orthogonal projection of z onto F lies* inside *F.*

The physical intuition gives an easy "solution": think of z as the center of mass of P. If not, use inhomogeneous material in P to make it so. Drop the polytope on a (hyper)plane. If the orthogonal projection of the center of mass is not inside the bottom facet F of the polytope, it will start rolling. Since it cannot roll indefinitely, the polytope eventually stabilizes, and we are done.

In order to convert the above argument into a honest proof we need to have a basic understanding of the physical nature of rolling. The following will suffice for the intuition: the height of center of mass drops at each turn. Now that we know what to look for, we can create a simple "infinite descent" style proof:

Proof. Let F be the facet such that the distance from z to a hyperplane spanned by F is the smallest. Suppose an orthogonal projection w of z on a hyperplane H containing F does not lie inside F. Let A be any $(d-2)$-dimensional face of F such that the subspace spanned by A separates H into two parts, one containing F and the other point w. Denote by $F' \neq F$ the other facet containing A. Observe that the distance from z to F' is smaller than the distance from z to F. This is clear from the 2-dimensional picture on an orthogonal projection along A (see Figure 9.1). This contradicts the assumptions. $\square$

Figure 9.1. A polytope that is about to roll twice; projections of z onto facets.

9.2. **Meditation on the rules and the role of rolling.** Let us further discuss the merits and demerits of the above "physical proof" vs. the formal proof of the admittedly easy Theorem 9.1. Basically, the "physical proof" comes with an algorithm for finding a desired facet F: start at any facet F_0 and move repeatedly to an adjacent facet that is closer to z, until this is no longer possible. This algorithm, of course, can be formalized, but some questions remain.

First, on the complexity of the algorithm. Clearly, not more than $f_{d-1}(P)$ (the total number of facets) of rolls is needed. In fact this is tight up to an additive constant[22]. On the other hand, the algorithm does not necessarily finds the closest facet used in the formal proof.

Now, note that the minimum plays a special role in the proof and cannot be replaced by the maximum. The situation is strikingly different in the smooth case. We will restrict ourselves to the dimension two case.

Let $C \subset \mathbb{R}^2$ be a smooth closed curve in the plane and let O be the origin. Throughout the section we will always assume that $O \notin C$. For a point $x \in C$ we say that the line segment (O, x) is *normal* if it is orthogonal to C at point x.

Proposition 9.2. *Every smooth closed curve C has at least two normals from every point $O \notin C$.*

Proof. Let $f(x)$ denote the distance from $x \in C$ to the origin O. Take any (local or global) extremal point x (minimum or maximum) of f. Observe that (O, x) is normal, since otherwise close points on the side with angle $> \pi/2$ are at distance $> |Ox|$, and on the side with angle $< \pi/2$ are at distance $< |Ox|$ (see Figure 9.2). $\square$

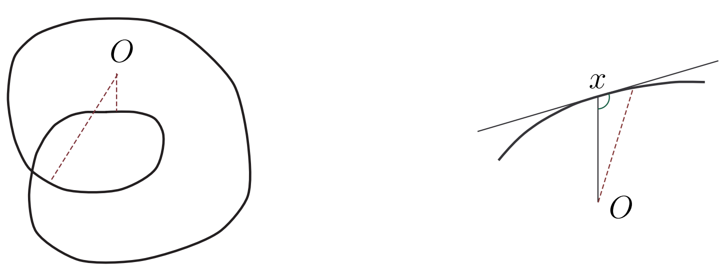

FIGURE 9.2. Normal points on a curve.

Note that we already used this argument in the proof of Theorem 3.3 (compare Figures 3.3 and 9.2). One can again give similar 'physical motivation': when C is convex, think of it as a two-dimensional 'die' filled to make any interior point its center of mass. Place it on a line and let it roll. The (local) minima and maxima correspond to stable and unstable equilibrium points, respectively.

Since we are really interested in discrete geometry, let us give a discrete version of Proposition 9.2. Let $Q \subset \mathbb{R}^2$ be a polygon in the plane, possibly with self-intersections. As before, we assume that $O \notin Q$. An interval (O, x), $x \in Q$, is

[22]Try to find an example in any fixed dimension!

called *quasi-normal* if the function $|Ox|$ is a local minimum or maximum at x.[23] When x lies on an edge e of Q this means that (O, x) is orthogonal to e, so in fact point x is normal. When x is a vertex of Q incident to edges e, e', this means that the angles between (O, x) and e, e' are either both acute or both obtuse. We immediately have the following result.

Proposition 9.3. *For every plane polygon Q, there exist at least two quasi-normals to Q from every point $O \notin Q$.*

Example 9.4. Proposition 9.3 implies that a convex polygon containing the origin has at least one normal and at least one quasi-normal. Figure 9.3 shows that these bounds cannot be improved.

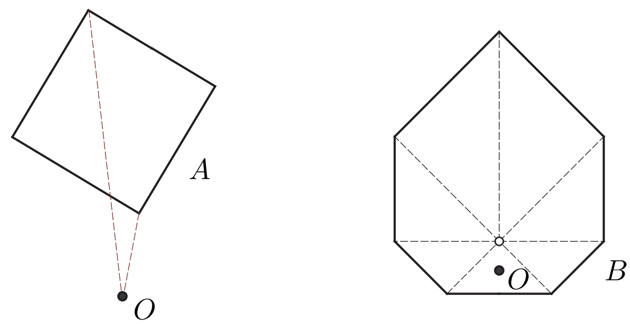

FIGURE 9.3. Square A and point $O \notin A$ with two quasi-normals; heptagon B and point $O \in B$ with one normal and one quasi-normal.

9.3. **Center of mass of polygons.** As the example above shows, there exists a convex polygon Q and an interior point O with a unique normal. The following result shows that this is impossible when O in the center of mass of the region inside Q.

Theorem 9.5. *Let $Q = \partial A$ be a convex polygon, $A \subset \mathbb{R}^2$, and let $z = \mathrm{cm}(A)$ be its center of mass. Then there exist at least two normals and at least two other quasi-normals from z to Q.*

In other words, if a convex polygon Q is filled uniformly and rolled on a line, then it can stand on two or more edges and has an unstable equilibrium at two or more vertices. The proof is based on the following elementary lemma.

Lemma 9.6. *Let $X, Y \subset \mathbb{R}^2$ be two convex sets of equal area such that their centers of mass coincide:* $\mathrm{cm}(X) = \mathrm{cm}(Y)$. *Then ∂X and ∂Y intersect in at least four points.*

Proof. Clearly, since $\mathrm{area}(X) = \mathrm{area}(Y)$, there are at least two points of intersection in $\partial X \cap \partial Y$. Assume that a and b are the only intersection points. Let $Z = X \cap Y$, $X' = X - Y$, $Y' = Y - X$. Observe that

$$\mathrm{cm}(X) \;=\; \alpha\, \mathrm{cm}(Z) \;+\; (1-\alpha)\, \mathrm{cm}(X'),$$

[23] Note that the notion of a quasi-normal can be applied to *all* convex sets, not just convex polygons. All results extend verbatim in this case. We will not use this generalization.

$$\mathrm{cm}(Y) \;=\; \alpha\,\mathrm{cm}(Z) \;+\; (1-\alpha)\,\mathrm{cm}(Y'),$$

where

$$\alpha \;=\; \frac{\mathrm{area}(Z)}{\mathrm{area}(X)} \;=\; \frac{\mathrm{area}(Z)}{\mathrm{area}(Y)}\,.$$

Since $\mathrm{cm}(X) = \mathrm{cm}(Y)$, we conclude that $\mathrm{cm}(X') = \mathrm{cm}(Y')$ which is impossible since X' and Y' lie on different sides of the line ℓ through a and b (see Figure 9.4). $\square$

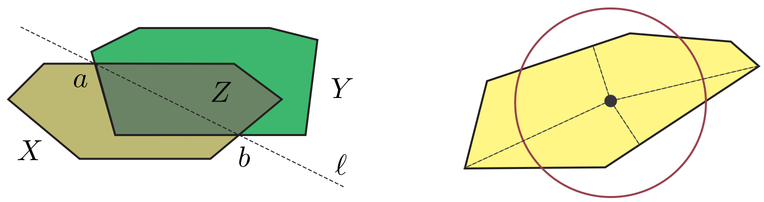

FIGURE 9.4. Two polygons of equal area with two boundary intersection points; a polygon and a circle of equal area with the same center of mass.

In a special case, the lemma gives the following corollary.

Corollary 9.7. *Let $X \subset \mathbb{R}^2$ be a convex set and let C be a circle centered at the center of mass* $\mathrm{cm}(X)$ *with area equal to the area of X. Then C and ∂X intersect in at least four points.*

Proof of Theorem 9.5. Let $Q = \partial A$ be a convex polygon as in the theorem and let C be a circle of equal area centered at $z = \mathrm{cm}(A)$ as in the corollary. Since there are at least four points of intersection of Q and C, the distance function $f(x) = |zx|$ has at least two different local maxima and two different local minima (see Figure 9.4). This implies the result. $\square$

Observe that by analogy with the previous section, we can now easily adapt the results to the smooth case:

Theorem 9.8. *Let $A \subset \mathbb{R}^2$ be a convex region bounded by a smooth convex curve $C = \partial A$, and let $z =$ $\mathrm{cm}(A)$ be its center of mass. Then there exist at least four normals from z to C.*

Example 9.9. (*Monostatic polytopes*) One can ask whether Theorem 9.5 remains true in higher dimensions. As it happens, it does not. Homogeneously filled polytopes in $\mathbb{R}^3$ which are stable on only one face are called *monostatic*. Here is the construction of one such polytope. Take a long cylinder with symmetrically slanted ends so that the resulting body can be static only in one position as in Figure 9.5. One can use a polyhedral approximation of the cylinder to obtain a monostatic polytope (see Exercise 9.12). Interestingly there are no monostatic tetrahedra in $\mathbb{R}^3$, but in higher dimensions there exist monostatic simplices (see Exercise 9.10).

In a different direction, let us mention that there is no natural generalization of Lemma 9.6 to higher dimensions. Namely, there exist convex polytopes P and P' of equal volume, such

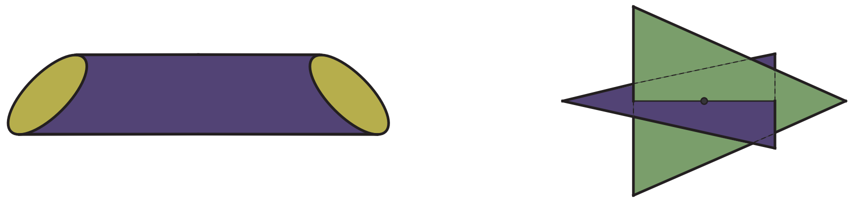

Figure 9.5. Monostatic slanted cylinder and two intersecting triangles.

that $\mathrm{cm}(P) = \mathrm{cm}(P')$ and the intersection $P \cap P'$ has only one connected component. For example, take two congruent triangles in orthogonal planes and arrange them as shown in Figure 9.5. Now replace triangles with thin triangular prisms.

9.4. **Special cuts of special cakes.** In the previous subsection we showed that for every smooth convex curve $C \subset \mathbb{R}^2$, there exists a point (the center of mass of the interior region) from which there exist at least four normals to C. On the other hand, as the smoothened version of the heptagon in Figure 9.3 shows, not all points z in the interior must have four normals to C. In fact, in most cases, there are infinitely many points which have four or more normals (cf. Exercise 9.7). Below we find another point z with four normals, special in its own way.

Let $C \subset \mathbb{R}^2$ be a smooth closed curve in the plane. A *cut* is an interval connecting two points on a curve. We say that a cut (x, y) is a *double normal* if C is orthogonal to (x, y) at both ends. Think of the region inside C as a cake and of double normals as of a way to cut the cake.

Theorem 9.10 (Double normals)**.** *Every smooth closed curve in the plane has at least two double normals.*

Let us discuss the result before proving it. First, finding one double normal is easy: take the most distant two points on a curve. By Proposition 9.2, this cut must be orthogonal to the curve at both ends. However, finding the second double normal is less straightforward. Even though it is intuitively clear that it has to correspond to some kind of minimum rather than maximum, one needs care setting this up.

Figure 9.6. Two double normals and a family of parallel cuts $K \in F_\theta$.

Proof. Denote by $C \subset \mathbb{R}^2$ the curve in the theorem. Consider families F_θ of cuts K of C, with the slope $\theta \in [0, 2\pi]$. Define

$$(\sharp) \quad c = \min_{\theta \in [0,2\pi]} \max_{K \in F_\theta} |K|,$$

where $|K|$ denotes the length of the cut. By compactness, the min max condition is achieved on some cut $K = (x, y)$, i.e., there exists an angle θ and a cut $K \in F_\theta$ such that $|K| = c$. We claim that this cut is a double normal.

First, observe that the incidence angles of (x, y) are the same at both ends, i.e., the tangent lines L_x and L_y at x and at y are parallel. Indeed, otherwise consider a parallel cut $K' \in F_\theta$ shifted in the direction where the sum of the angles is $> \pi$ (see Figure 9.7). By the smoothness of C, for small shifts the length $|K'| > |K|$, a contradiction.

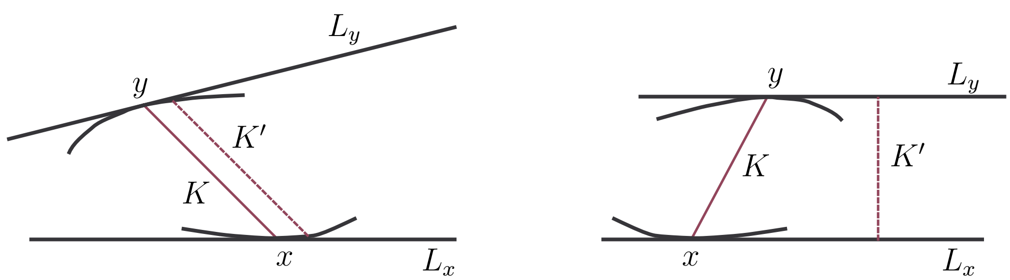

FIGURE 9.7. Checking whether cut K is a double normal.

Now suppose the tangent lines L_x and L_y are parallel. By convexity, the whole curve C lies between these lines. If $K = (x, y)$ is not orthogonal to the curve, take a family of cuts $F_{\theta'}$ of cuts perpendicular to the lines. Clearly, every cut, in particular the maximal cut $K' \in F_{\theta'}$, has length $< \ell(K)$. Therefore, cut K does not satisfy the min max condition of $(\sharp)$, a contradiction. $\square$

There is a simpler way to think of the proof: define width(C) to be the smallest distance between parallel lines which touch C on different sides. Then one can consider two such lines and proceed as in the last part of the proof above. However, the advantage of the longer proof is the insight it gives, which will be used in the next section. Let us mention also the following straightforward discrete analogue of the double normals theorem.

Theorem 9.11. *Every convex polygon in the plane has at least two double quasi-normals.*

The proof follows verbatim the proof of Theorem 9.10. Of course, unless the polygon has parallel edges, it does not have double normals.

9.5. **Playing pool on convex tables.** One can think of the double normals as of a billiard shot which returns back after two bounces off the curve boundary. Naturally, one can ask for more general shots with three or more bounces. Formally, a polygon inscribed into a smooth curve C is called a *periodic billiard trajectory* if the incidence angles of the adjacent sides are equal (see Figure 9.8). Note that given one periodic billiard trajectory, one can take a *power* of such trajectory, repeating it several times. Trajectories which cannot be obtained that way are called *irreducible.*

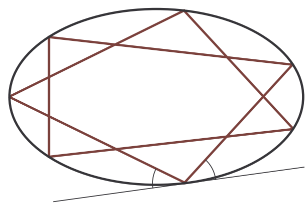

FIGURE 9.8. A periodic billiard trajectory.

Theorem 9.12 (Birkhoff)**.** *Every smooth convex curve has infinitely many irreducible periodic billiard trajectories.*

Proof. Suppose C is a smooth convex curve and let $n \geq 3$ be a prime. Let us show that C has a periodic billiard trajectory with exactly n sides. We use maximization over all polygons inscribed into C.

Denote by $\mathcal{R}_n$ the set of all (possibly self-intersecting) k-gons inscribed into C, with $k \leq n$. Let

$$(\circledast) \quad c = \max_{R \in \mathcal{R}_n} \ell(R),$$

where $\ell(R)$ is the sum of edge lengths of R. We claim that the maximum is achieved on an irreducible periodic billiard trajectory R with exactly n sides. First, note that $\mathcal{R}_n \subset C^n$ is a compact in $\mathbb{R}^{2n}$, so the maximum is well defined. Second, the maximum is achieved on an n-gon, since otherwise one can replace one edge (x, y) with any two edges (x, z) and (z, y), increasing the length of a polygon R. Finally, since n is prime, the maximum is not a power of a smaller trajectory.

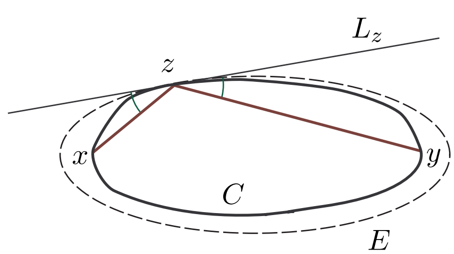

FIGURE 9.9. Edges (x, z) and (z, y) of the inscribed polygon R.

Now, for every two adjacent sides (x, z) and (z, y) of the polygon R, we will show that the incidence angles with the curve are equal. Indeed, denote by E the ellipse with x and y as its focal points and z on the boundary. In other words, ellipse E is the set of points $w \in \mathbb{R}^2$ with $|xw| + |wy| = |xz| + |zy|$, and for all points v outside E we have $|xv| + |vy| > |xz| + |zy|$. Since the polygon R is maximal, the ellipse E must contain the curve C; otherwise we can replace (x, z) and (z, y) with some (x, v) and (v, y). Since C is smooth, this implies that E and C must be touching each other at $z \in C, E$ with the same tangent line L_z. The result follows from the equal angle property of ellipses (see Figure 9.9). $\square$

Theorem 9.13 (Birkhoff)**.** *For every smooth convex curve $C = \partial X$ and every two points $x, y \in X$, there exist infinitely many distinct billiard trajectories from x to y.*

This result is formally not a corollary from Theorem 9.12, but follows directly from the argument in the proof above (see Exercise 9.3).

9.6. **Playing pool with sharp corners.** There are several ways to define billiard trajectories in polygons, since it is unclear what happens if the trajectory enters a vertex (the edge reflections are defined as before). One way is to simply forbid the trajectory to go through the vertices. We call this the (usual) *billiard trajectory* in a polygon.

Conjecture 9.14. *Every convex polygon has at least one periodic billiard trajectory.*

This conjecture is classical and holds in many special cases (see Exercises 9.13 and 9.14), but is open even for triangles.[24] Here is another closely related conjecture:

Conjecture 9.15. *For every convex polygon $X \subset \mathbb{R}^2$ and every two points $x, y \in X$, there exists infinitely many distinct billiard trajectories from x to y.*

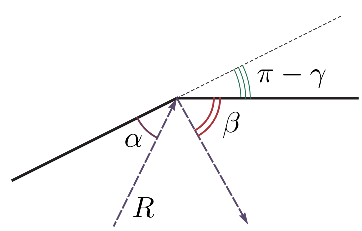

FIGURE 9.10. Angles of reflection in a quasi-billiard trajectory R.

Let us consider now another variation on the theme which will prove useful in the next section. A *quasi-billiard trajectory* in a convex polygon is defined by the following rule: when a trajectory enters a vertex of angle γ, it satisfies

$$(\angle) \qquad |\alpha - \beta| \leq \pi - \gamma,$$

where α and β are the reflection angles as in Figure 9.10. The following analogues of Birkhoff theorems follows along the same lines as the proof of Theorem 9.12.

[24]For triangles with angles at most 100 degrees this was proved by R. E. Schwartz (2005).

Theorem 9.16. *For every convex polygon $Q \subset \mathbb{R}^2$, there exists a periodic quasi-billiard trajectory inscribed into Q with exactly n sides, for every $n \geq 3$.*

Theorem 9.17. *For every convex polygon $Q = \partial A$, $A \subset \mathbb{R}^2$, and every two points $x, y \in A$, there exist infinitely many quasi-billiard trajectories from x to y.*

Note that the Birkhoff maximization construction when applied to polygons can never produce the usual billiard trajectories since the maximum will always be achieved at vertices. In other words, both theorems can be restated as existence results of vertex sequences which satisfy angle conditions as in $(\angle)$.

9.7. **Exercises.**

Exercise 9.1. [1-] *a*) Let $Q \subset \mathbb{R}^2$ be a closed polygon and let e be the longest edge. Prove that there exists a vertex $v \in Q$ which projects onto e.
b) [1-] Deduce part *a*) from the proof of Theorem 9.11.
c) [1] Find a convex polytope $P \subset \mathbb{R}^3$ such that for every face F and vertex v of P, $v \notin F$, vertex v projects outside of F.

Exercise 9.2. *a*) [1-] Prove that every smooth convex curve $C \subset \mathbb{R}^2$ has an inscribed triangle $\Delta = (xyz)$, such that a line tangent to C at each of these points is parallel to the opposite edge of Δ. For example, a line tangent to C at x must be parallel to (yz).
b) [1-] Show that every convex polygon $Q \subset \mathbb{R}^2$ has three vertices which form a triangle Δ, such that the line through either vertex of Δ parallel to the opposite edge is supporting Q.
c) [1] Generalize *a*) and *b*) to higher dimensions.

Exercise 9.3. $\diamondsuit$ *a*) [1-] Prove Theorem 9.13.
b) [1] Give a direct proof of Theorem 9.16.
c) [1] Give a direct proof of Theorem 9.17.
d) [1] Deduce Theorem 9.12 from Theorem 9.16 by a limit argument.
d) [1] Deduce Theorem 9.16 from Theorem 9.12 by a limit argument.

Exercise 9.4. Prove or disprove:
a) [1-] Every smooth simple closed curve in $\mathbb{R}^3$ has a plane which touches it at three or more points.
b) [1-] Every simple space polygon $Q \subset \mathbb{R}^3$ has a plane L which contains three or more vertices of C, such that all edges adjacent to these vertices either lie on L or on the same side of L.

Exercise 9.5. Let $X \subset \mathbb{R}^2$ be a convex set. We say that a point $z \subset X$ is *central* if every chord through z is divided by z inside X with a ratio at most $2:1$.
a) [1-] Prove that the center of mass $\mathrm{cm}(X)$ is central.
b) [1] Let Δ be a triangle inscribed into X with maximal area. Prove that $\mathrm{cm}(\Delta)$ is central.

Exercise 9.6. [2-] Let $Q = \partial A$ be a convex polygon, $A \subset \mathbb{R}^2$, and let $z \in A$ be an interior point. A triple of quasi-normals from z onto Q is called a *tripod at z* if they form an angle of $2\pi/3$ with each other (see Figure 9.11). Prove that every convex polygon has a tripod at some interior point.

Exercise 9.7. *a*) [1] Let $Q = \partial A$, $A \subset \mathbb{R}^2$, be a convex polygon. Denote by $x = \mathrm{cm}(Q)$ the center of mass of the curve with the uniform weights. Prove that there exist at least four quasi-normals from x onto Q.

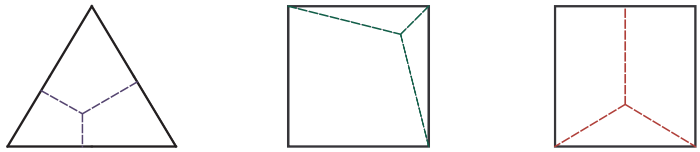

FIGURE 9.11. Tripods in an equilateral triangle and a square.

b) [1] Let V be the set of vertices of Q with the weights proportional to exterior angles $(\pi - \angle x)$. Denote by $y = \mathrm{cm}(V)$ the center of mass of the vertices with the weights as above. Prove that there exist at least four quasi-normals from y onto Q.
c) [1+] Let $B_r \subset \mathbb{R}^2$ a disk of radius r and let $X_r = X + B_r$, for all $r > 0$. Denote by $x_r = \mathrm{cm}(\partial X_r)$, $z_\varepsilon = \mathrm{cm}(X_r)$ the centers of mass as above. Prove that

$$\lim_{r\to\infty} x_r = \lim_{r\to\infty} z_r = y.$$

d) [1] Prove that there exist at least four normals from x_r onto Q, for all $r > 0$. Same for all z_r.

Exercise 9.8. Let $C \subset \mathbb{R}^2$ be a smooth convex curve.
a) [1+] Denote by x and y midpoints of the largest cut and the $\min\max$ cut (as in the proof of Theorem 9.10). Prove that there exist at least four normals from x onto C. Same for y.
b) [2-] Prove that the circle is the only convex curve with a unique point with four normals.

Exercise 9.9. Let $C \subset \mathbb{R}^2$ be a smooth convex curve. Denote by $\Pi(C)$ the set of interior points with exactly four normals to C.
a) $\diamondsuit$ [1] Compute $\Pi(C)$ for an ellipse C.
b) [1+] Prove or disprove: $\Pi(C)$ is always connected.

Exercise 9.10. (*Stability of simplices*) a) [2-] Prove that for every tetrahedron $\Delta \subset \mathbb{R}^3$ the center of mass $\mathrm{cm}(\Delta)$ projects onto at least two faces of P.
b) [1-] Construct a tetrahedron Δ which when placed on a plane may roll twice.
c) [1] Construct a tetrahedron Δ and an interior point $O \in \Delta$ such that O projects only on one of the faces.
d) [2-] Prove that for d large enough, there exists a d-simplex $\Delta \subset \mathbb{R}^d$ which can stand only on one facet, i.e., such that $\mathrm{cm}(\Delta)$ projects onto only one face of Δ.
e) [2] Prove that the dimension d in the previous part can be as small as 11, and must be at least 8.

Exercise 9.11. [1-] Find a parallelepiped where all edges have equal lengths, such that it can stand only on two opposite faces.

Exercise 9.12. (*Stability of polygons and polytopes*) a) [2-] Prove that every interior point O in a regular n-gon Q projects onto at least two faces of Q.
b) [1+] Construct explicitly a convex polytope $P \subset \mathbb{R}^3$ which can stand only on one face, i.e., such that $\mathrm{cm}(P)$ projects onto only one face of P (cf. Example 9.9).
c) [2+] Construct a convex polytope $P \subset \mathbb{R}^3$ such that the distance from $\mathrm{cm}(P)$ to ∂P has only one local minimum and one local maximum.
d) [1] Show that in the plane it is impossible to construct a polygon as in c).

Exercise 9.13. $\Diamond$ *a*) [1] Let $\Delta \subset \mathbb{R}^2$ be an acute triangle in the plane and let T be a triangle inscribed into T with vertices at the feet of the altitudes. Prove that T is a periodic billiard trajectory. Prove that of all triangles inscribed into Δ, triangle T has the smallest perimeter.
b) [2-] Prove that there exists a quadrilateral periodic billiard trajectory in every tetrahedron with acute dihedral angles.

Exercise 9.14. [2] Let $P \subset \mathbb{R}^2$ be a simple polygon with rational angles. Prove that P has infinitely many periodic billiard trajectories.

Exercise 9.15. [1] Prove that regular n-gon in the plane has exponentially many periodic simple quasi-billiard trajectories.

Exercise 9.16. (*Dense billiard trajectories*) A billiard trajectory R in a convex polygon $Q \subset \mathbb{R}^2$ is called *dense* if for every $x \in Q$ and $\varepsilon > 0$ there is a point $y \in R$ such that $|xy| < \varepsilon$.
a) [1] Prove that every billiard trajectory in a square is either dense or periodic.
b) [1+] Find a convex polygon $Q \subset \mathbb{R}^2$ and a billiard trajectory that is neither dense nor periodic.
c) [2+] Prove or disprove: in a regular pentagon every billiard trajectory is either dense or periodic.

Exercise 9.17. *a*) [1+] Prove that in every smooth convex curve C there exists at least two 3-sided periodic billiard trajectories inscribed into C.
b) [2-] Generalize this to two periodic billiard trajectories with larger number of sides.
c) [2-] Prove or disprove: in every convex polygon $Q \subset \mathbb{R}^2$ there exists at least two 3-sided periodic quasi-billiard trajectories.

Exercise 9.18. [2+] Let $S \subset \mathbb{R}^3$ be a smooth convex surface. Prove that there exists a point inside of S with at least four normals onto S.

Exercise 9.19 (*Double normals in* $\mathbb{R}^3$)**.** [2+] Let $S \subset \mathbb{R}^3$ be a smooth convex surface. Prove that it has at least three double normals.

Exercise 9.20. *a*) [1-] Let $C \subset \mathbb{R}^2$ be a convex cone. Prove that every billiard trajectory in C has a finite number of reflections.
b) [1] Generalize this to circular cones in $\mathbb{R}^3$.
c) [1+] Generalize this to convex polyhedral cones in $\mathbb{R}^3$.

Exercise 9.21. (*Dual billiards*) Let $Q \subset \mathbb{R}^2$ be a convex set and let x_1 be a point outside of Q. Let x_2 be a point such that (x_1x_2) is a line supporting Q at a point which is a midpoint of x_1 and x_2. Note that there are two such x_2; choose the one where Q is to the right of the line. Repeat the construction to obtain x_3, x_4, etc. A sequence $[x_1x_2x_3\ldots]$ is called a *dual billiard trajectory* (see Figure 9.12).
a) [1] Prove that when Q is a square, then every dual billiard trajectory is periodic.
b) [1] Same when Q is a triangle. Find all period lengths.
c) [1] Suppose Q is smooth and strictly convex. Prove that circumscribed n-gons of minimal area are periodic dual billiard trajectories. Conclude that there exist infinitely many periodic dual billiard trajectories.
d) [1] Extend part *c*) to self-intersecting n-gons. Show that for prime $n \geq 3$ there exist at least $\frac{n-1}{2}$ n-gonal dual billiard trajectories.

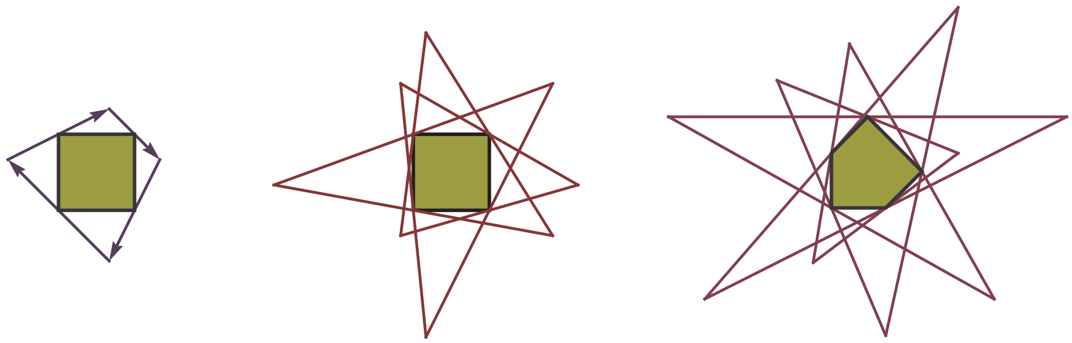

FIGURE 9.12. Periodic dual billiard trajectories.

e) [1] For a smooth Q, let T be an inscribed triangle of the maximal area. Show that the triangle with edges tangent to Q at vertices of T is a triangular dual billiard trajectory (cf. Exercise 9.5).
f) [1-] Extend *e*) to generic convex polygons Q (with no two parallel sides or diagonals).
g) [1+] Define the *generalized area* $A(Y)$ of a polygon $Y = [y_1 \ldots y_n]$ by the following formula:

$$A(Y) = \|\boldsymbol{w}\|, \quad \text{where } \boldsymbol{w} = \sum_{1 \le i < j \le n} (-1)^{j+i+1}\, \boldsymbol{y}_i \times \boldsymbol{y}_j\,, \ \ \boldsymbol{y}_i = \overrightarrow{Oy_i}.$$

For a smooth Q and odd $n \ge 3$, let Y be the inscribed n-gon of maximal generalized area. Show that the n-gon with edges tangent to Q at y_i is an n-gonal dual billiard trajectory. Extend this to generic polygons.
h) [2] Prove that for every regular polygon Q, all dual billiard trajectories are bounded (do not go to infinity).
i) [1+] Consider a quadrilateral Q with vertices $(0, \pm 1)$, $(-1, 0)$ and $(\theta, 0)$. Prove that for $\theta \in \mathbb{Q}$, all dual billiard trajectories are periodic.
j) [2] In notation of *i*), find a $\theta \notin \mathbb{Q}$ and an unbounded dual billiard trajectory.

9.8. **Final remarks.** Theorem 9.5 seems to be due to Arnold who proved it via reduction to the four vertex theorem (Theorem 21.1). See a sketch in [Arn2, §3] and a complete proof in [VD1]. In fact, Lemma 9.6 was used in [Tab1] to prove the four vertex theorem.

For more on double normals see [CFG, §$A3, 4$]. The number of special cuts in higher dimensions remains an open problem. Our proof of Theorem 9.12 follows the presentation in [KleW, §1.4]. The existence of monostatic polytopes in Example 9.9 was discovered in [CGG]. Monostatic simplices in higher dimensions we studied in [Daw2].

The theory of billiards plays an important role in ergodic theory. We refer to [Tab6] for a historical overview and an accessible introduction, and to [Tab2] for a good survey (see also [Schw]). We return to billiard trajectories in the next section (see also some exercises above).

Finally, let us mention that it is often difficult to recognize and compare the direct (variational) arguments vs. the indirect (existence) arguments. While the indirect arguments often have the advantage of being concise and amenable to generalizations, the direct arguments are often more elegant and can be implemented. We spend a great deal of the book discussing both methods. For now, let us leave you with a few words of wisdom by Sun Tzu [Sun]:

> iV.5. In all fighting, the direct method may be used for joining battle, but indirect methods will be needed in order to secure victory.
> V.10. In battle, there are not more than two methods of attack – the direct and the indirect; yet these two in combination give rise to an endless series of maneuvers.
> V.11. The direct and the indirect lead on to each other in turn. It is like moving in a circle – you never come to an end. Who can exhaust the possibilities of their combination?

These quotes are especially relevant to the proofs in Sections 35–37.

10. Geodesics and quasi-geodesics

In this section we study closed geodesics and quasi-geodesics on convex polytopes. In the first part, we derive a baby version of the general Cohn-Vossen's theorem which states that simple closed geodesics enclose half the curvature (see Subsection 25.6). In the second part, consider closed quasi-geodesics on convex polytopes. We use results on periodic billiard trajectories in the previous section to obtain a simple special case of the Lyusternik–Shnirelman theorem.

10.1. **Wise men never go to the top of the hill.** [25] Let $S \subset \mathbb{R}^3$ be a closed 2-dimensional polyhedral surface, e.g., the surface of a convex polytope. For two distinct points $x, y \in S$ consider all *shortest paths* γ between them. Of course, there can be many such paths, e.g., when S is a regular n-prism and x, y are centers of the n-gon faces. Note, however, that determining whether a path between x and y is shortest is a non-trivial "global" problem. Here is the corresponding local notion.

A piecewise linear path $\gamma \subset S$ is called a *geodesic* if for every point $z \in \gamma$ a sufficiently small neighborhood of γ around z is a shortest path. Clearly, every shortest path is a geodesic, but not vice versa (see Figure 10.1).

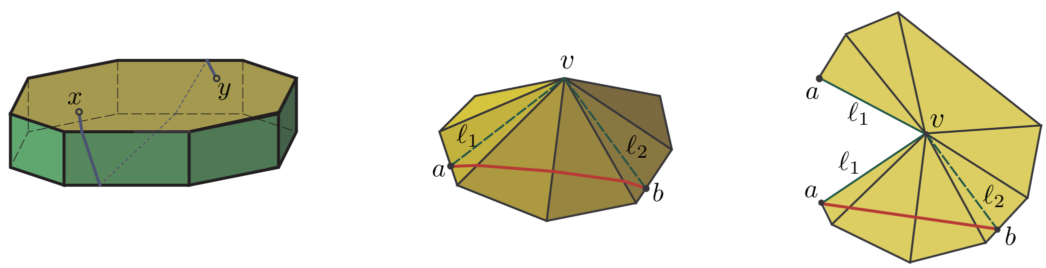

Figure 10.1. A geodesic that is not a shortest path and an unfolding along a geodesic.

Proposition 10.1. *No geodesic γ on a convex 2-dimensional polyhedral surface $S \subset \mathbb{R}^3$ has a vertex in its relative interior. In addition, whenever a geodesic γ crosses an edge e in S, it has equal angles with e in the faces adjacent to e.*

Proof. For the first part, suppose a geodesic γ between points x and y passes though vertex v. Choose points $a, b \in \gamma$ close enough to v, so that the $[ab]$ portion of γ is a shortest path which consists of two straight intervals ℓ_1 and ℓ_2 as in Figure 10.1 Since the surface S is convex, the angle $\angle avb < \pi$ on at least one of the sides of γ. Unfold S along the edges in this angle and note that $|ab| < |av| + |vb|$ in the unfolding, a contradiction.

For the second part, similarly unfold along edge e and observe that γ must be straight around the intersection point $z = \gamma \cap e$. The details are straightforward. $\square$

In other words, if you are on one side of the hill and you need to go to another side, it is always faster to go around than go all the way up to the top.

[25]This is a play on a Russian proverb "Умный в гору не пойдёт, умный гору обойдёт."

10.2. **Walking around and coming back.** Suppose now S is a smooth convex surface in $\mathbb{R}^3$. Does it have closed geodesics? Does it have a closed geodesics without self-intersections? (such geodesics are called *simple*).

Theorem 10.2 (Poincaré)**.** *Every smooth convex body has at least one simple closed geodesic.*

Poincaré proved this result in 1905 and conjectured that every convex body has at least three simple closed geodesics (see Figure 10.2). Birkhoff later proved existence of two such geodesics, but the conjecture was resolved by Lyusternik and Shnirelman in 1929.

Theorem 10.3 (Lyusternik–Shnirelman)**.** *Every smooth convex body has at least three simple closed geodesics.*

This result is highly technical and beyond the scope of this book.

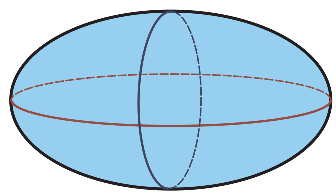

FIGURE 10.2. Three geodesics on an ellipsoid.

Remark 10.4. There are two ways to approach the Poincaré theorem. First, one can take the shortest of the curves which divide the surface area into two equal halves. This turns out to be a correct construction of a closed geodesic, but does not lead to generalizations.

Alternatively, imagine a solid convex body with surface S and no friction. Take a closed rubber band and place it on a solid. The idea is that the rubber band will slide out in one end or another unless it is in an equilibrium, in which case it is a geodesic. This basic idea lies at the heart of the Lyusternik–Shnirelman's approach.

10.3. **Back to billiard trajectories.** Let us give a connection between the double normals theorem (Theorem 9.10) and the Lyusternik-Shnirelman theorem. Consider a nearly flat convex body B with smooth boundary and an orthogonal projection C (think of B as a "cake" obtained by fattening C). The geodesics now look like billiard trajectories, as they alternate between the top and the bottom face of B (see Figure 10.3). In this sense, special cuts defined in the previous section (see Subsection 9.4) correspond to closed geodesics on B that alternate exactly once. The double normals theorem now implies that B has at least two such closed geodesics.

One can ask what happened to the third closed geodesic as in the Lyusternik-Shnirelman theorem. Well, it simply goes around the boundary of B.

Suppose now that $S = \partial P$ is the surface of a convex polytope $P \subset \mathbb{R}^3$. One can ask what is known about closed geodesics on S. Given the poor state of art for the periodic billiard trajectories on convex polygons, the following two conjectures are hardly surprising and unlikely to be easy.

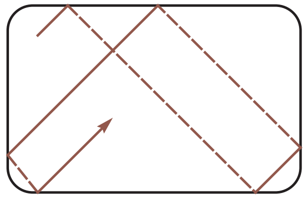

Figure 10.3. A geodesic on a convex body shaped as a pool table.

Conjecture 10.5. *The surface of every convex polytope has at least one closed geodesics.*

Conjecture 10.6. *Let S be the surface of a convex polytope $P \subset \mathbb{R}^3$. For every two points $x, y \in S$, there exist infinitely many geodesics from x to y.*

10.4. **Simple closed geodesics on convex polyhedra.** As before, let $S = \partial P$ be the surface of a convex polytope $P \subset \mathbb{R}^3$. One can ask if there is always at least one simple closed geodesic on S? The answer turns out to be negative. In fact, in a certain precise sense, almost no tetrahedra has a simple closed geodesics. On the other hand, a number of "natural" polytopes have many closed geodesics, including simple closed geodesics (see e.g., Figure 10.4 and Exercise 10.3).

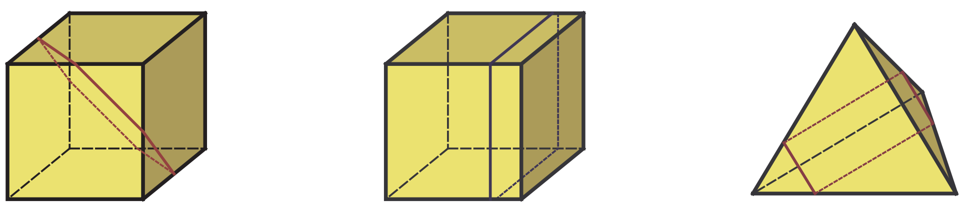

Figure 10.4. Closed geodesics on a cube and a regular tetrahedron.

Claim 10.7. *The surface of a random tetrahedron does not have simple closed geodesics.*

Of course, there are many different ways to define a random tetrahedron. For example, one can choose its vertices at random from a unit cube, a unit sphere, or some other natural distribution, as in Sylvester's problem (see Exercise 42.3). Alternatively, one can choose a random collection of face angles of a tetrahedron, conditioned that in each triangle they add up to π and the angle sums in vertices are $< 2\pi$ (see Exercise 37.1). Similarly, one can choose a tetrahedron uniformly at random from the compact set of all possible collections of edge lengths of tetrahedra with a unit total edge length (cf. Exercise 31.6). The claim holds in each case, and easily follows from the following simple observation.

Lemma 10.8. *Let Δ be a tetrahedron, and let α_i, $1 \le i \le 4$, denote the sums of the face angles at the i-th vertex. If $\alpha_i + \alpha_j \ne 2\pi$ for all $1 \le i < j \le 4$, then Δ does not have simple closed geodesics.*

Proof. First, observe that the total sum of all face angles in a tetrahedron is equal to $\alpha_1+\alpha_2+\alpha_3+\alpha_4 = 4\pi$. Clearly, a simple closed geodesic γ on Δ must separate either one or two vertices. Suppose $\gamma = (abc)$ goes around one vertex v_1 (see Figure 10.5). By Proposition 10.1, the angles at a, b and c are equal to π on each side. Therefore the sum of the face angles in a tetrahedron (v_1abc) is equal to $\alpha_1+4\pi$, a contradiction.

Figure 10.5. Potential closed geodesics on a tetrahedron.

Similarly, suppose $\gamma = [abcd]$ goes around two vertices v_1, v_2 as in Figure 10.5. Denote by Σ the total sum of angles in two 4-gons (v_1abv_2), (v_2cdv_1) and two triangles (dav_1), (bcv_2), i.e., all polygons on one side of γ. Clearly, $\Sigma = 6\pi$. On the other hand, since γ is a geodesic, the sum of angles at points a, b, c, and d is equal to π. Therefore, $\Sigma = \alpha_1 + \alpha_2 + 4\pi$, and we obtain $\alpha_1 + \alpha_2 = 2\pi$, a contradiction. $\square$

10.5. **Quasi-geodesics.** Let $S = \partial P$ be the surface of a convex polytope in $\mathbb{R}^3$. For a point $x \in S$, denote by $\alpha(x)$ the sum of face angles at x, if x is a vertex of P, and let $\alpha(x) = 2\pi$ otherwise.

For a piecewise linear path $\gamma \subset S$, at each vertex x of γ, consider the intrinsic angles $\alpha_+(x)$ and $\alpha_-(x)$ in S, one for each side of γ around x. By definition, $\alpha_+(x)+\alpha_-(x) = \alpha(x)$. Path γ is called a *quasi-geodesic* if $\alpha_+(x), \alpha_-(x) \leq \pi$ for every interior $x \in \gamma$. In other words, when γ crosses an edge in X, it behaves as a geodesics. On the other hand, path γ is now allowed to pass the vertices as long as the angles on both sides are $\leq \pi$. Examples of closed quasi-geodesics are shown in Figure 10.6.

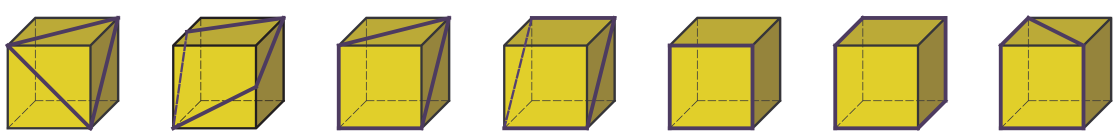

Figure 10.6. Seven closed quasi-geodesics on a cube.

The notion of a quasi-geodesic is an extension of the quasi-billiard trajectories from doubly covered polygons to general convex polyhedra. Observe that every even-sided periodic quasi-billiard trajectory in a convex polygon Q corresponds to a closed geodesic on doubly covered Q. Similarly, an even-sided periodic quasi-billiard trajectory can be doubled to give a closed geodesic. Theorem 9.16 then gives:

Proposition 10.9. *Every doubly covered convex polygon has infinitely many closed quasi-geodesics.*

Similarly, Theorem 9.11 and the discussion in Subsection 10.3, implies the following variation on the Lyusternik-Shnirelman theorem (Theorem 10.3).

Proposition 10.10. *Every doubly covered convex polygon has at least three simple closed quasi-geodesics.*

The most general result in this direction in the following theorem, also beyond our scope.

Theorem 10.11 (Pogorelov)**.** *Every convex polytope has at least three simple closed quasi-geodesics.*

10.6. **Exercises.**

Exercise 10.1. (*Explicit constructions*) $\Diamond$ *a*) [1-] Give an explicit construction of a polytope with no simple closed geodesics.
b) [1] Give an explicit construction of a polytope with no simple closed geodesics which has a (non-simple) closed geodesic.

Exercise 10.2. [1-] Find a polytope $P \subset \mathbb{R}^3$ with infinitely many non-periodic closed geodesics of distinct lengths. Further, find P such that the number of self-intersections of closed non-periodic geodesics on P is unbounded.

Exercise 10.3. *a*) [1+] Prove that the seven shortest geodesics on the unit cube have squared lengths 16, 18, 20, 90, 148, 208 and 212.
b) [1+] Prove that all simple closed geodesics on the regular octahedron with edge lengths 1 have squared lengths 8 and 9.
c) [1+] Prove that all simple closed geodesics on the regular icosahedron with edge lengths 1 have squared lengths 25, 27 and 28.

Exercise 10.4. (*Geodesics on equihedral tetrahedra*) *a*) [1-] Prove that a tetrahedron has three pairwise intersecting simple closed geodesics if an only if it is *equihedral*, i.e., has congruent faces (cf. Exercise 25.12).
b) [2-] Two geodesics are called *isotopic* if they intersect edges of the polyhedron in the same order. Prove that a tetrahedron has infinitely many non-isotopic geodesics if and only if it is equihedral.
c) [1] Show that one can perturb the equihedral tetrahedron so that the resulting tetrahedron has a finite number of non-isotopic geodesics, and this number can be arbitrary large.
d) [2-] Prove that a tetrahedron has infinite simple (non-self-intersecting) geodesics if and only if it is equihedral.
e) [2] Prove that a convex polytope $P \subset \mathbb{R}^3$ has an infinite simple geodesic if and only if P is the equihedral tetrahedron.

10.7. **Final remarks.** For the details, history and references on closed geodesics and the Lyusternik–Shnirelman theory (including a simple proof of their main result) see [Kli]. We refer to [Alb] for a readable account of the ideas. Unfortunately, both the Lyusternik–Shnirelman theorem and the proof in [Kli] are non-combinatorial in nature (and in some sense not even geometric) since they employ parameterized curves, and use the parameter in an essential way. Interestingly, in higher dimensions the number of simple closed geodesics on convex surfaces in $\mathbb{R}^d$ has yet to be shown to match $\binom{d}{2}$, the smallest number of geodesics the ellipsoids can have.

For Claim 10.7 and the related results see e.g., [Gal]. For more on quasi-geodesics see [A1] and [AZa]. Pogorelov's Theorem 10.11 is proved in [Pog1] in the generality of all convex surfaces (one can extend quasi-geodesics to this case). He uses approximation of general convex surfaces with smooth surfaces, which reduces the problem to the Lyusternik–Shnirelman theorem. We return to the study of closed geodesics on convex polytopes in Sections 25 and 40.

11. The Steinitz theorem and its extensions

The Steinitz theorem is one of the fundamental results in polyhedral combinatorics. It can be viewed as the converse of the result that all graphs of polytopes in $\mathbb{R}^3$ are planar and 3-connected. We present it in the first part of the book not so much because the proof is simple, but because it uses the variational principle approach, and the result plays a central role in the study of 3-dimensional convex polytopes. Among other things, it proves that all 3-dimensional polytopes are rational (cf. Section 12), and gives a basis for the algebraic approach, easily implying that almost all polyhedra in $\mathbb{R}^3$ are rigid (see Section 31).

11.1. **(Almost) every graph is a graph of a polytope.** Recall Balinski's theorem, which in the case of 3-dimensional convex polytopes states their graphs must be 3-connected. As it turns out, this and the planarity are the only conditions on graphs of polytopes.

Theorem 11.1 (Steinitz)**.** *Every* 3*-connected planar graph is a graph of a convex polytope in* $\mathbb{R}^3$.

This is an important result, and like many such results it has several proofs and interesting generalizations. We will mention three proof ideas, all somewhat relevant to the rest of the course. Unfortunately we will not include either of the proofs for two very different reasons: the first proof (graph theoretic) and the second proof (via circle packing) are so well presented in Ziegler's lectures that we see no room for an improvement. On the other hand, the third proof (via Tutte's theorem) is quite technical as its graph-theoretic part is yet to be sufficiently simplified for a leisurely style we favor in this book.

11.2. **Graph-theoretic proof.** The idea of this proof goes to the original proof by Steinitz, and is based on the following (difficult) lemma in graph theory:

Lemma 11.2. *Every* 3*-connected planar graph* Γ *can be reduced to a complete graph* K_4 *by a sequence of simple* $Y\Delta$ *transformations.*

The $Y\Delta$ transformations are shown in Figure 11.1 and can be used in either direction. The simple $Y\Delta$ transformations are the $Y\Delta$ transformations followed by removal of parallel and sequential edges (the P, S reductions shown in Figure 11.1).

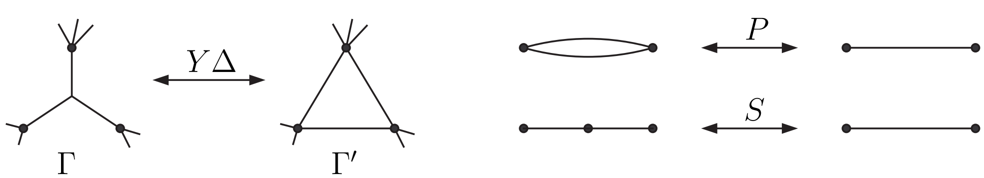

Figure 11.1. The $Y\Delta$ transformations $\Gamma \leftrightarrow \Gamma'$, and P, S reductions.

An example of transforming graph C_3 (graph of 3-dimensional cube) into a graph K_4 (graph of a tetrahedron) is given in Figure 11.2. Here we indicate all $Y\Delta$ transformations and P reductions (S reductions are obvious and omitted).

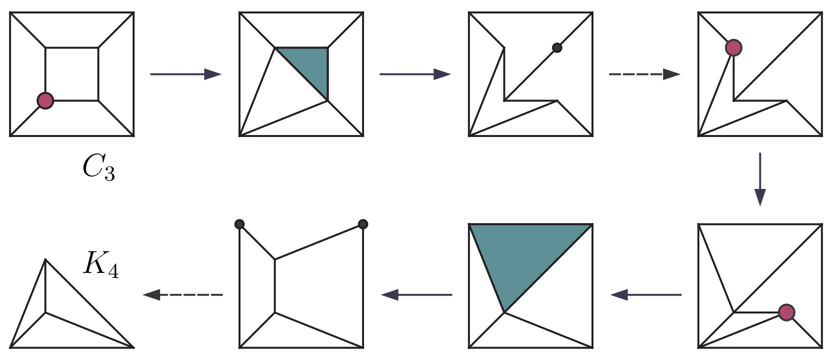

FIGURE 11.2. A sequence of simple $Y\Delta$ transformations from C_3 to K_4.

We will not prove Lemma 11.2, but refer to [Zie1, §4.3] for a simple and beautiful proof. Below we sketch the rest of the proof of the Steinitz theorem.

Given a 3-connected planar graph Γ, construct a sequence of simple $Y\Delta$ reductions as in the lemma:

$$\Gamma \to G_1 \to G_2 \to \cdots \to G_n = K_4 .$$

We will work backwards, given a polytope with graph G_i we will construct a polytope with graph G_{i-1}. Clearly, every tetrahedron has graph $G_n = K_4$, our starting point. Now observe that each $Y\Delta$ transformation corresponds to simply cutting a cone with a hyperplane (the $Y \to \Delta$ direction) or extending the three facets adjacent to a triangular face to meet at a vertex (the $\Delta \to Y$ direction). One can think of Figure 11.1 as depicting a 'view from the top' of this. The details are straightforward and left to the reader. $\square$

One important consequence form this proof of the Steinitz theorem is the ability to make all vertex coordinates to be rational. This property will prove useful in the next section.

Corollary 11.3. *Every 3-connected planar graph is a graph of a convex polytope in $\mathbb{R}^3$ with rational coordinates.*

For the proof simply observe that in the proof above we could have started with a rational tetrahedron and continued to cut it only with rational planes, thereby creating only rational vertices. Clearly, when we are doing extensions of rational planes we are also creating only rational vertices, which completes the argument. In fact, one can even blow up the resulting polytope and make all vertex coordinates integral.

11.3. **Circle packings are totally cool!** Consider the following circle packing theorem often attributed to Koebe, Andreev and Thurston:

Theorem 11.4 (Circle packing theorem)**.** *Every planar graph G can be represented in the plane by a collection of disjoint circles corresponding to vertices of G, with edges of G corresponding to touching circles.*

On the surface, this theorem has nothing to do with the Steinitz theorem. Here is the connection. Start by making a stereographic projection of the circles onto a sphere. This gives a corresponding collection of circles on a sphere. Now extend planes through each of the circles. This gives a polytope whose edges correspond to pairs of touching circles and in fact they touch a sphere at those points. In fact, this polytope has a graph G^*, dual to the graph G in the theorem. This leads to the following extension of the Steinitz theorem:

Theorem 11.5 (Schramm)**.** *Every 3-connected planar graph is a graph of a convex polytope in $\mathbb{R}^3$ whose edges are all tangent to a unit sphere. Moreover, if we require in addition that the origin is the barycenter of the contact points, such polytope is unique up to rotations and reflections.*

While the original proofs of both theorems are rather difficult, a recent proof by Bobenko and Springborn of Schramm's theorem via the variational principle is quite elegant and relatively simple. Again, we do not include the proof and refer to [Zie2, §1.3] for a beautiful exposition.

11.4. **Graph drawing is more than a game!** When we talk about planar graphs, one needs a little care as the edges of the graph can be drawn on a plane with lines, polygonal arcs, curves, etc., and one has to give an argument that the notion of planarity is in fact independent of the presentation. The following result is the best one can hope for:

Theorem 11.6 (Fáry)**.** *Every planar graph can be drawn on a plane with straight non-intersecting edges.*

It is not hard to prove the theorem directly, by starting with an outside edge and the adjacent face, making it a convex polygon and adding points one face at a time in such a way that the intermediate 'boundary' remains convex (see Figure 11.3). Slightly more carefully: take a dual graph, fix a vertex v on the boundary and orient all edges towards v, creating a partial order on vertices. Now extend this order to a linear order in any way and use this order to add the faces.

Note that this method of drawing graphs produces non-convex faces. The following result may come as an initial surprise:

Theorem 11.7 (Tutte's spring theorem)**.** *Every 3-connected planar graph can be drawn on a plane such that all interior faces are realized by non-overlapping convex polygons.*

In fact, the result follows immediately from the Steinitz theorem. Simply take a face F of the polytope P corresponding to the outside face, place it horizontally, and

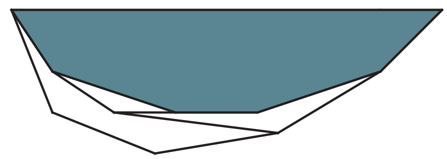

Figure 11.3. A simple way of proving the Fáry theorem.

set a point O just below the middle of the face. Now draw inside the face the view of the edges of P as they are seen through F (see Figure 11.4).

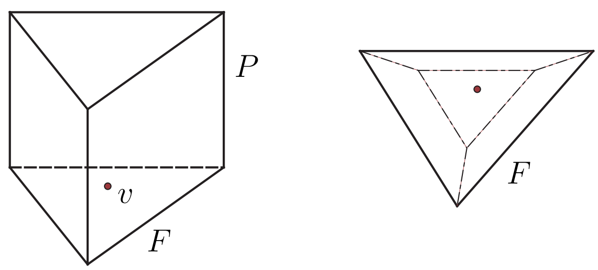

Figure 11.4. Fáry theorem from Steinitz theorem: polytope P as viewed from v though face F.

In fact, one can go the other way as well: first construct 'nice' graph drawings as in Tutte's theorem, and then lift it up to a polytope.

11.5. **Using stresses to draw graphs.** Imagine we physically build a graph with edges made out of springs. Take the outside face and stretch it far enough, fixing all the points in convex position. The remaining (interior) vertices will be forced to find some kind of equilibrium, which we show to be unique and satisfy some additional properties. In fact, this construction gives the drawing as in Tutte's theorem. Now all we need is to formalize and prove everything.

Let G be a 3-connected graph with a set of vertices V and a set of edges E. A *realization* of G is a map $f : V \to \mathbb{R}^2$. We say that vertex $v \in V$ is in an *equilibrium* if it is in a barycenter of its neighbors:

$$f(v) = \frac{1}{m}\big(f(w_1) + \ldots + f(w_m)\big),$$

where $N(v) = \{w_1, \ldots, w_m\}$ is the set of neighbors of v.

Lemma 11.8. *Let G be a connected graph with a set of vertices V, and let $f : A \to \mathbb{R}^2$ be fixed realization in the plane of nonempty subset $A \subset V$ of vertices. Then there exists a unique realization of the whole graph such that all vertices in $(V \smallsetminus A)$ are in equilibrium.*

Note that the lemma does not require the graph to 3-connected or planar. In fact, Figure 11.5 shows a (non-planar) Petersen's graph with all 5 interior vertices in equilibrium. Note that the resulting realization can he highly degenerate. For example, when A consists of one or two vertices, the whole graph is realized in that one point or on an interval between these two points, respectively.

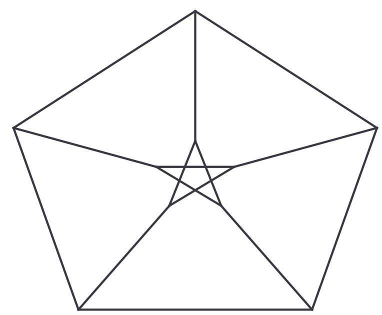

FIGURE 11.5. Petersen graph with five interior vertices in equilibrium.

The proof is straightforward and is suggested by the natural notion of the *energy function* coming from physics. Here is how we adapt it to our situation.

Proof. Let $A = \{a_1, \ldots, a_k\} \subset V$ be the fixed set of vertices. Without loss of generality we can assume that $f(a_1) = O$ is the origin. Let $V' = (V \smallsetminus A)$ denote the set of the remaining vertices, and let E' denote the set of edges with at least one vertex in V'. Define

$$\mathcal{E}(f) := \frac{1}{2} \sum_{(v,w)\in E'} \|f(v) - f(w)\|^2 = \frac{1}{2} \sum_{(v,w)\in E'} \big[(x_v - x_w)^2 + (y_v - y_w)^2\big],$$

where $f(v) = (x_v, y_w)$ denotes the coordinates of realization points. Note that $\mathcal{E} = \mathcal{E}(f)$ is a convex quadratic everywhere nonnegative function. Let us show that in fact $\mathcal{E}$ is *strictly convex*. Indeed, suppose one of the points, say $f(w)$, $w \in V'$, is far from the origin: $\|f(w) - O\| > C$. Connect w to a_1 by a path with at most $n = |V| - 1$ edges. Clearly, at least one of the edge lengths is longer than C/n, so $\mathcal{E} \geq C/n$. Therefore, for sufficiently large $C > 0$ we have $\mathcal{E}(f) > \mathcal{E}(f_0)$ for any fixed f_0 and $\|f - f_0\| > C$.

We conclude that $\mathcal{E}$ is strictly convex with a positive definite Hessian matrix. Therefore, $\mathcal{E}$ has a unique minimum. Writing the minimality condition for a critical realization $f : V \to \mathbb{R}^2$, we obtain:

$$\frac{\partial \mathcal{E}}{\partial x_v} = \sum_{w\in N(v)} (x_v - x_w) = 0, \qquad \frac{\partial \mathcal{E}}{\partial y_v} = \sum_{w\in N(v)} (y_v - y_w) = 0.$$

Adding these two equations, for every $v \in V'$, we obtain:

$$|N(v)| \cdot f(v) - \sum_{w\in N(v)} f(w) = O,$$

which is exactly the equilibrium condition. $\square$

Note that the variational principle approach quickly reduced the problem to a graph theoretic question. In fact, we do not even need the uniqueness part in the lemma — just the existence of the equilibrium configuration. Formally, the rest of the Tutte theorem follows from the following (difficult) lemma.

Lemma 11.9. *Let $G = (V, E)$ be a 3-connected planar graph and let $f : V \to \mathbb{R}^2$ be its realization. Suppose the outside face F forms a convex polygon and the remaining points in the realization are in equilibrium. Then all faces of G are disjoint convex polygons.*

The lemma is proved by a delicate and technical graph theoretic argument which we skip. Let us note, however, that Lemma 11.8 holds for all graphs, e.g., for a complete graph K_n. On the other hand, the result of Lemma 11.9 obviously cannot hold for non-planar graphs. Thus, planarity of G must be critical in the proof.

11.6. **Exercises.**

Exercise 11.1. Let $P \subset \mathbb{R}^3$ be a simple convex polytope whose faces are quadrilaterals.
a) [1-] Prove that it has eight vertices.
b) [1] Prove that P is combinatorially equivalent to a cube.
c) [1] Prove that if seven vertices of P lie on a sphere, then so does the eighth vertex.

Exercise 11.2. [1] Let $P \subset \mathbb{R}^3$ be a simple convex polytope such that every face can be inscribed into a circle. Prove that P can be inscribed into a sphere.

Exercise 11.3. [1-] Let $C \subset \mathbb{R}^3$ be a convex cone with four faces. Prove that C has an inscribed sphere if and only if the opposite face angles have equal sums.

Exercise 11.4. *a*) [1+] Let $P \subset \mathbb{R}^3$ be a convex polytope whose faces are colored in black and white such that no two black faces are adjacent. Suppose the total area of black faces is larger than the total area of white faces. Prove that P does not have an inscribed sphere.
b) [1-] Find an example of such polytope P.
c) [1-] Show that condition in part *a*) is necessary but not sufficient, even if there is a unique proper coloring of the faces.

Exercise 11.5. *a*) [1+] As above, let $P \subset \mathbb{R}^3$ be a convex polytope whose faces are colored in black and white such that no two black faces are adjacent. Suppose the number of white faces is smaller than the number of black faces. Prove that P does not have an inscribed sphere.
b) [1-] Let Q be a parallelepiped and let P be obtained from Q by cutting off the vertices. Use part a) to show that P does not have an inscribed sphere.

Exercise 11.6. *a*) [2] Let P be a convex polytope with graph Γ, and let C be a simple cycle in G. Prove that there exists a combinatorially equivalent polytope P', such that the edges of the boundary of a projection of P' correspond to C.
b) [1] Prove that the shape of the polygon in the projection cannot be prescribed in advance.

Exercise 11.7. *a*) [1] Prove that for all $k \geq 3$, a plane can be partitioned into k-gons (edge-to-edge).
b) [2] For every *simplicial* convex polytope $P \subset \mathbb{R}^3$, prove that the space $\mathbb{R}^3$ can be partitioned into polytopes combinatorially equivalent to P (face-to-face).

c) [2] For every convex polytope $P \subset \mathbb{R}^3$, prove that the space $\mathbb{R}^3$ can be *dissected* into polytopes combinatorially equivalent to P (not necessarily face-to-face).

Exercise 11.8. (*Infinite Fáry theorem*) [2-] Let G be an infinite planar graph of bounded degree. Prove that G can be drawn in the plane with straight edges. Check that this generalizes part *a*) of the previous exercise.

11.7. **Final remarks.** Steinitz theorem was discovered by Ernst Steinitz in the late 1920s and his manuscript was published posthumously in 1934 by Hans Rademacher. For a traditional graph theoretic proof of the Steinitz theorem see e.g., [Grü4, §13.1], and for the history of the Steinitz theorem see [Grü5]. An attractive presentation of the graph theoretic proof is given in [Zie1, §4.3]. For the Bobenko–Springborn proof see [Zie2, §1] and [Spr] (see also [Grub, §34]). An advanced generalization was obtained by Schramm, who showed that the unit sphere plays no special role in Theorem 11.5, but in fact one can realize a polytope with edges tangent to any smooth convex body [Schra]. For a complete proof of Tutte's spring theorem, the proof of the Steinitz theorem, references and details see [Ric] (see also [Tho2]).

The reason Theorem 11.4 is attributed to the three authors is because Koebe claimed it in 1936, but proved it only for triangulations. Almost half a century later, in 1985, Thurston gave a talk where he presented the theorem. He also noted in his book (which remained unpublished for years) that the result follows easily from Andreev's results of the 1970. Only then, with papers by Colin de Verdière (1988), Schramm (1991), and others, the result was brought into prominence. We refer to [Ber2] for an elementary introduction, to [PacA, Ch. 8] for a nicely written standard fixed-point type proof, and to [Spr, Zie2] for the references.

It is interesting to note a fundamental difference between the two variational proofs (of Theorems 11.5 and 11.7). While in the proof of Tutte's theorem proving existence of an equilibrium is easy, it is showing that the equilibrium is what we want that is hard. The opposite is true in the Bobenko-Springborn proof (as well as in Schramm's proof) it is proving the existence of an equilibrium that is hard; showing that the equilibrium produces a desired configuration is quite easy.

12. Universality of point and line configurations

This is the first of two sections on universality. In essence, it shows that the face combinatorics studied in Section 8 and the geometry of polyhedra are largely incompatible. To put it another way, we show that by making some restrictions on the face lattice one can ensure that the polytopes are irrational, as complicated as needed, or even non-existent. This is an extremely negative result showing that there is no hope for the Steinitz type theorem in higher dimensions (see Section 11).

12.1. **Universality is not for everyone.** In the following two sections we present several results with one unifying goal: to show that various constructions and configurations in discrete geometry can encode an extremely wide (universal!) range of objects. While this may seem an empowering proposition, from the postmodern point of view this is very disappointing, since we are basically saying that nothing interesting can be proved about all these general geometric objects. Indeed, what can possibly be *interesting* about *everything*?

12.2. **Ruler and compass constructions are so passé.** Recall that it is impossible to construct a regular 7-gon using a compass and a ruler. This classical result of Gauss is now considered iconic, but two hundred years ago the result took awhile to understand and appreciate. What was more impressive to mathematicians of the day was the construction of a regular 17-gon, unknown until Gauss. This constructive part of his work is considered elementary now, but his approach, in fact, can be viewed as the earliest universality type result.

Consider an increasing sequence of fields

$$K_0 \subset K_1 \subset K_2 \subset \ldots \subset K_n\,,$$

where $K_0 = \mathbb{Q}$, $K_i = K_{i-1}[\sqrt{a_i}]$, and $a_i \in K_{i-1}$, for all $1 \le i \le n$. We call a number $z \in \mathbb{C}$ *geometric* if there exist a chain of fields as above, such that $z \in K_n$, for some $n \in \mathbb{N}$.

Theorem 12.1 (Gauss)**.** *Every geometric number can be constructed by the ruler and the compass.*

We need to elaborate on what we mean by the ruler and compass construction (in the plane). We identify the plane with the field $\mathbb{C}$, and assume that in the beginning are given the set of two points $A_0 = \{0, 1\}$. Our algorithm increases the set of points by making the ruler and compass drawing, and adding new intersection points at each step.

Formally, suppose at the n-th step we have the set $A_n \subset \mathbb{C}$ of points in the plane. At the $(n+1)$-th step we can do two of the following operations:

- draw a line (a_i, a_j) through any two points $a_i, a_j \in A_n$, or
- draw a circle with a center at a_k and radius $|a_i a_j|$, for some $a_i, a_j, a_k \in A_n$.

Then we can add the set of intersection point(s) $B = \{b_1, b_2\}$ in the case of two circles, or a line and a circle, and $B = \{b\}$ in case two lines, to the set of points:

- let $A_{n+1} := A_n \cup B$.

We say that the above algorithm *constructs* all points in A_n. The Gauss theorem says that all geometric numbers (viewed as points in the plane) can be constructed this way.

Now, the reason why the 17-gon can be constructed by ruler and compass lies in the fact that $\sin\frac{\pi}{17}$ (equivalently, $\cos\frac{\pi}{17}$) is what we call a geometric number. We will skip the explanation of this and move to the proof of Theorem 12.1. The proof is, in fact, completely straightforward and is included here as a motivation for further results in this section.

Proof of Theorem 12.1. First, let us emulate the field operations. Starting with $A_0 = \{0,1\}$ we can obtain all of $K_0 = \mathbb{Q}$ by using a finite number of additions, multiplications, subtractions and divisions. Let us show how to compute $x+y$, $-x$, $x \cdot y$ and $1/x$ for every $x, y \in \mathbb{C}$. The constructions of $x+y$, $-x$ can be done easily (see Figure 12.1). To compute $1/x$ for complex x, first compute the argument $\overline{x}$ of $1/x$ (line on which $\overline{x}$ lies), and then the norm $1/a$, $a = |x|$. Similarly, for $x \cdot y$ first compute the argument, and then the norm $a \cdot b$, where $a = |x|$ and $b = |y|$.

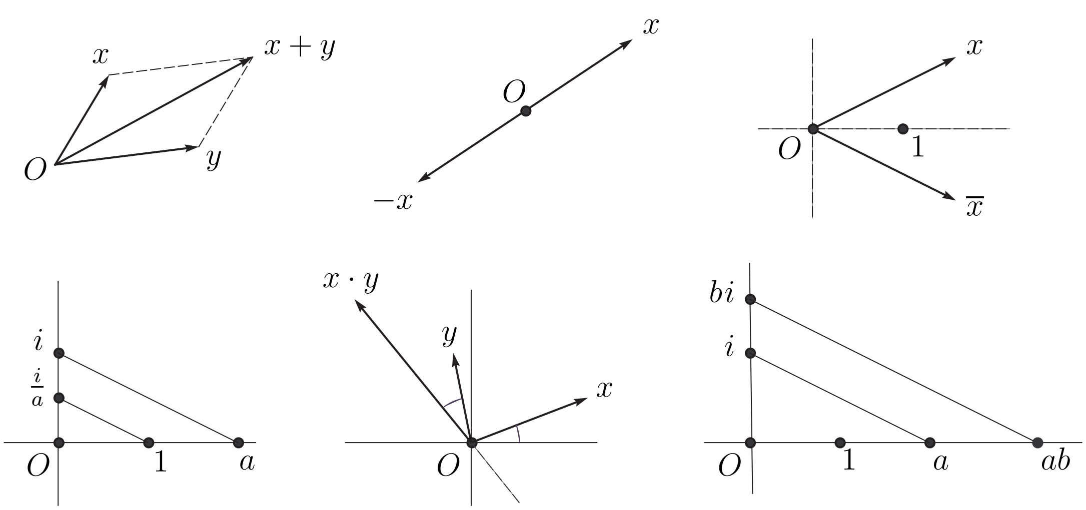

FIGURE 12.1. Computing $x+y$, $-x$, $x \cdot y$ and $1/x$, for $x, y \in \mathbb{C}$.

Now, by induction, given $\sqrt{a_i}$ we can now construct any element in $K_i = K_{i-1}[\sqrt{a_i}]$. Thus, it remains to show how to construct the roots $\sqrt{z}$ given any $z \in \mathbb{C}$ (one of the two complex roots will suffice, of course). As before, first construct the argument and then the norm $\sqrt{c}$, where $c = |z|$. The latter requires some care. If $c > 1$, construct the right triangle with sides $(c+1)$ and $(c-1)$, as in Figure 12.2. The length of the third side, $2\sqrt{c}$, divided by 2 gives the desired norm. If $c < 1$, compute the inverse $1/c$, then $\sqrt{1/c}$ by the method described above, and, finally, the inverse $\sqrt{c}$. $\square$

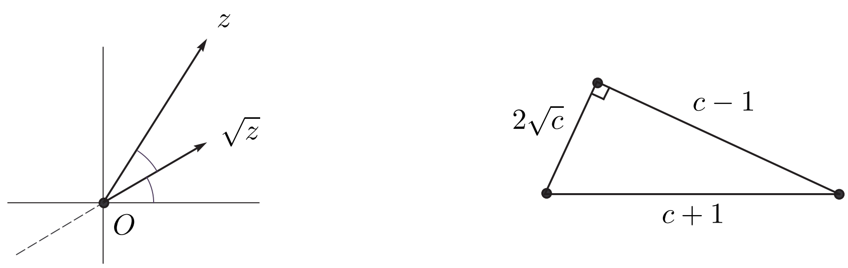

FIGURE 12.2. Computing $\sqrt{z}$, where $c = |z|$ and $z \in \mathbb{C}$.

12.3. **Rationality is a popular, but not universal, virtue.** Consider a finite set of points $A = \{a_1, \ldots, a_n\} \subset \mathbb{R}^2$ and lines $L = \{\ell_1, \ldots, \ell_m\}$ in the plane. A *point and line arrangement* is a pair (A, L). From each point and line arrangement we can record the set of incidence pairs

$$\mathcal{C} = \{(a_i, \ell_j) \mid a_i \in \ell_j, 1 \le i \le n, 1 \le j \le m\}$$

We call $\mathcal{C}$ a (*plane*) *configuration* and think of pair (A, L) as a *realization of* $\mathcal{C}$ *over* $\mathbb{R}$. It is important to emphasize that in a realization no additional incidences can be created. So, for example, two different points in a configuration cannot be realized by a single point in the plane.

Now, we can also consider realization of $\mathcal{C}$ over other fields, such as $\mathbb{Q}$, $\mathbb{C}$, or finite fields $\mathbb{F}_q$. Here is a natural question: Is it true that configurations realizable over $\mathbb{C}$ are also realizable over $\mathbb{R}$? What about $\mathbb{R}$ versus $\mathbb{F}_q$? What about different finite fields? The answers to all these questions turns out to be a “NO”, as we show below. Furthermore, in the next subsection we prove a “universality result” which shows that the situation is much worse than it seems.

Formally, write a configuration $\mathcal{C}$ as a pair (V, E) where $V = \{v_1, \ldots, v_n\}$ is a set of ‘points’, and $E = \{e_1, \ldots, e_m\} \subset 2^V$ is a set of ‘lines’ which can contain any subsets of at least two points. There are natural combinatorial conditions on E such as, e.g., two lines containing the same two points must coincide, but we will not be concerned with these. Also, we consider all projective realizations rather than affine realizations as they are easier to work with. Of course, this does not affect realizability as we can always take a projective linear transformation to move points away from the infinite line.

For a planar configuration $\mathcal{C} = (V, E)$ and a field K consider a map $f : V \to \mathsf{K}\mathbb{P}^2$. Denote by $f(e_j)$ a line spanned by some of its pairs points: $f(e_j) = \langle f(v_i), f(v_r)\rangle \subset \mathsf{K}^2$, where $v_i, v_r \in e_j$. We say that f is a (projective) *realization over* K if the following condition is satisfied:

$$f(v_i) \in f(e_j) \ \text{ if and only if } \ v_i \in e_j\,, \text{ for all } v_i \in V,\ e_j \in E.$$

The following two examples illustrate how existence of realizations depends on the field K.

Example 12.2. Consider the *Fano configuration* given in Figure 12.3. It can be easily realized over $\mathbb{F}_2$ by taking all points on the 2-dimensional projective plane. By the Gallai–Sylvester theorem (Theorem 2.3), for every finite set of point in $\mathbb{R}^2$ not all on the same line, there must be a line containing only two points. This immediately implies that the Fano configurations cannot be realized over $\mathbb{R}$. In other words, there is no way to "straighten" the red line (456) circled in the figure.

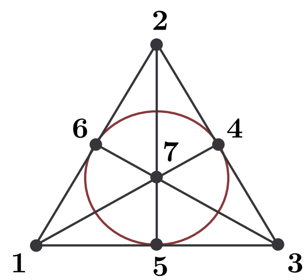

FIGURE 12.3. The Fano configuration.

Example 12.3. Consider *Perles configuration* $\mathcal{P}$ as shown in Figure 12.4. Note that it is realizable over $\mathbb{R}$ (when the pentagon in the middle is regular), but cannot be realized over $\mathbb{Q}$. In other words, we claim that it is impossible to make $\mathcal{P}$ rational. To prove this, first send the (1234) line to the line at infinity by a projective linear transformation. We obtain a pentagon [56798] whose four diagonals, all except for (57), are parallel to the opposite sides. Now use another projective linear transformation to send 5 to the origin $(0,0)$, $6 \to (0,1)$, $8 \to (1,0)$. Since $(78)\|(56)$ and $(69)\|(58)$, we have $7 \to (1,a)$, $9 \to (b,1)$. Also, since $(79)\|(68)$, we have $a = b = 1+t$ for some t. Finally, since $(59)\|(67)$, we get the following equation for the slopes:

$$\frac{t}{1} = \frac{1}{1+t},$$

and $t \in \{-\phi, \phi - 1\}$, where $\phi = \frac{\sqrt{5}+1}{2}$ is the *golden ratio*.

To summarize, we showed that if Perles configuration $\mathcal{P}$ had a rational realization, there would be a rational projective linear map into a nonrational arrangement, a contradiction. Thus $\mathcal{P}$ is not realizable over $\mathbb{Q}$.

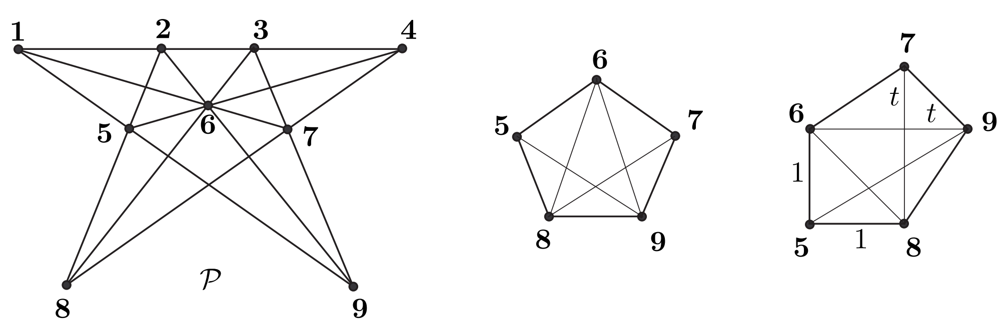

FIGURE 12.4. Three versions of the Perles configuration $\mathcal{P}$.

12.4. **Points and lines are really all you need.** We will show that there exist configurations whose realizations are not nonrational, but in fact can be as complicated as one wishes. To make this formal, consider all complex realizations $f : V \to \mathbb{CP}^2$ of a configuration $\mathcal{C} = (V, E)$

We consider finite products of ratios of the following type:

$$(\circledast) \qquad \theta := \frac{|v_{i_1} v_{i_2}|}{|v_{i_3} v_{i_4}|} \cdot \frac{|v_{i_5} v_{i_6}|}{|v_{i_7} v_{i_8}|} \cdot \ldots ,$$

where the points in the same ratio must belong to the same line. For a polynomial $P \in \mathbb{Q}[t]$, we say that configuration $\mathcal{C}$ *satisfies the law of* P if there exists a product $(\circledast)$ as above, such that $P(\theta) = 0$ for all complex realizations f of $\mathcal{C}$.

For example, in the Fano configuration without line (456) (see Figure 12.3) consider the ratio

$$\theta := \frac{|16|}{|62|} \cdot \frac{|24|}{|43|} \cdot \frac{|35|}{|51|} .$$

The *Ceva theorem* states that $\theta = 1$ in this case, the nicest law of all.

A different example is given by the Perles configuration. Consider the *cross-ratio* θ on a projective line (1234) defined as follows:

$$\theta := \frac{|12|}{|32|} \cdot \frac{|14|}{|34|}$$

Recall that cross-ratio is invariant under projective linear transformations. In Example 12.3, we showed that a projective linear transformation maps line (1234) into the infinite line, with points $1, 2, 3, 4$ mapped into $t, 0, -1, \infty$, respectively. Hence $\theta = -t$, where t is as in Figure 12.4. Thus, the calculations in the example above imply that the cross-ratio θ satisfies $\theta^2 - \theta - 1 = 0$, the "golden law".

Theorem 12.4 (Configuration universality)**.** *For every polynomial $P \in \mathbb{Q}[t]$, there exists a configuration $\mathcal{C}$ realizable over $\mathbb{C}$ which satisfies the law of P.*

Proof. We will draw the pictures over $\mathbb{R}$ for simplicity, but the whole construction can be done over $\mathbb{C}$. Fix a "coordinate system" by adding points and lines as in Figure 12.5. Any complex realization of this configuration can be moved into this one, with lines $x = (w_1 w_2 w_3)$ and $y = (w_1 w_4 w_6)$ being coordinate axes, an infinite line $(w_3 w_6)$, and $|w_1 w_2| = |w_1 w_4| = 1$. The main idea of the proof is to construct a configuration $\mathcal{C}_P$ starting with this coordinate system which would have points t and have $P(t)$ to be *identically* 0 (see Figure 12.5). Just like in the proof of Theorem 12.1, we will show that it is possible to add, subtract, multiply numbers and divide by an integer. At each step we add lines and points (including those at infinity if the lines are parallel) so that in every realization, the points of the configuration model the arithmetic operations. See Figure 12.6 for a step by step addition and multiplication.

For the addition of a and b make a line parallel to the y-axis (that is going through point a and an infinite point w_6) and take the intersection v with the $y = 1$ line, i.e., with the line $(w_4 w_5)$. Now, adding lines $(w_4 a)$ and a parallel line through v, i.e., a line though v and an intersection with the infinite line $(w_3 w_6)$, gives the desired point $(0, a{+}b)$ on the x-axis. Similarly, for multiplication, obtain a point $(b, 0)$ on the y-axis

FIGURE 12.5. Coordinate system for configurations.

by adding a line through point $(b, 0)$ parallel to $x + y = 1$ line, i.e. line (w_2w_4). Then make a line through point $(b, 0)$ parallel to a line through the points $(0, 1)$ and $(a, 0)$, obtaining an intersection point $(0, ab)$. Subtraction and division by an integer are completely analogous and will be skipped.

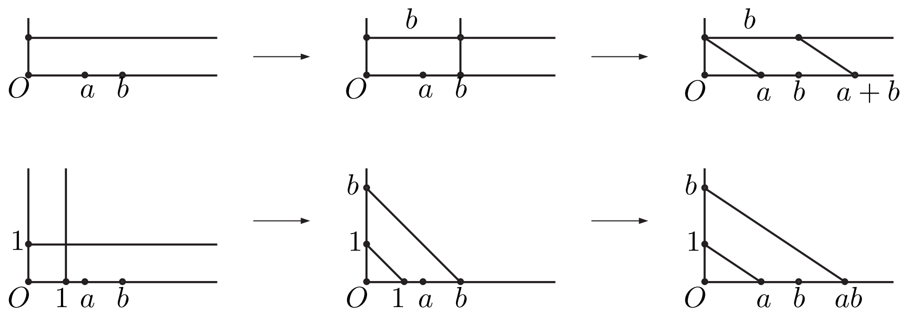

FIGURE 12.6. Addition and multiplication of a, b.

Now, suppose $P(t) = c_N t^N + \ldots + c_1 t + c_0$, $c_i \in \mathbb{Q}$, is the given polynomial. Start with a point t and construct all terms of the polynomial, eventually obtaining a configuration with t, $P(t)$ on the x axis, and one degree of freedom modulo coordinate system. Now consider a configuration where points $P(t)$ and $w_1 = O$ are identified, and so are some lines by implication. Naturally, such a configuration is realizable over $\mathbb{C}$ since every polynomial $P(t)$ as above has complex roots. On the other hand, the cross-ratio of the points w_1, w_2, w_3 and t on the x-axis line will satisfy the law of P. $\square$

Now that we have the result we can take the polynomial P to be as complicated as required. For example, if $P = t^2 - 2$, then the resulting configuration of points must be irrational. Perhaps even more surprising is the case $P = t^2 + 1$, when a configuration has complex, but no real, realizations.

Before we conclude, let us note that in place of one variable we could have chosen several variables and several polynomial equations. The same approach goes through, and at the end one obtains the configuration whose space of realizations satisfies any system of polynomial equations. A technical checking done by Mnëv shows that it is always possible to proceed with this without ever creating any extra equations. We leave all this aside. The main point of this subsection is to emphasize that in a certain sense (which can be made completely formal), all algebraic varieties over $\mathbb{Q}$ are encoded by finite configurations.

12.5. **Polytopes can also be irrational!** Let $P \subset \mathbb{R}^d$ be a convex polytope, and let L be the poset (lattice) of faces of P. We say that $P' \subset \mathbb{R}^d$ is *combinatorially equivalent* to P if they have the same poset L. We say that P is *realizable* over a subfield $\mathsf{K} \subset \mathbb{R}$ if there is a polytope P' combinatorially equivalent to P, such that all coordinates of P' lie in K. As it turns out, just like in case of realizations of points and lines, there exist polytopes which satisfy any prescribed law.

Theorem 12.5 (Perles, Mnëv)**.** *There exists a polytope $P \subset \mathbb{R}^d$ which cannot be realized over $\mathbb{Q}$. Moreover, there exists a polytope $P \subset \mathbb{R}^d$ which cannot be realized over any finite extension of $\mathbb{Q}$.*

Perles configuration (see Example 12.3) appeared in his original construction of an irrational polytope. This result may seem harder and unintuitive since in three dimensions this cannot happen: Corollary 11.3 of the Steinitz theorem asserts that all convex 3-dimensional polytopes are rational. Let us show first how to construct a polytope from a point and line configuration, and establish a direct map between their realization spaces.

The Lawrence construction. Think of a plane configuration $\mathcal{C}$ as a collection of vectors $V = \{v_1, \ldots, v_n\} \subset \mathbb{R}^3$, obtained by mapping points (x, y) into $(x, y, 1)$. Clearly, three points lie on a line if they are mapped into linearly dependent vectors. Now consider a vector space $\mathbb{R}^3 \times \mathbb{R}^n$ with the configuration $\mathcal{C}$ in $\mathbb{R}^3$, and an auxiliary space $\mathbb{R}^n$ spanned by vectors $w_1, \ldots, w_n$. Now consider a polytope $P_{\mathcal{C}}$ given by a convex hull of points

$$x_i = (v_i, w_i), \quad y_i = (v_i, 2w_i), \quad \text{where } x_i, y_i \in \mathbb{R}^3 \times \mathbb{R}^n,\ 1 \le i \le n.$$

The polytope $P_{\mathcal{C}}$ has a special combinatorial structure so that the linear relations (on vertices) inherit all linear relations of $\mathcal{C}$. To see this, notice that points $\{x_i, y_i \mid i \in I\}$ are *always* in convex position, and thus they lie on a facet if and only if

$$\sum_{i \in I} (2x_i - y_i) = \sum_{i \in I} (v_i, O) = (O, O),$$

i.e., when the points $\{v_i \mid i \in I\} \subset V$ lie on the same line in $\mathcal{C}$.

From the construction, we immediately conclude that if $\mathcal{C}$ is irrational (cannot be realized over $\mathbb{Q}$), as in Example 12.3, then so is the polytope $P_{\mathcal{C}}$. More generally, if a field K is a finite extension of $\mathbb{Q}$, then there exists a polynomial P with rational coefficients and roots not in K. By the configuration universality theorem there exists a configuration $\mathcal{C}$ which satisfies the law of P. Then the corresponding polytope $P_{\mathcal{C}}$ cannot be realized over K.

12.6. **Exercises.**

Exercise 12.1. [1] Prove that the center of a circle can be constructed by using only a compass (i.e., without a ruler).

Exercise 12.2. *a*) [1+] Prove that using the ruler, compass and angle-trisector one can construct a regular heptagon.
b) [2] Prove that a real cubic equation can be solved geometrically using a ruler, compass and angle-trisector if and only if its roots are all real.

Exercise 12.3. *a*) [1] Suppose $a_1, \ldots, a_n > 0$. Find the necessary and sufficient conditions for the existence of an n-gon with these edge lengths to be inscribed into a circle.
b) [2-] Prove that for $n \geq 5$, the radius of this circle cannot be constructed with a ruler and a compass (given edge lengths a_i).

Exercise 12.4. [1+] A geometry problem asks to use a ruler and compass to find a square with two vertices on each of the two given non-intersecting circles. Show how to solve it or prove that this is impossible in general.

Exercise 12.5. [1] The $y = x^2$ parabola is given, but the axes are erased. Reconstruct the axes using ruler and compass.

Exercise 12.6. [1] Prove that a regular pentagon cannot be realized in $\mathbb{Q}^3$. Conclude that the regular icosahedron cannot be realized in $\mathbb{Q}^3$.

Exercise 12.7. [1] The *Pappus theorem* states that for every two triples of points on two lines: $a, b, c \in L$ and $a', b', c' \in L'$ the points of intersection of $x = (a, b') \cap (a', b)$, $y = (a, c') \cap (a', c)$, $z = (b, c') \cap (b', c)$ must lie on a line. Remove this extra line to obtain a configuration with nine points and eight lines. Convert the Pappus theorem into a realizability result of this configuration over $\mathbb{R}$. Can this configuration be realized over $\mathbb{C}$ or over a finite field?

Exercise 12.8. [2] Find a (self-intersecting) polyhedral surface in $\mathbb{R}^3$ which cannot be realized over $\mathbb{Q}$.

Exercise 12.9. [2-] Find the analogue of Theorem 12.4 for polytopes. In other words, prove a universality type theorem which extends Theorem 12.5.

12.7. **Final remarks.** Thinking of Theorem 12.1 as a universality result is a classical idea (see [CouR, Ch. III §2]). Our writing was influenced by a terrific essay [Man]. Let us mention a well-known result by Mohr and Mascheroni, which asserts that every construction with a ruler and a compass is possible using a compass alone (see [CouR, Ch. III §5]). In a different direction, Steiner showed that given a fixed circle and its center, all the constructions in the plane can be carried out by the ruler alone. We refer to classical textbooks [Bieb, Leb2] for these and other results on geometric constructions (see also [Adl, Schr1]), and to [Hun, Kos, Mar, Smo] for an introduction (see also nice exercises in [CouR, Ch. IV §7]). Finally, an interesting complexity result comparing capabilities of two tools is given in [ABe].

Realizability of point and line configurations over fields is a part of the much more general subject of *matroid theory*. In a different direction, connections between various geometries go beyond realizability over fields, as there exist geometries without Desargues' theorem, etc. [Hil, Kag2].

What we call "Perles configuration" is the point and line configuration which appears in Perles's construction of the first non-rational polytope [Grü4, §3.5]. Theorem 12.4 is a weak version of the Mnëv Universality Theorem. For the introduction to the Mnëv theorem and the Lawrence construction see [Zie1, §6.6] and the original paper [Mnëv]. The full proof of the Mnëv theorem, further extensions and references can be found in [Ric].

13. Universality of linkages

This is the second of two sections on universality, where we investigate realizations of graphs in the plane with given edge lengths. The type of realization spaces we consider here will prove crucial in Section 31 where we study realization spaces of polyhedral surfaces. The universality of linkages, the main result in this section, also removes much of the mystery behind various constructions of flexible polyhedra which we discuss in Section 30.

13.1. **A machine can sign your name (as long as it is algebraic).** Imagine a configuration of bars and joints in the plane where some joints are fixed to a plane. Such configuration is called a *linkage*. If the linkage is not rigid, one can place a pen in one of the joints and let the configuration move. The result is a curve in the plane, and one can ask what curves can be obtained in this fashion.[26] For example, in a linkage shown in Figure 13.1, a pen placed in non-fixed joints $1, 2, 3$ give (parts of) the circles, while a pen in joint 4 gives a more complicated curve.

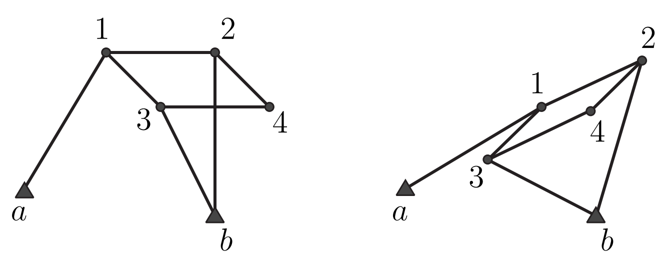

Figure 13.1. Two realizations of the same linkage with two fixed point a and b.

Let us make few remarks before we formalize the setup and state the main theorem. First, we always assume that the linkage is connected and has at least one fixed joint. Because of the metric constraints one cannot hope to obtain an unbounded curve, such as a parabola. Similarly, the resulting curve can be disconnected (this is in fact the case for the linkage in Figure 13.1, where joint 1 cannot be continuously pushed down); while the 'pen' analogy no longer makes sense, we will consider all connected components together as parts of the curve. Finally, in contrast with the point and line configurations, we will allow all possible degeneracies such as intersections of bars, overlap of joints, etc.

Formally, define a *linkage* $\mathcal{L}$ to be a graph $G = (V, E)$ and a *length function* $L : E \to \mathbb{R}_+$, where we denote the length of the edge $e = (v, w)$ by $\ell_e = L(e)$. Fixed points are the vertices $H \subset V$ and a function $h : H \to \mathbb{R}^2$. A *realization* of linkage $\mathcal{L}$ is a function $f : V \to \mathbb{R}^2$ such that $f(v) = h(v)$ for all $v \in H$, and the distance of the image of an edge $|f(v)\, f(w)| = \ell_e$ for all $e = (v, w) \in E$. By $\mathcal{M}_{\mathcal{L}}$ denote the *space of realizations* of $\mathcal{L}$.

[26]One can also talk about general semi-algebraic regions which can be "drawn" in the plane this way. We will restrict ourselves to curves for simplicity.

Suppose we are given an *algebraic curve* $C \subset \mathbb{R}^2$, defined by

$$F(x,y) = \sum_{i=0}^{n}\sum_{j=0}^{n} c_{i,j}\, x^i\, y^j = 0, \quad \text{where } c_{i,j} \in \mathbb{R},\, n \in \mathbb{N}.$$

We say that linkage $\mathcal{L}$ *draws* curve C if the space of realizations of some joint $v \in V$, defined as $\mathcal{M}_{\mathcal{L}}(v) = \{f(v) \mid f \in \mathcal{M}_{\mathcal{L}}\}$ lies on C, i.e. $F(x,y) = 0$ for all $(x,y) \in \mathcal{M}_{\mathcal{L}}(v)$. Finally, we say that curve C is *mechanical* if every compact subset $X \subset C$ lies in the realization space of a joint of a linkage which draws C: $X \subset \mathcal{M}_{\mathcal{L}}(v) \subset C$.

Theorem 13.1 (Linkage universality)**.** *Every algebraic curve $C \subset \mathbb{R}^2$ is mechanical.*

The idea of the proof is roughly similar to that of the configuration universality theorem (Theorem 12.4). We will construct a linkage with two degrees of freedom along axes x and y. We place the 'pen' into a joint which will always be located at (x,y). This linkage will also have a joint z located at $(F(x,y),0)$ on axis x. Once we fix the point z at the origin O, we obtain the desired linkage.

Outline of the proof. Let us start with some basic remarks. First, in the definition of linkages, the bars are allowed to connect only in the joints. On the other hand, by *rigidifying* the bars we can also connect them to the middle of bars as in Figure 13.2. Thus, from now on we can always allow joints in the middle of bars.

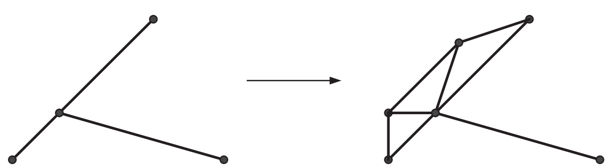

Figure 13.2. Rigidifying the bars with joints in the middle.

Second, we will ignore the connected components issue: there is always a way to add few extra bars to remove the undesired realizations. For example, in Figure 13.3 we first show how to ensure that the rigid triangle has all realizations oriented the same way, and then how to avoid the self-intersecting realizations in a parallelogram.

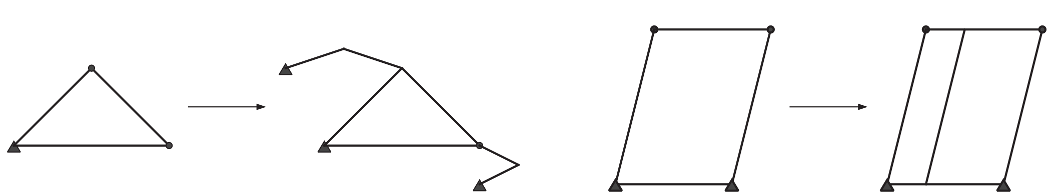

Figure 13.3. Avoiding extra realizations of linkages.

Finally, in our linkages we will never specify the absolute lengths of the bars, only their relative lengths to ensure the realizations lie on a given curve C. It is easy to see that the lengths can be taken large enough to satisfy the conditions of the theorem.

Let us start constructing a linkage by making a frame. Fix the joint at the origin O, and let x and y be the point at the end of the rhombi $(Oaxb)$ and $(Ocyd)$, as in Figure 13.4, with sides ℓ_1 and ℓ_2. Later we will ensure that these points are moved only along the corresponding axes. We will also need to obtain point y' along the x axis, and at distance $|yO|$ from the origin. We do this as in Figure 13.4, by adding an equal rhombus $(Oc'y'd')$ and adding bars (cc') and (dd') of length $\sqrt{2}\ell_2$, to form right equilateral triangles (cOc') and (dOd').

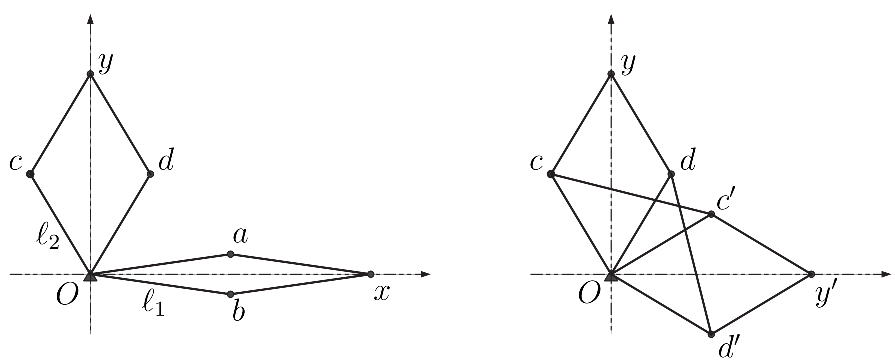

FIGURE 13.4. Obtaining axis joints x, y and the joint y'.

To obtain the desired joint $F(x, y)$ as above we need to be able to add and multiply distances of joints along the x axis, add constants, and multiply by constants.

The *translator* construction in Figure 13.5 is an easy way to make addition: $z = x + c$. For $c > 0$, simply take two fixed point a, b at distance c, parallel to x axis. Then connect them two points x, z through two parallelograms. Note that subtraction follows similarly, by switching roles of x and z. The case $c < 0$ can be made by a similar construction.

Similarly, the *pantograph* construction in is an easy way to make multiplication: $z = x \cdot c$, for any real $c \in \mathbb{R}$. Simply make homothetic triangles (Oux) and (Owz), the second with sides c times the first. To ensure they are homothetic add a parallelogram $(xuwv)$. Note that the pantograph construction uses joints in the middle of bars, which we can use by the observation above.

Adding two points by linkages is in fact easy to do in general, not only when the points are restricted to a line. The adder in Figure 13.6 models vector addition $z = x + y$. On the other hand, modelling multiplication by linkages is a delicate task which requires some preparation.

First, observe that squaring is enough to model multiplication since

$$x \cdot y = \frac{1}{4}(x+y)^2 - \frac{1}{4}(x-y)^2,$$

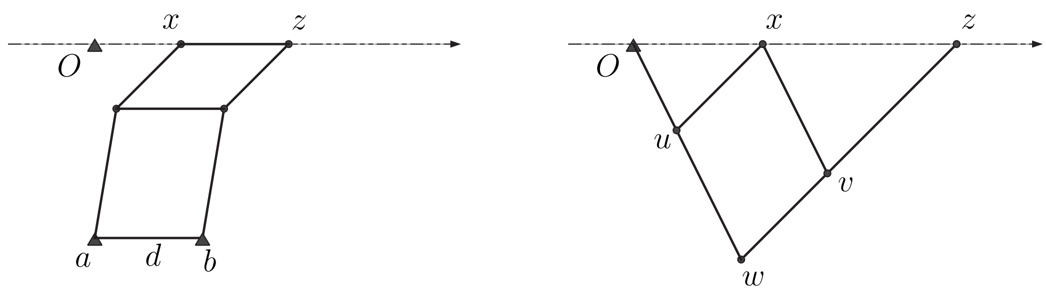

FIGURE 13.5. Translator models addition $z = x + d$, and pantograph models multiplication by a constant $z = x \cdot c$, where $c = |Ow|/|Ou|$.

and we already know how to add, subtract, and multiply by a constant. Similarly, squaring can be obtained from the inversion, since

$$\frac{1}{z-1} - \frac{1}{z+1} = \frac{2}{z^2-1}.$$

Finally, the celebrated *Peaucellier inversor* shown in Figure 13.6, models the inversion $z = (a^2 - b^2)/x$. Indeed, for the height of the triangle we have:

$$h^2 = a^2 - \left(\frac{x+z}{2}\right)^2 = b^2 - \left(\frac{x-z}{2}\right)^2.$$

Therefore,

$$x \cdot z = \left(\frac{x+z}{2}\right)^2 - \left(\frac{x-z}{2}\right)^2 = a^2 - b^2,$$

as claimed.[27] Let us summarize what we just did: through a sequence of above reductions, we obtained the desired multiplication by linkage constructions.

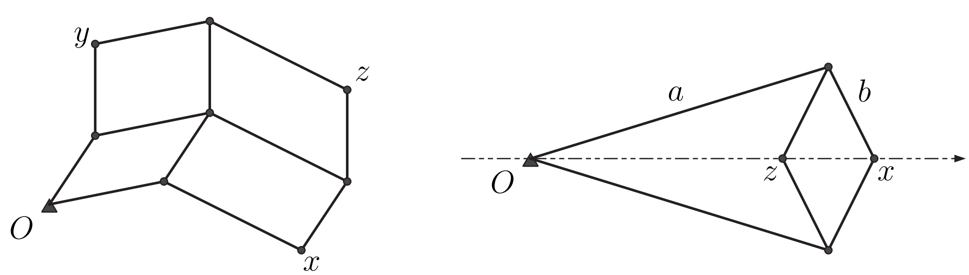

FIGURE 13.6. Adder $z = x + y$ and inversor $z = (a^2 - b^2)/x$.

[27]There is an often overlooked problem with the Peaucellier inversor which has to do with the central rhombus collapsing. Even if the rhombus is rigidified as above, one can still rotate the sides around the center. Attaching two extra edges to the opposite vertices in the rhombus prevents that; we leave the details to the reader.

It remains to show that we can have joints x and y move along the axis, an assumption we used in the beginning. In fact, drawing a straight line is historically the hardest part of the proof. This can be done by a linkage as in Figure 13.7. To see why the linkage draws a straight line, note that the main part of it is the Peaucellier inversor, which maps the dotted circle into a dotted line (i.e., a circle of 'infinite' radius).

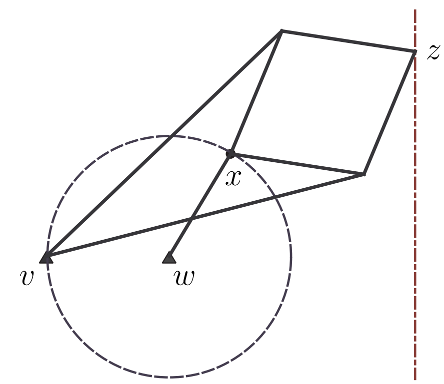

FIGURE 13.7. Drawing a straight line.

Now, after all these constructions we need to put everything together. Draw two orthogonal lines, representing the axes of the coordinate system as in Figure 13.4 above. This is our frame and the desired joint will be in position $v = (x, y)$. Rotate the joint y onto x-axis and obtain joint $F(x, y)$ via a series of operations (additions, multiplications, etc.) as above. Fix this joint $F(x, y)$ at the origin O. This restricts the space of realizations of v to the curve $F(x, y) = 0$, as desired.

Finally, the relative restrictions on the sizes of our bars will give limits on the compact subset of curve C. It remains to check that for every compact subset X of C there exist an appropriate collection of bar lengths so that the realization space would contain X. We omit this part. $\square$

13.2. **Spherical linkages.** It is natural to ask whether the configuration universality theorem (Theorem 12.4) is related to the linkage universality theorem (Theorem 13.1). As it turns out, the former is a special case of the spherical version of the latter. In other words, the world of linkages is much richer and contains planar configurations as a very special case. The following simple construction starts with a planar configuration and produces a spherical linkage with (essentially) the same realization space.

Spherical linkage construction. Start with a planar configuration $\mathcal{C} = (V, E)$ and make one joint on a sphere for every point $a_i \in V$ and every line $\ell_j \in E$. Whenever a point a_i lies on a line ℓ_j, attach between them a (spherical) bar of length $\pi/2$. Note that for every spherical realization of the resulting linkage $\mathcal{L}$, the points which lie on the same line $\ell_j \in E$ now lie on the same great circle in $\mathbb{S}^2$ and vice versa.

In order to make this construction completely rigorous and straightforward one needs to substitute $\mathbb{S}^2$ with $\mathbb{RP}^2$ and consider all projective realizations of $\mathcal{C}$. We leave the details to the reader.

13.3. **Exercises.**

Exercise 13.1. [1+] Give an explicit construction of a linkage which draws a regular pentagon.

Exercise 13.2. [1+] Give an explicit construction of a linkage which draws an ellipse with distinct axes.

Exercise 13.3. [2-] Prove that 3-dimensional algebraic curves and surfaces can be drawn by 3-dimensional linkages.

Exercise 13.4. Define a *spherical polygon* $[x_1 \ldots x_n] \subset \mathbb{S}^2$ to be the union of geodesic arcs $[x_i x_{i+1}]$ on a unit sphere. Denote by $\mathcal{L}$ the set of sequences of lengths of these arcs:

$$(\ell_1, \ldots, \ell_n), \quad \text{where} \ \ \ell_i = |x_i x_{i+1}|_{\mathbb{S}^2}\,.$$

a) [1-] Prove that if a sequence $(\ell_1, \ldots, \ell_n) \in \mathcal{L}$ then so does every permutation $(\ell_{\sigma(1)}, \ldots, \ell_{\sigma(n)})$, $\sigma \in S_n$.

b) [2-] Prove that $(\ell_1, \ldots, \ell_n) \in \mathcal{L}$ if and only if

$$0 \le \ell_i \le \pi, \quad \text{for all} \ \ 1 \le i \le n, \ \ \text{and}$$

$$\sum_{i \in I} \ell_i \, \le \, \sum_{j \notin I} \ell_j \, + \, \bigl(|I| - 1\bigr)\pi, \quad \text{for all} \ \ I \subset [n], \ |I| \ \text{odd}.$$

c) [1] Prove that the inequalities in part *b*) are non-redundant.

13.4. **Final remarks.** The study of linkages is long, classical and exciting. It became popular with Watt's discovery in 1784 of a linkage which approximately translates a forward into a circular motion. This mechanical linkage has been used in trains and is still in use in many car models. Chebyshev studied Watt's linkage for thirty years and introduced his own linkages. Interestingly, he believed that a linkage which draws a line is impossible to construct and introduced the Chebyshev polynomials in an effort to approximate the line with low degree polynomials. Somewhat unusually, Peaucellier announced his discovery in a letter in 1871 without a hint to the actual construction, so in 1873 it was independently rediscovered by Lipkin. We refer to [CouR, Ch. III §5.4] and [HilC, §40] for the introductory treatment, and to a celebrated lecture by Kempe [Kem] who essentially proved our linkage universality theorem (Theorem 13.1). For the story of interesting polyhedral linkages in three dimensions see [Gol1].

The universality of linkages idea was first proposed by Kempe and Lebesgue (see above), and was revived by Thurston who popularized it with a saying "*there exists a linkage which signs your name*" [King]. These results were recently extended and formalized in a powerful paper by Kapovich and Millson [KM4] (see also [King]). Our Theorem 13.1 is a toy version of their results, which are much more technical and precise. On the other hand, the basic idea can already be seen in the proof above. Finally, the connection between spherical linkages and planar configurations was found in [KM2].

Let us mention that throughout history *kinematics* has been an honorable field of study, and some of the linkages described above (such as the pantograph) have been used in practice. One can also use the linkages to make various computations and until very recently such linkage based machines were still in use in navigation and other fields [Svo].[28] From a theoretical point of view, the universality theorem shows that mechanical computers in a certain sense can find roots of all algebraic equations over $\mathbb{R}$. Fortunately, digital computers do a much better job.

[28] In 1948 the author of the Math. Review on [Svo] praised them as "the most economical, reliable and sturdy machines".

14. Triangulations

In this short section we introduce and study Voronoi diagrams and Delaunay triangulations, some of the most basic constructions in computational geometry. They will be used later in Section 21 in the proof of the four vertex theorem, and in Section 40 to study non-overlapping unfoldings of convex polyhedra. We also study the flips (local moves) on the planar and regular triangulations. Flips are an important tool which will prove useful later in Section 17.

14.1. **Flips on planar triangulations.** Let $X \subset \mathbb{R}^2$ be a set of n points in the plane. Denote by $\mathcal{T}(X)$ be the set of triangulations T of $\operatorname{conv}(X)$, such that vertices of triangles in T are all in X, and every point in X appears as a vertex of at least one triangle in T. We call these *full triangulations*. Define a *flip* (also called 2-*move*) to be a transformation $T_1 \leftrightarrow T_2$ on triangulations $T_1, T_2 \in \mathcal{T}(X)$ which replaces one diagonal of a convex quadrilateral with another one (see Figure 14.1).

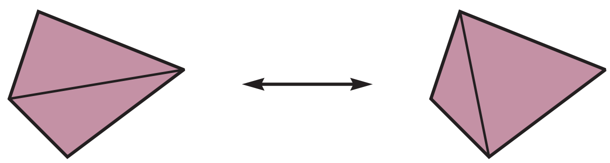

Figure 14.1. An example of a flip.

We say that points in X are in *general position* if no three points lie on a line and no four points lie on a circle.

Theorem 14.1. *Let $X \subset \mathbb{R}^2$ be a set of n points in the plane in general position. Then every two full triangulations of* $\operatorname{conv}(X)$ *are connected by a finite sequence of flips. Moreover, n^2 flips will always suffice.*

In other words, for every $T, T' \in \mathcal{T}(X)$ we have $T \leftrightarrow T_1 \leftrightarrow T_2 \leftrightarrow \ldots \leftrightarrow T_\ell \leftrightarrow T'$, for some $T_1, T_2, \ldots, T_\ell \in \mathcal{T}(X)$ and $\ell < n^2$ (see Figure 14.2). We prove the theorem later in this section.

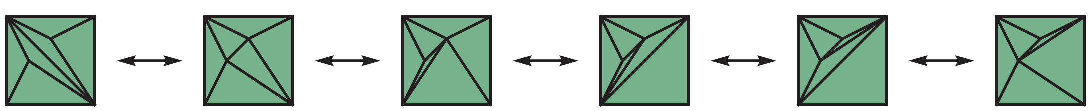

Figure 14.2. A sequence of moves connecting all triangulations with vertices at given six points.

Remark 14.2. (*Triangulations of a convex polygon*). When n points in $X \subset \mathbb{R}^2$ are in convex position, the number of triangulations $|\mathcal{T}(X)| = C_{n-2}$ is a Catalan number (see Example 8.5). The connectivity of the flip graph G_n on all triangulations of a convex polygon is an elementary result which can be seen directly, by connecting any triangulation to a star triangulation (a triangulation where all diagonals meet at a vertex, see Subsection 17.5), or by induction. Not only is this a special case of Theorem 14.1, but also of Theorem 14.10 presented below. Recall that the flips in this case correspond to edges of the associahedron, a simple $(n-3)$-dimensional convex polytope (see Example 8.5 and Exercise 8.2). By Balinski's theorem (Theorem 8.8), this implies that the flip graph G_n is $(n-3)$-connected.

14.2. **Voronoi diagrams and Delaunay triangulations.** Let $X = \{v_1, \ldots, v_n\}$ be a finite set of points in the plane. Define the *Voronoi diagram* $VD(X)$ as a collection of *Voronoi cells* D_i, defined as the set of points $z \in \mathbb{R}^2$ which are closer to v_i than to any other point in X:

$$D_i = \big\{ w \text{ such that } |wv_i| < |wv_j| \text{ for all } i \neq j \big\}.$$

By construction, cells D_i are open, possibly unbounded convex polygons (see Figure 14.3). Clearly, for X in general position no four cells of the Voronoi diagram $VD(X)$ meet at a point.

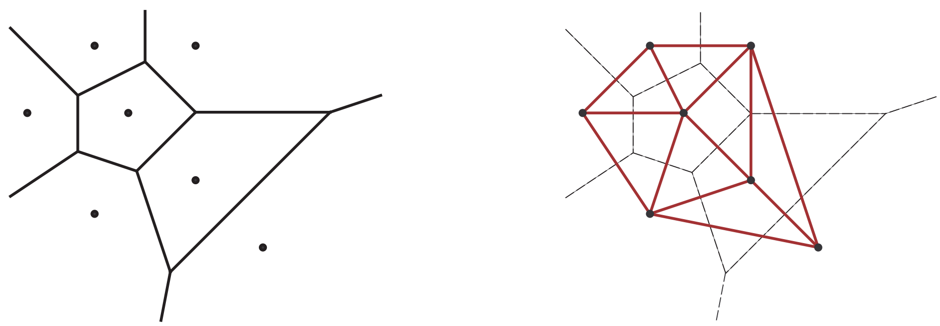

FIGURE 14.3. Voronoi diagram and the corresponding Delaunay triangulation.

Connect by a line segment every two points $v_i \neq v_j$, such that cells D_i and D_j are adjacent (see Figure 14.3). A *Delaunay triangulation* $DT(X)$ is defined by the resulting edges on X and can be viewed as a subdivision dual to $VD(X)$. The name is justified by the following result.

Proposition 14.3. *Let $X = \{v_1, \ldots, v_n\} \subset \mathbb{R}^2$ be a finite set of points in general position. Then the Delaunay triangulation $DT(X)$ defined above is a triangulation in $\mathcal{T}(X)$.*

Proof. First, let us show that for every vertex v_i of $\operatorname{conv}(X)$ the cell D_i is unbounded. Take a bisector ray R of the outside angle at v_i and observe that all points $w \in R$ lie in the Voronoi cell D_i. Similarly, let us show that for every edge (v_i, v_j) of $\operatorname{conv}(X)$, the cells D_i and D_j are adjacent. Take a ray R bisecting and perpendicular to (v_i, v_j). Observe that points $w \in R$ far enough from (v_i, v_j) have v_i and v_j as their closest points. This implies that $DT(X)$ contains all edges of $\operatorname{conv}(X)$. On the other hand, since X is in general position, no four cells of the Voronoi diagram meet at a point, which implies that $DT(X)$ is a triangulation. $\square$

Let $T \in \mathcal{T}(X)$ be a triangulation of $\operatorname{conv}(X)$. We say that a circle circumscribed around triangle $(v_i v_j v_k)$ in T is *empty*, if it contains no other vertices of X in its interior.

Proposition 14.4 (Empty circle condition)**.** *Let $DT(X)$ be the Delaunay triangulation of a polygon* $\operatorname{conv}(X)$*, where $X \subset \mathbb{R}^2$ is a finite set of points in general position. Then the circles circumscribed around triangles in $DT(X)$ are empty.*

The conclusion of the proposition is called the *empty circle condition.* Later in this section we prove that this condition is not only necessary, but also sufficient (for a triangulation to be the Delaunay triangulation).

Proof. In the Voronoi diagram, consider the circumscribed circle C around triangle $(v_i v_j v_k)$. By construction, the center O of C is a meeting point of cells D_i, D_j and D_k. Since X is generic, no other point can lie on C. If some v_r lies inside C, that means that O must be strictly closer to v_r than to v_i, a contradiction. □

Define a map $\eta : \mathbb{R}^2 \to \mathbb{R}^3$ by $\eta(x, y) = (x, y, x^2 + y^2)$. Consider a convex hull P of points $w_i = \eta(v_i)$. A face F of P is said to be in the *lower convex hull* if P lies above the plane spanned by F. Projections of all faces in the lower convex hull onto the plane $(x, y, 0)$ give a subdivision of $\operatorname{conv}(X)$. The following result shows that this subdivision is in fact a Delaunay triangulation.

Theorem 14.5 (Paraboloid construction)**.** *Let $X \subset \mathbb{R}^2$ be a finite set of points in general position. Then, the polytope P defined above is simplicial and the projections of the triangular faces in its lower convex hull is the Delaunay triangulation $DT(X)$.*

Note that we have some freedom of choice in the paraboloid construction, by making any parallel translation of it. The theorem implies that the construction is invariant under these transformations.

Proof. First, let us show that every intersection of a non-vertical plane $z = ax + by + c$ and a paraboloid $(x, y, x^2 + y^2)$ project onto a circle (see Figure 14.4). Indeed, the intersection satisfies $x^2 + y^2 = ax + by + c$, i.e. $\left(x - \frac{a}{2}\right)^2 + \left(y - \frac{b}{2}\right)^2 = c + \frac{a^2}{4} + \frac{b^2}{4}$, as desired. Now, if four points in $\eta(X)$ lie in the same plane, then their projections are four points in X which lie in the same circle. This is impossible since X is in general position. This implies that polytope P is simplicial.

Let F be a triangular face in the lower convex hull of P, let H be the plane spanned by F, let Δ be the projection of F onto the plane $(x, y, 0)$, and let C be the circumscribed circle around Δ. From above, if a point v_i lies inside C, then $\eta(v_i)$ lies below H, which contradicts the assumption that F is in the lower convex hull. Therefore, every circumscribed circle C as above is *empty*, i.e., contains no other vertices of X in its interior.

In the opposite direction, suppose $T \in \mathcal{T}(X)$ is a triangulation of $\operatorname{conv}(X)$, such that all circles circumscribed around triangles $\Delta = (v_i v_j v_k)$ in T, are empty. Let H be the plane spanned by the triangle $F = (w_i w_j w_k)$, where $w_s = \eta(v_s)$. By the argument above, points w_s lie above H, for all $s \notin \{i, j, k\}$, which implies that F is a face of P. By Proposition 14.4, in the Delaunay triangulation $T = DT(X)$ all circumscribed

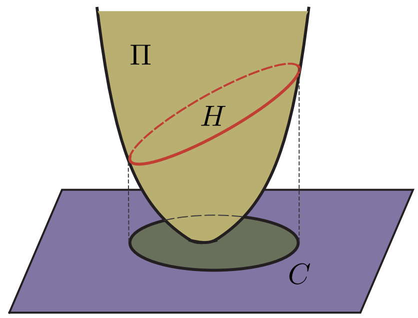

FIGURE 14.4. Intersection of a paraboloid Π with a plane H projects onto a circle C.

circles as above are empty. Therefore, $DT(X)$ is equal to the projection of the lower convex hull of P, as desired. □

Corollary 14.6 (Empty circle criterion)**.** *Triangulation $T \in \mathcal{T}(X)$ is the Delaunay triangulation $DT(X)$ if and only if it satisfies the empty circle condition in Proposition 14.4.*

This follows immediately from the last part in the proof of the theorem.

14.3. **Local convexity is just as good as global one.** To prove the main theorem (Theorem 14.1) we will need the following technical and intuitively obvious result.

Let $X = \{v_1, \ldots, v_n\} \subset \mathbb{R}^2$ be a set of points in general position, and let $T \in \mathcal{T}(X)$ be a triangulation of $A = \operatorname{conv}(X)$. Fix a function $f : X \to \mathbb{R}$ and extend function f to the whole A by linearity on each triangle in T. The resulting function $\xi : A \to \mathbb{R}$ can be viewed as a surface $S = S_f \subset \mathbb{R}^3$ which projects onto A.

Lemma 14.7 (Local convexity criterion)**.** *Suppose $\xi : \operatorname{conv}(X) \to \mathbb{R}$ and the surface S_ξ defined above is convex at every diagonal in T. Then S_ξ is a lower convex hull of points $\{(x, \xi(x)),\, x \in A\}$.*

In other words, the lemma is saying that the local convexity condition on S (at every diagonal) implies the global convexity of S. One can think of this result as a 3-dimensional generalization of Exercise 24.2, and the proof follows similar lines[29]

Proof. Let us first show that the cones C_x at every interior vertex $x = (v_i, \xi(v_i))$ of S_ξ are convex. Take x and intersect the neighborhood of x in S_ξ with a small sphere. The intersection is a simple spherical polygon with all angles $< \pi$. By the spherical analogue in Exercise 24.2, this implies that the cone C_x is convex.

Let $P = \{(x, z), x \in A, z \geq \xi(x)\}$ be the set of points above S_ξ. Proving that P is convex implies the result. Suppose P is not convex. Let us use a version of the argument as in the proof of Exercise 24.2. Then there exist two interior points $x, y \in P$ such that the shortest path γ between x and y inside P is not straight. Denote by y a point on the surface S_ξ where γ is locally not straight. Now note that neither y can

[29]The reader might want to solve the exercise first, or at least read the hint.

lie in the vertices of S_ξ because the cones are convex, nor on the edges of S_ξ, because all dihedral angles are $< \pi$, nor on faces of S_ξ, a contradiction. $\square$

14.4. **Increasing flips and the proof of Theorem 14.1.** Suppose $(v_i v_j v_k)$ and $(v_j v_k v_r)$ are triangles in a triangulation $T \in \mathcal{T}(X)$, such that v_r lies inside a circle circumscribed around $(v_i v_j v_k)$. In other words, more symmetrically, suppose $\angle v_j v_i v_r + \angle v_i v_r v_k < \angle v_i v_j v_k + \angle v_j v_k v_r$. A flip replacing (v_i, v_k) with (v_j, v_r) is called *increasing* (see Figure 14.5). Observe that we can make an increasing flip if an only if x_k lies in the circle circumscribed around $(v_i v_j v_r)$.

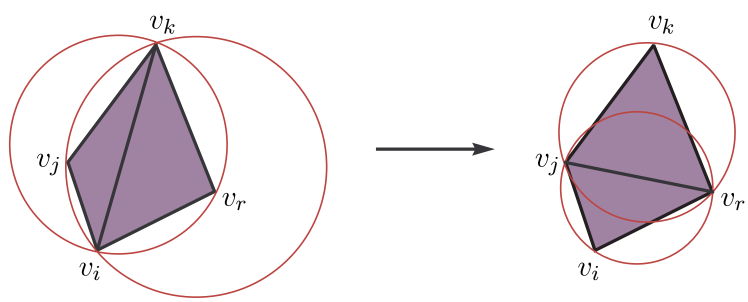

FIGURE 14.5. An increasing flip.

Suppose $T \in \mathcal{T}(X)$ is a triangulation different from the Delaunay triangulation $DT(X)$. From the empty circle criterion (Corollary 14.6), there must be a non-empty circumscribed circle. The following result is a stronger version of this observation.

Lemma 14.8 (Increasing flip condition)**.** *Let $X = \{v_1, \ldots, v_n\} \subset \mathbb{R}^2$ be a set of points in general position, and let $DT(X)$ be the Delaunay triangulation. Then every triangulation $T \neq DT(X)$ allows at least one increasing flip.*

We say that vertex v_k is *adjacent* to triangle $\Delta = (v_i v_j v_r)$ if it forms a triangle with one of the edges of Δ. The lemma then says that in order to check whether T is a Delaunay triangulation it suffices to check the empty circle condition only for the vertices adjacent to the triangles.

Proof. Consider a triangulation T of $A = \text{conv}(X)$, and let $\eta : \mathbb{R}^2 \to \mathbb{R}$ be as in the paraboloid construction. Denote by $S_\eta \subset \mathbb{R}^3$ the surface as in the proof of Lemma 14.7. If T has no increasing flips, then every circumscribed circle around triangle Δ in T does not have have vertices adjacent to Δ in its interior. Then, by the same argument as in the proof of Theorem 14.5, surface S_η is convex upward at every diagonal in T. Now the local convexity criterion (Lemma 14.7) implies that S_η is a lower convex hull. From Theorem 14.5, we conclude that $T = DT(X)$, a contradiction. $\square$

Proof of Theorem 14.1. Fix a triangulation $T \in \mathcal{T}(X)$. We will show that starting with T, after at most $\binom{n}{2}$ increasing flips, we can always obtain $DT(X)$. This implies that every two triangulations are connected by at most $2\binom{n}{2} < n^2$ flips.

Let $S = S_\eta$ be the surface defined as above. Suppose there exists an increasing flip $T \to T'$, replacing some (v_j, v_k) with some $(v_i v_r)$. Observe that the resulting surface S' lies directly below S, and, in particular, below the edge (v_j, v_k). Continue making increasing flips. From above, the edges of the flips are never repeated, and after at most $\binom{n}{2}$ flips we obtain a triangulation without increasing flips. By the lemma, this must be the Delaunay triangulation $DT(X)$. $\square$

Remark 14.9. (*Delaunay triangulations in higher dimensions*). For a set of points $X \subset \mathbb{R}^d$, $d \geq 3$, the definitions of Voronoi diagram and Delaunay triangulations extend nearly verbatim (see Exercise 14.2). The same goes for the *empty sphere criterion* (Corollary 14.6 and the Delaunay triangulation construction (Theorem 14.5).

Unfortunately, the increasing flips argument (Lemma 14.8) does not hold in full generality. Figuring out what goes wrong is difficult to see at first. That is because the intuitively obvious local convexity criterion no longer holds (see Exercise 14.3). In fact, the analogue of Theorem 14.1 is an open problem for triangulations of convex polytopes in $\mathbb{R}^3$. It fails for general sets $X \subset \mathbb{R}^3$ (that are not necessarily in convex position), and for convex X in higher dimensions (see Subsection 14.7).

14.5. **Regular triangulations.** While a direct generalization of Theorem 14.1 fails for triangulations in higher dimensions (see Remark 14.9 and Subsection 14.7), choosing a nice subset of triangulations is still connected by flips. The following variation on Theorem14.1 will prove useful in Section 17.

Let $P \subset \mathbb{R}^d$ be a simplicial convex polytope and let V be the set of its vertices. We say that P is *generic* if no $d+1$ vertices lie in the same hyperplane. A *triangulation* of P is a full triangulation with vertices in V. By a slight change of notation, denote $\mathcal{T}(P)$ the set of all triangulations of P; these triangulations are exactly full triangulations on the set of vertices V. For the rest of this section, we will consider only full triangulations of polytopes.

For every function $\xi : V \to \mathbb{R}$, let S_ξ be the lower convex hull surface of points $\big(v, \xi(v)\big) in \mathbb{R}^{d+1}$ defined as above. The last coordinate of a point $(x, \xi) \in S$ is called the *height* of $x \in P$, and the function ξ is called the *height function.*

Observe that the projection of S_ξ onto P gives a polyhedral subdivision Q_ξ of P (see Figure 14.6). We call Q_ξ the subdivision *associated* with the function ξ. We say that ξ is *generic* if no $d+2$ vertices of S_ξ lie in the same hyperplane in $\mathbb{R}^{d+1}$. Note that when ξ is generic, Q_ξ is a full triangulation: $Q_\xi \in \mathcal{T}(P)$.

A full triangulation $T \in \mathcal{T}(P)$ is called *regular* if there exists a height function ξ, such that $T = Q_\xi$ is a triangulation associated with ξ. Denote by $\mathcal{T}_*(P)$ the set of all regular triangulations $T \in \mathcal{T}(P)$. For example, by the paraboloid construction (Theorem 14.5), the Delaunay triangulation of a convex polygon is always regular. In fact, part *a*) of Exercise 14.10 says that every triangulation of a convex polygon is regular.

For the rest of this section we take $d = 3$; already in $\mathbb{R}^3$, the flips on triangulations are no longer obvious. Take a bipyramid with two triangulations T_1 and T_2 as in Figure 14.6, with two and three tetrahedra, respectively. Define the 2–3 *move* $T_1 \leftrightarrow T_2$ to be a local transformation between them.

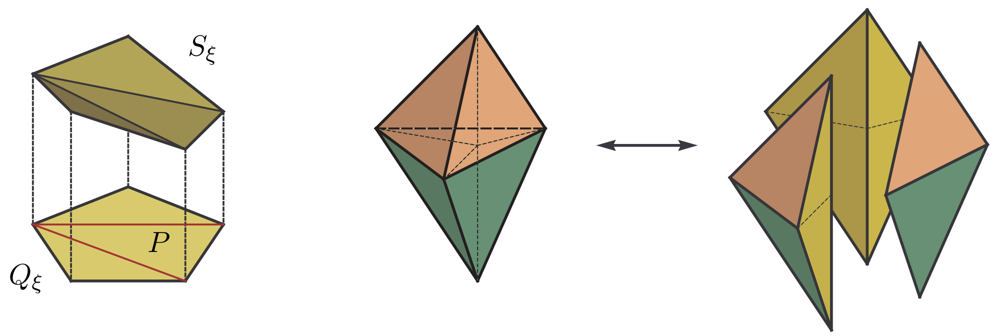

FIGURE 14.6. Pentagon P, surface S_ξ and a triangulation Q_ξ associated with ξ; a 2–3 move.

Theorem 14.10 (Connectivity of regular triangulations)**.** *Let $P \subset \mathbb{R}^3$ be a generic convex polytope. Then every two regular triangulations $T_0, T_1 \in \mathcal{T}_*(P)$ are connected by a finite sequence of* 2–3 *moves.*

Proof. Before we proceed with the proof, let us make some preliminary observations. Denote by $\Phi(P)$ the set of all generic height functions on P. Because P is also generic, the condition that $\Phi \in \Xi(P)$ is equivalent to an inequality, for every 5-tuple of vertices of P. Geometrically, in the space $W = \mathbb{R}^V$ of functions on the vertices, this means that $\Phi(P) \subset W$ is a complement to a finite union of hyperplanes.

Suppose T_0 and T_1 are two regular triangulations of P, associated with the height functions ξ_0 and ξ_1, respectively. We assume that ξ_0 and ξ_1 are generic, since otherwise we can perturb them without changing triangulations.

Consider now a family of height functions $\xi_t = (1-t)\xi_0 + t\xi_1$, where $t \in [0,1]$. Again, we can assume that all ξ_t are generic, at all but a finite set of values $Z \subset [0,1]$. Indeed, if we view ξ_0 and ξ_1 as points in W, we can always perturb them in such a way that the straight interval between them crosses hyperplanes discussed above one at a time. In other words, the interval $[\xi_0, \xi_1]$ will lie in $\Phi(P)$ everywhere, except for finitely many points corresponding to these intersections.

Denote by $S_t = S_{\xi_t} \subset \mathbb{R}^4$ the lower convex hull surface defined above, and by Q_t the associated subdivision of P, for all $t \in [0,1]$. From above, the surfaces S_t are simplicial for all $t \notin Z$, and Q_t are regular triangulations. Note also that between different values in Z, the triangulations are unchanged, and the change can happen only at Z.

Consider now what happens at $z \in Z$. By construction, S_z is simplicial everywhere except at one facet F with five vertices. Therefore, Q_z is a triangulation except for a 5-vertex polytope. Recall that the only generic 5-vertex polytope in $\mathbb{R}^3$ is a bipyramid, and that the latter have only two triangulations corresponding to a 2–3 move. Therefore, by continuity, we conclude that Q_t where $t \to z-$, and Q_t where $t \to z+$, correspond to two triangulations which differ by a 2–3 move. Since Z is finite, this implies that there is a finite sequence of 2–3 moves corresponding to all $z \in Z$, which connects T_0 and T_1. $\square$

14.6. **Exercises.**

Exercise 14.1. $\Diamond$ *a)* [1] State precisely what generality assumption need to be used on ξ_0 and ξ_1 in the proof of Theorem 14.10.
b) [1] Suppose polytope $P \subset \mathbb{R}^3$ has n vertices. Give a polynomial upper bound on the number of 2–3 moves between any two full triangulations of P.
c) [1] Generalize Theorem 14.10 to higher dimensions.

Exercise 14.2. (*Delaunay triangulations in higher dimensions*) $\Diamond$ Define the Voronoi diagram and Delaunay triangulation of a finite sets of points $X \subset \mathbb{R}^d$ in general position, $d \geq 1$.
a) [1] Prove the *empty sphere criterion*, a generalization of Proposition 14.4 to all $d \geq 1$.
b) [1+] Generalize Theorem 14.5 to all $d \geq 1$.

Exercise 14.3. (*Local convexity criterion in* $\mathbb{R}^4$) $\Diamond$ [1+] Find a set $X \subset \mathbb{R}^3$, a triangulation of $A = \mathrm{conv}(X)$, and a function $f : \mathbb{R}^3 \to \mathbb{R}$, such that the local convexity criterion fails on $S_f \subset \mathbb{R}^4$, i.e., show that S_f can be convex at all 2-faces, but not globally convex.

Exercise 14.4. (*Inverse Voronoi diagrams and inverse Delaunay triangulations*) $\Diamond$ Let $X \subset \mathbb{R}^2$ be a finite set of points in general position. Subdivide the plane into cells according to the *furthest* point in X. For this *inverse Voronoi diagram* define the corresponding *inverse Delaunay triangulation* $IDT(X)$.
a) [1] Prove the *full circle condition*, the analogue of Proposition 14.4.
b) [1] Give a paraboloid style construction of $IDT(X)$, the analogue of Theorem 14.5.
c) [1] Define *decreasing flips* to be the inverse of increasing flips. Prove that every triangulation $T \in \mathcal{T}(X)$ is connected to $IDT(X)$ by at most $\binom{n}{2}$ decreasing flips.

Exercise 14.5. (*Double chain configuration*) $\Diamond$ *a)* [1] Let $X_n \subset \mathbb{R}^2$ be the set of $n = 2k$ points as in Figure 14.7. Prove that the edges as in the figure must be present in every full triangulation of $\mathrm{conv}(X_n)$. Conclude that the number of full triangulations is equal to

$$|\mathcal{T}(X_n)| = \binom{2k-2}{k-1} C_{k-2}^2,$$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the *Catalan number* (see Example 8.5).
b) [1+] Prove that the diameter of the flip graph on $\mathcal{T}(X_n)$ is $\Omega(n^2)$.

Exercise 14.6. [1] Let T be a full triangulation of a convex n-gon with k interior points $x_1, \ldots, x_i$. Suppose every interior point x_i is adjacent to at least 6 triangles in T. Prove that $k = O(n^2)$.

Exercise 14.7. (*Number of planar triangulations*) *a)* [2-] Prove that for every $X \subset \mathbb{R}^2$ in general position, the number of full triangulations is at least exponential: $|\mathcal{T}(X)| = e^{\Omega(n)}$.
b) [2] Prove that for every $X \subset \mathbb{R}^2$, the number of full triangulations is at most exponential: $|\mathcal{T}(X)| = e^{O(n)}$.

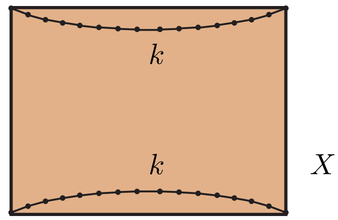

Figure 14.7. Double chain configuration of points.

Exercise 14.8. Let $X \subset \mathbb{R}^2$ be the set of n points in general position.
a) [2-] Prove that every triangulation $T \in \mathcal{T}(X)$ allows at least $(n-4)/2$ flips.
b) [2-] Prove that every triangulation $T \in \mathcal{T}(X)$ allows at least $(n-4)/6$ *simultaneous flips* (flips along pairwise non-adjacent diagonals).
c) [1+] Show that part b) cannot be improved to $n/5$ simultaneous flips.

Exercise 14.9. $\lozenge$ [1] Find a set $X \subset \mathbb{R}^2$ of n points and a triangulation T, such that
(i) there is an increasing flip sequence from T to $DT(X)$ of length $\theta(n)$, and
(ii) there is an increasing flip sequence from T to $DT(X)$ of length $\theta(n^2)$.

Exercise 14.10. (*Flips in convex polygons*) $\lozenge$ a) [1-] Prove that every two triangulations of a convex n-gon are connected by at most $2n$ flips.
b) [2] Prove the lower bound $2n - O(\sqrt{n})$ in the convex case.

Exercise 14.11. Let $X \subset \mathbb{R}^2$ be a finite set of points in general position.
a) [1] Consider a complete graph Γ on X with weights on edges given by Euclidean distances. Prove that the Delaunay triangulation $DT(X)$ contains the *minimum spanning tree* in Γ.
b) [1-] Prove or disprove: $DT(X)$ minimizes the sum of edge lengths over all triangulations $T \in \mathcal{T}(X)$.

Exercise 14.12. (*Delaunay triangulations*) $\lozenge$ Let $X \subset \mathbb{R}^2$ be a finite set of points in general position, let $\mathcal{T}(X)$ be the set of full triangulations, and let $DT = DT(X)$ be the Delaunay triangulation.
a) [1-] Suppose a triangulation $T \in \mathcal{T}(X)$ has no obtuse triangles. Prove that $T = DT$.
b) [1] Denote by $\alpha(T)$ the maximum angle in all triangles of a triangulation $T \in \mathcal{T}(X)$. Prove that DT minimizes α over all $T \in \mathcal{T}(X)$.
c) [1-] Denote by $\beta(T)$ the minimum angle in all triangles of a triangulation $T \in \mathcal{T}(X)$. Check that DT does not necessarily maximize β over all $T \in \mathcal{T}(X)$.
d) [1] Prove that the inverse Delaunay triangulation IDT (see Exercise 14.4) maximizes β over all $T \in \mathcal{T}(X)$.
e) [1] Denote by $\mu(T)$ the largest circumradius of all triangles in T. Prove that DT minimizes $\mu(T)$.
f) [1+] Denote by $\rho(T)$ the mean circumradius of all triangles in T. Prove that DT minimizes $\rho(T)$.
g) [1+] Denote by $\eta(T)$ the mean inradius of all triangles in T. Prove that IDT maximizes $\eta(T)$.

Exercise 14.13. (*Tutte's formula*) [2] Denote by a_n the number of rooted planar triangulations with $2n$ triangles (and thus with $n+2$ vertices, $3n$ edges). Prove that

$$a_n = \frac{2(4n-3)!}{n!\,(3n-1)!}\,.$$

Exercise 14.14. (*Acute triangulations*) Let $Q \subset \mathbb{R}^2$ be a convex polygon. An *acute* (*non-obtuse*) *triangulation* of a polygon $Q \subset \mathbb{R}^2$ is a triangulation of Q into acute (acute or right) triangles (see Figure 14.8). Define the acute dissections similarly.

FIGURE 14.8. An acute triangulation and an acute dissection of a square.

a) [1-] Prove that every triangle has an acute triangulation. Conclude that every polygon in the plane has an acute dissection.
b) [1] Prove that every circumscribed convex polygon has an acute triangulation.
c) [1+] Prove that every convex polygon has a non-obtuse triangulation.
d) [1+] Prove that every convex polygon $Q \subset \mathbb{R}^2$ which has a non-obtuse triangulation also has an acute triangulation.
e) [2-] Prove that the surface of every convex polytope $P \subset \mathbb{R}^3$ has an acute triangulation.
f) [$*$] Construct a triangulation of a cube into tetrahedra with acute dihedral angles.
g) [2] A simplex is called *acute* if all angles between the facets (generalized dihedral angles) are acute. Let $C_d \subset \mathbb{R}^d$ be a hypercube. Prove that C_d has no triangulations into acute simplices, for all $d \geq 5$.

Exercise 14.15. A space polygon $X = [x_1 \dots x_n] \subset \mathbb{R}^3$ is called *triangulable* if there exists an embedded triangulated surface homeomorphic to a disk, with the boundary X and vertices at x_i, $1 \leq i \leq n$.
a) [1-] Suppose X has a simple projection onto a plane. Prove that X is triangulable.
b) [1] Prove that every space pentagon is triangulable.
c) [1-] Find a space hexagon which is not triangulable.
d) [1+] Find an unknotted space polygon which is not triangulable.
e) [2] Show that deciding whether a space polygon is triangulable is NP-hard.

Exercise 14.16. (*Convex polygons*) $\diamondsuit$ Let $X = [w_1 \dots w_n] \subset \mathbb{R}^3$ be a space polygon which projects onto a convex n-gon Q in the plane. Denote by P the convex hull of X and suppose P is simplicial. Two full triangulations T_1 and T_2 are associated with the top and the bottom part of the surface of P, respectively.
a) [1-] Prove that for all Q and a triangulation T_1 of Q, there exists a polygon X which projects onto Q, and and has a shadow T_1. In other words, prove that every triangulation of a convex polygon is regular: $\mathcal{T}(Q) = \mathcal{T}_*(Q)$.
b) [1] Note that the shadows $T_1, T_2 \vdash Q$ must have distinct diagonals. Prove or disprove: for every convex polygon Q and every pair of full triangulations T_1 and T_2 of Q with distinct diagonals, there exists a polygon X with shadows T_1 and T_2, respectively.
c) [2-] A *combinatorial triangulation* of an n-gon is defined by pairs of vertices corresponding to its diagonals. Prove that for every two combinatorial triangulations T_1, T_2 of an n-gon with distinct diagonals, there exists a polygon $Q \subset L$ and a space polygon $X \subset \mathbb{R}^3$ which projects onto Q and has shadows T_1 and T_2. In other words, prove that T_1, T_2 can be realized as shadows by a choice of Q and X.

Exercise 14.17. (*Six bricks problem*) A *brick* is a rectangular parallelepiped. A collection of bricks is called *disjoint* if no two bricks intersect (even along the boundary). We assume that bricks are non-transparent.
a) [1-] Can you have six disjoint bricks not containing the origin O, so no brick vertex is visible from O?
b) [1-] Can you have six disjoint bricks not containing the origin, so that no point far enough is visible?

Exercise 14.18. (*Triangulations of non-convex polytopes*) $\diamondsuit$ *a*) [1] Let Q be a simple polygon in $\mathbb{R}^3$. Prove that Q can be subdivided into triangles without adding new vertices.
b) [1] Show that part *a* does not extend to non-convex polyhedra in $\mathbb{R}^2$. Formally, find an embedded (not self-intersecting) polyhedron $P \subset \mathbb{R}^3$ homeomorphic to a sphere which cannot be subdivided into simplices without adding new vertices.
c) [1+] Construct an embedded simplicial polyhedron $P \subset \mathbb{R}^3$ homeomorphic to a sphere, such that no tetrahedron spanned by vertices of P lies inside P.
d) [1+] Construct an embedded simplicial polyhedron $P \subset \mathbb{R}^3$ homeomorphic to a sphere, such that for some point O *inside* P, none of the intervals (Ov) lie in P, for all vertices v of P.
e) [1+] Find an embedded simplicial polyhedron $P \subset \mathbb{R}^3$ homeomorphic to a sphere, such that for some point O *outside* P, all intervals (Ov) have points inside P, for all vertices v of P.

Exercise 14.19. [2] Show that the problem whether a given non-convex polyhedron can be subdivided into tetrahedra without adding vertices is NP-hard.

Exercise 14.20. (*Regular triangulations*) $\diamondsuit$ Let $P \subset \mathbb{R}^d$ be a convex polytope. A triangulation $\mathcal{D}$ of P is called *regular* if there exists a height function ξ such that $\mathcal{D}$ is associated with ξ. For example, part *a*) of Exercise 14.10 says that every full triangulation of a convex polygon is regular. Similarly, from the proof of Lemma 17.15 we know that every star triangulation of P is regular.
a) [1] Find a (general) triangulation of a convex polygon $Q \subset \mathbb{R}^2$ which is not regular.
b) [1+] Find a non-regular full triangulation of a convex polytope in $\mathbb{R}^3$.
c) [1+] Let P be the 3-dimensional cube $-1 \le x, y, z \le 1$. Find a non-regular triangulation of P, such the vertices of tetrahedra in the triangulation are at the vertices of P or at the origin O.
d) [1+] Find a non-regular full triangulation of a 4-dimensional cube.

Exercise 14.21. *a*) [2-] Prove that every combinatorial triangulation (see Exercise 14.16) is a Delaunay triangulation of some convex polygon.

Exercise 14.22. *a*) [1+] In a full triangulation of a d-dimensional cube, split the simplices into d groups according to the number of vertices at level 0 and 1 of the first coordinate. Prove that the sum of the volumes of simplices in each group is the same.
b) [1+] Find a full triangulation of a d-dimensional cube with $o(n!)$ simplices.

Exercise 14.23. (*Dominoes*) Let $\Gamma \subset \mathbb{Z}^2$ be a finite simply connected region on a square grid. A *domino tiling* T of the region Γ, write $T \vdash \Gamma$, is a tiling of Γ by the copies of 1×2 and 2×1 rectangles. Define a 2-*move* to be a flip of 2 vertical dominoes forming a 2×2 square to 2 horizontal dominoes, or vice versa.
a) [1] Prove directly that every two domino tilings of Γ are connected by a finite sequence of 2-moves.

b) [1−] Show that part b) is false for non-simply connected regions. Define by analogy the domino tilings in $\mathbb{R}^3$ and construct a counterexample to 2-move connectivity for simply connected 3-dimensional regions. Moreover, show that for every k there exist a simply connected region in $\mathbb{Z}^3$ with exactly two domino tilings, and which requires the moves of at least k dominoes.
c) [1] Color the squares in Γ in a checkerboard fashion and orient $\partial\Gamma$ counterclockwise. Define a height function $h : \partial\Gamma \to \mathbb{Z}$ by the following rule. Start at any fixed point $a \in \partial\Gamma$ and let $h(a) = 0$. Now, when going around a black square add 1, when going around a white square subtract 1 (see Figure 14.9). Prove that for every tileable region Γ function h is well defined. Show that for every tiling it extends to all (integer) points in Γ. Check what happens to the height function when a 2-move is applied.

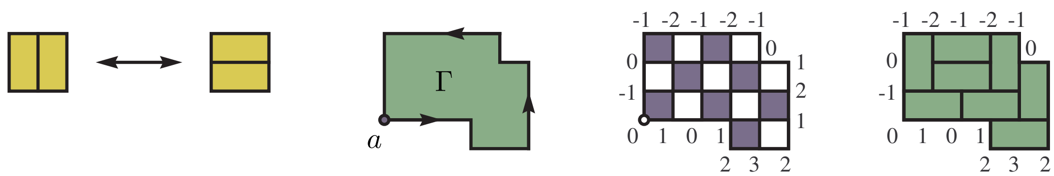

FIGURE 14.9. Domino tilings: 2-moves, region Γ, the height function and the minimal tiling of Γ.

d) [1] For two domino tilings T_1, T_2 of Γ we say that $T_1 \prec T_2$ if for their height function we have $h_1(x) \le h_2(x)$ for all $x \in \Gamma$. Suppose $T_\circ$ satisfies $T \not\prec T_\circ$ for all $T \vdash \Gamma$. Prove that the maximal value of $h_\circ(x)$ of the height function corresponding to $T_\circ$ appears on the boundary.
e) [1+] Suppose Γ is tileable by dominoes. Construct a tiling $T_{\min}$ of Γ as follows. Compute the height function of Γ and find $x \in \partial\Gamma$, such that $h(x)$ is maximal on $\partial\Gamma$. Place a domino inside Γ so that x is adjacent to both squares. Repeat. Prove that this algorithm always works, i.e., produces the same domino tiling for all choices of x.
f) [1+] Start with tiling T and repeatedly apply 2-moves until a local minimum $T_\circ$ is reached. Prove that $T_\circ = T_{\min}$. Conclude that $T_{\min}$ is global minimum: $T_{\min} \prec T$ for all $T \vdash \Gamma$. Conclude that every two domino tilings of Γ are connected by 2-moves, as in part a).
g) [1] Show that when Γ is simply connected and not tileable by dominoes, the algorithm in d) fails. Design a linear time algorithm (in the area) for testing the tileability of Γ by dominoes.
h) [1+] Show that for all $T_1, T_2 \vdash \Gamma$ there exist unique $T_\vee$ and $T_\wedge$ such that

$$h_\vee(x) = \max\{h_1(h), h_2(x)\}, \quad h_\wedge(x) = \min\{h_1(h), h_2(x)\}, \quad \text{for all } x \in \Gamma.$$

Prove that this defines a lattice structure on all domino tilings of Γ.
i) [1−] When Γ_n is a $2n\times 2n$ square, find the only two tilings which have admit exactly one 2-move. Check that these are the max and min tilings defined above.
j) [1] When Γ_n is a $2n\times 2n$ square, conclude that the number of 2-moves required to go from one tiling to another is $\theta(n^3)$.
k) [2+] Generalize these results to tilings by $k\times 1$ and $1\times\ell$ rectangles, for every fixed $k, \ell \ge 2$.

Exercise 14.24. (*Ribbon trominoes*) Consider all tilings of a finite simply connected region $\Gamma \subset \mathbb{Z}^2$ by copies of the four *ribbon trominoes* shown in Figure 14.10 (only translations are allowed).

a) [2] Define 2-moves on the tilings to be exchanges of two trominoes by another two. Prove that all ribbon tromino tilings of a rectangle are connected by 2-moves.
b) [2] Extend *a*) to all simply connected regions Γ.
b) [1] Conclude from *a*) that in any tiling of Γ, the number of ribbon trominoes of the second type minus the number of ribbon trominoes of the third type is a constant $c(\Gamma)$ independent of the tiling.
c) [1] Denote by Ψ_n the staircase shaped region as in Figure 14.10. Show that whenever $\mathrm{area}(\Psi_n)$ is divisible by 3, the region Ψ_n is tileable by ribbon trominoes.
d) [1] Find all n for which Ψ_n is tileable by ribbon trominoes of the second and the third type.

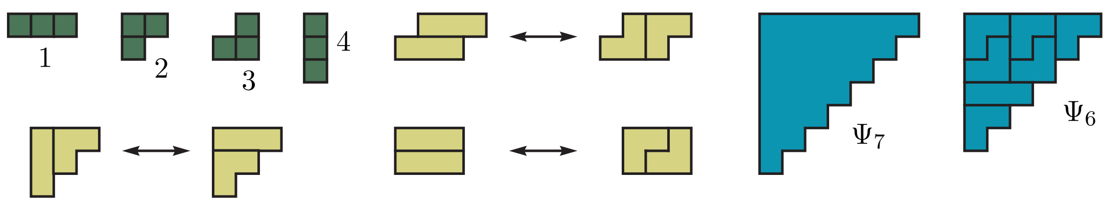

FIGURE 14.10. Ribbon trominoes, their local moves and a tiling of Ψ_6.

Exercise 14.25. (*Mosaics*) $\diamondsuit$ Let P be a centrally symmetric convex polygon with $2n$ sides. A *mosaic* M is a subdivision of P into parallelograms with sides parallel to the sides of P.[30] Denote by $\mathcal{R} = \mathcal{R}(P)$ the set of mosaics in the polygon P.
a) [1] Prove that every such mosaic has exactly $\binom{n}{2}$ parallelograms, each corresponding to a pair of non-parallel sides.
b) [1+] Prove that the number of mosaics $|\mathcal{R}(P)|$ depends only on n and is at most $\binom{n}{2}!$.

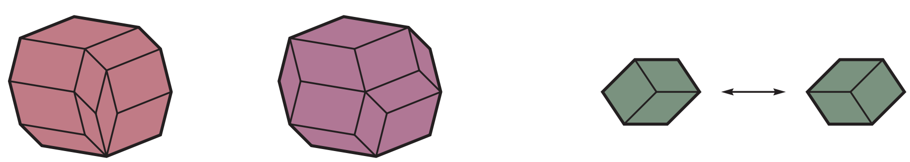

FIGURE 14.11. Two mosaics of a decagon and a flip.

c) [2-] Define a *flip* on $\mathcal{R}$ to be a transformation of triples of parallelograms forming a hexagon into another (see Figure 14.11). Prove that every two mosaics are connected to each other by a finite sequence of flips. Prove that every mosaic of P allows at least $n-2$ flips.
d) [2] Define the height function of a mosaic M to be a piecewise linear convex function $\xi : P \to \mathbb{R}$ whose flats give M when projected. Mosaic M is called *regular* in this case. Denote by $\mathcal{R}^*(P) \subset \mathcal{R}(P)$ the set of all regular mosaics of P. Prove that when P is generic, the number of regular mosaics depends only on n. Furthermore, prove that as $n \to \infty$ a random mosaic is not regular.

[30]A more standard name for these subdivisions is *zonotopal tilings*, since they extend to general zonotopes (see Exercise 7.16).

e) [2-] When P is deformed, one can also deform a mosaic. Prove that there exists a mosaic which is not regular for all deformations.

Exercise 14.26. [1] Let P be a centrally symmetric convex polygon. Consider a subdivision of P into finitely many centrally symmetric polygons. Use Exercise 2.3 to prove that this tiling has at least 3 parallelograms.

14.7. **Final remarks.** Voronoi diagrams are named after the Georgy Voronoy who introduced them in 1908. In a different context they were studied much earlier, most notably by Descartes (1644) and Dirichlet (1850). Similarly, Delaunay triangulations are named after Boris Delone (note another difference in spelling), who proved the empty sphere criterion (Exercise 14.2) in 1934. According to Dolbilin, this result was repeatedly rediscovered in the West, but after Coxeter received a letter from Delone, he read and popularized the original papers of Voronoy and Delone. For more on Voronoi diagrams and Delaunay triangulations, their history, applications and references see [Aur, AurK].

The inverse Voronoi diagrams (and dual to them inverse Delaunay triangulations) defined in Exercise 14.4, go back to the foundational paper [ShaH]. They are usually called the "furthest site" or the "farthest point" Voronoi diagrams [AurK].

The local convexity criterion (Lemma 14.7) is a discrete version of a classical result by Hadamard (1897), further generalized by Tietze (1928), Nakajima (1928) and others. We refer to [KarB] for further references.

The flip (local move) connectivity on full triangulations has been also extensively studied in the literature [San2]. In particular, in dimension $d \geq 5$ the flip graph can be disconnected, and there is an evidence that this might be true even for $d = 3$. For more on various classes of triangulations, in particular regular triangulations, and connections to other fields see [DRS] (see also [San2]).

The full triangulations (triangulations with a fixed set of vertices) have also been studied at length in connection with A-discriminants and variations on the theme. The height functions and regular triangulations (see Exercise 14.20) play important roles in the field. We refer the reader to [GKZ], where many of these fundamental results were first summarized (a number of connections and applications have appeared since [GKZ]).

Let us mention here the mosaics (zonotopal tilings) defined in Exercise 14.25. In a certain precise sense, the regular mosaics are projections of the "top" 2-dimensional faces of a hypercube. In a different direction, the mosaics correspond to the *pseudoline arrangements*, while regular mosaics are dual to stretchable pseudoline arrangements. We refer to [Bjö+] for further results and references.

Finally, the local move connectivity plays an important role in the modern study of finite tilings. In some special cases one can use the height functions and combinatorial group theory to establish that certain local moves connect all tilings of a given region by a fixed set of tiles. Some examples are given in Exercises 14.23 and 14.24. See [Pak3] for these and other tiling results, and the references.

15. Hilbert's third problem

The name *scissor congruence* comes from cutting polygons on a plane (hopefully, with scissors) and rearranging the pieces. Can one always obtain a square this way? What happens in three dimensions? Can one take a regular tetrahedron, cut it into pieces and rearrange them to form a cube? This problem has a glorious history and a distinction of being the subject of one of Hilbert's problems. Perhaps surprisingly, the solutions are elementary, but use a bit of algebra and a bit of number theory.

In this section we give an introduction to scissor congruence in the plane and in space. Our main result is the proof of Bricard's condition which resolves (negatively) Hilbert's original problem. In the next section we present an algebraic approach and prove further results on scissor congruence. Among other things, we show that if a polytope is scissor congruent to a union of two or more similar polytopes, then it must be scissor congruent to a cube. In Section 17 we use a different combinatorial approach for proving general scissor congruence results. The idea of *valuations* employed there will reappear later on, in the proof of the *bellows conjecture* (Section 34). Finally, in Section 18 we study a related problem on Monge equivalence. Despite the similarities, that problem turns out to be much simpler and we completely resolve it.

15.1. **Scissor congruence.** Let $P, Q \subset \mathbb{R}^d$ be two convex polytopes.[31] We say that they are *scissor congruent*, write $P \sim Q$, if P can be cut into finitely many smaller polytopes which can be rearranged and assembled into Q. Formally, polytopes $A, B \subset \mathbb{R}^d$ are *congruent*, write $A \simeq B$, if B can be obtained from A by a *rigid motion*: combination of translations and rotations.[32] We say that $P \sim Q$ if each polytope is a disjoint union[33] of congruent polytopes:

$$P = \cup_{i=1}^m P_i\,, \qquad Q = \cup_{i=1}^m Q_i\,,$$

where $P_i \simeq Q_i$. Of course, if P and Q are scissor congruent, then $\mathrm{vol}(P) = \mathrm{vol}(Q)$.

Let us first note that scissor congruence is an equivalence relation: if $P \sim Q$ and $Q \sim R$, then $P \sim R$. Indeed, simply take the intersection of two corresponding decompositions of Q. These smaller polytopes can be reassembled into both P and R, proving the claim.

Now, the study of scissor congruence starts with the following intuitively clear result, which was rediscovered on several occasions.

Theorem 15.1 (Bolyai, Gerwien)**.** *Two convex polygons in the plane are scissor congruent if and only if they have equal area.*

---

[31]Throughout this and next section, convexity plays no role and can be weakened to any finite union of convex polytopes. The results and proofs extend verbatim. Still, we decided to keep this condition for the clarity of exposition.

[32]Throughout the section we assume that congruent polytopes must have the same orientation. In fact, every two mirror symmetric polytopes are scissor congruent, so the reader can ignore the difference (see Exercise 15.3).

[33]Here and throughout the next three sections, when we speak of a disjoint union of polytopes we ignore the boundary.

Motivated by his study of foundations of geometry Hilbert noted that this result can be used to define the area of polygons in a combinatorial way, by setting the area of a square and axiomatizing the scissor congruence. On the other hand, he recalled that the Archimedes' method of exhaustion for computing the volume of a pyramid involved essentially taking the limits (cf. Exercise 15.1). In his famous list of open problems, the *third problem* asked whether one can avoid taking limits and use scissor congruence once again. The same year Hilbert posed the problem, it was resolved by Dehn in the negative:

Theorem 15.2 (Dehn)**.** *A cube and a regular tetrahedron of the same volume are not scissor congruent.*

In this section we give elementary proofs of both theorems.

15.2. **Even seamsters could use some group theory.** While one can view the Bolyai–Gerwien theorem as a reasonably hard exercise in elementary geometry, and find one of the numerous ad hoc proofs, we decided to employ a more enlightening (even if a bit postmodern) approach to obtain this classical result. Let us start with the following easy observation.

Lemma 15.3. *Let* Γ *be a discrete group acting on* $\mathbb{R}^d$*, and suppose convex polytopes* P *and* Q *are fundamental regions of the action of* Γ*. Then the polytopes are scissor congruent:* $P \sim Q$*.*

Here by a *fundamental region* $\overline{X} \subset \mathbb{R}^d$ we mean the closure of a region X which satisfies the following property: for every $z \in \mathbb{R}^d$ there exists a unique point $x \in X$ such that x lies in the orbit of z under action of Γ. The proof of Lemma 15.3 is straightforward from definition.

Proof. Consider the tiling of $\mathbb{R}^d$ obtained by the action of Γ on Q. This tiling subdivides P into smaller convex polytopes $P_i = P \cap (g_i \cdot Q)$, for some $g_i \in \Gamma$. Since Q is a fundamental region, these polytopes cover the whole P, and since Γ is discrete there is only a finite number of g_i giving nonempty intersections P_i. Now, reassemble polytopes P_i into Q as follows: $Q = \cup_i (g_i^{-1} \cdot P_i)$. Since P is a fundamental region, this indeed covers the whole polytope Q, so that polytopes $Q_i = g_i^{-1} \cdot P_i$ intersect only at the boundary. This proves that $P \sim Q$. $\square$

15.3. **Cutting polygons into pieces.** Now we are ready to prove Theorem 15.1. We need the following steps:

0) triangulate every convex polygon;
1) show that every triangle with side a and height h is scissor congruent to a parallelogram with side a and height $h/2$;
2) show that every two parallelograms with the same side and height are scissor congruent;
3) show that a union of two squares with sides a and b is scissor congruent to a square with side $\sqrt{a^2 + b^2}$.

Let us show that these steps suffice. Indeed, after step 0) we can start with a triangle T and convert it into a parallelogram B with sides $a \geq b$. Denote by h the height of B, so that $\text{area}(B) = ha$, and let $t = \sqrt{ha}$. Clearly, $h \leq b \leq t$. By 2), parallelogram B is scissor congruent to a parallelogram D with sides a and t, and height h. Similarly, D is scissor congruent to a square with side t. Finally, using 3) repeatedly one can assemble all the squares into one big square S with sides $\sqrt{\text{area}(P)}$. Therefore, $P \sim S \sim Q$, which implies the result.

The proof of 1), 2), 3) is easy to understand from the next three figures. In Figure 15.1 let $\Gamma_1 \simeq \mathbb{Z}^2$ be a group of translations preserving the colored triangles (or parallelograms), and let Γ_2 be a $\mathbb{Z}_2$-extension of Γ_1 obtained by adding a central reflection with respect to the origin, which switches colors. Now both the white triangle and the white parallelogram are fundamental regions of Γ_2, and the lemma implies that they are scissor congruent. The last of the three pictures in Figure 15.1 shows the working of the lemma in this case.

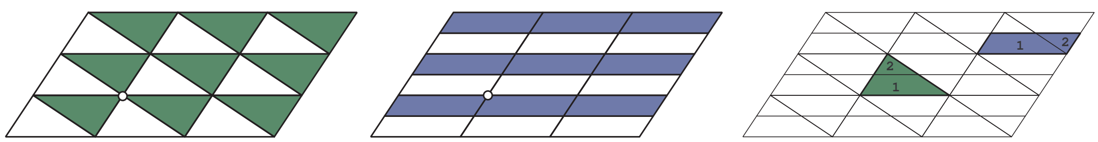

FIGURE 15.1. Converting a triangle into a parallelogram.

Similarly, any two parallelograms with the same side and height are fundamental regions of the action of same group $\Gamma \simeq \mathbb{Z}^2$ on $\mathbb{R}^2$ by translations. Now the lemma implies their scissor congruence (see Figure 15.2).

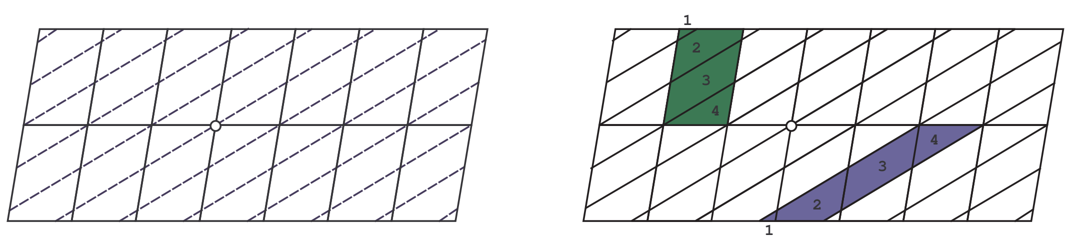

FIGURE 15.2. Converting a parallelogram into another parallelogram.

Finally, any two squares can be attached to each other to form a fundamental regions of the action of the group of translation $\Gamma \simeq \mathbb{Z}^2$ along vectors (a, b) and $(b, -a)$. The square with side $\sqrt{a^2 + b^2}$ is another fundamental region of Γ. Now the lemma implies their scissor congruence (see Figure 15.3).

To summarize, we just showed all three congruences $1), 2)$, and 3) as above can be obtained from Lemma 15.3. This completes the proof of Theorem 15.1.

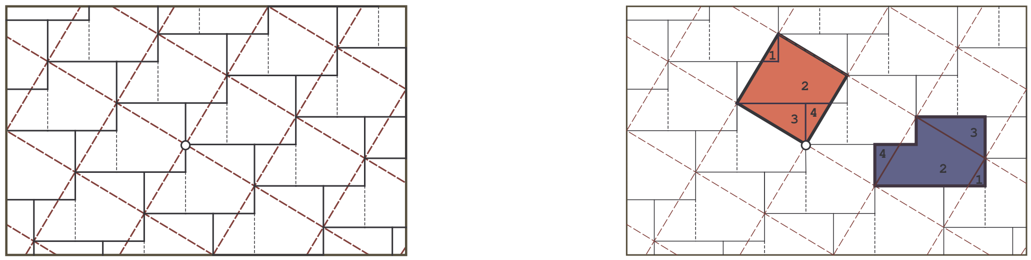

Figure 15.3. Converting two squares into a bigger one.

15.4. **Chain saw is really not as powerful as scissors.** Our proof of Dehn's theorem is somewhat weaker than the original version of Dehn, but more elementary and sufficient for our purposes.

Let $E = \{e_1, \ldots, e_l\}$ be the set of edges of a convex polytope $P \subset \mathbb{R}^3$. Denote by α_i the dihedral angle at edge e_i, where $1 \le i \le N$. We say that polytope P is *fortunate* if π can be written as a rational combination of angles α_i:

$$c_1\alpha_i + \ldots + c_N\alpha_N = \pi, \quad \text{where} \;\; c_i \in \mathbb{Q}_+, \quad \text{for all} \;\; 1 \le i \le N.$$

Otherwise, we say that P is *unfortunate*.

Lemma 15.4 (Bricard's condition)**.** *An unfortunate polytope in $\mathbb{R}^3$ is not scissor congruent to a cube.*

Dehn's theorem (Theorem 15.2) follows immediately form Bricard's condition. Indeed, all dihedral angles of a regular tetrahedron Δ are equal to $\alpha = 2 \arcsin \frac{1}{\sqrt{3}} = \arccos \frac{1}{3}$ (see the calculation below Figure 20.4). It is not hard to show that $(\alpha/\pi) \notin \mathbb{Q}$ (see Appendix 41.3), which implies that P is unfortunate. Thus, from the lemma we conclude that Δ is not scissor congruent to a cube of the same volume, as desired.

Before we begin the proof, let us start consider some special cases of increasing generality. Start with polytopes $P \sim Q$, and take the corresponding polytope decompositions $P = \cup_i P_i$ and $Q = \cup_i Q_i$, where $P_i \simeq Q_i$. We say that these are *polyhedral subdivisions* if no vertex of P_i (Q_i) lies in the interior of an edge or a face of P_j (Q_j).

Sublemma 15.5. *Suppose $P = \cup_i P_i$ and $Q = \cup_i Q_i$ are polyhedral subdivisions into congruent polytopes $P_i \simeq Q_i$, and Q is a cube. Then P is fortunate.*

Proof of Sublemma 15.5. Denote by $\sigma(X)$ the sum of the dihedral angles of polytope $X \subset \mathbb{R}^3$, and let

$$\Sigma = \sum_i \sigma(P_i) = \sum_i \sigma(Q_i),$$

where the second equality follows from the congruences $P_i \simeq Q_i$. We will calculate Σ in two different ways and compare the results.

Denote by $\mathcal{E}$ the set of all edges in the subdivisions of P and Q. Let $\alpha_i(e)$ be the dihedral angle in polytope P_i, at edge $e \in \mathcal{E}$. Define by $\sigma(e) = \sum_i \alpha_i(e)$. We have:

$$\Sigma = \sum_{i=1}^{m} \sum_{e \in \mathcal{E}} \alpha_i(e) = \sum_{e \in \mathcal{E}} \sigma(e).$$

Now observe that $\sigma(e) = 2\pi$ when e is an interior edge, $\sigma(e) = \pi$ when e is on the boundary of P, and $\sigma(e) = \alpha_s$ when $e \subset e_s$ is the (usual) edge of P. We conclude:

$$\Sigma = \sum_{s=1}^{N} k_s \alpha_s + n\pi, \quad \text{for some } k_1, \dots, k_N \in \mathbb{N}, \quad \text{and } n \in \mathbb{Z}_+.$$

By the same argument for the decomposition of Q, since all dihedral angles are equal to $\pi/2$ in this case, we obtain that $\Sigma = l\pi/2$ for some $l \in \mathbb{N}$. This implies that P is fortunate, as desired. □

In the general case, suppose we have vertices which subdivide the edges of P_i into intervals. By analogy with the previous case, denote these intervals by e, and the set of intervals by $\mathcal{E}$. Let $\ell_e = |e|$ denote the lengths of the intervals. Similarly, for the decomposition of Q, denote by $\mathcal{E}'$ the resulting set of intervals e', and by $\ell_{e'}$ their lengths.

Sublemma 15.6. *Let $P = \cup_i P_i$ and $Q = \cup_i Q_i$ are polytope decompositions into congruent polytopes $P_i \simeq Q_i$, and Q is a cube. Suppose the lengths of all intervals are rational: $\ell_e, \ell_{e'} \in \mathbb{Q}$ for all $e \in \mathcal{E}$, $e' \in \mathcal{E}'$. Then P is fortunate.*

Proof of Sublemma 15.6. Taking the previous proof as guidance, we now modify the definition of Σ to be as follows:

$$\Sigma = \sum_{i=1}^{m} \sum_{e \in \mathcal{E}} \ell_e \, \alpha_i(e), \quad \Sigma' = \sum_{i=1}^{m} \sum_{e' \in \mathcal{E}'} \ell_{e'} \, \alpha_i(e'),$$

where $\alpha_i(e)$ is the dihedral angle in polytope P_i at interval $e \in \mathcal{E}$, and $\alpha_i(e')$ is the dihedral angle in polytope Q_i at interval $e' \in \mathcal{E}'$.

Let us first show that $\Sigma = \Sigma'$. For a polytope X with the set of edges $U = E(X)$, let $\sigma(X) = \sum_{u \in U} \ell_u \cdot \alpha(u)$ be the sum of the dihedral angles in X weighted by the length ℓ_u of the edges u. We have

$$\Sigma = \sum_{i=1}^{m} \sum_{u \in E(P_i)} \sum_{e \in \mathcal{E}, \, e \subset u} \ell_e \, \alpha_i(e) = \sum_{i=1}^{m} \sum_{u \in E(P_i)} \ell_u \, \alpha_i(u) = \sum_{i=1}^{m} \sigma(P_i),$$

where the second equality follows by additivity of the interval length along the edge, and because the dihedral angles are equal along the same edge. We conclude:

$$\Sigma = \sum_{i=1}^{m} \sigma(P_i) = \sum_{i=1}^{m} \sigma(Q_i) = \Sigma',$$

as desired. By analogy with the proof above, we can redefine $\sigma(e) = \sum_i \ell_e \alpha_i(e)$. We can then rewrite Σ as

$$\Sigma = \sum_{e \in \mathcal{E}} \sigma(e).$$

Since all lengths ℓ_e are rational, by the same argument as above we conclude that

$$\Sigma = \sum_{s=1}^{N} c_s \alpha_s + r\pi, \quad \text{for some } c_1, \dots, c_N, r \in \mathbb{Q}_+.$$

The same argument for the cube Q gives $\Sigma'/\pi \in \mathbb{Q}$, as desired. $\square$

Proof of Lemma 15.4. In full generality, suppose now the interval lengths ℓ_e are not necessarily rational. We now change the definition of Σ once again, substituting values of a general function $f : \mathcal{E} \to \mathbb{R}$ and $g : \mathcal{E}' \to \mathbb{R}$ in place of the interval lengths:

$$\Sigma_f \,=\, \sum_{i=1}^{m} \sum_{e \subset \mathcal{E}} f(e)\, \alpha_i(e)\,, \quad \Sigma'_g \,=\, \sum_{i=1}^{m} \sum_{e' \subset \mathcal{E}'} g(e)\, \beta_i(e')\,.$$

If we can find positive functions f, g as above so that $f(e), g(e') \in \mathbb{Q}_+$ for all $e \in \mathcal{E}$ and $e' \in \mathcal{E}'$, and such that $\Sigma = \Sigma'$, the result of the lemma then follows by the same argument as in the proof above.[34]

Let us write the conditions the functions f, g need to satisfy to guarantee $\Sigma_f = \Sigma'_g$. Again, by the argument in the proof above, it suffices to check that the sum along the edge $u \in E(P_i)$ is equal to that along the corresponding edge $u' \in E(Q_i)$:

$$(\Im) \qquad \sum_{e \in \mathcal{E},\, e \subset u} f(e) \,=\, \sum_{e' \in \mathcal{E}',\, e' \subset u'} g(e')$$

Think of $(\Im)$ as a set of linear equations for the values $f(e)$, $g(e')$ which must hold for all polytopes P_i and all edges $u \in E(P_i)$. Because ℓ_e is an obvious positive *real* solution of $(\Im)$, there is also a positive *rational* solution of $(\Im)$. From above, this finishes the proof of the lemma. $\square$

15.5. **Exercises.**

Exercise 15.1. (*Method of exhaustion*) $\Diamond$ [1-] Let $P, Q \subset \mathbb{R}^3$ be two convex polytopes such that $\mathrm{vol}(P) < \mathrm{vol}(Q)$. Prove that there exists a polytope $P' \subset Q$, such that $P \sim P'$.

Exercise 15.2. (*Extended Bricard's condition*) $\Diamond$ [1-] Let $P, Q \subset \mathbb{R}^3$ be scissor congruent convex polytopes. Denote by $\alpha_1, \ldots, \alpha_m$ and $\beta_1, \ldots, \beta_n$ their dihedral angles. Prove that

$$r_1 \alpha_1 + \ldots + r_m \alpha_m \,=\, s_1 \beta_1 + \ldots + s_n \beta_n + k\pi\,,$$

for some $r_i, s_j \in \mathbb{N}$ and $k \in \mathbb{Z}$.

Exercise 15.3. (*Mirror symmetry*) $\Diamond$ [1] Let $P, P' \subset \mathbb{R}^3$ be two polyhedra which are congruent up to a mirror symmetry. Prove that $P \sim P'$.

Exercise 15.4. (*Parallelepipeds*) $\Diamond$ *a*) [1] Let R be a parallelepiped in $\mathbb{R}^3$. Prove that R is *rectifiable* (i.e., scissor congruent to a cube of equal volume).
b) [1-] Prove that the union of two rectifiable polytopes is also rectifiable.
c) [1] Extends parts *a*) and *b*) to higher dimensions.

Exercise 15.5. (*Prisms*) $\Diamond$ Define a *prism* in $\mathbb{R}^3$ to be the Minkowski sum of a convex polygon in a plane and a line segment.
a) [1-] Prove that every prism is rectifiable (scissor congruent to a cube of equal volume).
b) [1] Prove that polytopes B_i in the proof of Lemma 16.6 can be decomposed into two prisms. This gives an alternative proof that B_i is rectifiable.

[34] We need positivity to ensure that all coefficients $c_i > 0$ in the definition of a fortunate polytope.

Exercise 15.6. (*Zonotopes*) [1-] By Exercise 7.16, recall that every zonotope can be subdivided into parallelepipeds. Use Exercise 15.4 to conclude that every zonotope is scissor congruent to a cube.

Exercise 15.7. (*Hadwiger–Glur*) Two polytopes $P, Q \subset \mathbb{R}^d$ are called *T-congruent* if they are unions of a finite number of simplices which are equal up to translations. They are called *S-congruent* if central symmetries are also allowed.
a) [1] Prove the following extension of Theorem 15.1: every two polygons $P, Q \subset \mathbb{R}^2$ of the same area are S-equivalent.
b) [1+] Prove that a convex polygon $Q \subset \mathbb{R}^2$ is T-congruent to a square of equal area if and only if Q is centrally symmetric.
c) [1+] Prove that every parallelepiped in $\mathbb{R}^3$ is T-congruent to a cube of equal volume.
d) [2-] Prove that a convex polytope $P \subset \mathbb{R}^3$ is T-congruent to a cube of equal volume is and only if P is a zonotope.

Exercise 15.8. Two polytopes $P, Q \subset \mathbb{R}^d$ are called *D-congruent* if P can be decomposed into smaller polytopes which can then rearranged to Q, such that the boundary points remain on the boundary. Think of the boundary ∂P as painted, so the goal is to keep the paint outside (see Figure 15.4).[35]

FIGURE 15.4. Two examples of D-congruent polygons.

a) [1] In the plane, prove that every two polygons with the same area and perimeter are D-congruent.
b) [1+] In $\mathbb{R}^3$, prove that two scissor congruent polytopes with the same volume and surface area are D-congruent.
c) [1-] In $\mathbb{R}^4$, find two scissor congruent polytopes with the same volume and surface area, which are *not* D-congruent.

Exercise 15.9. Two unbounded convex polyhedra are called *scissor congruent* if one can be decomposed to a finite number of (bounded or unbounded) convex polyhedra which can be assembled into the other.
a) [1-] Prove that this is an equivalence relation.
b) [1-] In the plane, show that infinite cones with different angles are not scissor congruent.
c) [1+] In the plane, let $C \subset C'$ be two unbounded convex polygons such that $C' \smallsetminus C$ is bounded. Prove that C and C' are scissor congruent.
d) [$*$] Prove or disprove the 3-dimensional analogue of part *c*).

Exercise 15.10. (*Tiling with crosses*) [1] A *cross* in $\mathbb{R}^d$ is a union of $2d + 1$ unit cubes, where one cube in the center is attached to others by a facet. Prove that crosses can tile the whole space $\mathbb{R}^d$.

[35]Just as in the definition of scissor congruence, the boundary condition requirement only concerns the $(d-1)$-dimensional faces of P; we ignore the lower dimensional boundary.

Exercise 15.11. (*Tiling with notched cubes*) A *notched cube* $R(a_1,\ldots,a_d) \subset \mathbb{R}^d$, where $0 < a_1,\ldots,a_d < 1$, is defined as follows:

$$R(a_1,\ldots,a_d) \,=\, \{\,(x_1,\ldots,x_d) \in \mathbb{R}^d \mid 0 \le x_i \le 1, \text{ and } x_i \le a_i \text{ for some } i\,\}.$$

a) [1+] Prove that notched cube $R(a_1,\ldots,a_d)$ can tile the whole space $\mathbb{R}^d$, for all a_i as above.
b) [2-] Find all such tilings.
c) [1] Prove that no notched cube can tile a brick.

Exercise 15.12. (*Reptiles*) A (non-convex) polytope $P \subset \mathbb{R}^d$ is called a *reptile* if it can tile a copy cP of the same tile, for some $c > 1$. The *order* of a reptile is the smallest number $N = c^d$ needed for such a tiling.
a) [1-] Prove that every triangle is a reptile of order 4.
b) [1-] Prove that a trapezoid with sides $2, 1, 1$ and 1 is a reptile of order 4. Similarly, prove that a trapezoid with sides $3, 2, 1$ and 2 is a reptile of order 9.
c) [1] Find all rectangular reptiles of order at most 5.
d) [1] Show that notched cube $R(\frac{1}{2},\ldots,\frac{1}{2})$ is a reptile of order $2^d - 1$.
e) [1] Suppose polytope $P \subset \mathbb{R}^d$ is a reptile of order N. Iterate the tiling construction by tiling bigger and bigger regions with more and more copies of P. Show that this construction defines a limit tiling of the whole space, called a *substitution tiling*.
f) [1-] For reptiles in parts *a*) and *b*), check whether the resulting tiling of the plane is periodic or aperiodic.[36]
g) [1] Check that the substitution tiling of $\mathbb{R}^d$ with notched cube is periodic (cf. Exercise 15.11).

Exercise 15.13. (*Knotted tiles*) [2] For every knot $K \subset \mathbb{R}^3$, prove that there exists a (non-convex) polytope isotopic to K, which can tile the whole space $\mathbb{R}^3$.

Exercise 15.14. Let $Q \subset \mathbb{R}^2$ be a convex set of unit area, $\text{area}(Q) = 1$, and let $Q_1,\ldots,Q_n$ be its translations. Suppose n is odd. Denote by A the set of points covered by an odd number of Q_i's.
a) [1-] Suppose Q is a square. Prove that $\text{area}(A) \ge 1$.
b) [1-] Same for the regular hexagon.
c) [1] Show that convexity is necessary: if Q is non-convex, we can have $\text{area}(A) < 1$.
d) [1+] Show that if Q is a regular triangle, we can have $\text{area}(A) < 1$. For $n = 3$, find the smallest possible such area.
e) [$*$] What happens for a regular octagon? What about when Q is a circle? How about general centrally symmetric convex sets?

Exercise 15.15. (*φ-congruence*) Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be a linear function non-constant on the axis. Two polytopes $P, Q \subset \mathbb{R}^d$ are called *φ-congruent* if they are unions of a finite number of simplices which are equal up to φ-invariant translations.
a) [1] Suppose $\varphi(x, y, z) = ax + by + cz$, is a linear function on $\mathbb{R}^3$, where $a, b, c \in \mathbb{N}$. Prove that the following two bricks are φ-congruent:

$$B_1 = \{(x, y, z) \mid 0 \le x \le b,\, 0 \le y \le c,\, 0 \le z \le a\},$$
$$B_2 = \{(x, y, z) \mid 0 \le x \le c,\, 0 \le y \le a,\, 0 \le z \le b\}.$$

[36]There is certain degree of ambiguity in the definition of the substitution tiling. Thus, in each case, check whether one *can* obtain an aperiodic tiling.

b) [1] For an axis brick $B = \{(x_1, \ldots, x_d) \mid 0 \le x_i \le a_i, 1 \le i \le d\}$ denote by $\varphi[B]$ the multiset $\{\varphi(a_1), \ldots, \varphi(a_d)\}$. Prove that two axis bricks B_1 and B_2 are φ-congruent if and only if $\varphi[B_1]$ and $\varphi[B_2]$ are equal multisets of rational numbers.

15.6. **Final remarks.** Although Hilbert motivated his third problem by the foundations of geometry, the basic ideas of scissor congruence go back to Ancient Greece (see e.g., Archimedes' "Stomachion" manuscript). For the comprehensive history of the subject and an elementary exposition of the scissor congruence see [Bolt, Sah]. The group theoretic approach is a variation on the classical tessellation technique and is formalized in [Mül]. The proof we present in Subsection 15.2 is somewhat similar to the proof given in [AFF]. Of course, the congruences resulted in steps 1), 2) and 3) are completely standard.

Dehn's original paper (1902) is based on an earlier paper by Bricard [Bri1] who proved Bricard's condition for subdivisions rather than all decompositions of polytopes (see [Ben]). Boltyansky speculates that although Hilbert never mentioned Bricard's paper, he was "undoubtedly influenced by it". Our proof starts with Bricard's original approach and then follows the argument originally due to V. F. Kagan [Kag1] (see also [Hop2, §4.3] for a similar proof and the references). The proof of scissor congruence of mirror symmetric polyhedra (Exercise 15.3) also follows Bricard's paper.

Bricard's condition (Lemma 15.4) is a corollary of Dehn's general result on *Dehn invariants*, which are global invariants involving edge lengths and dihedral angles (see Subsection 17.3). These Dehn invariants generalize to a series of Hadwiger invariants in higher dimensions (see [Sah]). While in $\mathbb{R}^3$ and $\mathbb{R}^4$ these invariants are proved to be both necessary and sufficient conditions for scissor congruence (these are results of Sydler and Jessen), in higher dimensions this remains an open problem [Bolt]. For a modern treatment, extensions to higher dimensions, spherical and hyperbolic space, and further references, see [Car, Dup].

16. Polytope algebra

This is the second of the three sections on *scissor congruence*. Here we develop an algebraic approach, by introducing certain algebraic operations on the polytopes which preserve scissor congruence. This approach allows us to prove that certain families of polytopes are scissor congruent, while others are not, all without actually exhibiting the decompositions or computing the dihedral angles. Curiously, the algebraic approach is almost completely independent of the previous section. In this section we do not fully develop the theory; instead, we present only the most basic tools and concentrate on attractive examples. Among other things, we prove that all polytopes which tile the space periodically are scissor congruent to a cube, a result of particular interest, related to classical problems on tilings.

16.1. **The power of negative thinking.** While the Bricard condition in Lemma 15.4 (see also a straightforward extension in Exercise 15.2) resolves Hilbert's original problem by giving necessary conditions for the scissor congruence, it is very restrictive and does not give a sufficient condition. Nor is it easy to use, as one is asked to compute the dihedral angles and verify their rational independence. In fact, even the statement of Bricard's condition is somewhat misleading, as it suggests that scissor congruence is a property of dihedral angles.[37] The following result is an easy corollary of the main results in this section, and gives a flavor of things to come.

Theorem 16.1 (Sydler)**.** *A regular tetrahedron is not scissor congruent to a disjoint union of $k \geq 2$ regular tetrahedra, not necessarily of the same size.*

Let us emphasize that the dihedral angles are all equal in this case, so one needs to find other tools to prove this result. Let us note that Dehn's theorem combined with the above Sydler's theorem implies that the cube, the regular tetrahedron and the regular triangular bipyramid (see Figure 19.1) of the same volume are not scissor congruent. In other words, there are at least three equivalence classes of polytopes of the same volume. In Exercise 16.1 we will show that there is a continuum of such equivalence classes.

We first develop and prove several technical algebraic tools and only then return to further examples and applications.

16.2. **Rectifiable polytopes and operations on polytopes.** We say that a polytope $P \subset \mathbb{R}^3$ is *rectifiable* if it is scissor congruent to a cube of equal volume. For example, all parallelepipeds are rectifiable by Exercise 15.4. Denote by $P \oplus Q$ the disjoint union of polytopes $P, Q \subset \mathbb{R}^3$. One can show that the disjoint union of rectifiable polytopes is also rectifiable (see Exercise 15.4). Let $n!\times!P = P \oplus \ldots \oplus P$ (n times) denotes the disjoint union of n copies of P.

Polytopes cP obtained by an expansion of P by a factor $c > 0$, are called *similar* to P. Obviously, polytopes similar to a rectifiable polytope are also rectifiable. Denote by $\mathcal{R}$ the set of rectifiable polytopes.

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[37]The proof of Bricard's condition given in the previous section is somewhat more revealing, as it is crucially based on the edge lengths.

We say that a polytope $P \subset \mathbb{R}^3$ is *self-similar* if it is scissor congruent to a disjoint union of two or more polytopes similar to P. For the rest of this section we use $\oplus$ to denote the disjoint union of polytopes, and write this as

$$P \sim c_1 P \oplus \ldots \oplus c_k P, \quad \text{where } c_i > 0, \quad \text{k} \geq 2.$$

The main result of this section is the following theorem.

Theorem 16.2 (Sydler's criteria)**.** *A polytope $P \subset \mathbb{R}^3$ is rectifiable if and only if P is self-similar. Alternatively, P is rectifiable if and only if $P \sim cP \oplus R$, for some $R \in \mathcal{R}$ and $c < 1$.*

Sydler's theorem (Theorem 16.1) follows immediately from here. Indeed, if a regular tetrahedron is self-similar, then by the above theorem it must be rectifiable, which contradicts Dehn's theorem (Theorem 15.2).

Before we move to the proof of Sydler's criteria, let us present two other useful properties of scissor congruence. They can be viewed as saying that rectifiability of polytopes is invariant under "subtraction" and "division". As we show later in this section, these properties allow a remarkable degree of flexibility in deciding whether a given polytope is rectifiable.

Theorem 16.3 (Complementarity lemma)**.** *Suppose polytopes $A, B, C, D \subset \mathbb{R}^3$ satisfy: $A \oplus B \sim C \oplus D$ and $B \sim D$. Then $A \sim C$.*

In particular, the complement of a rectifiable polytope inside a brick is also rectifiable.

Theorem 16.4 (Tiling lemma)**.** *Let $P \subset \mathbb{R}^3$ be a polytope such that $P_1 \oplus \ldots \oplus P_m$ is rectifiable, where every P_i is either congruent to P or a mirror image of P, $1 \leq i \leq m$. Then P is also rectifiable.*

In other words, if we can tile a parallelepiped with copies of a given polytope, then this polytope is rectifiable. We say that a tiling of the space $\mathbb{R}^3$ is *periodic* if there exist three independent vectors such that the tiling is invariant under translations by these vectors. Obviously, if a polytope tiles a parallelepiped, it has a periodic tiling of the space.

Corollary 16.5. *Let $P \subset \mathbb{R}^3$ be a polytope which tiles the space periodically. Then P is rectifiable.*

Proof. Fix a periodic tiling of $\mathbb{R}^3$ by P and let Q be a fundamental parallelepiped of the translation group which preserves the tiling. By Lemma 15.3, parallelepiped Q is scissor congruent to a disjoint union of copies of P and mirror symmetric copies of P. Since $Q \in \mathcal{R}$, by the tiling lemma (Theorem 16.4) so is P. □

16.3. **Proofs of the theorems.** We start with an important technical result.

Lemma 16.6. *Let $P \subset \mathbb{R}^3$ be a polytope and let $\alpha_1, \ldots, \alpha_k > 0$ be fixed positive real numbers, such that $\alpha_1 + \ldots + \alpha_k = 1$. Then*

$$P \sim \alpha_1 P \oplus \ldots \oplus \alpha_k P \oplus R,$$

for some $R \in \mathcal{R}$.

Proof. First, let us show that it suffices to prove the result for the tetrahedra. Indeed, suppose $P = \cup_{i=1}^m \Delta_i$ and

$$\Delta_i \sim \alpha_1\Delta_i \oplus \ldots \oplus \alpha_k\Delta_i \oplus R_i \quad \text{for all } 1 \le i \le m.$$

Then

$$P \sim \bigoplus_{j=1}^{k} \left[\bigoplus_{i=1}^{m} \alpha_j \Delta_i \right] \oplus \left[\bigoplus_{i=1}^{m} R_i \right] \sim \alpha_1 P \oplus \ldots \oplus \alpha_k P \oplus R,$$

where we are using the fact $R_1 \oplus \ldots \oplus R_m \in \mathcal{R}$ (see Exercise 15.4).

Let us prove the result in a special case. Let $\Delta \subset \mathbb{R}^3$ be a *standard tetrahedron* defined by the equations $x, y, z \ge 0$ and $x+y+z \le 1$. Cut the tetrahedron into layers $0 \le z \le \alpha_1$, $\alpha_1 \le z \le \alpha_1 + \alpha_2$, etc. Remove the tetrahedra $\alpha_i\Delta_i$, $1 \le i \le k$, from the corners of the corresponding layers, as shown in Figure 16.1. Denote by B_i the remaining hexahedron (polytope with 6 faces) in the i-th layer, for all $1 \le i \le k-1$ (the k-th layer is equal to $\alpha_k\Delta$). Observe that each B_i is scissor congruent to a brick (see Figure 16.1). By Exercise 15.4, this implies that $B_1 \oplus B_2 \oplus \ldots \oplus B_{k-1}$ is rectifiable, and finishes the proof.

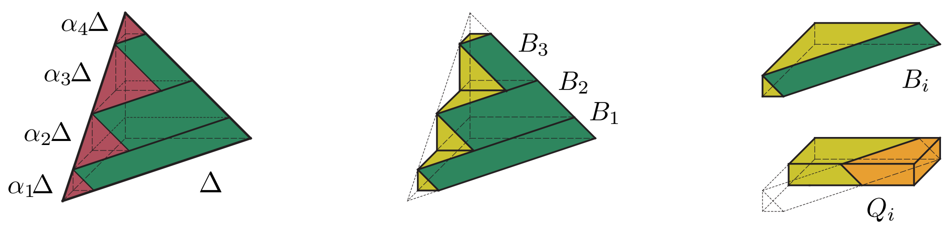

FIGURE 16.1. Layers $B_i \cup \alpha_i\Delta$ and scissor congruence $B_i \sim Q_i$.

For the general tetrahedra, use a construction obtained via a linear transformation. The image of a brick is a parallelepiped which are rectifiable by Exercise 15.4. $\square$

Proof of Theorem 16.4. Since the mirror image of polytope P is scissor congruent to P (see Exercise 15.3), it suffices to prove the result for congruent polytopes $P_i \sim P$. The theorem and Lemma 16.6 can be written as:

$$n{\times}P \sim R_1, \quad nP \sim (n{\times}P) \oplus R_2,$$

for some $R_1, R_2 \in \mathcal{R}$. We conclude:

$$nP \sim (n{\times}P) \oplus R_2 \sim R_1 \oplus R_2 \in \mathcal{R},$$

as desired. $\square$

Proof of Theorem 16.3. First, observe that

$$\text{vol}(A) = \text{vol}(A \oplus B) - \text{vol}(A) = \text{vol}(C \oplus D) - \text{vol}(D) = \text{vol}(C).$$

Fix a large enough integer n and define

$$A' = \frac{1}{n}A, \quad B' = \frac{1}{n}B, \quad C' = \frac{1}{n}C, \quad D' = \frac{1}{n}D.$$

By Lemma 16.6, we have:

$$A \sim (n\times A') \oplus R_1, \quad C \sim (n\times C') \oplus R_2,$$

for some $R_1, R_2 \in \mathcal{R}$. From above,

$$\operatorname{vol}(R_1) = \Bigl(1 - \frac{n}{n^3}\Bigr)\operatorname{vol}(A) = \Bigl(1 - \frac{n}{n^3}\Bigr)\operatorname{vol}(C) = \operatorname{vol}(R_2).$$

This implies that $R_1 \sim R_2$ and that $\operatorname{vol}(R_1) = (1-1/n^2)\operatorname{vol}(A) \to \operatorname{vol}(A)$ as $n \to \infty$. Therefore, $\operatorname{vol}(n \times B')/\operatorname{vol}(A) \to 0$ as $n \to \infty$, and for sufficiently large n we can arrange n copies of B' inside R_1.[38] In other words, for large enough n, we have

$$R \sim (n\times B') \oplus S,$$

for some polytope $S \subset \mathbb{R}^3$. We then have:

$$\begin{aligned} A &\sim (n\times A') \oplus R_1 \sim (n\times A') \oplus (n\times B') \oplus S \sim n\times(A' \oplus B') \oplus S \\ &\sim n\times(C' \oplus D') \oplus S \sim (n\times C') \oplus (n\times D') \oplus S \\ &\sim (n\times C') \oplus (n\times B') \oplus S \sim (n\times C') \oplus R_1 \sim (n\times C') \oplus R_2 \sim C, \end{aligned}$$

as desired. □

Proof of Theorem 16.2. The "only if" part is trivial in both parts. For the second criterion, by Lemma 16.6, we have:

$$cP \oplus R \sim P \sim cP \oplus (1-c)P \oplus R',$$

for some $R, R' \in \mathcal{R}$. By the complementarity lemma (Theorem 16.3), we get $(1-c)P \oplus R' \sim R$. Write $R \sim R' \oplus R''$ for some $R'' \in \mathcal{R}$. After another application of the complementarity lemma, we obtain $(1-c)P \in \mathcal{R}$, as desired.

For the first part, $P \sim c_1P \oplus \ldots \oplus c_kP$ and let $c = c_1 + \ldots + c_k$. By Lemma 16.6, we have:

$$cP \sim c_1P \oplus \ldots \oplus c_kP \oplus R \sim P \oplus R,$$

for some $R \in \mathcal{R}$. Now the second criterion we just proved implies that $P \in \mathcal{R}$. □

16.4. **Examples and special cases.** Let $O = (0,0,0)$, $a = (1,0,0)$, $b = (0,1,0)$, $c = (0,0,1)$, and $d = (1,1,0)$. Consider the *standard tetrahedron* $\Delta_1 = \operatorname{conv}\{O,a,b,c\}$ and the *Hill tetrahedron* $\Delta_2 = \operatorname{conv}\{O,b,c,d\}$. Clearly, $\operatorname{vol}(\Delta_1) = \operatorname{vol}(\Delta_2) = 1/6$.

Proposition 16.7. *The standard tetrahedron, Hill tetrahedron and the regular tetrahedron of equal volume are not scissor congruent.*

[38] Formally speaking, this still requires an argument (see Exercise 15.1).

Proof. First, let us show that Δ_2 is rectifiable. To see this, observe that six copies of Δ_2 tile the unit cube $C = [0,1]^3$. Indeed, take each copy to be a convex hull of a path from O to $(1,1,1)$ as shown in Figure 16.2 (there are six such paths), and use the tiling lemma. By Dehn's theorem (Theorem 15.2), this implies that $\Delta_0 \not\sim \Delta_2$, where Δ_0 is the regular tetrahedron of volume $1/6$.

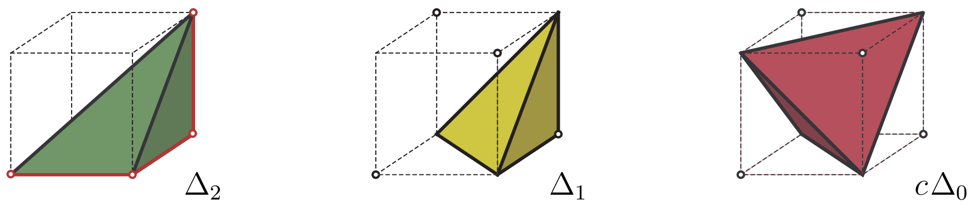

FIGURE 16.2. Hill tetrahedron, standard tetrahedron and the regular tetrahedron.

To see that Δ_1 is not rectifiable, remove four copies of Δ_1 from the cube C as shown in Figure 16.2. We are left with a regular tetrahedron $c\Delta_0$, $c = \sqrt[3]{2}$, which by Dehn's theorem is not rectifiable. By the complementarity lemma, we conclude that the above union of four copies of Δ_1 is not rectifiable, and by the tiling lemma so is Δ_1. This implies that $\Delta_1 \not\sim \Delta_2$.

Finally, the same construction shows that $\Delta_0 \not\sim \Delta_1$. Otherwise, four copies of Δ_0 and one copy of $c\Delta_1$ are scissor congruent to a cube, which contradicts the first Sydler's criterion (Theorem 16.2). This completes the proof of Proposition 16.7. $\square$

By analogy with the case of a regular tetrahedron, one can use Lemma 15.4 to prove directly that the regular octahedron is not rectifiable. In fact, one can even use the same proof of the irrationality of dihedral angles (see Subsection 41.2). However, if one wants to avoid calculating dihedral angles this time, here is a neat geometric round about argument.

Proposition 16.8. *The regular octahedron is not scissor congruent to the standard, Hill, or regular tetrahedra of the same volume.*

Proof. First, observe that the regular octahedron Q can be tiled with eight copies of a standard tetrahedron. By the tiling lemma this immediately immediately implies that Q is not rectifiable, and by the first Sydler's criterion this implies that Q is not scissor congruent to a standard tetrahedron. We need a separate argument to show that $Q \not\sim c\Delta_0$.

Take a regular tetrahedron Δ_0 and remove four corner tetrahedra $\Delta_0' = \frac{1}{2}\Delta_0$. We are left with a regular octahedron Q (see Figure 16.3). Take a disjoint union of Q with any two of tetrahedra Δ_0'. Now, if $Q \sim c\Delta_0$, we have:

$$\Delta_0 \sim Q \oplus 4 \times \Delta_0' \sim c\Delta_0 \oplus 4 \times \Delta_0'.$$

This again contradicts the first Sydler's criterion (Theorem 16.2). $\square$

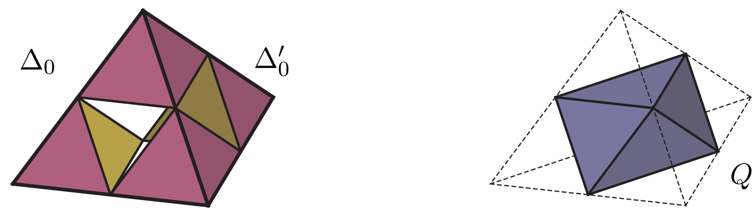

FIGURE 16.3. Octahedron Q and four tetrahedra Δ_0' tile tetrahedron Δ_0.

16.5. **What if you had a magic wand to make things bigger and smaller?** To conclude this section, we consider the following question: suppose in addition to translation and rotations in $\mathbb{R}^3$ we are also allowed to inflate and deflate the polytope. Perhaps surprisingly, one can transform every polytope into every other polytope in this case, in stark contrast with Exercise 16.1.

Formally, we say that polytopes P and Q are Π-*congruent*, write $P \asymp Q$, if each polytope is a disjoint union of similar polytopes:

$$P = \cup_{i=1}^m P_i\,, \qquad Q = \cup_{i=1}^m Q_i\,,$$

where $P_i \simeq cQ_i$, for some $c > 0$. Of course, all scissor congruent polytopes are also Π-congruent, but the inverse is not true even if the polytopes have the same volume.

Theorem 16.9 (Zylev)**.** *Every two polytopes $P, Q \subset \mathbb{R}^3$ are Π-congruent: $P \asymp Q$.*

Proof. First, note that Π-congruence is an equivalence relation, so it suffices to prove that every polytope P is Π-congruent to a unit cube C. Second, it is easy to see that the complementarity lemma (Theorem 16.3) holds for Π as well, and the proof extends verbatim. Now use Lemma 16.6 to obtain

$$3\times\Delta \oplus R_1 \asymp \Delta \asymp 2\times\Delta \oplus R_2\,,$$

for some $R_1, R_2 \in \mathcal{R}$. Now the complementarity lemma implies the result. $\square$

16.6. **Exercises.**

Exercise 16.1. (*Continuum of scissor congruence equivalence classes*) $\Diamond$ [1] For every $\lambda \in \left[0, \frac{1}{2}\right]$ define a λ-*truncated cube* $Q(\lambda) \subset \mathbb{R}^3$ by the inequalities $|x|, |y|, |z| \le 1$, $|x| + |y| + |z| \le 3 - \lambda$. For example, $Q(0)$ is the unit cube, $Q(\frac{1}{\sqrt{8}})$ is the *truncated cube*, and $Q(\frac{1}{2})$ is the *cuboctahedron* (see Figure 16.4). Prove that $Q(\lambda) \nsim cQ(\mu)$, for all $0 \le \lambda < \mu \le \frac{1}{2}$ and $c > 0$.

Exercise 16.2. *a*) [1-] Prove that the plane can be tiled with copies of a convex pentagon.
b) [1-] Prove that the space R^3 can be tiled with copies of a convex heptahedron (polytope $P \subset \mathbb{R}^3$ with exactly 7 faces).

Exercise 16.3. *a*) [1+] Prove that the plane cannot be tiled with copies of a convex octagon.
b) [1-] Show that this is possible when the octagon is non-convex.

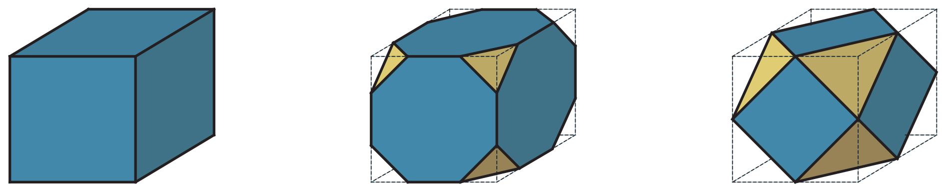

Figure 16.4. Cube, truncated cube and cuboctahedron.

Exercise 16.4. [2-] Prove that an unfortunate convex polytope cannot tile $\mathbb{R}^3$.

Exercise 16.5. *a*) [1+] Find a convex polytope in $\mathbb{R}^3$ which tiles the space, but only aperiodically.
b) [2-] Find a polygon in the plane which tiles the plane, but does not have a tiling with a transitive symmetry group.
c) [$*$] Find a polygon in the plane which tiles the plane, but only aperiodically.

Exercise 16.6. (*Permutohedron*) Let P_n be a permutohedron defined in Example 8.4.
a) [1-] Prove that P_n is vertex-transitive and all its 2-dimensional faces are squares or regular hexagons. Conclude that P_4 is an Archimedean solid called the *truncated octahedron*.
b) [1-] Prove that P_n is inscribed into a sphere.
c) [1] Prove that P_n is a fundamental region of a discrete group acting on $\mathbb{R}^{n-1}$, and generated by reflections.
d) [1-] Prove that P_4 tiles the space $\mathbb{R}^3$ periodically. Conclude from here that P_4 is rectifiable.
e) [1] Prove that P_n is a zonotope (see Exercises 7.16 and 15.6). Conclude from here that P_n is rectifiable.
f) [1+] Subdivide P_n into n^{n-2} parallelepipeds of equal volume, whose edges are parallel to the edges of P_n (use Cayley's formula). Use this to compute the volume of P_n.

Exercise 16.7. Let $Q \subset \mathbb{R}^3$ be a (non-regular) octahedron defined as a convex hull of points $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, and $(0, 0, \pm\frac{1}{\sqrt{2}})$.
a) [1-] Prove that Q tiles $\mathbb{R}^3$ periodically. Deduce from here that Q is rectifiable.
b) [1-] Prove that Q can be tiled with sixteen copies of Hill tetrahedron. Deduce from here that Q is rectifiable.

Exercise 16.8. (*Föppl polytope*) Consider the *truncated tetrahedron* P, defined as an Archimedean solid with four triangular and four hexagonal faces (see Figure 39.3).
a) [1-] Prove that two copies of P and two regular tetrahedra Δ tile a parallelepiped.
b) [1] Cut Δ into four triangular pyramids and attach them to triangular faces of P. Prove that the resulting polytope Q tiles the space.
c) [1-] Conclude that Q is rectifiable.

Exercise 16.9. Consider the *truncated cuboctahedron* P, defined as an Archimedean solid with a square, hexagonal and octagonal faces meeting at each vertex.
a) [1] Prove that copies of P and 8-prisms tile the space.
b) [1-] Use *a*) to prove that P is rectifiable. Alternatively, check that P is a zonotope and conclude that P is rectifiable (see Exercise 15.6).
c) [1] Find a periodic tiling of $\mathbb{R}^3$ with copies of P, cubes and permutohedra (see Example 8.4).

d) [1] Find a periodic tiling of the space with copies of P, truncated cube (see Subsection 16.4) and truncated tetrahedron (see Exercise 16.8).

Exercise 16.10. [1] Give an explicit construction of scissor congruence of the Hill tetrahedron Δ_2 and a prism.

Exercise 16.11. (*Rectifiable tetrahedra*) Prove that the following tetrahedra $(xyzw) \subset \mathbb{R}^3$ are rectifiable:
a) [1-] $x = O$, $y = (1, 0, 0)$, $z = (1, 1, 1)$, $w = (1, -1, 1)$;
b) [1-] $x = O$, $y = (1, -1, -1)$, $z = (1, 1, 1)$, $w = (1, -1, 1)$;
c) [1-] $x = O$, $y = (2, 0, 0)$, $z = (1, 1, 1)$, $w = (1, -1, 1)$;
d) [1-] $x = O$, $y = (1, 0, 1/2)$, $z = (1, 1, 1)$, $w = (1, -1, 1)$;
e) [1] $|xz| = |yw| = 2$, $|xy| = |xw| = |yz| = |zw| = \sqrt{3}$;
f) [1+] $|xy| = |zw| = |yz| = a$, $|xz| = |yw| = b$, $|zw| = 3c$, where $b^2 = a^2 + 3c^2$.

Exercise 16.12. (*Hill tetrahedra*) For every $\alpha \in (0, 2\pi/3)$, let $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3 \in \mathbb{R}^3$ be three unit vectors with angle α between every two of them. Define the *Hill tetrahedron* $Q(\alpha)$ as follows:

$$Q(\alpha) \, = \, \{c_1\boldsymbol{v} + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3 \mid 0 \le c_1 \le c_2 \le c_3 \le 1\}.$$

For example, $Q(\pi/2)$ is the Hill tetrahedron Δ_2 defined earlier.
a) [1] Decompose $Q(\alpha)$ into three pieces which assemble into a prism. Conclude that all Hill tetrahedra are rectifiable.
b) [1+] Find a periodic tiling of space with $Q(\alpha)$. Again, conclude that all Hill tetrahedra are rectifiable.

Exercise 16.13. (*Golden tetrahedron*) Denote by $T \subset \mathbb{R}^3$ the *golden tetrahedron* defined by

$$x, y, z \ge 0, \quad \phi\, x + y + \frac{z}{\phi} \le 1, \quad \text{where} \;\; \phi = \frac{1+\sqrt{5}}{2}\,.$$

a) [1] Prove that all dihedral angles of T are rational multiples of π. Does T tile the space?
b) [1+] Prove that T is rectifiable by an explicit construction.
c) [1+] Prove that the Hill tetrahedron $Q(2\pi/5)$ can be decomposed into four golden tetrahedra (possibly, of different size).
d) [1+] Generalize the tiling lemma (Theorem 16.4) to similar polytopes. Use this and part *c*) to give another proof that T is rectifiable.

Exercise 16.14. (*Schläfli simplices*) For a simplex $\Delta = (v_1 v_2 \dots v_{n+1}) \subset \mathbb{R}^n$ denote by F_i the facet containing all vertices except v_i. Let G_Δ be a graph with the vertices $\{1, 2, \dots, n+1\}$ and edges (i, j) corresponding to acute dihedral angles between facets F_i and F_j.
a) [1+] Prove that G_Δ contains a spanning tree. Conclude that Δ has at least n acute dihedral angles.
b) [1] Show that for every spanning tree T in a complete graph K_{n+1}, there exists a simplex $\Delta \subset \mathbb{R}^n$ with $\binom{n}{2}$ right dihedral angles and n acute dihedral angles corresponding to this spanning tree: $G_\Delta = T$. Such Δ are called *Schläfli simplices*.[39]
c) [1] Show that every Schläfli simplex corresponding to tree T is a convex hull of mutually orthogonal intervals which form a tree isomorphic to T.
d) [1] Denote by ℓ_{ij} the length of the edge $(v_i v_j)$, and by a_i the length of the altitude in Δ from vertex v_i. Prove that $\cos \gamma_{ij} = a_i a_j / \ell_{ij}$, where γ_{ij} is the dihedral angle at $(v_i v_j)$.

[39] H. S. M. Coxeter calls these simplices *orthogonal trees*.

e) [1] Prove that

$$\frac{1}{a_i^2} = \sum_{(i,j)\in T} \frac{1}{\ell_{ij}^2}\,.$$

Deduce from here a formula for $\cos^2 \gamma_{ij}$ as a polynomial in the squared edge lengths ℓ_{ij}^2.

Exercise 16.15. *a*) [1-] Prove that there exists a unique circumscribed brick B around every Schläfli simplex Δ, so that all vertices of Δ are also vertices of B.
b) [1-] Prove that the midpoint of the longest edge of a Schläfli simplex is the center of a circumscribed sphere.
c) [1] Prove that all faces of Schläfli simplices (of all dimensions) are also Schläfli simplices.
d) [1+] Prove that every simplex all of whose 2-dimensional faces are all right triangles must be a Schläfli simplex. Deduce from here part *c*).

Exercise 16.16. (*Orthoschemes*) [1] Consider Schläfli simplices, corresponding to paths. These simplices are called *orthoschemes*.[40] These can be defined as convex hulls of pairwise orthogonal intervals in $\mathbb{R}^d$ forming a path. Prove that every orthoscheme in $\mathbb{R}^d$ can be dissected into $d+1$ orthoschemes.

Exercise 16.17. (*Coxeter simplices*) Consider Schläfli simplices corresponding to trees in Figure 16.5 with interval lengths written next to the edges.
a) [1-] Use parts *c*) and *d*) of Exercise 16.14 to compute all dihedral angles of these Schläfli simplices.
b) [1+] Prove that these simplices are fundamental regions of the natural action by the corresponding affine Coxeter groups.[41]
c) [1-] Conclude that these simplices are rectifiable.

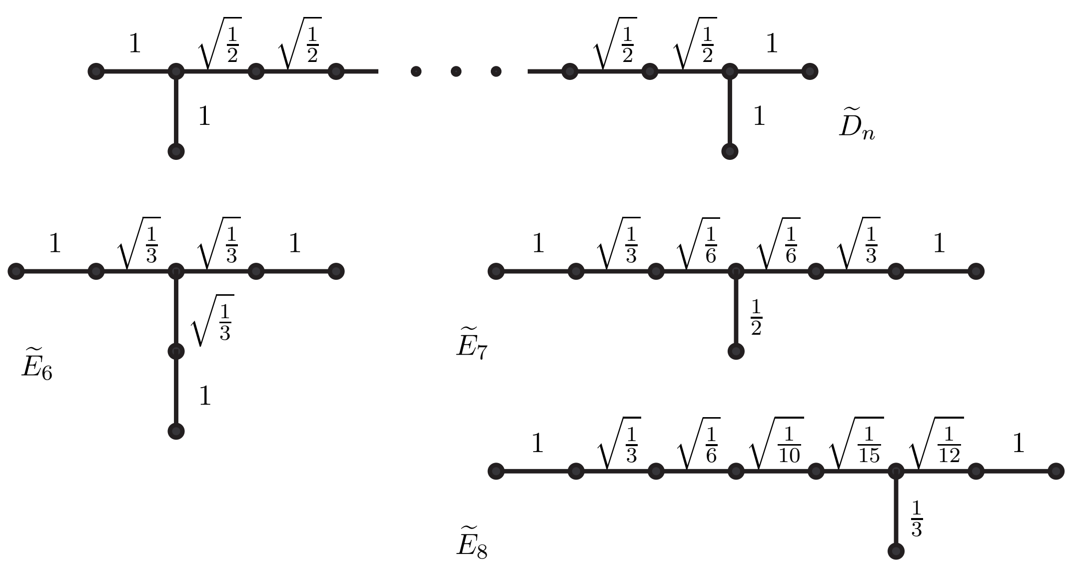

FIGURE 16.5. Schläfli simplices corresponding to affine Coxeter groups.

[40]In computer science literature, these are called *path-simplices*.
[41]There is a subclass of discrete groups generated by reflections: `http://tinyurl.com/26fbqeu`

Exercise 16.18. (*Sommerville simplices*) [2] Classify all tetrahedra $\Delta \subset \mathbb{R}^3$ which tile the space face-to-face (mirror symmetry of Δ is not allowed).
b) [$*$] Same problem, but now mirror symmetry is not allowed.

Exercise 16.19. [1+] Let P be Jessen's orthogonal icosahedron defined in part *b*) of Exercise 19.17. Prove that P is rectifiable.

Exercise 16.20. *a*) [1] Use Bricard's condition (Lemma 15.4) to show that no two Platonic solids are scissor congruent.
b) [2-] Denote by Δ, Q, I and D the regular tetrahedron, octahedron, icosahedron and dodecahedron with unit edge lengths. Find all $m_i \geq 0$, such that $(m_1 \times \Delta) \oplus (m_2 \times Q) \oplus (m_3 \times I) \oplus (m_4 \times D)$ is rectifiable.
c) [2-] Denote by W the *icosidodecahedron* with unit edge length, an Archimedean solid where two pentagons and two triangles meet at every vertex. Prove that $I \oplus D \oplus W$ is rectifiable.

Exercise 16.21. [1] Prove that the regular cross-polytope Q in $\mathbb{R}^4$ can tile the whole space. Deduce from here that Q is rectifiable.

Exercise 16.22. *a*) [1-] Let $\Delta, Q \subset \mathbb{R}^8$ be a regular simplex and a cross-polytope with edge lengths $\sqrt{2}$. Prove that

$$1920 \cdot \mathrm{vol}(\Delta) \, + \, 135 \cdot \mathrm{vol}(Q) \, = \, 1 \, .$$

b) [1+] Extend the above equation by showing that $1920 \times \Delta \, \oplus \, 135 \times Q$ is scissor congruent to a unit cube.
c) [2-] Prove that copies of Δ and Q can periodically tile the whole space $\mathbb{R}^8$. Deduce from here part *b*).
d) [2-] Decide in what other dimensions there are analogues of *a*) and *b*).

Exercise 16.23. *a*) [1+] Find an explicit decomposition of a regular tetrahedron into polytopes such that similar polytopes can be arranged into a cube, i.e., prove $\Delta_0 \asymp C$ directly.
b) [2] Suppose only translations and no rotations are allowed between similar polytopes in the definition of Π-congruence (cf. Exercise 15.7). Extend Theorem 16.9 to this case.
b) [2-] Generalize Theorem 16.9 to higher dimensions.

Exercise 16.24. (*Sydler*) [2] Let $P \subset \mathbb{R}^3$ be a tetrahedron with all dihedral angles rational multiples of π. Prove that P is rectifiable.

Exercise 16.25. (*Jessen*) [2] Let P be a polytope in $\mathbb{R}^4$. Prove that there exists a polytope $Q \subset \mathbb{R}^3$ such that $P \sim Q \times [0, 1]$.

16.7. **Final remarks.** Most results and many proofs in this section go back to Sydler's original paper [Syd1]. In particular, the crucial idea to use Lemma 16.6 to prove the complementarity lemma is also due to Sydler. Theorem 16.9 was proved by in [Zyl] (see also [Had2]). Exercises 16.24 and 16.25 are partial results in the Sydler–Jessen's theory. We refer to [Bolt] for complete proofs, generalizations and references.

Let us mention here several connections of scissor congruence and tileability of the space by copies of the same polytope. Already in the plane there are many examples of such polygons, including some interesting pentagons, which tile plane. Now, if polytope $P \subset \mathbb{R}^3$ can tile the space periodically, then P is rectifiable (Corollary 16.5). Similarly, if P admits a substitution tiling of the space (see Exercise 15.12), then P is also rectifiable (this follows immediately from Theorem 16.2). The most general result was obtained by Debrunner (see [LM]), who showed that every polytope which tiles $\mathbb{R}^3$ is rectifiable. On the other hand, Conway has a simple construction of a polytope which has only aperiodic tilings of the space (Exercise 16.5). Unfortunately, the proof by Debrunner is indirect and uses Sydler's theorem on Dehn invariants (Theorem 17.7). Exercise 16.4 gives an idea of the proof in a special case. We refer to [Sene] for an accessible introduction to the tiling of the space by tetrahedra and general convex polyhedra (cf. Exercises 16.7, 16.11 and 16.6).

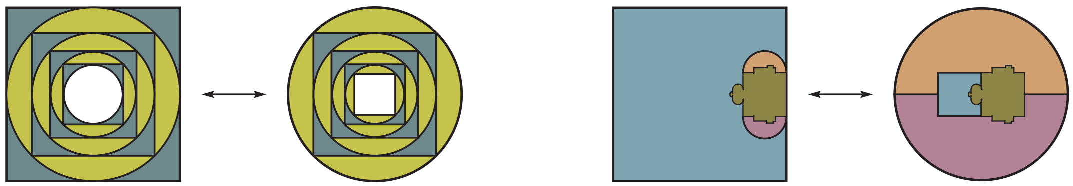

FIGURE 16.6. Generalized Π-congruences between a square and a circle.

Finally, in an effort to extend the notion of scissor congruence to general regions one has to be careful not to over-extend. Not only do there exist various Banach–Tarski paradox type results, but even with a restriction to "almost nice" regions, just about everything is scissor congruent. In Figure 16.6 we show two ways of "squaring the circle", both using homotheties, one with infinitely many piecewise smooth regions (or two disconnected regions), and another with just four simply connected regions. We refer to [HerR, KleR] for more on these results and references.

17. Dissections and valuations

This is the last of three sections on *scissor congruence*, where we finally define the *Dehn invariants* mentioned in Section 15. We begin by showing that all dissections of a polytope are connected by certain local moves. These are the smallest possible transformations between dissections, switching between one and two simplices. The idea of local moves is to be able to prove various properties of *all* dissections by checking that the properties are preserved under a local move. This is now a standard approach in discrete geometry, which we repeatedly use in Section 23 as well as in various other examples and exercises.

17.1. **Slicing and dicing the polytopes.** In this section we consider *dissections* of convex polytopes, defined as simplicial decompositions. It is important to distinguish dissections from *triangulations* defined as simplicial subdivisions and considered earlier in Section 2. Clearly, every triangulation is a dissection and every polytope has infinitely many dissections.

Let $P \subset \mathbb{R}^d$ be a convex polytope and let $\mathcal{D}$ be its dissection, write $\mathcal{D} \vdash P$. Define local moves on dissections as follows. We say $\mathcal{D}_1, \mathcal{D}_2 \vdash P$ are connected by an *elementary move*, write $\mathcal{D}_1 \Leftrightarrow \mathcal{D}_2$, if they coincide everywhere except at one simplex in $\mathcal{D}_1$ which is subdivided into two in $\mathcal{D}_2$, or if one simplex in $\mathcal{D}_2$ which is subdivided into two in $\mathcal{D}_1$. In other words, one can cut one of the tetrahedra with a hyperplane through all but two vertices (the only way to decompose a tetrahedron into two) or do the inverse operation (see Figure 17.1).

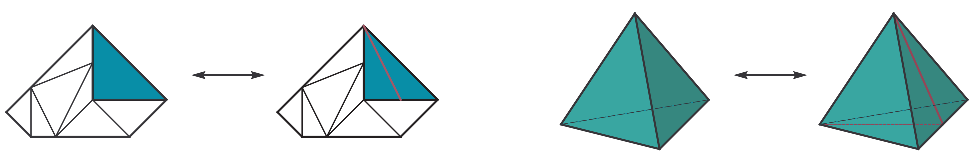

Figure 17.1. Examples of elementary moves on dissections.

We say that two dissections $\mathcal{D}, \mathcal{D}' \vdash P$ are *elementary move equivalent*, write $\mathcal{D} \leftrightarrow \mathcal{D}'$, if there exists a sequence $\mathcal{D}_1, \ldots, \mathcal{D}_\ell$ of dissections of P such that

$$\mathcal{D} \Leftrightarrow \mathcal{D}_1 \Leftrightarrow \mathcal{D}_2 \Leftrightarrow \ldots \Leftrightarrow \mathcal{D}_\ell \Leftrightarrow \mathcal{D}'.$$

In other words, the elementary move equivalent dissections are those connected by a finite sequence of elementary moves.

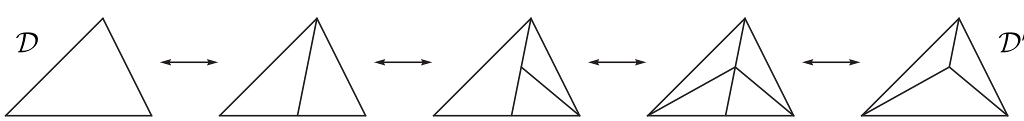

Figure 17.2. A sequence of elementary moves on dissections: $\mathcal{D} \leftrightarrow \mathcal{D}'$.

Theorem 17.1 (Elementary move connectivity)**.** *Every two dissections $\mathcal{D}, \mathcal{D}' \vdash P$ of a convex polytope $P \subset \mathbb{R}^d$ are elementary move equivalent: $\mathcal{D} \leftrightarrow \mathcal{D}'$.*

We prove the theorem only for $d = 2, 3$, leaving the higher dimensional cases to the reader (see Exercise 17.1). The case of space dissections ($d = 3$) is particularly important in connection with scissor congruence.

17.2. **Symmetric valuations.** The connection between Theorem 17.1 and scissor congruence may not be immediately apparent, so let us start with some definitions.

Let $\varphi : \{\Delta\} \to \mathbb{R}$ be a map from all tetrahedra $\Delta \subset \mathbb{R}^d$ into real numbers. We say that φ is a *valuation* if it is invariant under the elementary moves: $\varphi(\Delta_1 \cup \Delta_2) = \varphi(\Delta_1) + \varphi(\Delta_2)$. Valuation φ is called *symmetric* if it is invariant under the rigid motions (rotations and translations). For example, $\mathrm{vol}(\Delta)$ is a natural symmetric valuation. An *extension* of a valuation is a function $\varphi : \{P\} \to \mathbb{R}$ on all convex polytopes $P \subset \mathbb{R}^d$, which satisfies

$$\varphi(P) = \sum_{\Delta \in \mathcal{D}} \varphi(\Delta), \quad \text{for all } \mathcal{D} \vdash P \text{ and } P \subset \mathbb{R}^d.$$

Here and everywhere below we denote the extension also by φ. We first need to show that the extensions always exist.

Lemma 17.2 (Valuation extension lemma)**.** *Every symmetric valuation φ on simplices in $\mathbb{R}^d$ has a unique extension to all convex polytopes $P \subset \mathbb{R}^d$. Moreover, this extension is also symmetric, i.e., invariant under rigid motions.*

Proof. For the existence, take any dissection $\mathcal{D} \vdash P$ (take, e.g., the triangulation constructed in Subsection 2.1). The uniqueness follows from the fact that the summation in the definition is equal for all dissections $\mathcal{D} \vdash P$. Indeed, by Theorem 17.1 every two valuations are connected by elementary moves and the valuation is invariant under elementary moves. Finally, the symmetry of the extension follows immediately from the symmetry of φ. □

Corollary 17.3. *Let φ be a symmetric valuation on simplices in $\mathbb{R}^d$ and let $P, Q \subset \mathbb{R}^d$ be convex polytopes, such that $\varphi(P) \neq \varphi(Q)$. Then these two polytopes are not scissor congruent: $P \nsim Q$.*

Proof. To the contrary, suppose $P \sim Q$. Then there exist a decomposition of both polytopes into the same set of simplices:

$$P \,=\, \cup_{i=1}^m \Delta_i\,, \quad Q \,=\, \cup_{i=1}^m \Delta_i'\,, \quad \text{where } \Delta_i \simeq \Delta_i'.$$

But then

$$\varphi(P) \,=\, \varphi(\Delta_1) \,+\, \ldots \,+\, \varphi(\Delta_m) \,=\, \varphi(\Delta_1') \,+\, \ldots \,+\, \varphi(\Delta_m') \,=\, \varphi(Q),$$

a contradiction. □

For example, when φ is the volume and $\mathrm{vol}(P) \neq \mathrm{vol}(Q)$, we trivially have $P \nsim Q$. In the next section we give some further examples of symmetric valuations.

17.3. **Dehn invariants.** We say that $f:\mathbb{R}\to\mathbb{R}$ is an *additive function* if $f(a+b)=f(a)+f(b)$. Define the *Dehn invariant* to be the following function on tetrahedra $\Delta\subset\mathbb{R}^3$:

$$\varphi(\Delta) \,=\, \sum_{e\in\Delta} \ell_e\, f(\gamma_e)\,,$$

where γ_e is the dihedral angle at edge $e\subset\Delta$, ℓ_e is the length of e, and $f:\mathbb{R}\to\mathbb{R}$ is an additive function which satisfies $f(\pi)=0$. The following two results show that Dehn invariants can be defined on all convex polytopes $P\subset\mathbb{R}^3$ and give an explicit formula for $\varphi(P)$.

Lemma 17.4. *Every Dehn invariant on tetrahedra in $\mathbb{R}^3$ is a symmetric valuation.*

Proof. The symmetry follows immediately by the definition of Dehn invariants. Consider now a 2-move $\Delta\Leftrightarrow\Delta_1\cup\Delta_2$. By definition, two new edges e,e' are created with corresponding dihedral angles α, $\pi-\alpha$, β and $\pi-\beta$ (see Figure 17.3). Therefore,

$$\begin{aligned}\varphi(\Delta) \,-\, \varphi(\Delta_1) \,-\, \varphi(\Delta_2) \,&=\, \ell_e f(\alpha) \,+\, \ell_e f(\pi-\alpha) \,+\, \ell_{e'} f(\beta) \,+\, \ell_{e'} f(\pi-\beta)\\ &=\, \ell_e\big(f(\alpha)+f(\pi-\alpha)\big) \,+\, \ell_{e'}\big(f(\beta)+f(\pi-\beta)\big) \,=\, \ell_e f(\pi) \,+\, \ell_{e'} f(\pi) \,=\, 0,\end{aligned}$$

which implies that φ is a valuation. $\square$

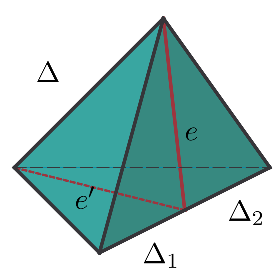

FIGURE 17.3. Decomposition of a tetrahedron $\Delta=\Delta_1\cup\Delta_2$.

Lemma 17.5. *The Dehn invariant defined as above has the following extension to all convex polytopes:*

$$\varphi(P) \,=\, \sum_{e\in P} \ell_e\, f(\gamma_e)\,,$$

where γ_e is the dihedral angle at edge $e\subset P$.

Proof. Fix a triangulation $P=\cup_{i=1}^m\Delta_i$. As in the proof of the lemma above or in the proof of Bricard's condition for subdivisions (Lemma 15.4), for every interior edge e, the sum of dihedral angles at e is equal to 2π. Similarly, for an edge inside a face, the sum of dihedral angles at e is equal to π. We have then

$$\varphi(P) \,=\, \sum_{i=1}^m \varphi(\Delta_i) \,=\, \sum_{i=1}^m \sum_{e\in\Delta_i} \ell_e\, f(\gamma_e) \,=\, \sum_{e\in P} \ell_e\, f(\gamma_e),$$

as desired. $\square$

We can now summarize the results as follows.

Theorem 17.6 (Dehn)**.** *Define the* Dehn invariant *on all convex polytopes* $P \subset \mathbb{R}^3$ *by the following formula:*

$$\varphi(P) \, = \, \sum_{e \in P} \ell_e \, f(\gamma_e) \, ,$$

where γ_e *is the dihedral angle at edge* $e \in P$, ℓ_e *is the length of* e, *and where* $f : \mathbb{R} \to \mathbb{R}$ *is an additive function such that* $f(\pi) = 0$. *Suppose now* $P, Q \subset \mathbb{R}^3$ *are two convex polytopes such that* $\varphi(P) \neq \varphi(Q)$. *Then* P *and* Q *are not scissor congruent:* $P \nsim Q$.

17.4. **From abstraction to applications.** The Dehn invariants defined above may not seem like much. In fact, at first glance it might seem too abstract to be useful in actual applications. Before giving several examples, let us state the key result in the field.

Theorem 17.7 (Sydler)**.** *Every two convex polytopes* $P, Q \subset \mathbb{R}^3$ *are scissor congruent if and only if* $\operatorname{vol}(P) = \operatorname{vol}(Q)$ *and for every Dehn invariant* φ *as above we have* $\varphi(P) = \varphi(Q)$.

In other words, Sydler's theorem implies that whenever $P \nsim Q$, we should be able to prove this by finding an appropriate Dehn invariant. Unfortunately the proof is a bit too technical to be included. Here are two basic examples.

Example 17.8. Let $P = \Delta \subset \mathbb{R}^3$ be the regular tetrahedron with unit edge length, and let $Q \subset \mathbb{R}^3$ be a cube of the same volume. Let $\alpha = \arccos \frac{1}{3}$ denotes the dihedral angle in Δ. Consider an additive function $f : \mathbb{R} \to \mathbb{R}$ which satisfies

$$f(\alpha s) = s \quad \text{and} \quad f(\pi s) = 0, \quad \text{for all } \; s \in \mathbb{Q}.$$

It is easy to see that such functions exist,[42] since $\alpha/\pi \notin \mathbb{Q}$ as shown in Subsection 41.3. Now, for the Dehn invariant we have:

$$\varphi(P) \, = \, 6 \cdot 1 \cdot \varphi(\alpha) = 6 \quad \text{and} \quad \varphi(Q) \, = \, 12 \cdot c \cdot \varphi\left(\frac{\pi}{2}\right) \, = \, 0,$$

where c is the edge length of Q. Since $\varphi(P) \neq \varphi(Q)$, by Theorem 17.6 we conclude that P and Q are not scissor congruent. This gives another proof of Dehn's Theorem 15.2.

Example 17.9. As before, let $P = \Delta \subset \mathbb{R}^3$ be the regular tetrahedron with unit edge length, and let $Q = c_1\Delta \oplus \ldots \oplus c_k\Delta$, $k \geq 2$, be a disjoint union of regular tetrahedra of equal volume. Let φ be the Dehn invariant as in the previous example. We have

$$\varphi(Q) \, = \, 6 \cdot c_1 \cdot \varphi(\alpha) \, + \, \ldots \, + \, 6 \cdot c_k \cdot \varphi(\alpha) \, = \, 6\,(c_1 + \ldots + c_k).$$

Since $k \geq 2$ and $\operatorname{vol}(P) = \operatorname{vol}(Q)$, we have $c_1^3 + \ldots + c_k^3 = 1$ and $0 < c_i < 1$. From here, $c_1 + \ldots + c_k > 1$, and $\varphi(P) \neq \varphi(Q)$. Again, by Theorem 17.6 we conclude that P and Q are not scissor congruent, which gives another proof of Sydler's Theorem 16.1.

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[42]Well, no, it is not *that* easy to see. To formalize this, one needs to use the transfinite induction. However, for our purposes, a weaker statement suffices (see Exercise 17.3).

17.5. **Dissecting the plane.** The proof of Theorem 17.1 will occupy much of the rest of this section. We start with polygons in the plane and then extend the proof to higher dimensions. First, let us start with a few definitions.

A *(full) triangulation* $\mathcal{T} \vdash P$ of a convex polygon P is a triangulation of P with vertices in the vertices of P. Define a *star triangulation* $\mathcal{T} \vdash P$ to be a triangulation where all triangles have a common vertex a. We call a the *center* of the star triangulation.

Lemma 17.10. *Every two star triangulations of a convex polygon are elementary move equivalent.*

Proof. First, let us show that every two triangulations of a convex polygon in the plane are elementary move equivalent. Define a 2*-move* by choosing two adjacent triangles $\Delta, \Delta' \in \mathcal{T}$ and switching one diagonal in a convex quadrilateral $\Delta \cup \Delta'$ to another. Observe that all triangulations of a convex polygon are connected by 2-moves (see Theorem 14.1 and Remark 14.2). Since every 2-move can be obtained by four elementary moves (see Figure 17.4), we conclude that star triangulations are elementary move equivalent. □

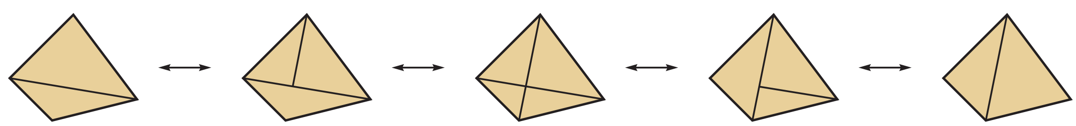

FIGURE 17.4. Making a 2-move using four elementary moves.

Let L be a line intersecting a convex polygon P and dividing it into two polygons P_1 and P_2. For every two star triangulations $\mathcal{D}_1 \vdash P_1$ and $\mathcal{D}_2 \vdash P_2$, their union $\mathcal{D}_1 \cup \mathcal{D}_2$ is a dissection of $\mathcal{D}$.

Lemma 17.11. *The union of star triangulations $\mathcal{D}_1$ and $\mathcal{D}_2$ of convex polygons P_1 and P_2, respectively, is elementary move equivalent to a star triangulation $\mathcal{D}$ of $P = P_1 \cup P_2$.*

Proof. Denote by a, b the points of intersection of the line L and the boundary of P. By Lemma 17.10, triangulations $\mathcal{D}_1$ and $\mathcal{D}_2$ are elementary move equivalent to star triangulations centered at a. If b is not a vertex, remove the edge (a, b) which separates two triangles (see Figure 17.5). If a is a vertex, we are done. If it lies on edge e of P, we can choose a vertex v of P not on e. For each of the polygons separated by (a, v), use Lemma 17.10 connect their triangulations to star triangulations centered at v. Now remove the (a, v) edge. This give a star triangulation centered at v (see Figure 17.5). Again, by Lemma 17.10, this triangulation is elementary move equivalent to every star triangulation $\mathcal{D}$ of the polygon P. □

We are now ready to prove Theorem 17.1 in the plane.

Proposition 17.12. *Theorem 17.1 holds for $d = 2$, i.e., every two dissections of a convex polygon in the plane are elementary move equivalent.*

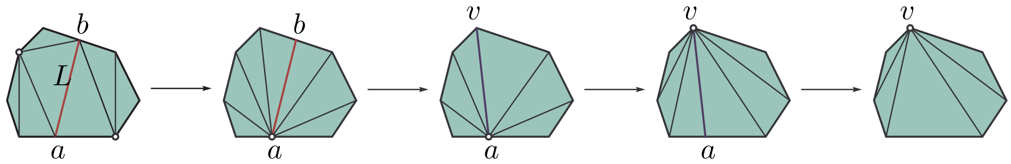

FIGURE 17.5. From two star triangulations to one.

Proof. Consider a decomposition $\mathcal{Q}$ of P into convex polygons $P = \cup_{i=1}^m P_i$. A dissection $\mathcal{D}$ is called a *star refinement* of $\mathcal{Q}$, write $\mathcal{D} \prec \mathcal{Q}$, if $\mathcal{D} = \cup_{i=1}^m \mathcal{D}_i$, where $\mathcal{D}_i$ is a star triangulation of P_i. We call $\mathcal{Q}$ *reducible* if a star refinement $\mathcal{D} \prec \mathcal{Q}$ is elementary move equivalent to a star triangulation of P. By Lemma 17.10, every star refinement $\mathcal{D} \prec \mathcal{Q}$ is elementary move equivalent to every star triangulation of P.

We prove by induction on the number m of polygons that every decomposition $\mathcal{Q}$ as above is reducible. There is nothing to prove when $m = 1$. For $m > 1$, take any interior edge $e \subset P$ and cut P by a line ℓ spanned by e. Denote by P, P' the resulting two polygons, and let $P_i' = P_i \cap P'$, $P_i'' = P_i \cap P''$ for all $1 \le i \le m$. By Lemma 17.11, star triangulations of P_i are elementary move equivalent to a union of star triangulations of P_i' and P_i''. In other words, if $P' = \cup P_i'$ and $P'' = \cup P_i''$ are reducible decompositions, then so is $\mathcal{Q}$.

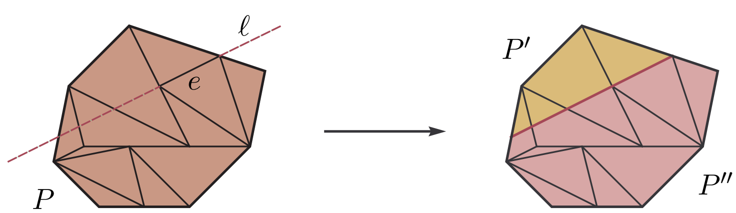

FIGURE 17.6. Cutting a decomposition by a line in the induction step.

Now observe that by convexity, the polygons on both sides of edge e are not cut by ℓ. Thus, both decompositions have fewer than m polygons, and the inductive assumption applies. From above, the decomposition $\mathcal{Q}$ is reducible, which completes the proof. $\square$

17.6. **Dissecting the space.** Here we extend the argument in the previous subsection to convex polytopes in $\mathbb{R}^3$. The heart of the proof (the proof of Proposition 17.12) extends nearly verbatim, while the straightforward proof of Lemma 17.10 extends only to regular triangulations (see Subsection 14.10).

Lemma 17.13. *Let $\mathcal{D}_1, \mathcal{D}_2$ be dissections of a convex polygon P and let Q be a cone over P. Denote by $\mathcal{D}_1', \mathcal{D}_2'$ the dissections of Q obtained as union of cones over triangles in $\mathcal{D}_1, \mathcal{D}_2$. Then $\mathcal{D}_1' \sim \mathcal{D}_2'$.*

Proof. Taking a cone over an elementary move in P corresponds to an elementary move in Q. $\square$

Let $P \subset \mathbb{R}^3$ be a convex polytope and let V be the set of its vertices. As before, a *(full) triangulation* of P is a triangulation with vertices in V (we will skip the adjective full throughout the section whenever possible). A *star triangulation* centered at a vertex $a \in V$ is a triangulation of P with all simplices containing a and giving star triangulations of all faces of P. Recall the definition of *regular triangulations* in Subsection 14.5.

Lemma 17.14. *Every star triangulation of a convex polytope in $\mathbb{R}^3$ is regular.*

Proof. We need to show that every star triangulation $\mathcal{D}$ of P is associated to some height function $\xi : V \to \mathbb{R}_+$. Suppose $\mathcal{D}$ is centered at $a \in V$. Let $\xi_\circ(a) = 1$ and $\xi_\circ(v) = 0$ for all $v \neq a$. The subdivision associated with ξ consists of cones over the faces of P. By definition, the restriction of $\mathcal{D}$ to a face F gives a triangulation of F. Fix $\varepsilon > 0$ very small. Consider a height function $\xi(a) = 1$, $\xi(b) = \varepsilon$, if b is a center of a triangulation in some face of P, and $\xi(v) = 0$ otherwise. When ε is small enough, in the surface S every cone over a face of P is then triangulated according to $\mathcal{D}$, as desired. $\square$

Lemma 17.15. *Every two star triangulations of a convex polytope in $\mathbb{R}^3$ are elementary move equivalent.*

Proof. Let us first show that every 2–3 move (see Subsection 14.5) can be obtained by elementary moves. For each of the two tetrahedra in a bipyramid we need to have a dissection into three cones. By Lemma 17.13 and Proposition 17.12, we can make these dissections using elementary moves (see also Figure 17.2). Now attach together the corresponding three pairs of resulting tetrahedra.

Now, by Exercise 17.6 extending Theorem 14.10 to all convex polytopes in $\mathbb{R}^3$, it suffices to check that remaining stellar flips can be obtained by elementary moves. The proof is along similar lines and is left as (an easy) part d) of that exercise. $\square$

To continue with the proof as in the case $d = 2$ we need an analogue of Lemma 17.11. Let L be a plane intersecting a convex polytope $P \subset \mathbb{R}^3$, dividing it into two polytopes P_1 and P_2. For every two star triangulations $\mathcal{D}_1 \vdash P_1$ and $\mathcal{D}_2 \vdash P_2$, their union $\mathcal{D}_1 \cup \mathcal{D}_2$ is a dissection of $\mathcal{D}$.

Lemma 17.16. *The union of star triangulations $\mathcal{D}_1$ and $\mathcal{D}_2$ of convex polytopes P_1 and P_2, respectively, is elementary move equivalent to a star triangulation $\mathcal{D}$ of $P = P_1 \cup P_2$.*

Proof. The idea is to combine the approach in the proof of Lemma 17.11 with the height function technique as above. Denote by A the set of points of intersection of L and edges in P. Fix a point $a \in A$, and denote by $\mathcal{D}_\circ$ the star triangulation of P with vertices in $V \cup A$, centered at a and such that the center of triangulation of every face lies on L. By construction, the triangulation $\mathcal{D}_\circ$ is a union of star triangulations of P_1 and P_2, which are elementary move equivalent to $\mathcal{D}_1$ and $\mathcal{D}_2$ by Lemma 17.15.

Now, suppose L intersects face $F \subset P$, $a \notin F$, by an edge (b, c). The cone from a over F is a dissection, which, by Lemma 17.13, is elementary move equivalent to a star triangulation with vertices in $V \cup A \smallsetminus \{b, c\}$ (see Figure 17.7). Repeating this over every such face F we obtain a star triangulation $\mathcal{D}$ of P centered at a and with vertices in $V \cup \{a\}$.

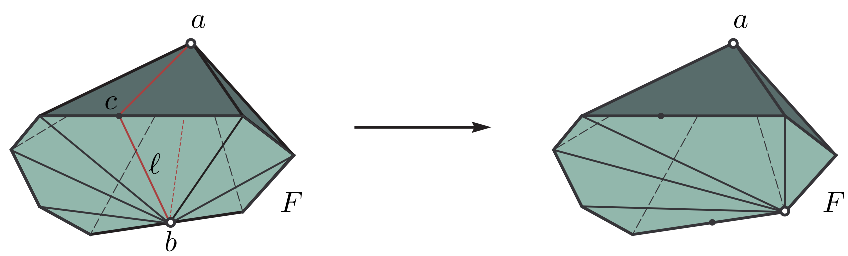

FIGURE 17.7. star triangulation of a cone over F (view from the bottom).

Finally, let us use the height function approach to show that $\mathcal{D}$ is elementary move equivalent to a star triangulation of P with vertices in V. Suppose a lies on the edge (v, w). Take the height function $\xi : V \cup \{a\} \to \mathbb{R}_+$ defined in the proof of Lemma 17.14. Recall that $\xi(a) = 1$ and $\xi(v), \xi(w) > 0$ are small. Continue as in the proof of Theorem 14.10. Start increasing the values of v and w generically, until $\xi(v), \xi(w) > 1$, when a is no longer a vertex in a triangulation. Other than 2–3 moves, at some point, when ξ is linear on (v, w) a single elementary move will be used once, when the height function ξ is linear on (v, w). Note that the resulting triangulation is full, but not necessarily a star triangulation. By Theorem 14.10, we can connect it to a star triangulation by a sequence of 2–3 moves, which implies the result. □

Proposition 17.17. *Theorem 17.1 holds for* $d = 3$*, i.e., every two dissections of a convex polytope in* $\mathbb{R}^3$ *are elementary move equivalent.*

As we mentioned above, the proof follows from the lemmas above and the same inductive argument.

Remark 17.18. As we mentioned in Remark 14.9, it is an open problem whether all full triangulations of a convex polytope in $\mathbb{R}^3$ are connected by 2–3 moves [San2]. The corresponding result in dimensions 5 and higher is false. On the other hand, the graph of all regular triangulations is not only connected, but is a graph of a convex polytope (see Example 8.5 for a special case).

17.7. **Wait, there is more!** Yes, indeed. Remember we only proved the elementary move connectivity theorem (Theorem 17.1) in the plane and in 3-dimensional space. In higher dimensions the proof follows roughly the same lines as long as one defines star triangulation in higher dimension (see Exercise 17.1). As a consolation prize, we prove that the following generalization follows easily from Theorem 17.1.

Theorem 17.19. *For every (possibly non-convex) polytope $P \subset \mathbb{R}^d$, every two dissections $\mathcal{D}, \mathcal{D}' \vdash P$ are elementary move equivalent: $\mathcal{D} \leftrightarrow \mathcal{D}'$.*

Here by a polytope we mean any finite union of convex polytopes. Note that we implicitly used convexity, for example, in the proof of the induction step in Proposition 17.12. Interestingly, the reduction to convex polytopes is now completely straightforward.

Proof of Theorem 17.19. Take the intersection of the dissections (i.e., superimpose them as decompositions). We obtain a decomposition into convex polytopes. Refine this decomposition by triangulating each polytope and denote by $\mathcal{D}_\circ$ the resulting dissection of P. Now observe that every simplex Δ in $\mathcal{D}$ is connected by elementary moves to its dissection in $\mathcal{D}_\circ$. Therefore, $\mathcal{D}$ is elementary move equivalent to $\mathcal{D}_\circ$, and we conclude: $\mathcal{D} \leftrightarrow \mathcal{D}_\circ \leftrightarrow \mathcal{D}'$. □

17.8. **Exercises.**

Exercise 17.1. $\Diamond$ [1+] Finish the proof of Theorem 17.1 by extending the proof in Subsection 17.6 to higher dimensions.

Exercise 17.2. [1-] Show that Exercise 16.24 is a simple corollary of Sydler's theorem (Theorem 17.7). More generally, show that Sydler's theorem implies that all polytopes with rational dihedral angles are rectifiable.

Exercise 17.3. $\Diamond$ *a*) [1] Suppose Δ and Q are as in Example 17.8. As in the example, assume that $\Delta \sim Q$ and consider the decompositions into tetrahedra proving this. Check that it suffices define the additive function f only on the (finitely many) dihedral angles that appear in the decomposition and obtain a contradiction.
b) [1+] Prove existence of the additive functions f in Example 17.8.

Exercise 17.4. (*Tverberg's theorem*) Let $P \subset \mathbb{R}^3$ be a convex polytope. Suppose we are allowed to cut P with a plane. Two parts are then separated, and each is then allowed to be separately cut with a new plane, etc.[43]
a) [1-] Prove that a regular octahedron can be cut into tetrahedral pieces with only three cuts. Similarly, prove that for a cube four cuts suffice.
b) [1-] Prove by an explicit construction that both regular icosahedron and regular dodecahedron need at most 100 cuts.
c) [1] Prove that every convex polytope in $\mathbb{R}^3$ can be cut into tetrahedral pieces with finitely many cuts.
d) [1+] Generalize this to $\mathbb{R}^d$.

Exercise 17.5. *a*) $\Diamond$ [1-] Let $Q = \partial A$ be a simple polygon in $\mathbb{R}^2$. Prove that every triangulation of A contains a triangle T with two sides in Q, i.e., such that $A \smallsetminus T$ is homeomorphic to a disk.
b) [1+] In the plane, prove that for every decomposition $Q = \cup_i Q_i$ of a simply connected polygon Q into simply connected polygons Q_i, there exist Q_i such that $Q \smallsetminus Q_i$ is also simply connected.
c) [1+] Show that part *b*) does not generalize to $\mathbb{R}^3$.

[43]Think of chopping polytopes as if they were vegetables.

d) [2] Find a simplicial subdivision $\mathcal{D}$ of a tetrahedron Δ, such that for every tetrahedron T in $\mathcal{D}$, the closure of $\Delta \smallsetminus T$ is not homeomorphic to Δ.

Exercise 17.6. (*Stellar flips*) $\diamondsuit$ Consider the following flips on triangulations (simplicial subdivisions) of a polytope in $\mathbb{R}^3$:

- the 2–3 move defined above,
- the 1–4 move defined as a subdivision of a tetrahedron into four smaller tetrahedra,
- the 4–8 move defined as a flip from one triangulations of an octahedron to another, as shown in Figure 17.8,
- the 1–3 move, a degenerate case of the 1–4 move when the new point is on the boundary,
- the 3–5 move, a degenerate case of the 4–8 move when the new point is on the diagonal separating two coplanar triangles.

a) [2] Prove that every two triangulations of a convex polytope $P \subset \mathbb{R}^3$ are connected by these flips.
b) [2] Generalize this to non-convex polytopes in $\mathbb{R}^3$.
c) [2+] Generalize this to higher dimensions.
d) [1-] Check that all stellar flips can be obtained by elementary moves.

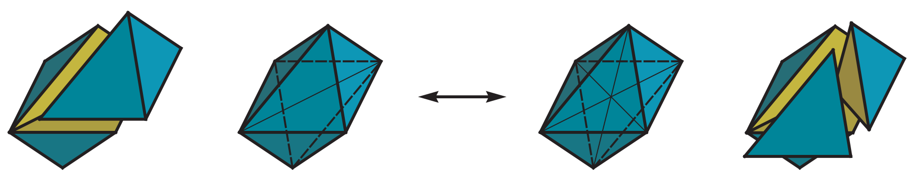

FIGURE 17.8. A 4–8 move on triangulations of an octahedron.

17.9. **Final remarks.** The study of valuations has a long history and a number of applications which go outside of the scissor congruence. We changed some of the definitions to streamline the connections and at the expense of generality. Traditionally, valuations are defined as maps $\varphi : \{P\} \to G$, where $P \subset \mathbb{R}^d$ are convex polytopes, G is an abelian group (say, $\mathbb{R}$ by addition), and such that

$$\varphi(P) + \varphi(Q) \, = \, \varphi(P \cup Q) + \varphi(P \cap Q), \quad \text{for all } \ P, Q \subset \mathbb{R}^d.$$

Under this definition one needs to be careful with the boundary of polytopes and define the valuations on degenerate polytopes, which we largely ignore for simplicity. The advantage is the wealth of other examples of valuations, such as the surface area, the Euler characteristic, and the mean curvature (see Section 28). We refer to [Grub, §7] for these and further results, and to [McM2] for a detailed survey and references.

The study of local moves on triangulations is also classical and in the context of combinatorial topology goes back to J. W. Alexander (1930) and M. H. A. Newman (1926, 1931). Here one aims to obtain a topological invariant of a manifold using simplicial subdivisions. Thus, one wants to make sure they are indeed invariant under certain local moves. The result then follows from the connectivity of all triangulations under such local moves. We refer to [Lic] for the survey and references.

The geometric study of dissections and triangulations is more recent, more delicate and has a number of negative results and open problems (see Remark 17.18). Theorem 17.19 in this form is given in [LudR]. Although our proof uses several different ideas, such as height functions, the proofs share a similar blueprint. The triangulation counterpart of the elementary move connectivity theorem is also known, and uses the so-called *stellar flips* or *Pachner moves*. These flips play an important role in both geometric combinatorics and algebraic geometry. While many natural questions turn out to have negative answers, an important positive result (see Exercise 17.6) was established by Morelli (1996) and Włodarczyk (1997). We refer to [San2] for a survey of these result and to [IzmS] for an elegant presentation of the Morelli–Włodarczyk theorem.

18. Monge problem for polytopes

This short section presents a little known variation on the scissor congruence problem. Although we do not apply the main theorem elsewhere, it has a pleasant flavor of universality, while the proof uses a version of the local move connectivity, fitting well with several previous sections.

18.1. **Stretching the polytopes.** Let $P, Q \subset \mathbb{R}^d$ be two polytopes of equal volume: $\operatorname{vol}(P) = \operatorname{vol}(Q)$. We say that map $\varphi : P \to Q$ is a *Monge map* if it is continuous, piecewise linear (PL), and volume-preserving. Two polytopes P and Q are called *Monge equivalent*, written $P \bowtie Q$, if there exists a Monge map $\varphi : P \to Q$. Observe that "$\bowtie$" is an equivalence relation since the composition of Monge maps, and the inverse of a Monge map are again Monge map.

Here is another way to think of the Monge equivalence: there must exist two triangulations (simplicial subdivisions) of polytopes $P = \cup_{i=1}^n P_i$ and $Q = \cup_{i=1}^n Q_i$ and a continuous map $\varphi : P \to Q$, such that $\varphi(P_i) = Q_i$, the map φ is linear on each P_i, and $\operatorname{vol}(P_i) = \operatorname{vol}(Q_i)$, for all $1 \le i \le n$. In this setting Monge equivalence is somewhat similar to scissor congruence: we no longer require the map $\varphi : P_i \to Q_i$ to be orthogonal, just volume-preserving.[44] If that was all, establishing the equivalence would be easy (see below), but the added continuity requirement makes matters more complicated. Here is the main result of this section.

Theorem 18.1. *Every two convex polytopes in $\mathbb{R}^d$ of equal volume are Monge equivalent.*

As the reader shall see, the result is nontrivial already in the plane. The reader might want to look for an independent proof in this case before proceeding to the next section. Let us mention here that the theorem has a number of far-reaching generalizations, for example to polygons in the plane with the same area and the same number of holes (see Exercise 18.2).

18.2. **Pre-proof analysis.** Before we present the proof of the theorem, let us first consider several special cases and weak versions of the theorem.

Example 18.2. (*Monge equivalence in the plane*) Let $P, Q \subset \mathbb{R}^2$ be two convex polygons in the plane of equal area: $\operatorname{area}(P) = \operatorname{area}(Q)$. Take an n-gon P and any vertex v in P. Consider a triangle formed by v and its neighbors u and w. Let z be the second neighboring vertex of w (other than v). Now transform the triangle (uvw) into $(uv'w)$ by shifting v along the line parallel to (uw), such that v' now lies on a line containing (wz). Keep the rest of the polygon unchanged. We obtain a convex $(n-1)$-gon P' (see Figure 18.1).

Proceed in this manner until P is mapped into a triangle: $\varphi_1 : P \to \Delta_1$. Proceed similarly with the polygon Q to obtain $\varphi_2 : Q \to \Delta_2$. Since the resulting triangles Δ_1, Δ_2 have the same area, there exist a volume-preserving linear map $\gamma : \Delta_1 \to \Delta_2$. We conclude: $P \bowtie \Delta_1 \bowtie \Delta_2 \bowtie Q$.

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[44] In other words, we replace the smaller group $\mathrm{SO}(d, \mathbb{R})$ of rigid motions of simplices, with a bigger group $\mathrm{SL}(d, \mathbb{R})$.

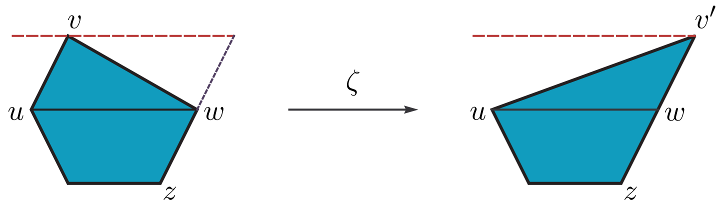

FIGURE 18.1. Monge map $\zeta : P \to P'$.

Example 18.3. (*Monge equivalence without the continuity condition*) Let us weaken the Monge restriction on a map in the theorem by removing the continuity condition. In other words, we prove that between every two polytopes $P, Q \subset \mathbb{R}^d$ of equal volume there exists a volume-preserving PL-map $\varphi : P \to Q$.

Without loss of generality we can assume that $\mathrm{vol}(P) = \mathrm{vol}(Q) = 1$. Consider any two simplicial subdivisions $P = \cup_{i=1}^m P_i$, $Q = \cup_{j=1}^n Q_j$, and let $\alpha_i = \mathrm{vol}(P_i)$, $\beta_j = \mathrm{vol}(Q_j)$. Subdivide further each of these simplices into smaller simplices: $P_i = \cup_{j=1}^n P_{ij}$, $Q_j = \cup_{i=1}^m Q_{ij}$, such that $\mathrm{vol}(P_{ij}) = \mathrm{vol}(Q_{ij}) = \alpha_i \beta_j$. Since every simplex P_{ij} can be mapped into simplex Q_{ij} by a volume-preserving map, this implies the claim.

Example 18.4. (*Monge equivalence without the volume-preserving condition*) Let us remove the volume-preserving restriction in the Monge map, and prove a weaker version of the theorem. In other words, let us show that between every two polytopes $P, Q \subset \mathbb{R}^d$ of equal volume there exists a continuous PL-map $\varphi : P \to Q$.

We can assume that polytopes P, Q are simplicial; otherwise subdivide each facet into simplices. Move the polytopes so that the origin $O \in \mathbb{R}^d$ lies in the interior of both polytopes: $O \in P, Q$.

Now, consider a simplicial fan $F = \cup_{i=1}^m F_i \in \mathbb{R}^d$ defined as a union of infinite cones F_i which have a vertex at O and are spanned over the facets in P. Similarly, consider a fan $G = \cup_{j=1}^n G_i \in \mathbb{R}^d$ over facet simplices of Q. Let C be the 'union fan' which consists of cones $F_i \cap G_j$, and denote by $\widetilde{C} = \cup_r \widetilde{C}_r$ a subdivision of C into simple cones (see Exercise 2.13). Finally, define simplicial subdivisions $P = \cup_r P_r$, $Q = \cup_r Q_r$ by intersecting the fan $\widetilde{C}$ with polytopes P and Q: $P_r = P \cap \widetilde{C}_r$, $Q_r = Q \cap \widetilde{C}_r$ (see Figure 18.2).

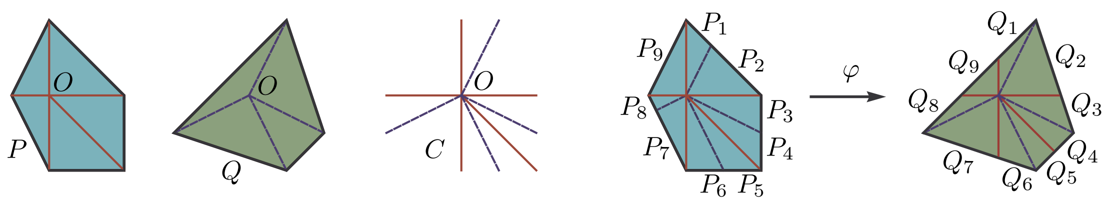

FIGURE 18.2. Polygons P, Q with fans F, G, the union fan $C = \widetilde{C}$, and the continuous PL-map $\varphi : P \to Q$.

Fix r and consider the simplices P_r and Q_r. Denote by $v_1, \ldots, v_d$ and $w_1, \ldots, w_d$ their vertices other than O. From above, $w_i = \alpha_i v_i$ for some $\alpha_1, \ldots, \alpha_d > 0$. Define a PL-map $\varphi : P \to Q$ to be linear on each P_r and map each P_r into Q_r:

$$\varphi : \sum x_i v_i \mapsto \sum x_i w_i \,, \quad \text{where} \;\; x_i \geq 0.$$

The map φ is clearly continuous and piecewise linear, which proves the claim.

Example 18.5. (*Monge equivalence for bipyramids*) While Theorem 18.1 is trivial for simplices, already for bipyramids it is less obvious, so it makes sense to start by proving the result in this case.

Formally, let $P, Q \subset \mathbb{R}^d$ be two d-dimensional bipyramids of equal volume: $\mathrm{vol}(P) = \mathrm{vol}(Q)$. Let $u_1, u_2, x_1, \ldots, x_d$ and $v_1, v_2, y_1, \ldots, y_d$ be the vertices of P and Q respectively, where u_1, u_2 and v_1, v_2 are the simple vertices. Let us prove that there exists a continuous volume-preserving PL-map $\varphi : P \to Q$ which is linear on each facet and which sends u_i to v_i and x_j to y_j, for all $i = 1, 2$ and $1 \le j \le d$.

Let us start with a volume-preserving affine transformation of $\mathbb{R}^d$ which maps the vertices $x_1, \ldots, x_d$ into the vertices $x'_1, \ldots, x'_d$ of a regular $(d-1)$-dimensional simplex S. Denote by z the barycenter of S, and by ℓ the line going through z and orthogonal to S. Let u'_1 and u'_2 be the orthogonal projections of u_1 and u_2 onto ℓ.

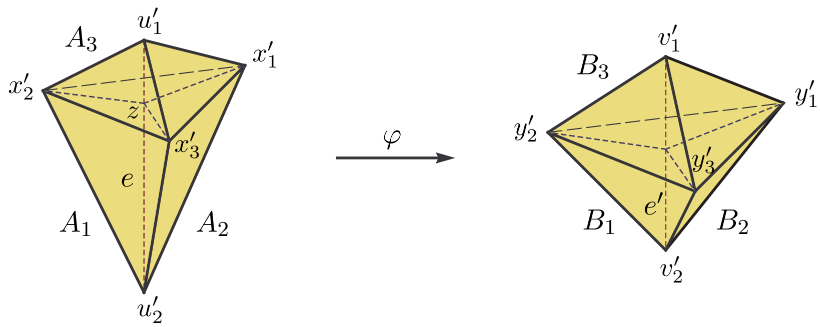

FIGURE 18.3. The map $\varphi : P' \to Q'$, where $P = A_1 \cup A_2 \cup A_3$ and $Q' = B_1 \cup B_2 \cup B_3$.

Apply the volume-preserving linear map $(u_1, x'_1, \ldots, x'_d) \to (u'_1, x'_1, \ldots, x'_d)$ which fixes S and sends u_1 to u'_1. Similarly, map $(u_2, x'_1, \ldots, x'_d)$ to $(u'_2, x'_1, \ldots, x'_d)$. The resulting bipyramid is now symmetric with respect to the diagonal $e = (u'_1, u'_2)$. Therefore, the simplices $A_1 = (u_1, u_2, x_2, \ldots, x_d)$, $A_2 = (u_1, u_2, x_1, x_3 \ldots, x_d)$, $\ldots$, $A_d = (u_1, u_2, x_1, \ldots, x_{d-1})$ have equal volumes. They each contain the edge e, and form a simplicial subdivision $P' = \cup_{i=1}^d A_i$. Denote by $\psi : P \to P'$ the resulting PL-map.

Now, apply the analogues piecewise linear transformations to Q, to obtain a simplicial subdivision $Q' = \cup_{i=1}^d B_i$, where all simplices B_i contain an edge $e' = (v'_1, v'_2)$. Denote by $\chi : Q \to Q'$ the corresponding PL-map.

There is a natural linear transformation $\varphi_i : A_i \to B_i$ which maps $e \to e'$ by sending u'_1 to v'_1 and u'_2 to v'_2, and maps the boundary of P' into the boundary of Q'.[45] Since $\mathrm{vol}(A_i) = \mathrm{vol}(B_i)$, these maps are volume-preserving and combine into a continuous volume-preserving PL-map $\varphi : P' \to Q'$ (see Figure 18.3). All together, $\chi^{-1} \circ \varphi \circ \psi : P \to Q$ is the desired PL-map.

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[45]Strictly speaking, φ_i are *affine* transformations of $\mathbb{R}^d$ which are *linear* on A_i.

18.3. **Monge maps by volume sharing.** We are now ready to prove Theorem 18.1. Start with a continuous PL-map $\varphi : P \to Q$ constructed in Example 18.5. This map creates simplicial subdivisions $P = \cup_{i=1}^n P_i$, $Q = \cup_{i=1}^n Q_i$, which will be fixed from here on. For every $i \in [n]$ compute $a_i = \mathrm{vol}(P_i)/\mathrm{vol}(Q_i)$. If all $a_i = 1$, the map φ is also volume-preserving, i.e., the Monge map is constructed. If not (which is more likely), let us "correct" map φ with a series of "local" PL-maps $\gamma : Q \to Q$.

Think of the numbers a_i as *contraction ratios* of the linear maps $\varphi_i : P_i \to Q_i$. Formally, a_i is the determinant of the inverse map at points in Q_i. The maps we construct below will be continuous PL-maps $\psi : P \to Q$ which will have contraction ratios b_i at all points $z \in Q_i$. In other words, the maps will neither be linear on P_i, nor take P_i to Q_i. The maps we construct will be piecewise linear on all on P_i, and the determinant of the inverse map ψ^{-1} will be a constant on Q_i, for all i.

We begin with the obvious equation for the volume of simplices involved:

$$(\pounds) \quad \mathrm{vol}(P) \,=\, \sum_{i=1}^{n} a_i \cdot \mathrm{vol}(Q_i) \,=\, \sum_{i=1}^{n} \mathrm{vol}(Q_i).$$

Consider two adjacent simplices Q_i and Q_j and real numbers a_i', a_j' such that

$$(\yen) \quad a_i \cdot \mathrm{vol}(Q_i) + a_j \cdot \mathrm{vol}(Q_j) \,=\, a_i' \cdot \mathrm{vol}(Q_i) + a_j' \cdot \mathrm{vol}(Q_j).$$

Let us show how to construct a continuous PL-map $\varphi' : P \to Q$ which is going to have contraction ratios a_i' and a_j' on Q_i and Q_j, and unchanged contraction ratios elsewhere.

Suppose simplices Q_i, Q_j have a common face F. Expand the height of both simplices by a factor of a_i and a_j, respectively. The resulting simplices $\widetilde{Q}_i$ and $\widetilde{Q}_j$ form a bipyramid A of volume $\bigl(a_i \cdot \mathrm{vol}(Q_i) + a_j \cdot \mathrm{vol}(Q_j)\bigr)$. Similarly, expand the height of both simplices by a factor of a_i' and a_j', respectively, to obtain simplices $\widetilde{Q}_i'$ and $\widetilde{Q}_j'$ which form a bipyramid B of the same volume. Let $Q_{ij} = Q_i \cup Q_j$, and denote by $\rho_1 : Q_{ij} \to A$, $\rho_2 : Q_{ij} \to B$ the two maps defined above. Let $\zeta : A \to B$ be a volume-preserving PL-map between these two bipyramids constructed in Example 18.5.

Now, let $\gamma : Q_{ij} \to Q_{ij}$ be defined as the composition $\gamma = \rho_2^{-1} \circ \zeta \circ \rho_1$ and extend γ to a PL-map $Q \to Q$ by the identity map outside of the bipyramid Q_{ij}. This is possible since the map ζ in the example was linear on the corresponding faces of the bipyramids (see Figure 18.4). Finally, let $\varphi' = \varphi \circ \gamma$. By construction, $\varphi^{-1} = (\varphi')^{-1}$ outside of Q_{ij}, and the contraction ratios on Q_i and Q_j are now a_i' and a_j' because ζ is volume-preserving and ρ_1 and ρ_2 expand by exactly these factors.

Now that we know how to change the contraction ratios locally, let us make the global change. Loosely speaking, we claim that we can make local changes as in ($\yen$) to obtain a change as in the second equality in ($\pounds$). Formally, consider a set $\mathcal{A}$ of all sequences $(a_1, \ldots, a_n)$ such that $\sum_i a_i \mathrm{vol}(Q_i) = \mathrm{vol}(Q)$ and $a_i > 0$. Let E be the set of pairs (i, j) such that simplex Q_i is adjacent to Q_j (the intersection is a $(d-1)$-dimensional face). We are allowed to change

$$(\ldots, a_i, \ldots, a_j, \ldots) \to (\ldots, a_i', \ldots, a_j', \ldots) \quad \text{whenever} \quad (\yen) \quad \text{is satisfied.}$$

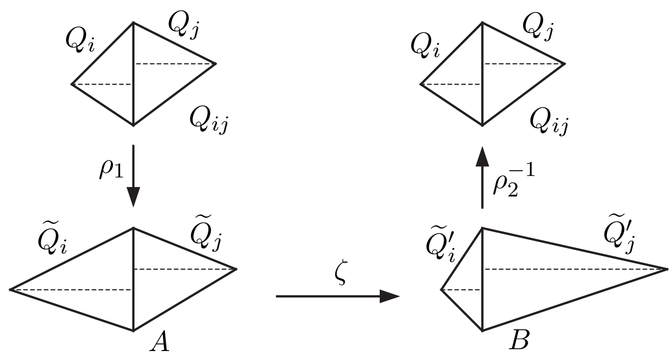

FIGURE 18.4. Map $\gamma : Q_{ij} \to Q_{ij}$, where $\gamma = \rho_2^{-1} \circ \zeta \circ \rho_1$.

We need to show that $(a_1, \ldots, a_n)$ is connected to $(1, \ldots, 1)$ through a series of changes.

Denote by G the graph on $[n]$ with E as its edges. Since G is a dual graph to a simplicial subdivision of Q, we conclude that G is connected. The desired series of changes is possible by Exercise 18.1. This completes the proof of Theorem 18.1. $\square$

18.4. **Exercises.**

Exercise 18.1. (*Money exchange*) $\Diamond$ [1] Finish the proof of Theorem 18.1 by showing that global change can be made via a finite number of local changes, as in the proof above.

Exercise 18.2. (*Polygons with holes*) $\Diamond$ While Theorem 18.1 does not cover polygons with holes, it is in fact true that the theorem extends to this case. Formally, let A and B be two convex polygons and let $S_1, \ldots, S_k \subset A$ and $T_1, \ldots, T_k \subset B$ be non-intersecting convex polygons. Suppose

$$\operatorname{area}(A) - \sum_{i=1}^{k} \operatorname{area}(S_i) \;=\; \operatorname{area}(B) - \sum_{i=1}^{k} \operatorname{area}(T_i).$$

Consider two non-convex polygons $P = A - \cup_i S_i$ and $Q = B - \cup_i T_i$ with the same area and the same number of holes.

a) [1] Prove by a direct argument that there exists a continuous piecewise linear map $\varphi : P \to Q$.

b) [1] Prove that there exists a Monge map $\varphi : P \to Q$.

c) [1] Suppose $A = B$. Prove that there exists a Monge map $\varphi : P \to Q$ which is identity on the boundary ∂A.

d) [1+] Let $A \simeq B$ be 2×2 squares, $k = 1$, and let $S \simeq T$ be 1×1 squares, such that S is in the center of A, and T is at distance $z > 0$ from the boundary. Denote by $f(z)$ the minimal number of triangles needed to define a Monge map $\varphi : (A \smallsetminus S) \to (B \smallsetminus T)$. Find the lower and the upper bounds on $f(z)$ as $z \to 0$. An example proving that $f(\frac{1}{4}) \le 14$ is given in Figure 18.5.

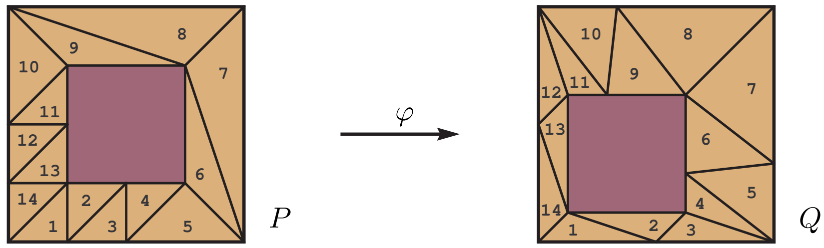

FIGURE 18.5. Monge map $\varphi : P \to Q$ which is linear on 14 triangles.

Exercise 18.3. (*Poset polytope*) Let $\mathcal{P}$ be a poset with the partial order $\prec$ on a finite set X, where $|X| = n$.[46] Define *poset polytope* $\mathrm{P}_{\mathcal{P}} \subset \mathbb{R}^n$ as a set of functions $f : X \to \mathbb{R}$, such that $0 \le f(x) \le 1$ for all $x \in X$, and $f(y) \le f(z)$ for all $y \prec z$, $y, z \in X$.
a) [1-] Prove that the vertices of $\mathrm{P}_{\mathcal{P}}$ correspond to order ideals of $\mathcal{P}$.
b) [1-] Prove that $\mathrm{vol}(\mathrm{P}_{\mathcal{P}}) = e(\mathcal{P})/n!$, where $e(\mathcal{P})$ is the number of *linear extensions* of $\mathcal{P}$.
c) [1] Define the *Fibonacci polytope* $\mathrm{FP}_n \subset \mathbb{R}^n$ to be the convex hull of *Fibonacci sequences* $(z_1, \ldots, z_n)$, where $z_i \in \{0, 1\}$ and no two ones are adjacent. Show that FP_n is congruent to a certain poset polytope. Conclude that $\mathrm{vol}(\mathrm{FP}_n) = a_n/n!$, where a_n is the number of *alternating permutations* $\sigma(1) < \sigma(2) > \sigma(3) < \ldots$

Exercise 18.4. (*Chain polytope*) ♢ As in the previous exercise, let $\mathcal{P}$ be a poset on a finite set X. Define *chain polytope* $\mathrm{Q}_{\mathcal{P}}$ to be the set of functions $g : X \to \mathbb{R}$ such that $g(x) \ge 0$ for every $x \in X$, and $g(x_1) + \ldots + g(x_i) \le 1$ for every antichain $\{x_1, \ldots, x_n\} \subset X$.
a) [1-] Prove that the vertices of $\mathrm{Q}_{\mathcal{P}}$ correspond to antichains of $\mathcal{P}$.
b) [1] Suppose $\mathcal{P}$ has no 3-element chains. In notation of the previous exercise, prove that $\mathrm{P}_{\mathcal{P}}$ is congruent to $\mathrm{Q}_{\mathcal{P}}$.
c) [1] Let $\phi : \mathrm{P}_{\mathcal{P}} \to \mathrm{Q}_{\mathcal{P}}$ be a map defined by

$$\big[\phi(f)\big](x) \, = \, \min_{y \prec x, \, y \in X} \, f(x) - f(y).$$

Check that ϕ is a Monge map. Conclude that $\mathrm{vol}(\mathrm{Q}_{\mathcal{P}}) = e(\mathcal{P})/n!$.
d) [1] Use the map ϕ to show that $i(c\mathrm{P}_{\mathcal{P}}) = i(c\mathrm{Q}_{\mathcal{P}})$, for all $c \in \mathbb{N}$, where $i(Z)$ denotes the number of integral points in Z.
e) [1-] Check that $\mathrm{P}_{\mathcal{P}}$ and $\mathrm{Q}_{\mathcal{P}}$ have equal number of vertices. Give a direct bijective proof of this fact. Explain what is special about ϕ, since not every Monge map preserves the number of vertices.

Exercise 18.5. Define the *plane partitions polytope* R_n by the following equations and inequalities:

$$\left\{ \begin{array}{l} \displaystyle\sum_{k=1}^{n-i} b_{i+k,k} = n - i \, , \quad \sum_{k=1}^{n-j} b_{k,j+k} = n - j \, , \\ b_{i,j} \ge 0, \quad b_{i,j} \ge b_{i,j+1} \, , \quad b_{i,j} \ge b_{i+1,j} \, , \end{array} \right. \quad \text{for all} \quad 1 \le i, j \le n.$$

a) [2-] Prove that R_n has exactly $n!$ integral points.

[46] For poset terminology and various related results, see [Sta3, §3].

b) [2-] Recall the definition of the Birkhoff polytope $B_n \subset \mathbb{R}^{n^2}$ (see Exercise 8.19). Prove that the number of symmetric integer matrices $(a_{ij}) \in B_n$ is equal to the number of symmetric integer matrices $(b_{ij}) \in R_n$.
c) [2] Give an explicit Monge map $\varphi : B_n \to R_n$. Conclude that $\mathrm{vol}(B_n) = \mathrm{vol}(R_n)$ and $i(cB_n) = e(cR_n)$, for all $c \in \mathbb{N}$, where $i(Z)$ is as in the previous exercise.

18.5. **Final remarks.** This section is based on papers [HenP, Kup2]. The main theorem and the outline of the proof is due to Kuperberg, while various details in the presentation follow [HenP]. Both papers are motivated by a theorem of Jürgen Moser, which states that if two manifolds are diffeomorphic and have equal volume, then there is a volume-preserving diffeomorphism between them. In a different direction, the explicit Monge maps often appear in algebraic combinatorics to encode the bijections between integral points in certain "combinatorial polytopes" (see Exercises 18.5 and 18.4). We refer to [HenP] for an extensive discussion, for connections to Moser's theorem and piecewise linear combinatorics, and for the references.

We name the maps in this section after Gaspard Monge, a 19-th century French geometer who in fact never studied piecewise linear maps in this context. We do this in part due to their importance and in part for lack of a better name. Also, in the case of general convex bodies, continuous volume-preserving maps do appear in connection with the Monge–Kantorovich mass transportation (or optimal transportation) problems (see [Barv, § 4.14]) and the Monge–Ampère equations (see [Caf]). The uniqueness of solutions in that case seems to have no analogue in our (polyhedral) situation, which is not surprising in the absence of a natural minimization functional and the non-compactness of the space of PL-maps.

We should mention that Exercise 18.2 is a special case of a general result which states that all PL-homeomorphic PL-manifolds of the same volume have a volume-preserving PL-homeomorphism [HenP, Kup2]. The proof is nearly unchanged from the proof of Theorem 18.1.

A small warning to the reader: not all homeomorphic PL-manifolds are PL-homeomorphic. While this holds for 2- and 3-dimensional manifolds, this fails for 5-dimensional manifolds (Milnor, 1961), and the problem is known under the name "Hauptvermutung". Thus, one should be careful generalizing the result in the exercise. We refer to [RouS] for a good introduction to the subject.

19. Regular polytopes

It is often believed that the regular polytopes are classical, elementary, and trivial to understand. In fact, nothing can be further from the truth. In fact, various questions about them lead us to the Cauchy theorem (Section 26) and the Alexandrov existence theorem (Section 37). Although this section does not have a single theorem, it is useful as a motivation to these crucial results in the second part.

19.1. **Definition is an issue.** While everyone seems to know the *classification* of *regular polytopes* (tetrahedron, cube, octahedron, dodecahedron and icosahedron), few people tend to know *what are they*, i.e., how the set of regular polytopes are defined. This leads to several confusions and misperceptions which we will try to dispel. In fact, even the most basic incorrect definitions of regular polytopes are sufficiently enlightening to justify their study.

Incorrect Definition 1. *Regular polytopes* = convex polytopes where all sides are regular polygons.

Why not: this definition is not restrictive enough. For example, a triangular prism and square antiprism fit into this category.

Incorrect Definition 2. *Regular polytopes* = convex polytopes where all sides are regular polygons with the same number s of sides.

Why not: again not restrictive enough; For example, a triangular bipyramid fits into this category (see Figure 19.1).

Correct yet unsatisfactory Definition 3. *Regular polytopes* = convex polytopes where all sides are regular polygons with the same number s of sides, and where every vertex has the same degree (i.e., adjacent to the same number r of edges).

Why unsatisfactory: while the set of such polytopes is exactly what we want, there are two problems with this definition. First, this definition does not generalize well to higher dimensions. Second, it does not explain the large group of symmetries of regular polytopes.

Figure 19.1. Triangular prism and bipyramid.

We are now ready to give a correct and fully satisfactory definition.

Definition 19.1. *A polytope $P \subset \mathbb{R}^d$ is called* regular *if its group of symmetries acts transitively on complete flags of P.*

Here by *complete flags* we mean a sequence: vertex $\subset$ edge $\subset$ 2-dimensional face $\subset \ldots \subset P$. Now consider the numerical consequences of the definition when $d = 3$.

Denote by Γ the group of symmetries of P. Let $\mathcal{A}_F$ be the set of edges contained in face F. Since Γ acts transitively on complete flags, polytope P is *face-transitive*. Thus, $|\mathcal{A}_F| = |\mathcal{A}_{F'}|$, for all faces F and F'. In other words, all faces are polygons with the same number s of sides.

Similarly, denote by E_v the set of edges containing vertex v of the polytope P. Again, by transitivity of the action of Γ we conclude that P is *vertex-transitive*. Thus, $|E_v| = |E_{v'}|$, for all vertices v and v'. In other words, all vertices are adjacent to the same number r of edges. This gives the conditions in "definition 3". Note that P must also be *edge-transitive*, restricting the possibilities even further.

19.2. **Classification of regular polytopes in $\mathbb{R}^3$.** Let $P \subset \mathbb{R}^3$ be a regular convex polytope, where the sides are regular s-gons, and all vertices are adjacent to r edges. Denote by $n = |V|$ the number of vertices, $m = |E|$ the number of edges, and $f = |\mathcal{F}|$ the number of faces. Counting all edges via vertices and faces we obtain:

$$2 \cdot m \, = \, n \cdot r \, = \, f \cdot s \, .$$

Writing n and f through m, we rewrite Euler's formula $n - m + f = 2$ as follows:

$$\frac{2m}{r} - m + \frac{2m}{s} \, = \, 2 \, ,$$

and finally

$$\frac{1}{r} + \frac{1}{s} = \frac{1}{m} + \frac{1}{2} \, .$$

From here either $1/r$ or $1/s$ is $> 1/4$, and since $r, s, m \geq 3$, we can check all cases to conclude that

$$(r, s) \, \in \, \{(3,3), \, (3,4), \, (4,3), \, (3,5), \, (5,3)\}.$$

These five cases correspond to the five platonic solids in the same order as listed in the beginning of the previous subsection. It would seem that we are finished, except for one annoying little problem:

How do we know these regular polytopes actually exist?

We discuss this "little problem" at length and eventually resolve it completely in the next subsection.

19.3. **Constructing polytopes is harder than you think.** Well, we all *know* that these polytopes exist. Having seen them all over the place it is easy to take things for granted. Of course, there must be a theoretical reason. After all, that was a crown achievement of Euclid. But are you really sure? Let us go over the list to see what is really going on.

Naturally, there is little doubt the regular tetrahedron and the cube exist. The centers of faces of a cube give vertices of an octahedron. Same for the dodecahedron: it is formed by taking centers of faces of the icosahedron. Now, when we get to icosahedron we are in trouble. A priori, there is no theoretical reason why it should exist. Can we explicitly construct it?

(i) *Coordinate approach.* The icosahedron can be constructed by coordinates of all 12 vertices: $(0, \pm 1, \pm\phi)$, $(\pm 1, \pm\phi, 0)$, and $(\pm\phi, 0, \pm 1)$, where $\phi = \frac{1+\sqrt{5}}{2}$ is the *golden ratio* (see [Hada, §564]). It takes some minor checking to see that the convex hull of these points forms an icosahedron. First, observe that the vertices lie on a sphere centered at the origin, and thus in convex position. Cyclically permuting coordinates and changing signs defines an action of the group $\mathbb{Z}_3 \ltimes \mathbb{Z}_2^3$ (of order 24) on all $120 = 12 \cdot 5 \cdot 2$ complete flags. With one additional rotation around a diagonal (see Exercise 19.1), one can obtain the full group of symmetries $\Gamma \simeq A_5 \times \mathbb{Z}_2$ of order 120, and prove existence of the icosahedron.

It is important to observe that the above argument was indirect: we did not really construct the icosahedron – we just checked someone else's construction. Also, Euclid clearly could not have conceived this construction. So here is another approach.

(ii) *Continuity argument.* Start with a regular pentagon Q with side 1 and a pentagon Q' obtained from Q by rotation (see Figure 19.2). Start lifting Q up until at some point the side edges connecting two levels become equal to 1. This will happen by continuity. Finally, add a regular pentagonal cap on top and on the bottom to obtain an icosahedron (see Figure 19.2). Although, by construction, all faces are equilateral triangles, this does not immediately imply that the resulting polytope is regular.

Now, consider the group of symmetries of the resulting polytope P. Clearly, we have all symmetries which preserve the main diagonal (North to South Pole). This gives a group of symmetries with 20 elements. However, the action of the full symmetry group on vertices is not obviously transitive: to prove that P has all 120 symmetries we need to exhibit an orthogonal transformation which maps the main diagonal into another diagonals.

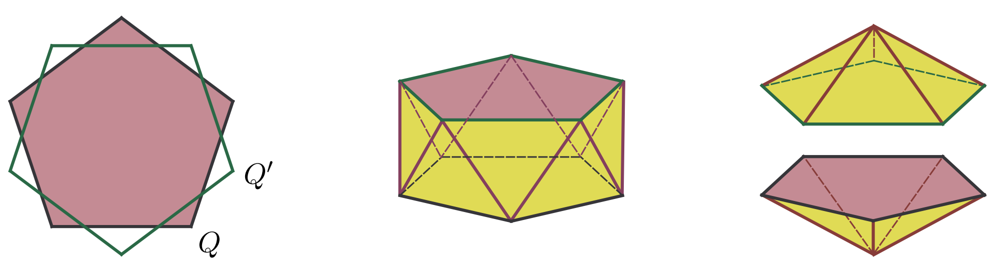

FIGURE 19.2. Construction of the regular icosahedron, step by step.

(iii) *Inscribing into a cube.* Here is another approach, arguably more elegant. Draw intervals of length x symmetrically on each face of the square as in Figure 19.3. Denote by P_x a convex hull of these intervals. Beginning with a small length x, increase x until all edges of P_x have the same length. Again, the resulting polytope P has all faces equilateral triangles, but that in itself does not a priori prove that P is regular.

By construction, the group of symmetries of P is transitive on all vertices. In fact, already the subgroup of joint symmetries of P and the cube is transitive on the vertices of P and has order 24. However, the action on faces is not obviously transitive: triangles containing original intervals and those that do not may a priori lie in different orbits.

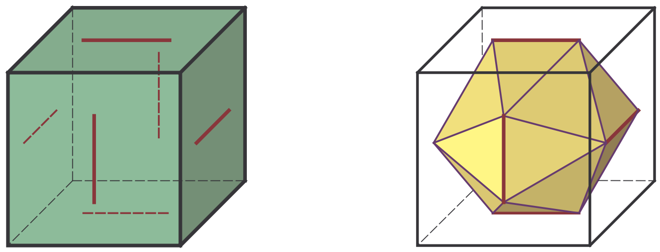

FIGURE 19.3. Another construction of the regular icosahedron.

Unfortunately, there seem to be no "soft" argument which would give a construction of the icosahedron and prove that it has the full group of symmetries. Either an additional calculation or an advance result is needed.

Let us the classical Cauchy theorem to prove that polytope P has the full group of symmetries Γ. Fix two flags $v \subset e \subset F \subset P$ and $v' \subset e' \subset F' \subset P$ in polytope P. Let P' be a polytope obtain from P by relabeling of faces with respect to these flags. Formally, let P' be a polytope combinatorially equivalent to P, where the map in the definition of combinatorial equivalence is given by $v \to v'$, $e \to e'$, and $F \to F'$. By the Cauchy theorem, polytope P' can be moved into P by a rigid motion, thus giving the desired symmetry between the flags. Therefore, the symmetry group Γ acts transitively on complete flags of P, and the polytope P is a regular icosahedron.

There are two more constructions of the icosahedron that we would like to mention. First, one can start with an unfolding of 20 equilateral triangles and glue the polytope (see Figure 19.4). The Alexandrov existence theorem (Theorem 37.1) implies that this is possible. Still, to prove uniqueness in this case one needs to use the Alexandrov uniqueness theorem (Theorem 27.7), an extension of the Cauchy theorem, and the edges a priori are not guaranteed to be at the right positions.

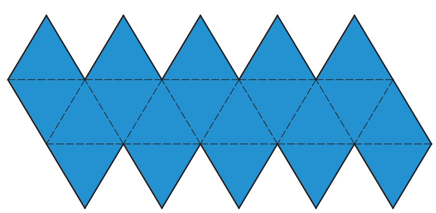

FIGURE 19.4. An unfolding of the regular icosahedron.

(iv) *The original construction.* Finally, there is a construction that was essentially used by Euclid. Let us construct a dodecahedron rather then an icosahedron. It is a classical fact (but still remarkable for those seeing it the first time), that a cube can be inscribed into a dodecahedron such that the edges become diagonals of the pentagonal faces. In Figure 19.5 we show the view from the top: the diagonals are equal because each face is a regular pentagon and the angles are straight by the symmetry.

Now, one can start with a cube and build a dodecahedron by attaching 'caps' to all six sides (see Figure 19.5 for a cube and two caps). Note that the Cauchy theorem is obvious in this case as the polytope is simple, and thus three pentagons completely determine the dihedral angles in all vertices. By the argument as above, this implies that our polytope is regular. We leave it as an exercise to see why this construction produces a regular dodecahedron seemingly without any extra calculations (see Exercise 19.2).

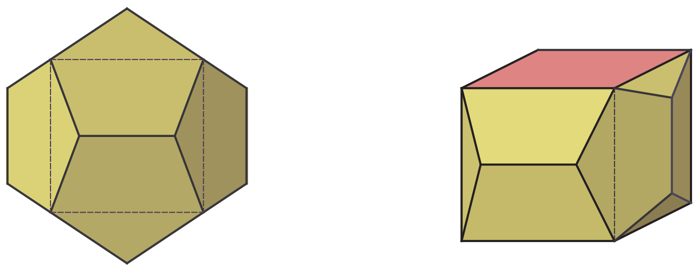

Figure 19.5. Construction a regular dodecahedron, after Euclid.

19.4. **Exercises.**

Exercise 19.1. ♢ [1] Find an explicit 3×3 matrix for a rotational symmetry of order 5 to show that the icosahedron constructed via coordinates is indeed a regular polytope.

Exercise 19.2. ♢ [1-] Compute the dihedral angles in Euclid's construction of the dodecahedron to show that the pentagons are indeed flat.

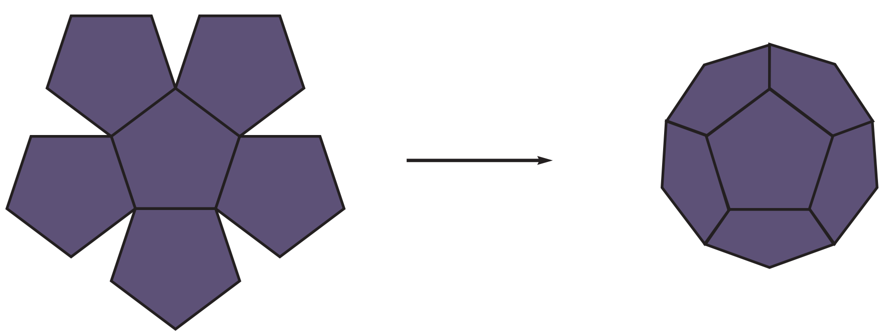

Figure 19.6. Folding six pentagons to form a half of the dodecahedron.

Exercise 19.3. ♢ *a*) [1-] Start with five regular pentagons attached to one. Fold them all along the edges continuously until they form a half of the surface of the regular dodecahedron (see Figure 19.6). Note that there is a unique way to attach five regular pentagons to extend this surface. Check that by the symmetry these pentagons fit together. Again, check that

by the symmetry, the remaining hole in the surface is a regular pentagon. Is this an honest calculation-free construction of the regular dodecahedron?
b) [1-] Start the same way by gluing together 6 regular pentagons. Consider their projection as in Figure 19.6. Use Euclidean geometry to show that the projection is a regular 10-gon. Conclude that two such surfaces fit together to form the surface of the dodecahedron.
c) [1-] Use the rigidity of simple polytopes, to show that the resulting polyhedron has the full group of symmetries.
d) [1-] Use this to construct the regular icosahedron.

Exercise 19.4. $\diamondsuit$ *a*) [1] Decide on theoretical grounds (as opposed to explicit construction) which of the three unfoldings in Figure 19.7 are unfoldings of convex polytope whose faces are equilateral triangles and squares. Is there a unique way of gluing them together?
b) [1+] Prove that these polytopes exist, by an explicit construction.

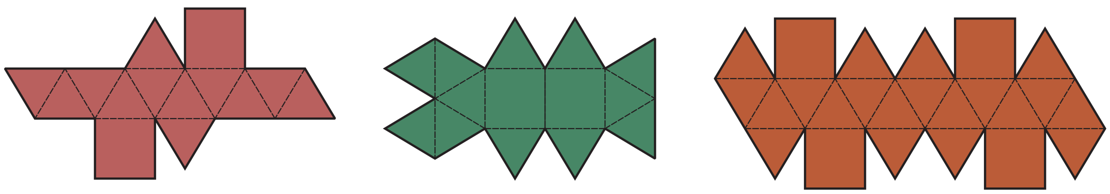

FIGURE 19.7. Unfoldings of three convex polytopes.

Exercise 19.5. Let P be a convex polytope with at least one *symmetry line*, defined as an axis of rotational symmetry by $180°$.
a) [1-] Prove that all symmetry lines pass through the same point.
b) [1] Prove that P has an odd number of symmetry lines.

Exercise 19.6. [1] Show that a regular octahedron Q can be inscribed into a cube $C \subset \mathbb{R}^3$ so that every vertex of Q lies on the edge of C.

Exercise 19.7. *a*) [1] Prove that there exists a 3-coloring of edges of the icosahedron such that every face has all three colors. Moreover, show that such 3-colorings are combinatorially equivalent (up to the symmetry of the icosahedron and relabeling of the colors).
b) [1] The same question for 5-colorings of vertices of a 600-*cell* (a regular simplicial polytope in $\mathbb{R}^4$ with 600 facets; see Exercise 19.15).

Exercise 19.8. A polytope $P \subset \mathbb{R}^3$ is called *face-transitive* if for every two faces $F, F' \subset P$ there exists a symmetry of P which maps F into F'.
a) [1-] Find a face-transitive polytope with n triangular faces, for every even $n \geq 4$.
b) [1] Find a face-transitive polytope with n quadrilateral faces, for every even $n \geq 4$.
c) [1+] Show that there are no face-transitive polytopes in $\mathbb{R}^3$ with an odd number of faces.

Exercise 19.9. Let $P \subset \mathbb{R}^3$ be a convex polytope. Think of P as a playing die. Denote by $O = \mathrm{cm}(P)$ the center of mass of P. For every face F in P, denote by C_F a cone from O over F. We say that P is *fair* if the solid angles $\sigma(C_F)$ are equal (see Subsection 25.4). We say that P is *super fair* if O projects orthogonally onto centers of mass $\mathrm{cm}(F)$ of all faces, and all faces $F \subset P$ are centrally symmetric.
a) [1] Prove that there exists a fair die with n faces, for all $n \geq 4$.
b) [1] Prove that there exist a super fair die with n faces, for all $n \geq 4$.

Exercise 19.10. (*Grünbaum–Shephard*) [2] Let $S \subset \mathbb{R}^3$ be an embedded polyhedral surface of genus of $g > 0$. Prove that S cannot be face-transitive.

Exercise 19.11. [1] Suppose $P \subset \mathbb{R}^3$ is a simple polytope such that all faces are inscribed polygons. Prove that P is inscribed into a sphere.

Exercise 19.12. *a*) [1-] Find a convex polytope in $\mathbb{R}^3$ with equal edge lengths, which is midscribed (has all edges touch the sphere) and circumscribed, but not inscribed into a sphere.
b) [1-] Find a convex polytope in $\mathbb{R}^3$ with equal edge lengths, which is midscribed and inscribed into sphere, but not circumscribed around a sphere.

Exercise 19.13. Let $P \subset \mathbb{R}^3$ be a convex polytope which is midscribed and has edges of equal length.
a) Suppose one of the faces of P is odd-sided. Prove that P is also inscribed.
b) Suppose all faces of P have the same number of sides. Prove that P is circumscribed.

Exercise 19.14. *a*) [1] Find all convex polytopes whose faces are equilateral triangles.
b) [1] Prove that there is only a finite number of combinatorially different convex polytopes whose faces are regular convex polygons with at most 100 sides.
c) [1] Prove that for every $n > 100$ the number of such polytopes with n faces is two if n is even, and one if n is odd.

Exercise 19.15. (*Regular polytopes in higher dimensions*) $\diamondsuit$ *a*) [2-] Show that the only regular convex polytopes in $\mathbb{R}^d$, $d \geq 5$ are the simplex, the hypercube and the cross-polytope.
b) [1+] Prove that in $\mathbb{R}^4$ there are three additional regular convex polytopes, with 24, 120, and 600 vertices. Give a construction of these polytopes in the same manner as in the section.
c) [1+] Prove that in $\mathbb{R}^4$, cross-polytopes tile the space in a regular fashion, i.e. the resulting tiling is flag-transitive.

Exercise 19.16. (*Catalan zonotopes*) Define the *rhombic dodecahedron* to be a face-transitive polytope dual to the cuboctahedron (see Figure 16.4). Similarly, define *rhombic triacontahedron* to be a face-transitive polytope dual to the icosidodecahedron (see Exercise 16.20).
a) [1] Prove that in addition to the Platonic solids and icosidodecahedron, these are the only convex polytopes in $\mathbb{R}^3$ which are edge-transitive.
b) [1-] Prove that both polyhedra are zonotopes (see Exercise 7.16).
c) [1] Find an explicit decomposition of both polyhedra into parallelepipeds.

Exercise 19.17. (*Jessen's orthogonal icosahedron*) $\diamondsuit$ *a*) [1-] Start with a regular icosahedron positioned in a cube as in Figure 19.3. For every (red) edge e of the icosahedron lying on the cube remove the tetrahedron spanned by two triangles containing e (see Figure 19.8). Prove that the resulting (non-convex) polyhedron is vertex-transitive and combinatorially equivalent to an icosahedron.
b) [1] Construct a combinatorial icosahedron as in Figure 19.3 where the distance x is equal to half the edge length of the cube. Starting with this convex polytope construct a non-convex polyhedron as in part *a*). Prove that all dihedral angles are either $\pi/2$ or $3\pi/2$.

Exercise 19.18. (*Equifacetal simplices*) *a*) [1] Prove that there exist $\delta > 0$ small enough, such that for every collection of lengths $\ell_{ij} \in [1-\delta, 1+\delta]$, there exists a simplex $\Delta = [v_0 v_1 \ldots v_d] \subset \mathbb{R}^d$ with edge lengths $\ell_{ij} = |v_i v_j|$.

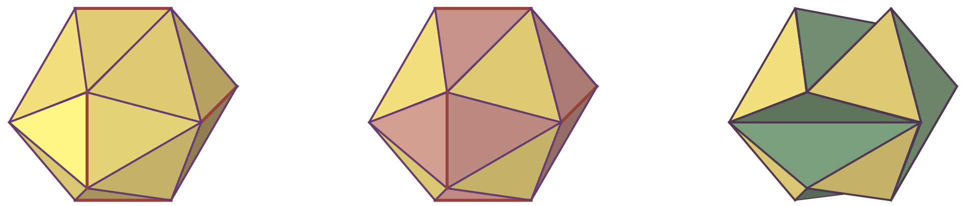

FIGURE 19.8. Construction of Jessen's orthogonal icosahedron.

b) [1-] A simplex with congruent facets is called *equifacetal*. Construct an equifacetal simplex Δ which is not regular.
c) [1+] Prove that every equifacetal simplex is vertex-transitive.
a) [1+] Find the largest possible δ in part a).

19.5. **Final remarks.** Regular polytopes in $\mathbb{R}^3$ are called *Platonic solids*, while pairs (r, s) defined above are called their *Schläfli symbols*. Note that values $(r, s) \in \{(3, 6), (4, 4), (6, 3)\}$ also make sense: they correspond to infinite plane tessellations (honeycombs): the hexagonal, square and triangular lattices. In dimension 4 there are three special regular polytopes (see Exercise 19.15) which are even harder to construct without coordinates (their uniqueness, however, is much easier given the 3-dimensional case; cf. Subsection 27.2). For a comprehensive overview of regular polytopes in all dimensions see classical monographs [Cox1, Fej2], and a recent opus magnum [McS]. It is worth noting that already (perhaps, especially) in dimension 4, the regular polytopes and tessellations are unintuitive. The reader can take solace in the following warning by Coxeter [Cox1, §7.1]:

> Only one or two people have ever attained the ability to visualize hyper-solids as simply and naturally as we ordinary mortals visualize solids.

Although perhaps overly pessimistic, in all fairness, this quote is taken somewhat out of context, as Coxeter dedicates the whole section to this discussion (aided, in turn, with a nice quote by Poincaré).

Let us mention here the 13 *Archimedean solids*, a related family of polytopes which are all vertex-transitive and have faces regular polygons. Similarly, the face transitive polytopes dual to Archimedean solids are called *Catalan solids*. Finally, all convex polytopes with regular faces are called *Johnson solids* (see Exercise 19.14), and are completely classified by Johnson and Zalgaller (see [Crom, Joh, Zal3]).

Various constructions of the icosahedron and connection to rigidity are described in [BorB, §20]. Note that our constructions (i) and (iii) essentially coincide - we chose to emphasize the difference in the geometric argument rather than the outcome. As for the original construction (part (iv)), by itself it does not produce a regular polytope: one has to check that the parts of pentagonal faces align properly (see Exercise 19.2). However, there are other constructions which seem to require fewer computation (see Exercise 19.3), albeit with some sort of weak rigidity argument. The connection to rigidity was first discovered by Legendre in his effort to rewrite Euclid's "Elements" (see [Sab6]).

20. Kissing numbers

The problem of determining *kissing numbers* is one of the most celebrated problems in discrete geometry. The problem is surprisingly difficult, so we prove only the most basic results in small dimensions. While these results are completely unrelated to the rest of the book, we found them an excellent way to give a quick and easy introduction to spherical geometry. As the reader will see, the basic spherical geometry is used throughout the second part.

20.1. **Basic results.** Denote by K_d the d-dimensional *kissing number*, defined as the largest number of unit balls touching a fixed unit ball in $\mathbb{R}^d$. Observe that K_d can be also viewed as the largest number of points on a unit sphere $\mathbb{S}^{d-1} \subset \mathbb{R}^d$ with pairwise distances ≥ 1.

Proposition 20.1. $K_1 = 2,\ K_2 = 6.$

Proof. The first part is clear. For the second part, observe that $K_2 \geq 6$ as shown in Figure 20.1. To see that $K_2 \leq 6$, consider k points $a_1, \ldots, a_k$ on a unit circle with center at the origin O, and with pairwise distances ≥ 1. Take their convex hull. Since from the edge lengths of (a_1, a_2), (a_2, a_3), $\ldots$, (a_k, a_1) are ≥ 1, we conclude that $\angle a_1 O a_2$, $\angle a_2 O a_3$, $\ldots$, $\angle a_k O a_1 \geq \pi/3$. Therefore

$$2\pi \,=\, \angle a_1 O a_2 + \angle a_2 O a_3 + \,\ldots\, + \angle a_k O a_1 \,\geq\, k \cdot \frac{\pi}{3}\,,$$

and $k \leq 6$. $\square$

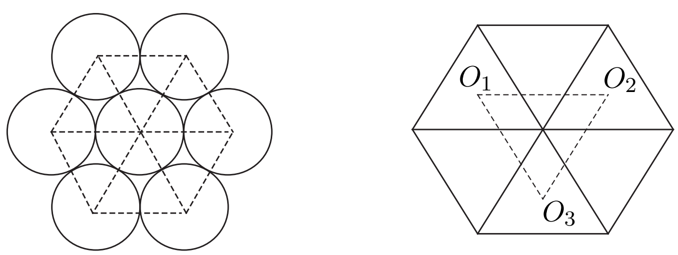

Figure 20.1. Six touching circles and centers of twelve touching spheres.

It is natural to use the idea of the proof of Proposition 20.1 to obtain the bounds in dimension three.

Proposition 20.2. $12 \leq K_3 \leq 14.$

Proof. The lower bound follows from a "grocery style" sphere packing. Consider six unit spheres with centers on a horizontal plane as above, and start lowering the three new unit spheres so that their centers are projected orthogonally onto centers O_1, O_2, O_3 of triangles, as shown in Figure 20.1. By the symmetry, at one point all three spheres will be touching the spheres below. Now observe that these three new spheres are non-overlapping (in fact touching themselves) since the distance between their centers is equal to 2. Now adding in a similar fashion the three spheres from below we obtain the desired 12 spheres.

For the upper bound, consider k unit spheres $S_1, \ldots, S_k$ kissing a fixed unit sphere $S_0 \subset \mathbb{R}^3$ centered at the origin. Now let C_i be a cone over S_i centered at the origin O, and let $R_i = S_0 \cap C_i$ be the spherical caps on a sphere. Since the centers O_i of spheres S_i are all at the same distance from O and their pairwise distances are ≥ 2, we conclude that the cones C_i must be disjoint and thus so are caps R_i. We conclude that $\text{area}(R_i) \leq \sigma/k$, where $\sigma = \text{area}(S) = 4\pi$ denotes the area of a unit sphere.

To compute the $\text{area}(R_i)$, recall that area of a spherical cap is proportional to the height of the cap. From Figure 20.2, we have $\angle yOz = \angle wOO_i = \arcsin\frac{1}{2} = \frac{\pi}{6}$. Therefore, for the height h of the cap R_i we have:

$$h = |zx| = 1 - |Oz| = 1 - \cos\frac{\pi}{6} = 1 - \frac{\sqrt{3}}{2}.$$

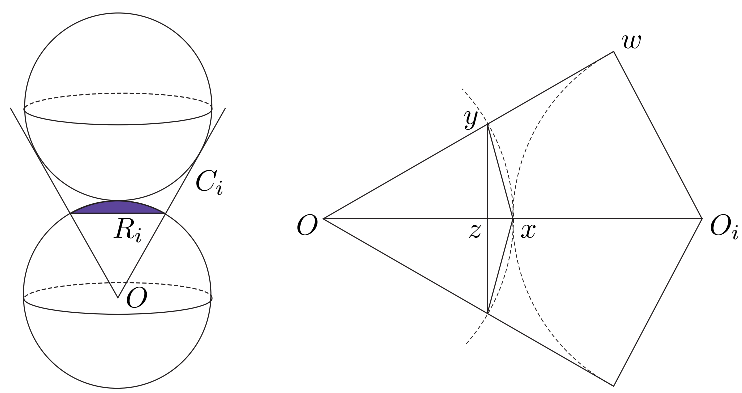

FIGURE 20.2. Computing the height of a cap C_i on a sphere.

Finally, we compute the area of a cap:

$$\text{area}(R_i) = \sigma \cdot \frac{h}{2} = \sigma \cdot \frac{1 - \frac{\sqrt{3}}{2}}{2} > \frac{\sigma}{14.9283}.$$

Since from above $\text{area}(R_i) \leq \sigma/k$, we conclude that $k \leq 14$. $\square$

20.2. **The problem of fourteen spheres.** Let us employ a somewhat sharper (but still elementary) technique to show that $K_3 \leq 13$. For that, we will need only two elementary results: Euler's formula (see Section 25) and Girard's formula for the area of spherical triangles (see Appendix 41.1). In fact, it is known that $K_3 = 12$, but all known proofs are much more involved.

Let us first discuss the limitations of the approach in the proof above. Note that we significantly underestimated the area of a sphere by adding the areas of the caps, leaving all regions between the cap unaccounted for. From a combinatorial point of view, it is natural to make the caps larger so as to count more of the area, and then use the inclusion-exclusion principle to account for overlap of the caps. The argument below is exactly of this kind: we first overestimate the area of a sphere and then subtract the intersection areas.

Theorem 20.3. $K_3 \leq 13$.

Proof. Suppose one can place points $a_1, \ldots, a_{14}$ on the unit sphere $\mathbb{S}^2$ so that the pairwise distances $|a_i a_j| \geq 1$, for all $i \neq j$. Denote by $P = \mathrm{conv}\{a_1, \ldots, a_{14}\}$ the convex polytope obtained as a convex hull of points a_i. Note that all a_i are vertices of P.

We will assume that the origin O is the center of $\mathbb{S}^2$, and that $O \in P$ since otherwise all a_i lie in the same half-space, and one can add an extra point a_{15} contradicting Proposition 20.2. Finally, we will assume that P is *simplicial*, i.e., each face of P is a triangle. Otherwise, triangulate each face and proceed as follows without any change.

By Euler's Theorem, the polytope P with $n = 14$ vertices will have $3n - 6 = 36$ edges and $2n - 4 = 24$ triangular faces. Now, for each face $(a_i a_j a_k)$ consider a circle C_{ijk} around the vertices; one can think of C_{ijk} as an intersection of $\mathbb{S}^2$ and the plane which goes thorough a_i, a_j and a_k.

Let us consider the 14 caps $R_1, \ldots, R_{14}$ with centers at points a_i and such that the circles of the caps $C_i = \partial R_i$ have the same radius $\rho = 1/\sqrt{3}$. We need several simple results on geometry of caps R_i. As before, let $\sigma = \mathrm{area}(\mathbb{S}^2) = 4\pi$.

Lemma 20.4. *For the caps $R_1, \ldots, R_{14}$ defined as above, we have:*
1) $\mathrm{area}(R_i) > 0.0918\,\sigma$, *for all* $1 \leq i \leq 14$;
2) $\mathrm{area}(R_i \cap R_j) < 0.0068\,\sigma$, *for all* $1 \leq i < j \leq 14$;
3) $\mathrm{area}(R_i \cap R_j \cap R_k) = 0$, *for all* $1 \leq i < j < k \leq 14$.

We prove the lemma in the next subsection, after we finish the proof of the theorem. Observe that the only pairs (i, j) for which we can have $\mathrm{area}(R_i \cap R_j) \neq 0$ are those corresponding to 36 edges of the polytope P. Using all three parts of the lemma we get:

$$\sigma \,=\, \mathrm{area}(S) \,\geq\, \sum_{i=1}^{14} \mathrm{area}(R_i) \;-\; \sum_{1\leq i<j\leq 14} \mathrm{area}(R_i \cap R_j)$$
$$> \, 14 \cdot (0.0918\,\sigma) \,-\, 36 \cdot (0.0068\,\sigma) \,>\, 1.03\,\sigma,$$

a contradiction. Therefore, $K_3 < 14$ as desired. □

20.3. **Proof of Lemma 20.4.** For part 1), recall that $\mathrm{radius}(R_i) = \rho = 1/\sqrt{3}$. From Figure 20.3, the height h of the cap R_i is given by[47]

$$h \,=\, 1 - \sqrt{1 - \rho^2} \,=\, 1 - \sqrt{1 - \frac{1}{3}} \,=\, 1 - \sqrt{\frac{2}{3}}\,.$$

[47]Note that in the proof of Proposition 20.2 we used spherical cap of radius $1/2$. Loosely speaking, we are *undercounting* in the proof of the proposition, and are *overcounting* in the proof of the theorem.

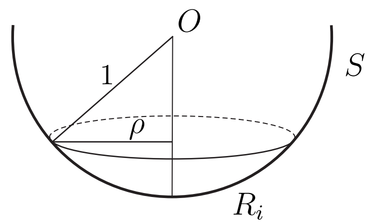

FIGURE 20.3. Computing the area of spherical caps R_i.

Therefore, by the same reasoning as in the proof of Proposition 20.2, for the area of R_i we have:

$$\operatorname{area}(R_i) = \sigma \cdot \frac{h}{2} = \frac{1 - \sqrt{\frac{2}{3}}}{2}\,\sigma > 0.0918\,\sigma.$$

For part 2), we can assume that the distance $|a_i a_j| = 1$ since for larger distances the area of the *link* $L := R_i \cap R_j$ is smaller. Denote by b the remaining vertex of the equilateral spherical triangle $(a_i a_j b)$ and by u, v the corners of the link as in Figure 20.4.

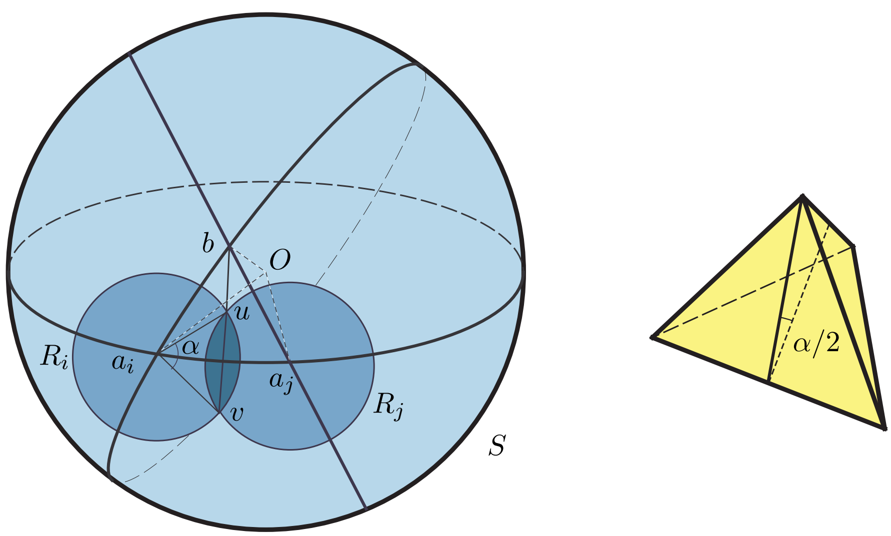

FIGURE 20.4. Computing the area of lenses $R_i \cap R_j$.

By the symmetry, the point $u \in S$ is the center of the triangle $(a_i a_j b)$, and we have equal spherical angles $\alpha = \sphericalangle u a_i v = \sphericalangle u a_j v = \sphericalangle b a_i a_j$. Now, for the area of L we have

$$\operatorname{area}(L) = 2\,\big(\operatorname{area}(Q) - \operatorname{area}(T)\big),$$

where Q denotes the sector $[a_i uv] \subset R_i$ and T denotes the spherical triangle $(a_i uv) \subset Q$. Again, by the symmetry and from Girard's formula we have:

$$\operatorname{area}(Q) \,=\, \frac{\alpha}{2\pi}\cdot\operatorname{area}(R_i) \,=\, \frac{\alpha}{2\pi}\cdot\frac{1-\sqrt{2/3}}{2}\cdot(4\pi) \,=\, \alpha - \sqrt{2/3}\cdot\alpha\,,$$
$$\operatorname{area}(T) \,=\, \alpha + \frac{\pi}{3} + \frac{\pi}{3} - \pi \,=\, \alpha - \frac{\pi}{3}\,.$$

Now, observe that α is a dihedral angle in the regular tetrahedron (Oa_ia_jb) with side 1. Computing it as shown on Figure 20.4, we obtain $\alpha = 2\arcsin\frac{1/2}{\sqrt{3}/2} = 2\arcsin\frac{1}{\sqrt{3}}$. We conclude:

$$\operatorname{area}(L) \,=\, 2\big(\operatorname{area}(Q) - \operatorname{area}(T)\big) \,=\, 2\,(\pi/3 - \sqrt{2/3}\cdot\alpha)$$
$$=\, \frac{2\pi}{3} - 4\sqrt{\frac{2}{3}}\,\arcsin\frac{1}{\sqrt{3}} \,<\, 0.0843 \,<\, 0.0068\,\sigma.$$

For part 3), we begin with the result which follows directly from the argument used in the proof of Corollary 1.8, with all inequalities reversed.

Lemma 20.5. *For all faces $(a_ia_ja_k)$ in P, the* $\operatorname{radius}(C_{ijk}) \geq \rho$, *where $\rho = \frac{1}{\sqrt{3}}$.*

Suppose now there is a point $z \in S$ which lies in the interior of three caps R_i, R_j and R_k. By definition, the distances from z to vertices a_i, a_j and a_k are strictly smaller than ρ. Thus, the same is true for the projection z' on a plane spanned by a_i, a_j and a_k. This implies that the triangle $(a_ia_ja_k)$ can be enclosed into a circle C with center z' and radius $r < \rho$. Expand the triangle $(a_ia_ja_k)$ to a triangle $(a_i'a_j'a_k')$ inscribed into C with bigger edge lengths, as shown in Figure 20.5. By Lemma 20.5, the circumradius of a triangle with edge lengths ≥ 1 is $\geq \rho$, a contradiction. This completes the proof of Theorem 20.3.

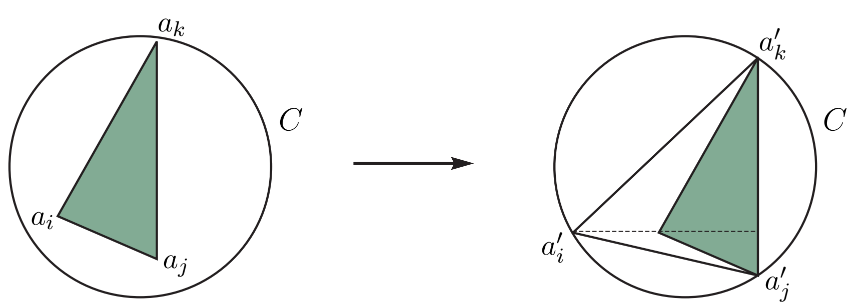

Figure 20.5. Expanding triangle $(a_ia_ja_k) \to (a_i'a_j'a_k')$.

20.4. **Exercises.**

Exercise 20.1. (*Asymptotics*) *a*) [1] Prove that $K_d = \exp O(d)$.
b) [1] Prove that $K_d = \exp \Omega(d)$.

Exercise 20.2. [1] Find the largest number of lines in $\mathbb{R}^3$, which pass through the origin and have equal angles between them.

Exercise 20.3. [1+] Let $P \subset \mathbb{R}^3$ be a convex polytope with 19 faces. Suppose P is circumscribed around a sphere of radius 10. Prove that $\operatorname{diam}(P) > 21$.

Exercise 20.4. [1] Let $x_1, \ldots, x_{30} \in \mathbb{S}^2$ be points on a unit sphere. Prove that there exist $1 \le i < j \le 30$, such that

$$\left|x_i x_j\right|_{\mathbb{S}^2} < \frac{\pi}{4}.$$

In other words, show that some two of the vectors $\overrightarrow{Ox_i}$ have angle $< \pi/4$ between them.

Exercise 20.5. *a*) [1] Suppose n points on a unit sphere have pairwise spherical distances $\ge \pi/2$. Prove that $n \le 6$, and that for $n = 6$ these points must be vertices of an inscribed regular octahedron.
b) [1+] Prove an icosahedral analogue of *a*).
c) [2-] Suppose $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are two configurations of points in the unit sphere $\mathbb{S}^2$, such that $|x_i x_j| \ge |y_i y_j|$ for all $i, j \in [n]$. Suppose further that $O \in \operatorname{conv}\{x_1, \ldots, x_n\}$, where O is the center of $\mathbb{S}^2$. Prove that $|x_i x_j| = |y_i y_j|$ for all $i, j \in [n]$.

Exercise 20.6. *a*) [1-] Is it possible to cover the plane by a union of parabolas, i.e., regions obtain by translations and rotations of $y \ge a x^2$, where $a > 0$.
b) [1] Suppose in the space $\mathbb{R}^3$ there is a finite number of disjoint cones, i.e., regions obtain by translations and rotations of $x^2 + y^2 \le a z$, where $a > 0$. Prove that these cones cannot be moved to cover the whole space.

Exercise 20.7. On a unit sphere, define an α-arc to be an arc of a great circle of length α.
a) [1-] For every $\alpha < \pi$, prove that there are infinitely many not self-intersecting α-arcs on a unit sphere.
b) [1] For $\alpha = 5/3\pi$, prove that there are at most two not self-intersecting α-arcs on a unit sphere.
c) [1] Prove that there is an unbounded number of not self-intersecting π-arcs on a unit sphere.
d) [1+] For every $\alpha > \pi$, prove that there is at most a bounded number $n = n(\alpha)$ of not self-intersecting α-arcs on a unit sphere.

Exercise 20.8. Prove that the following triangles (given by their angles) can tile the sphere $\mathbb{S}^2$ without overlap:
a) [1-] $(90°, 90°, 1°)$, $(90°, 60°, 60°)$, $(120°, 60°, 60°)$,
b) [1] $(120°, 45°, 45°)$, $(72°, 60°, 60°)$, $(150°, 60°, 60°)$,
c) [1+] $(80°, 60°, 60°)$, $(100°, 60°, 60°)$, $(100°, 80°, 60°)$.

Exercise 20.9. $\diamondsuit$ *a*) [1-] Place the centers of twelve unit spheres in $\mathbb{R}^3$ at the vertices of the icosahedron inscribed into a sphere of radius 2. Check that these spheres are not overlapping and, moreover, no two of them touch each other. This gives an alternative proof of the lower bound in Proposition 20.2.
b) [2-] In *a*), the outside spheres can be continuously moved while they remain non-overlapping and touching the center sphere. Is it possible to switch any two spheres that way?
c) [2+] Can these twelve spheres be moved into twelve spheres in the "grocery style" sphere arrangement?

Exercise 20.10. Denote by KC_d the number of infinite cylinders of unit radius in $\mathbb{R}^d$, which are non-overlapping and touching the unit sphere. Such configuration is called *kissing cylinders*.

a) [1-] Prove that $KC_d \geq K_{d-1}$.
b) [1-] Find a continuous family of different configurations of six kissing cylinders in $\mathbb{R}^3$ (different up to rotations).
c) [2-] Prove that $KC_3 \leq 7$.
d) [*] Prove that $KC_3 \leq 6$.

Exercise 20.11. For a finite arrangement $\mathcal{B}$ of balls in $\mathbb{R}^d$ with disjoint interior (not necessarily of the same radii), denote by $a(\mathcal{B})$ their *average kissing number*. Define AK_d to be the supremum of $a(\mathcal{B})$ over all finite ball arrangements.
a) [1-] Show that the $AK_d > a(\mathcal{B})$ for any finite $\mathcal{B}$.
b) [1] Prove that $AK_3 > 12$.
c) [1] Prove that $AK_d \leq 2K_d$.
d) [2] Prove that $AK_3 \leq 8 + 4\sqrt{3}$.

Exercise 20.12. (*Pairwise kissing spheres*) Denote by PK_d the maximal number of pairwise kissing spheres in $\mathbb{R}^d$ (possibly, of different radius).
a) [1-] Use the previous exercise to show that $PK_d = \exp O(n)$. Explain the difference with Exercise 42.45.
b) [1] Use the algebraic approach (see Section 31) to show that $PK_d = O(d)$.
c) [1] Prove that $PK_d = d + 2$.

20.5. **Final remarks.** Finding kissing numbers K_d is a classical problem connected to the study of error-correcting codes, lattices, number theory and many problems in geometry [ConS, Fej2, Zon1] (see also [Bez] for a recent survey). Proving that $K_3 = 12$ is known as the *Newton–Gregory problem* named after their celebrated exchange in 1694. The problem was resolved by Schütte and van der Waerden only in 1953, but most known proofs are quite technical. In fact, this section can be viewed as an introduction to Leech's proof, an elementary presentation of which recently appeared in [Mae2].

We refer to [Mus6] for a recent interesting proof of $K_3 = 12$. Let us also note that while the exact value $K_4 = 24$ was determined recently by Musin [Mus5], the kissing numbers $K_8 = 240$ and $K_{24} = 196560$ have been known since 1979 (see [ConS, Mus5, PfeZ]). The proof of Theorem 20.3 presented in this section is a minor reworking of the proof in [Yag2].

Part II

Discrete Geometry of Curves and Surfaces

21. The four vertex theorem

In this section we present a number of results on the geometry of planar and space polygons, all related to the *four vertex theorem*. While motivation for some of these results lies in differential geometry, the piecewise linear results are often more powerful and at the same time easier to prove. On the other hand, some continuous results seem to have no discrete analogues, while others have several. To give a complete picture, in the beginning and at the end of this section we present (without proof) a number of continuous results as a motivation. Interestingly, the proof of the main theorem in this section uses Voronoi diagrams (see Section 14).

Although the results in this section are similar in spirit to various results in the first part (see Sections 5 and 9), there is a common thread of results in the second part. Namely, in the next section we obtain several results on relative geometry of convex polygons, further extending the four vertex theorem. One of these extensions, the *Legendre-Cauchy lemma* is then used to prove the Cauchy theorem (see Section 26). Much of the rest of the book is dedicated to various extensions and generalizations, including *Dehn's theorem*, the *Alexandrov theorem*, etc. The relative geometry of polygons reappears several times, often in disguise, but still at the heart of many proofs.

21.1. **The flavor of things to come.** Let $C \subset \mathbb{R}^2$ be a smooth curve in the plane. Throughout this section, all curves will be closed. We say that a curve is *simple* if it is not self-intersecting. A point $x \in C$ is called a *vertex* if it is an extremum (local maximum or minimum) of the curvature κ. Here by a *curvature* $\kappa(x)$ we mean the inverse of the radius $R(x)$ of the osculating circle at point x, so $\kappa(x) = 0$ when x is a flat point.[48]

Observe that every smooth closed curve has at least two vertices. The following classical result shows that if C is simple the number of vertices is at least four.

Theorem 21.1 (Four vertex theorem)**.** *Every smooth simple curve has at least four vertices.*

A few quick examples: an ellipse with distinct axes has exactly four vertices, while all points on a circle are vertices. On the other hand, for non-simple curves the claim is easily false (see Figure 21.1).

The next extension of the four vertex theorem is based on restricting the notion of vertices. Let C be a smooth convex curve. An osculating circle R tangent to C at point $x \in C$ is called *empty* if it lies entirely inside C. Of course, then κ is a local maximum at x, but the inverse does not necessarily hold. Similarly, an osculating circle R tangent to C at point $y \in C$ is called *full* if C lies entirely inside R. Then κ has a local minimum at y. The empty and full osculating circles are called *extremal circles*. By definition, the number of extremal circles is at most the number of vertices of curve C.

[48]If the curve is parameterized by arc length then the curvature is the length of the second derivative vector.

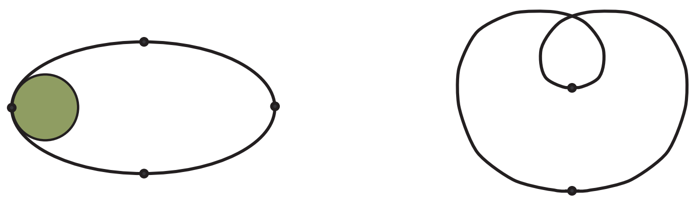

FIGURE 21.1. An osculating circle in a convex curve with four vertices and a non-simple curve with two vertices.

Theorem 21.2. *Every smooth convex curve has at least four extremal circles.*

Note that extremal circles have a "global" nature in contrast with a "local" nature of vertices. Here is a quantitative extension of Theorem 21.2 due to Bose (1932). We say that C is *generic* if it is not tangent to any circle at more than three points.

Theorem 21.3. *Let $C \subset \mathbb{R}^2$ be generic smooth convex curve in the plane. Denote by s_+ and s_- the number of full and empty osculating circles, respectively. Denote by t_+ and t_- the number of full and empty circles tangent to C at three points. Then*

$$s_+ - t_+ = s_- - t_- = 2.$$

The theorem immediately implies Theorem 21.2, which in this language states that $s_+ + s_- \geq 4$. We postpone other extensions and generalizations of the four vertex theorem till Subsection 21.8.

21.2. **Discretizing is ambiguous and harder than it looks.** There are several natural discrete analogues of the four vertex theorem (Theorem 21.1). We present a few of them in this section, but more will appear later on. Remember that the word *vertex* when applied to convex polygons has the usual meaning.

Let $Q = [x_1 \dots x_n] \subset \mathbb{R}^2$ be a convex polygon. We say that Q is *generic* if no four vertices of Q lie on a circle. Denote by R_i a circle circumscribed around triangle $(x_{i-1}x_ix_{i+1})$, where all indices are taken modulo n. We say that a vertex x_i is *extremal* if x_{i-2} and x_{i+2} lie on the same side of R_i, i.e., either both vertices lie inside or both vertices lie outside of the circle.

Theorem 21.4 (Discrete four vertex theorem)**.** *Every generic convex polygon with at least four vertices has at least four extremal vertices.*

The following quantitative version of the four vertex theorem follows easily from the theorem (see Exercise 21.1). We say that Q is *coherent* if the center of R_i lies inside the cone of Q at x_i.

Corollary 21.5. *Let $Q = [x_1 \dots x_n] \subset \mathbb{R}^2$ be a generic coherent convex polygon, $n \geq 4$, and let r_i denotes the radius of R_i, $1 \leq i \leq n$. Then there are at least four sign changes in the cyclic sequence $(r_1 - r_2, r_2 - r_3, \dots, r_n - r_1)$.*

The idea is that circles R_i for convex polygons play the role of the osculating circles for the curves, and their inverse radii correspond to the curvature. In this language, the corollary implies that convex polygons have at least two maximal and at least two minimal radii.

Let us note that for non-coherent polygons the corollary does not hold (see Exercise 21.1). On the other hand, all obtuse polygons (convex polygons where all angles are right or obtuse) are coherent. Thus Corollary 21.5 for obtuse polygons can be viewed as a direct extension of the (usual) four vertex theorem (Theorem 21.1).

Consider the following further reduction of the theorem. Let Q be an *equilateral* convex polygon, i.e., a polygon with equal edge lengths a. Note that all equilateral convex polygons are coherent. Denote by $\alpha_i = \angle x_{i-1}x_ix_{i+1}$ the angle in vertex x_i. Since $a = 2r_i \cos(\alpha_i/2)$, the larger angles α_i correspond to larger radii r_i, and we obtain the following.

Corollary 21.6. *Let $Q = [x_1 \ldots x_n] \subset \mathbb{R}^2$ be a generic equilateral convex polygon, $n \geq 4$, and let $\alpha_i = \angle x_{i-1}x_ix_{i+1}$ denote the angle in Q, $1 \leq i \leq n$. Then there are at least four sign changes in the cyclic sequence $(\alpha_1 - \alpha_2, \alpha_2 - \alpha_3, \ldots, \alpha_n - \alpha_1)$.*

Here is a natural analogue of the four extremal circles theorem. Let $Q = [x_1 \ldots x_n]$ be a convex polygon as above. A circle R_{ijk} through a triple of vertices x_i, x_j, x_k, $i < j < k$, is called *disjoint* if no two vertices are adjacent; it is called *neighboring* if two of the vertices are adjacent to the third. The remaining circles, with only one pair of adjacent vertices, are called *intermediate*. The circle R_{ijk} is called *empty* if no other vertices x_r are inside, and it is called *full* if all other vertices x_r are inside. We are now ready to discretize Theorem 21.3.

Theorem 21.7. *Let $Q \subset \mathbb{R}^2$ be a generic convex polygon with at least four vertices. Denote by s_+, t_+ and u_+ the number of full circles that are neighboring, disjoint and intermediate, respectively. Similarly, denote by s_-, t_- and u_- the number of empty circles that are neighboring, disjoint and intermediate, respectively. Then*

$$s_+ - t_+ = s_- - t_- = 2,$$

$$s_+ + t_+ + u_+ = s_- + t_- + u_- = n - 2.$$

We prove the theorem in the next subsection. Until then, let us obtain a stronger version of the discrete four vertex theorem (Theorem 21.4). 5 Define an *extremal circle* in Q to be a neighboring full circle or a neighboring empty circle.

Corollary 21.8. *Every generic convex polygon with at least four vertices has at least four extremal circles.*

Clearly, the corollary immediately implies Theorem 21.4. Therefore, all results in this section follow from Theorem 21.7.

Example 21.9. (*Non-convex polygons*) Recall that the original four vertex theorem (Theorem 21.1) holds for all simple curves, while the results in this section are stated for convex polygons. In fact, without extra conditions the discrete four vertex theorem (Theorem 21.4) is false for non-convex polygons (see Figure 21.2). We present a non-convex version later in this section (see Theorem 21.17).

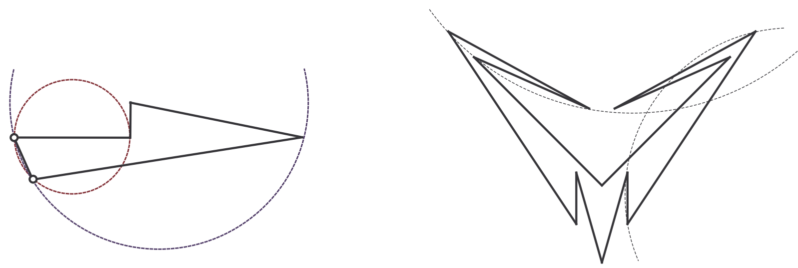

FIGURE 21.2. A (non-convex) pentagon with two extremal vertices and a 12-gon with no extremal circles.

21.3. **Proof of Theorem 21.7 via Voronoi diagrams.** Let $Q = [x_1 \ldots x_n]$ be a generic convex polygon. Consider the *Voronoi diagram* $VD(V)$ of the set of vertices $V = \{x_1, \ldots, x_n\}$ (see Figure 21.3 and Subsection 14.2). The cells of $VD(V)$ are unbounded convex polygons which consist points which are strictly closer to one vertex x_i than to all others.

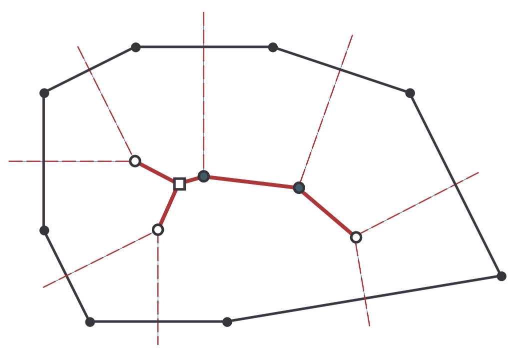

FIGURE 21.3. Voronoi diagram of vertices of a convex polygon, where broken lines correspond to infinite rays. White nodes and squares correspond to centers of neighboring and disjoint circles, respectively.

Define the *cut locus* $\mathcal{C}(V)$ to be the complement to $VD(V)$, i.e., the set of points y in the plane which are equidistant to some two vertices x_i and x_j of Q: $|yx_i| = |yx_j|$. By construction, the cut locus lies in the union of a $\binom{n}{2}$ lines equidistant from pairs of points.

Let us prove that $\mathcal{C}(V)$ is a plane binary tree (a tree where all vertices have degree 0 or 3) with n infinite rays. For every path γ from x_i to x_j, when a point moves from x_i along γ, by continuity there exists a point in $\mathcal{C}(V)$. This implies that the cut locus is connected. Observe that the cell containing a vertex x_i is unbounded since it must contain the outer bisector at x_i. Moreover, $\mathcal{C}(V)$ has no other cycles since every point $z \in VD(V)$ is closest to some x_i, and thus (z, x_i) does not intersect the cut locus. Therefore, $\mathcal{C}(V)$ is a tree. Note that $\mathcal{C}(V)$ has rays which go to infinity and are orthogonal at midpoints to intervals (x_i, x_j). Since Q is convex, these intervals must be edges of Q, so there are exactly n of them.[49] Since Q is generic, no point in the plane is at equal distance to four or more vertices, which implies that $\mathcal{C}(V)$ is a binary tree.

There are three types of vertices of the cut locus tree: vertices which are connected to two, one, or zero infinite rays (there are no vertices connected to three infinite rays since $n \geq 3$). Observe that they correspond to empty neighboring, intermediate and disjoint circles, respectively. The result now follows by induction.

We need to prove that $s_- - t_- = 2$ and $s_- + t_- + u_- = n - 2$. The base $n = 4$ is trivial: we have $s_- = 2$ and $t_- = u_- = 0$. For larger n, remove one of the endpoints. There are two cases to consider, depending on what kind of vertex the removed vertex is adjacent to (see Figure 21.4). In both cases the relations are easily satisfied.

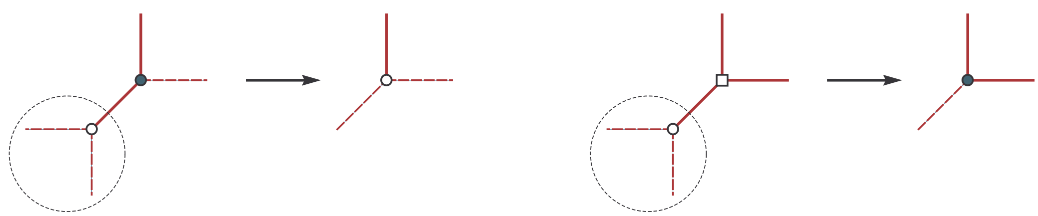

FIGURE 21.4. Inductive step of the proof: removing an endpoint of the cut locus tree.

Finally, to prove the corresponding relations for full circles, we need to consider the *inverse Voronoi diagram*, subdividing points in $\mathbb{R}^2$ according to the farthest vertex x_i (see Exercise 14.4). Again, by convexity, there are n infinite rays corresponding to edges of Q and pointing in the opposite direction to edge normals. The proof extends verbatim to this case, and we obtain equations on s_+, t_+ and u_+ as in the theorem. This completes the proof of Theorem 21.7. $\square$

21.4. **A dual version.** Note that from a continuous point of view there is no reason to prefer circumscribed circles to inscribed circles—both are the same in the limit. As we show below, there is in fact a precise analogue of the discrete four vertex theorem and its extensions to this case. The basic idea is to use the standard "points to lines" duality on a projective plane. We will not formalize it, but rather use it implicitly whenever needed (cf. Exercise 2.2).

Let $Q = [x_1 \ldots x_n] \subset \mathbb{R}^2$, be a convex polygon with edges $e_1 = (x_1, x_2)$, ..., $e_n = (x_n, x_1)$. Denote by ℓ_i, $1 \leq i \leq n$ the lines spanned by the edges e_i. We say that Q is (*dually*) *generic* if no point $z \in \mathbb{R}^2$ lies at equal distance to four of these lines. A circle R^*_{ijk} is called *inscribed* if it is tangent to three lines ℓ_i, ℓ_j and ℓ_k and lies on the same side of these lines as the polygon.

Denote by $R^*_i = R^*_{(i-1)\,i\,(i+1)}$ a circle inscribed into three consecutive lines. We say that edge e_i is *extremal* if R^*_i either intersects both edges e_{i-2} and e_{i+2}, or intersects neither of the two. Now we are ready to state a dual analogue of Theorem 21.4.

[49] This is an easy but crucial point in the proof (cf. the proof of Proposition 14.3).

Theorem 21.10 (Four edge theorem)**.** *Every dually generic convex polygon with at least four edges has at least four extremal edges.*

Corollary 21.11. *Let* $Q = [x_1 \dots x_n] \subset \mathbb{R}^2$ *be a generic convex polygon,* $n \geq 4$*, and let* r_i^* *denotes the radius of* R_i^**,* $1 \leq i \leq n$*. Then there are at least four sign changes in the cyclic sequence* $(r_1^* - r_2^*, r_2^* - r_3^*, \dots, r_n^* - r_1^*)$*.*

Now suppose Q is an *equiangular* convex polygon, i.e., a polygon with equal angle $\pi(n-2)/n$. Since larger lengths $|e_i|$ correspond to larger radii r_i^*, we obtain the following.

Corollary 21.12. *Let* $Q = [x_1 \dots x_n] \subset \mathbb{R}^2$ *be a generic equiangular convex polygon,* $n \geq 4$*, with edges* $e_i = (x_i, x_{i+1})$*,* $1 \leq i \leq n$*. Then there are at least four sign changes in the cyclic sequence* $\bigl(|e_1| - |e_2|, |e_2| - |e_3|, \dots, |e_n| - |e_1|\bigr)$*.*

By analogy with the four vertex theorem, Theorem 21.10 can be proved by a counting all inscribed circles argument (see Exercise 21.4). We will prove the result later in this section by a different argument.

21.5. **A picture is worth a thousand words.** Another way to understand the discrete four vertex theorem (Theorem 21.4) is to interpret it by using the notion of evolute (sometimes called caustic).

Let $Q = [x_1 \dots x_n] \subset \mathbb{R}^2$ be a generic convex polygon as above, and let R_i be the circumscribed circles around the triangle $(x_{i-1}x_ix_{i+1})$, $1 \leq i \leq n$. Denote by r_i the radius and by O_i the center of R_i. The (*discrete*) *evolute* of Q is a polygon $\Upsilon = [O_1 \dots O_n]$. Clearly, the edge (O_{i-1}, O_i) of Υ lies on a line perpendicular to edge $e_i = (x_i, x_{i+1})$ of Q. Orient the edges of Υ away from e_i. The vertex O_i of the evolute is called a *cusp* if both edges adjacent to O_i are oriented to O_i, or both edges oriented from O_i.

Corollary 21.13. *The evolute of a generic coherent convex polygon with at least four vertices has at least four cusps.*

The corollary follows immediately from Corollary 21.5. To see this, simply observe that the circle radii increase in the direction of edges in the evolute. Note that from this point of view it is obvious that the evolute of every (not necessarily convex) generic polygon has at least two cusps.

Now, one reason to consider the evolute is the visually appealing property that the between the cusps the evolute is concave, so the cusps are visually distinctive (see Figure 21.5. In the continuous case this phenomenon is even more pronounced, with sharp angles at the cusps.

The *dual evolute* is defined to be a polygon $\Upsilon^* = [O_1^* \dots O_n^*]$ with vertices at centers of inscribed circles. Other details are similar: centers O_i^* lie on angle bisectors, which are oriented away from the vertices and the *cusps* are defined as vertices of Υ^* with outdegree zero or two.

Corollary 21.14. *The dual evolute of a dually generic convex polygon with at least four vertices has at least four cusps.*

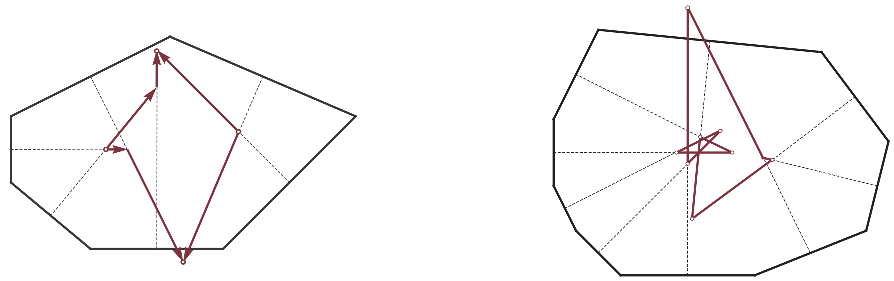

FIGURE 21.5. Evolutes of two convex polygons with four and eight cusps.

In the next subsection we generalize Corollary 21.14 in a rather unexpected direction.

21.6. **Relative evolutes of parallel polygons.** Let $Q = [x_1 \ldots x_n]$ be a convex polygon in a plane. We say that a polygon $W = [w_1 \ldots w_n]$ is *parallel* to Q, if the edges (w_i, w_{i+1}) are parallel to (x_i, x_{i+1}), for all $1 \le i \le n$. We say that W *surrounds* Q if Q is inside W, and vectors $u_i = (x_i w_i)$ turn in the same direction as points x_i as in Figure 21.6.

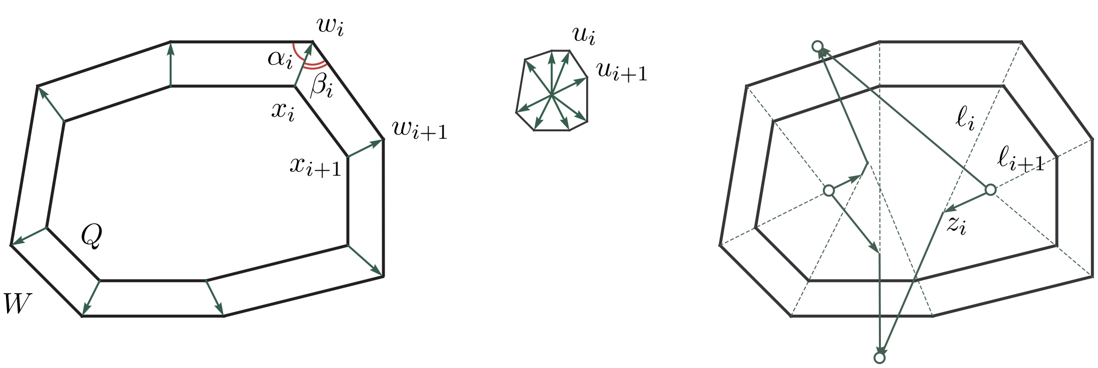

FIGURE 21.6. Polygon W surrounds Q and their relative evolute with four cusps.

Denote by ℓ_i the lines spanned by (x_i, w_i) and let z_i be the intersection of ℓ_i and ℓ_{i+1}, $1 \le i \le n$. Define the *relative evolute* as the polygon $\Upsilon = [z_1 \ldots z_n]$. Orient the edges of Υ away from the vertices x_i. As before, define the *cusps* to be points z_i with with either two ingoing or two outgoing edges. Finally, we say that Q and W are (*relatively*) *generic* if no three lines ℓ_i, ℓ_j and ℓ_k intersect.

Theorem 21.15. *Let Q and W be parallel convex polygons which are relatively generic. Suppose also that W surrounds Q. Then the relative evolute Υ defined above has at least four cusps.*

Let us first show that Theorem 21.15 is a direct generalization of Corollary 21.14. Denote by α_i and β_i the angles of ℓ_i with edges in Q. Observe that the condition that (w_i, w_{i+1}) is parallel to (x_i, x_{i+1}) is equivalent to $|u_{i+1}|/|u_i| = \sin\beta_i/\sin\alpha_{i+1}$. Therefore, for a polygon Q and fixed lines ℓ_i with angles (α_i, β_i), $1 \le i \le n$, a similar polygon W exists if and only if

$$\sin\alpha_1 \cdot \ldots \cdot \sin\alpha_n = \sin\beta_1 \cdot \ldots \cdot \sin\beta_n .$$

Similarly, such W surrounds Q if and only if

$$\beta_i + \alpha_{i+1} < \pi, \quad \text{for all } \ 1 \le i \le n .$$

Thus when ℓ_i are bisectors in Q, there exists a parallel polygon W which surrounds Q, and the theorem applies. Let us note also that as a consequence of Theorem 21.15, we obtain a new proof of Theorem 21.10 via the Corollary 21.14.

Proof of Theorem 21.15. Let $\tau_i = |x_i w_i|/|z_i x_i|$. Denote by T_i a triangle spanned by u_i and u_{i+1} and let Δ_i be a triangle $(z_i x_i x_{i+1})$. Now observe that Δ_i and T_i are similar with similarity coefficient τ_i. Therefore, the number of relative cusps as in the theorem is equal to the number of sign changes in the cyclic sequence $(\tau_1 - \tau_2, \tau_2 - \tau_3, \ldots, \tau_n - \tau_1)$. Since Q and W are generic, the τ_i are distinct and there are at least two sign changes. Suppose now that there are exactly two sign changes. We can assume that $\tau_i - \tau_{i+1} > 0$ for $1 \le i \le k$ and < 0 for $k+1 \le i \le n$. Observe that for every point y inside Q we have:

$$\sum_{i=1}^{n} (\tau_i - \tau_{i+1})\, \overrightarrow{yx_i} = \sum_{i=1}^{n} \tau_i \left(\overrightarrow{yx_{i+1}} - \overrightarrow{yx_i} \right) = \sum_{i=1}^{n} \tau_i \cdot \overrightarrow{x_i x_{i+1}} = \sum_{i=1}^{n} \left(u_{i+1} - u_i \right) = \mathbf{0}.$$

Set O to be a point on a line L crossing (x_n, x_1) and (x_k, x_{k+1}). Then the sum on the l.h.s. lies on one side of L, a contradiction. □

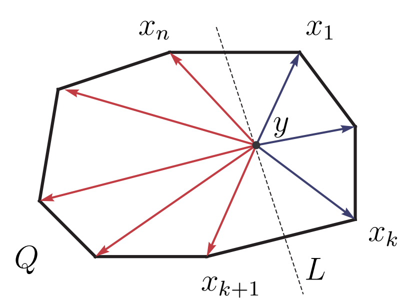

FIGURE 21.7. Line L separating positive and negative coefficients $(\tau_i - \tau_{i+1})$.

21.7. **Going into space.** The main result of this subsection can be viewed as a generalization of the four vertex theorem. Although simple, it is surprisingly powerful as it implies an extension of the discrete four vertex theorem in the plane (Theorem 21.4).

Let $Q = [x_1 \ldots x_n] \subset \mathbb{R}^3$, be a simple space polygon. We say that Q is *generic* if no four vertices of Q lie on the same plane. We say that Q is *weakly convex*[50] if it

[50] This notion is different and in higher dimensions is less restrictive than the notion of convexity given in Exercise 2.15.

lies on the surface of the convex polytope $P = \mathrm{conv}(Q)$. We say that vertex x_i is a *support vertex* if (x_{i-1}, x_i, x_{i+1}) is a face in P. In other words, the plane spanned by edges (x_{i-1}, x_i) and (x_i, x_{i+1}) is a supporting plane (see Figure 21.8).

Theorem 21.16 (Four support vertex theorem)**.** *Every generic weakly convex simple space polygon with at least four vertices has at least four support vertices.*

Proof. Let $Q \subset \mathbb{R}^3$ be the polygon as in the theorem. Since Q is generic, the polytope $P = \mathrm{conv}(Q)$ is simplicial. Observe that Q is a Hamiltonian cycle in the graph Γ of P. Thus, it divides the surface of $S = \partial P$ into two triangulations T_1 and T_2, neither of which have interior vertices. If Q has $n \geq 4$ vertices, triangulations T_1, T_2 have $(n-2) \geq 2$ triangles. Since the graph dual to T_1 is a tree, it has at least two endpoints, i.e., T_1 has at least two triangles $(x_{i-1}x_ix_{i+1})$ with two edges Q. This implies that T_1 has at least two support vertices. Similarly, T_2 also has two support vertices. Finally, note that these four support vertices must be distinct since otherwise they have degree two in Γ. $\square$

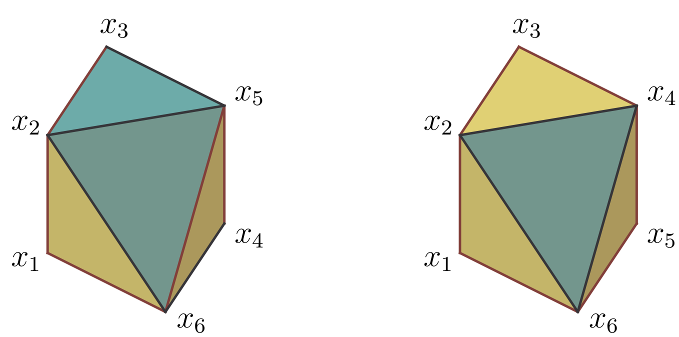

FIGURE 21.8. Two weakly convex space polygons $[x_1 \dots x_6] \subset \mathbb{R}^3$, one with four and another with six support vertices.

To see the connection to extremal circles in polygons, assume that a simplicial convex polytope $P \subset \mathbb{R}^3$ as above is inscribed into a sphere. Then a circumscribed spherical circle around every triangular face (x_{i-1}, x_i, x_{i+1}) is extremal, i.e., contains all remaining vertices on one side. Thus, one can think of Corollary 21.8 as a limiting case of Theorem 21.16, when the circumscribed sphere becomes a plane.

Recall that a polygon $Q \subset \mathbb{R}^2$ is generic if no four points lie on a circle. We say that Q is a *Delaunay polygon* if it is simple, generic, and for every edge $(x, y) \in Q$ there exists a circle R through x and y, such that the remaining vertices are either all inside of R or all outside of R.[51] For example, every generic convex polygon is a Delaunay polygon.

Theorem 21.17. *Every Delaunay polygon in the plane with at least four vertices has at least four extremal circles.*

[51]The name comes from Delaunay triangulations and the empty circle condition (see Section 14).

Proof. Let $Q = [x_1 \dots x_n] \subset \mathbb{R}^2$ be a polygon as in the theorem and let $Q' = [x'_1 \dots x'_n] \subset \mathbb{S}^2$ be a stereographic projection of Q onto a sphere. Since no four vertices in Q lie on a line or a circle, we conclude that no four vertices in Q' lie on a spherical circle, and thus on the same plane in $\mathbb{R}^2$. Since Q is a Delaunay polygon, there exists a circle around every edge which is then mapped into a circle on a sphere, such that all vertices of Q' are on one side of it. If you think of $Q' \subset \mathbb{R}^3$ as of a space polygon now, this implies that Q' is a generic weakly convex space polygon. Similarly, if two edges (x'_i, x'_{i+1}) and (x'_j, x'_{j+1}) intersect, then all four points lie in the plane and thus on a spherical circle, which implies that x_i, x_{i+1}, x_j, and x_{j+1} lie on a circle or a line. Since this is impossible by the assumption that polygon Q is generic, we conclude that Q' is simple. The result now follows from Theorem 21.16. □

While Theorem 21.17 is a direct extension of the four vertex theorem (Theorem 21.4), and Corollary 21.8, the analogue of Corollary 21.5 is less obvious (see Exercise 21.7).

Let $Q = [x_1 \dots x_n]$ be a simple polygon in the plane and let r_i, $1 \le i \le n$ denotes the radius of the circumscribed circle R_i as above. Define the *curvature* κ_i at x_i to be $1/r_i$ if x_i is convex and $-1/r_i$ if x_i is concave.

Corollary 21.18. *Let $Q = [x_1 \dots x_n] \subset \mathbb{R}^2$ be a coherent Delaunay polygon, $n \ge 4$. Then there are at least four sign changes in the cyclic sequence of the curvature values: $(\kappa_1 - \kappa_2, \kappa_2 - \kappa_3, \dots, \kappa_n - \kappa_1)$.*

Now, recall that all equilateral polygons are coherent. We conclude that Corollary 21.6 holds verbatim for all (not necessarily weakly convex) simple polygons:

Corollary 21.19. *Let $Q = [x_1 \dots x_n] \subset \mathbb{R}^2$ be an equilateral Delaunay polygon, $n \ge 4$. Denote by $\alpha_i = \angle x_{i-1} x_i x_{i+1}$ the interior angle in Q. Then there are at least four sign changes in the cyclic sequence $(\alpha_1 - \alpha_2, \alpha_2 - \alpha_3, \dots, \alpha_n - \alpha_1)$.*

Finally, let us give a non-convex analogue of Theorem 21.7. Note that the triangular faces of $P = \mathrm{conv}(Q)$ correspond to both empty and full extremal circles. Thus it is natural to consider circle numbers $s = s_+ + s_-$, $t = t_+ + t_-$ and $u = u_+ + u_-$ in this case.

Theorem 21.20. *Let $Q \subset \mathbb{R}^2$ be a coherent Delaunay polygon with at least four vertices. Denote by s, t and u the number of neighboring, disjoint and intermediate circles, respectively. Then*

$$s - t = 4, \quad s + t + u = 2n - 4.$$

Sketch of proof. Start as in the proof of Theorems 21.16 and 21.17, by considering triangulations T_1, T_2 on two sides of Q of the surface $S = \partial P$, where $P = \mathrm{conv}(Q)$. In each triangulation there are no interior vertices and triangles of three types: with zero, one and two edges in Q. These triangles correspond to disjoint, intermediate and neighboring circles, respectively. Since the dual graph to T_1 is a binary tree and the same is true for T_2, we can proceed as in the proof of Theorem 21.7 (compare Figure 21.3 and Figure 21.9). □

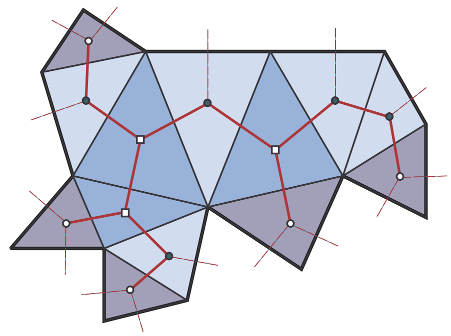

Figure 21.9. Triangles of three types in a triangulation.

21.8. **Further generalizations.** The (usual) four vertex theorem has a number of other interesting generalizations, some of them natural and some quite surprising. Here are some of them. First, for spherical curves one can define *spherical* vertices in a similar manner, by considering osculating spherical circles.

Theorem 21.21 (Spherical four vertex theorem)**.** *Every simple smooth curve C on a sphere $\mathbb{S}^2$ has at least four spherical vertices.*

The theorem has a straightforward discrete analogue which will be proved and generalized in the next section. Note that the (spherical) curvature can be strictly positive on a curve C in a hemisphere $\mathbb{S}^2_+$. The following result is an extension of the spherical four vertex theorem, proving that there are at least two (spherical) vertices with positive curvature and two vertices with negative curvature under a certain area condition.

Theorem 21.22 (Tennis ball theorem)**.** *Every simple smooth curve C on a sphere $\mathbb{S}^2$ which divides the area into two equal parts has at least four inflection points.*

The next result is an even stronger generalization, showing that the area condition is too restrictive.

Theorem 21.23 (Four inflection points theorem)**.** *Every simple smooth curve C on a sphere $\mathbb{S}^2$ which is not contained in a closed hemisphere has at least four inflection points.*

If one considers centrally symmetric curves, an even stronger result is known.

Theorem 21.24 (Möbius)**.** *Every centrally symmetric simple smooth curve C on a sphere $\mathbb{S}^2$ has at least six inflection points.*

Back to the plane, the next result improves the bound on the number of vertices.

Theorem 21.25. *Every smooth convex curve $C \subset \mathbb{R}^2$ intersecting a circle at least k times has at least k vertices. In particular, if C touches the smallest circumscribed circle at least m times, it has at least $2m$ vertices.*

Clearly, a circumscribed circle touches the curve in at least two points, so this result is stronger that the four vertex theorem. To see that the second part follows from the first part, take a circumscribed circle and shrink it by a small $\varepsilon > 0$. In fact, a circle in the theorem can be substituted with any smooth convex curve. Thus one can view Theorem 21.25 as another example of a "relative result", similar to Lemma 9.6 and Theorem 21.15. Further results of this type will be given in the next section.

Our next extension is an extension to non-simple curves. Think of a simple curve as the boundary of a 2-dimensional disk embedded into $\mathbb{R}^2$. We say that a curve $C \subset \mathbb{R}^2$ *bounds an immersed disc* if there exists a 2-dimensional disc D immersed into $\mathbb{R}^2$ with boundary $C = \partial D$.

Theorem 21.26. *Every smooth curve $C \subset \mathbb{R}^2$ which bounds an immersed disk has at least four vertices.*

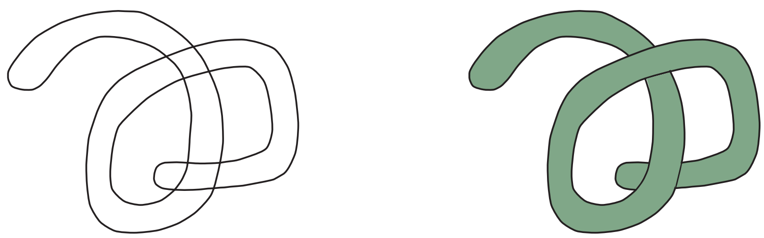

FIGURE 21.10. A smooth curve which bounds an immersed disk.

In a different direction, the four vertex theorem generalizes to contact numbers of higher order. For a point x on a smooth convex curve $C \subset \mathbb{R}^2$ we can consider an *osculating conic*, defined to have contact with C of order 4 at point x. We say that x is a *sextactic vertex* if the osculating conic has contact with C of order 5 at x.

Theorem 21.27 (Six vertex theorem)**.** *Every smooth convex curve has at least six sextactic vertices.*

In conclusion, let us mention that the four vertex theorem has a natural converse.

Theorem 21.28 (Converse four vertex theorem)**.** *Let $\kappa : \mathbb{S}^1 \to \mathbb{R}$ be a continuous function with at least two local maxima and two local minima. Then there is an embedding $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$ whose curvature at the point $\gamma(t)$ is equal to $\kappa(t)$.*

The results of this type are called *existence theorems* and play a special role later in the book. In fact, Theorems 35.4, 36.2, and 37.1 can be viewed as discrete generalizations of the convex case of Theorem 21.28.

21.9. **Exercises.**

Exercise 21.1. $\diamondsuit$ *a*) [1-] Deduce Corollary 21.5 from Theorem 21.4.
b) [1-] Show that the coherence condition in the corollary is necessary.
c) [1] Deduce Corollary 21.18 from Theorem 21.17.

Exercise 21.2. [1-] Let $Q = [v_1 \dots v_n] \subset \mathbb{R}^2$ be a convex polygon with equal angles. Suppose

$$|v_1v_2| \leq |v_2v_3| \leq \dots \leq |v_{n-1}v_n| \leq |v_nv_1|$$

Prove that Q is a regular polygon.

Exercise 21.3. $\Diamond$ [1] Use limit argument to deduce Theorem 21.1 in full generality from Theorem 21.17.

Exercise 21.4. [1+] The lines ℓ_i and ℓ_j are called *adjacent* if $|i-j| = 1$, i.e., they are spanned by adjacent edges. A circle R^*_{ijk}, $i < j < k$, is called *distant* if no two lines ℓ_i, ℓ_j, ℓ_k are adjacent, it is called *mediocre* if exactly one pair of lines is adjacent, and it is called *close* if two pairs of lines are adjacent. The circle R^*_{ijk} is called *clear* if it does not intersect any other lines, and it is called *crossing* if it intersects all other lines ℓ_m, $m \neq i, j, k$.

Let $Q \subset \mathbb{R}^2$ be a dually generic convex polygon with at least four vertices. Denote by s_+, t_+ and u_+ the number of full close, distant and mediocre crossing circles, respectively. Similarly, denote by s_-, t_- and u_- the number of close, distant and mediocre clear circles, respectively. Prove the following linear relations:

$$s_+ - t_+ = s_- - t_- = 2,$$

$$s_+ + t_+ + u_+ = s_- + t_- + u_- = n - 2.$$

Exercise 21.5. $\Diamond$ [1+] Find a dual version to Theorem 21.15, generalizing Corollary 21.13.

Exercise 21.6. [1] Show that when the number of vertices is odd, the dual evolute and orientation of its edges uniquely determine the convex polygon.

Exercise 21.7. $\Diamond$ [1] Deduce Corollary 21.18 from Theorem 21.17.

Exercise 21.8. $\Diamond$ [1] Use a limit argument to deduce Theorem 21.1 from Corollary 21.6 when the curve is convex. Similarly, use Corollary 21.19 to deduce Theorem 21.1 in full generality.

Exercise 21.9. [2-] Find and prove a discrete analogue of Theorem 21.21.

Exercise 21.10. [2-] Find and prove a discrete analogue of Theorem 21.22. Show that the result is false for self-intersecting curves.

Exercise 21.11. [2] Find and prove a discrete analogue of Theorem 21.27.

Exercise 21.12. [2] Recall the *cross-ratio* of an ordered 4-tuple of distinct numbers, defined as

$$[a, b, c, d] = \frac{(a-c)(b-d)}{(a-b)(c-d)}.$$

Let $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ be distinct real numbers, and let

$$\alpha_i = [x_i, x_{i+1}, x_{i+2}, x_{i+3}], \quad \beta_i = [y_i, y_{i+1}, y_{i+2}, y_{i+3}],$$

where, all indices are taken modulo n. Prove that a cyclic sequence $(\alpha_1 - \beta_1, \dots, \alpha_n - \beta_n)$ has at least four sign changes.

Exercise 21.13. *a*) [2-] Prove that every smooth convex curve C in the plane has at least three pairs of opposite points (points with parallel tangents) which have equal curvature.
b) [$*$] Find a discrete analogue of part *a*).

21.10. **Final remarks.** The four vertex theorem (Theorem 21.1) is a classical result in Differential Geometry, available in numerous textbooks and survey articles (see e.g., [Cher, Gug1]). It was proved by Mukhopadhyaya for convex curves (1909) and by A. Kneser in full generality (1912). Theorem 21.2 is due to H. Kneser (1922), the son of A. Kneser, and Theorem 21.3 is due to Bose (1932).

Interestingly, a related result of Möbius (Theorem 21.24) is much older (1852). For the history of the four vertex theorem, various extensions and references see [DGPV, Mus4, Ume, Weg2] and [OT2, Chapter 4].

Corollary 21.6 and Corollary 21.5 for obtuse polygons are due to S. Bilinski (1961, 1963). The general version of Corollary 21.5 is due to Musin (see [Mus1] for an introduction and an elementary proof). Theorem 21.7 can be viewed as the most general discrete Bose theorem, and is a variation on several known results (see e.g., Theorem 1.7 in [Mus4]). The proof of Theorem 21.7 is new in this form. It uses the cut locus idea in the René Thom's proof (1972) of the four vertex theorem (see [Ume]), subsequently repeatedly rediscovered and adapted to polygons (see [BanG, Weg2, Weg4]). The four edge theorem (Theorem 21.10) seems to be new. In computational geometry the cut locus is well studied under the name *medial axis* [AurK].

Theorem 21.15 and the proof in Subsection 21.5 are due to Tabachnikov [Tab4]. Theorems 21.16 and 21.17 are due to Sedykh [Sed] (a version of Corollary 21.19 was also discovered by Dahlberg). Interestingly, there is also a *two vertex theorem* for general Jordan curves [Hau].

Theorem 21.23 was first proved by Segre (1968), but a version of it was stated by Blaschke as an exercise in [Bla1]. Its corollary, the tennis ball theorem (Theorem 21.22) was discovered, aptly named and popularized by Arnold [Arn1, §20]. There is a great deal of literature on these results and their various generalizations (see [Arn3, pp. 99, 553] for a short survey and further references).

The first part of Theorem 21.25 goes back to Blaschke (1916) and Mukhopadhyaya (1931) (see also [Bla3]). It was rediscovered several times with the definitive version due to Jackson [Jac][52]. The second part is an easy corollary of the first and was popularized in [Oss]). Similarly, the six vertex theorem (Theorem 21.27) was proved by Mukhopadhyaya in his original paper (1909), and was repeatedly rediscovered. For more on the history and advanced generalizations, including common generalizations with the Möbius theorem and connections to classical Cayley's results, see [ThoU].

Theorem 21.26 is due to Pinkall [Pink] who proved it in the generality of all immersed surfaces. The converse of the four vertex theorem (Theorem 21.28) was proved by Gluck for convex curves (1971), and by Dahlberg in full generality (1997, published posthumously in 2005, see [DGPV]). We should mention that much of the difficulty in the proof is analytical and we are not aware of a nontrivial discrete version.

[52]References to older results, mostly in German, can be found in the AMS Math. Reviews on [Jac].

22. Relative geometry of convex polygons

Here we continue our study of four vertex theorems, shifting in the direction of their relative geometry. We limit our scope and concentrate only on results which will be used later on, in the proof of rigidity results of convex polyhedra.

22.1. **The Legendre–Cauchy lemma.** We start with the following classical result which can be viewed as *relative versions* of the four vertex theorem (Theorem 21.4).

Theorem 22.1 (Legendre–Cauchy lemma)**.** *Let $Q = [v_1 \dots v_n]$ and $Q' = [v'_1 \dots v'_n]$ be two convex polygons in the plane with equal corresponding edge lengths: $|v_i v_{i+1}| = |v'_i v'_{i+1}|$. Denote by $\alpha_i = \angle\, v_{i-1} v_i v_{i+1}$ and $\beta_i = \angle\, v'_{i-1} v'_i v'_{i+1}$ the angles in the polygons, where $1 \le i \le n$ and the indices are taken modulo n. Then either there are at least four sign changes in the cyclic sequence $(\alpha_1 - \beta_1, \dots, \alpha_n - \beta_n)$, or the sequence is zero.*

To see the connection to the four vertex theorem, consider the case of equilateral polygons $Q = [v_1 v_2 \dots v_n]$ and $Q' = [v_2 \dots v_n v_1]$. The theorem implies in this case that either all corresponding angles are equal, or there are at least four sign changes in the sequence $(\alpha_1 - \alpha_2, \dots, \alpha_n - \alpha_1)$. This is a minor variation on Corollary 21.6.

Let us immediately state a spherical version of the theorem, which will prove useful in the proof of the Cauchy theorem (see Section 26).

Theorem 22.2 (Spherical Legendre–Cauchy lemma)**.** *Let $Q = [x_1 \dots x_n]$ and $Q' = [x'_1 \dots x'_n]$ be two spherical convex polygons on a hemisphere $\mathbb{S}_+$ with equal corresponding edge lengths: $|x_i x_{i+1}|_{\mathbb{S}^2} = |x'_i x'_{i+1}|_{\mathbb{S}^2}$. Denote by $\alpha_i = \sphericalangle\, v_{i-1} v_i v_{i+1}$ and $\beta_i = \sphericalangle\, v'_{i-1} v'_i v'_{i+1}$ the spherical angles in the polygons, $1 \le i \le n$. Then either there are at least four sign changes in the cyclic sequence $(\alpha_1 - \beta_1, \dots, \alpha_n - \beta_n)$, or the sequence is zero.*

One can think of this result as a generalization of Theorem 22.1 since the plane polygons are the limits of spherical polygons, as the radius of a sphere tends to infinity. Later in this section we prove both result by the same general argument.

Note that the condition that both polygons lie inside a hemisphere $\mathbb{S}_+$ is necessary, since otherwise one can find two spherical triangles with equal corresponding sides and all angles of the first strictly smaller than the corresponding angles of the second triangle. For example, take any small spherical triangle Δ and its complement $\Delta' = \mathbb{S}^2 \smallsetminus \Delta$. The angles of Δ are $< \pi$, while the angles of Δ' are $> \pi$. Similarly, as can be seen already for two quadrilaterals in Figure 22.1, the result is false for non-convex polygons.

22.2. **Making errors at all the right places.** The proof of the Legendre-Cauchy lemma is elementary, but delicate at one point. The original proof, which by now has become standard, contained a famous error which eluded discovery for nearly a century and has led to a number of interesting (and correct) results. Thus we start with the original proof, then point out the famous error, and then correct it. The reader might want to pay extra attention to the figures and think it over before learning the answer.

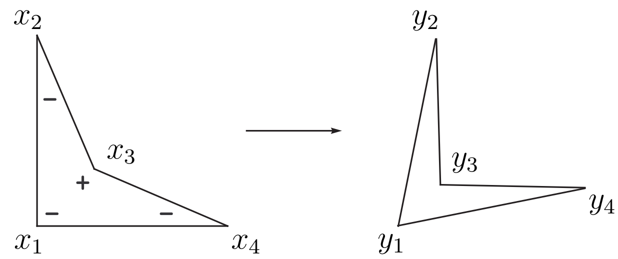

FIGURE 22.1. Non-convex polygons with two sign changes.

We present the proof of only the spherical Legendre–Cauchy lemma (Theorem 22.2). The plane version (Theorem 22.1) follows by essentially the same argument. Throughout the proof we will treat spherical geometry almost the same as plane geometry, relying on reader's intuition. This is not where the mistake lies.[53]

Proof of the Legendre–Cauchy lemma. Think of the signs as being placed in the vertices of the polygon Q. Now suppose a spherical polygon has exactly two sign changes. Then there exists a diagonal (y, z) such that on one side of it there are only $(+)$ and (0) labels, with at least one $(+)$, and on the other side only $(-)$ and (0) labels, with at least one $(-)$, as in Figure 22.2.

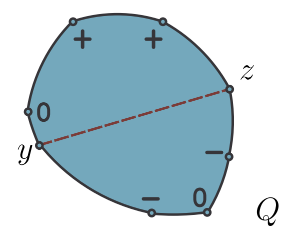

FIGURE 22.2. A spherical polygon Q with exactly two sign changes.

The idea of the proof is to show that if the edge lengths are the same and angles on one side are increasing, the length of diagonal (y, z) is also increasing. Similarly, if angles on the other side are decreasing, the length of (y, z) is also decreasing, a contradiction. Formally, we prove the following result.

Lemma 22.3 (Arm lemma)**.** *Let $X = [x_1 x_2 \ldots x_n]$ and $X' = [x'_1 x'_2 \ldots x'_n]$ be two convex spherical polygons in the hemisphere $\mathbb{S}^2_+$, such that:*

$$\sphericalangle\, x_1x_2x_3 \le \sphericalangle\, x'_1x'_2x'_3\,, \ \sphericalangle\, x_2x_3x_4 \le \sphericalangle\, x'_2x'_3x'_4\,, \ \ldots\ , \sphericalangle\, x_{n-2}x_{n-1}x_n \le \sphericalangle\, x'_{n-2}x'_{n-1}x'_n\,,$$

and

$$|x_1x_2| = |x'_1x'_2|, \quad |x_2x_3| = |x'_2x'_3|, \quad \ldots\ , |x_{n-1}x_n| = |x'_{n-1}x'_n|.$$

[53]In fact, if one uses the spherical law of cosines (see Appendix 41.2) in place of the usual law of cosines that was implicitly used in the proof, one proof easily translates into the other.

Then $|x_1x_n| \le |x'_{n-1}x'_n|$, *and the equality holds only if all inequalities between the angles are equalities. Similarly, if* X *and* X' *are plane convex polygons with equal corresponding edge lengths and inequalities on the angles, the same conclusion holds.*

Let us first deduce the Legendre–Cauchy lemma from the arm lemma. Let X be a polygon on the side of the diagonal (y, z) in Q with positive and zero signs, and let X' be the corresponding polygon in Q', with the diagonal (y', z') as one of its sides. Since not all labels are zero, by the arm lemma we have $|yz| < |y'z'|$.

Similarly, reverse the role of X and X' for the other side of the diagonal. Let X' be the polygon on the side of the diagonal (y, z) in Q with negative and zero signs, and let X be the corresponding polygon in Q. By the arm lemma we have $|yz| > |y'z'|$, a contradiction. Therefore, having exactly two sign changes is impossible.

It remains to prove that zero sign changes is impossible unless all labels are zero. Clearly, if all labels are $(+)$ or (0), with at least one $(+)$, we can apply the arm lemma to any edge to get a contradiction.[54] Use the same argument for $(-)$ and (0) labels. This completes the proof of the Legendre-Cauchy lemma. $\square$

22.3. **An incorrect proof of the arm lemma (be vigilant!)** Use induction on the number n of sides of the polygons. The claim is clear for $n = 3$, when X and X' are spherical triangles in the upper hemisphere. It follows easily from the cosine law on a sphere (see Proposition 41.3 in the Appendix), and the observation that all angles are $\le \pi$ in this case.

Suppose now that $n > 3$, and assume we have an equality between some of the angles: $\sphericalangle x_{i-1}x_ix_{i+1} = \sphericalangle x'_{i-1}x'_ix'_{i+1}$, for some $1 < i < n$. We can simply remove triangles $\Delta = (x_{i-1}x_ix_{i+1})$ and $\Delta' = (x'_{i-1}x'_ix'_{i+1})$ from X and X' and consider the remaining polygons (see Figure 22.3). Note that the side lengths and the angle between them determine the triangles, so $\Delta = \Delta'$. Thus, the remaining $(n-1)$-gons satisfy conditions of the lemma, and the claim follows by the inductive assumption in this case.

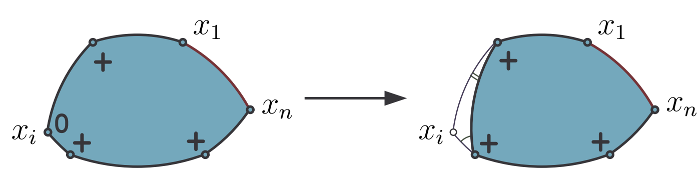

FIGURE 22.3. Removing a zero label in vertex x_i of a polygon.

Now suppose all inequalities between the corresponding angles are strict. Start increasing angle $\sphericalangle x_1x_2x_3$ until the angle is equal to the desired value $\sphericalangle x'_1x'_2x'_3$ (see Figure 22.4). Denote by $Y = [y_1x_2x_3 \dots x_n]$ the resulting polygon. In a triangle $(x_1x_2x_3)$, the lengths of the side (x_1x_2) and the diagonal (x_2x_n) remain the same, so

[54] Of course, in the plane we can use equality of the angle sums of X and X', which removes the need for the arm lemma in this case.

the length $|x_1x_n|$ increases: $|x_1x_n| < |y_1x_n|$. Let us compare polygons Y and X'. By construction, $\sphericalangle\, y_1x_2x_3 = \sphericalangle\, x_1'x_2'x_3'$, by the previous argument $|y_1x_n| \leq |x_1'x_n'|$. This proves the induction step and establishes the arm lemma. $\square$

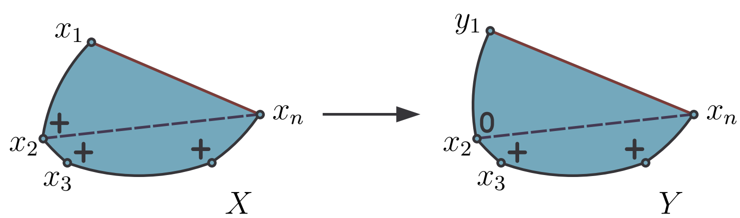

FIGURE 22.4. Increasing the angle $\sphericalangle\, x_1x_2x_3$ in a spherical polygon.

22.4. **An explanation, a discussion, and an idea of correction.** Now, before we show the mistake in the proof let us point out why the mistake must exist. Note that throughout the proof we occasionally state and always implicitly assume that the spherical polygons are convex. This, of course, is essential to the proof: without convexity the claim in the arm lemma claim is false, as shown in Figure 22.5. However, in the inductive proof above we never used the fact that X' is convex. Since the claim is wrong for non-convex polygons, the proof must also be incorrect.

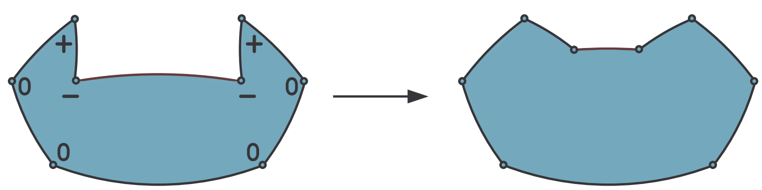

FIGURE 22.5. Diagonal length $|x_1x_n|$ may decrease in a non-convex polygon.

To see the error we need to get into details of the inductive argument. The problem arises after we increase the angle $\sphericalangle\, x_1x_2x_3$ (see Figure 22.4). Unfortunately, the resulting polygon Y does not have to be convex, and the induction step fails. In Figure 22.6 we show a convex spherical polygon which after increase of $\sphericalangle\, x_1x_2x_3$ becomes non-convex, and after subsequent increase of $\sphericalangle\, x_nx_{n-1}x_{n-2}$ becomes convex once again. By the symmetry, there is no order in which these two angles can be increased without producing a non-convex polygon in between.

Now that we know the mistake, correcting it is not difficult. Here is a natural way, somewhat involved and educational at the same time. We start to increase the angles in a convex polygon $X = [x_1x_2 \ldots x_n]$. Observe that we can increase the angle in x_2 all the way until vertex x_1 lies on a line $(x_{n-1}x_n)$. After that, further increase may give a non-convex polygon, so we do not do it. Now increase the angle in x_{n-1} until it lies on a line (x_1x_2). Then increase the angle in x_2, etc. Keep increasing until one of the angles reaches the desired value. Since all intermediate polygons are convex by

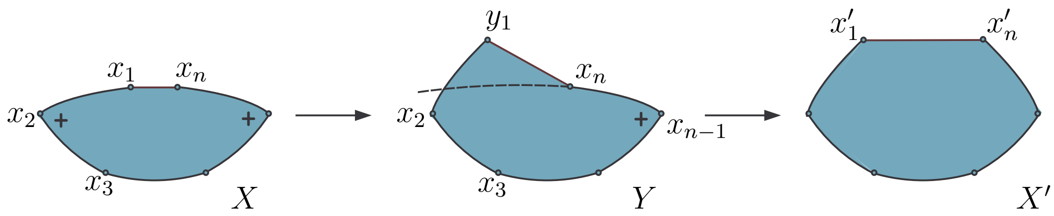

FIGURE 22.6. Increasing angle $\sphericalangle x_1x_2x_3$ may result in a non-convex polygon.

construction, the distance $|x_1x_n|$ will only increase by the argument as in the proof above.

The trouble with this argument is that it requires an unbounded number of steps (as long as this number is finite it is not really a problem), and that it takes some work to show that the increase in the angles does not converge to a value lower than necessary (see Figure 22.7). This latter part is trickier and uses convexity of X'; it is left to the reader. Once all the details are set and done, one would really want a different proof of the arm lemma, somewhat simpler even if perhaps not as straightforward (see also Subsection 23.5).

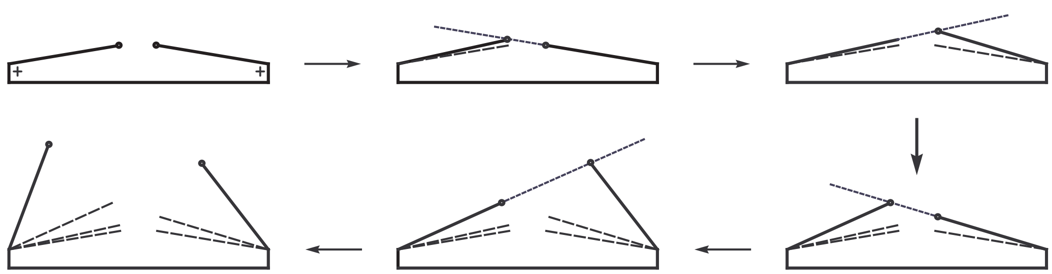

FIGURE 22.7. The iterative process of increasing two angles.

22.5. **All's well that ends well.** Based on the first impression one would assume that the arm lemma is obvious. After some thinking and working out the iterative argument above one can conclude that it is inherently complicated. Well, one would be wrong again. Here is a simple and ingenious inductive proof of the arm lemma.

Proof of the arm lemma (*for real now!*). Use induction on n. As in the argument above, we can assume that all angles of X as in the lemma are strictly increasing. The base $n = 3$ for triangles is also established. The argument in the original proof fails if there exists a polygon $Y = [y_1x_2 \ldots x_n]$ where $\sphericalangle x'_1x'_2x'_3 > \sphericalangle y_1x_2x_3 \geq \sphericalangle x_1x_2x_3$, and point y_1 lies on the $(x_{n-1}x_n)$ line. Apply the base of induction to go from X to Y: in the triangle $(x_1x_2x_3)$ we have $|y_1x_n| \geq |x_1x_n|$.

Think of Y as $(n-1)$-gon with one side comprised of two: $|y_1x_{n-1}| = |y_1x_n| + |x_{n-1}x_n|$. Let $Z = [x'_1x'_2 \ldots x'_{n-1}]$ be a $(n-1)$-gon with sides and angles as in the lemma. By inductive assumption, going from Y to Z, we have $|x'_1x'_{n-1}| \geq |y_1x_{n-1}|$.

Now take point z_n on the edge (x'_1, x'_{n-1}), such that $|z_n x'_{n-1}| = |x_{n-1} x_n|$, and think of $Z = [x'_1 x'_2 \ldots x'_{n-1} z_n]$ as of an n-gon. Clearly, $|x'_1 x'_{n-1}| = |x'_1 z_n| + |z_n x'_{n-1}|$, and

$$|x'_1 z_n| \,=\, |x'_1 x'_{n-1}| \,-\, |z_n x'_{n-1}| \,\geq\, |y_1 x_{n-1}| \,-\, |x_n x_{n-1}| \,=\, |y_1 x_n|$$

Since X' is convex, after comparing it with Z we have $\sphericalangle\, x'_{n-2} x'_{n-1} x'_n \geq \sphericalangle\, x'_{n-2} x'_{n-1} z'_n$. Thus the angle in x'_{n-1} should be increased. Apply the base of induction to go from Z to X': in the triangle $(x'_1 x'_{n-1} z_n)$ we have $|x'_1 x'_n| > |x'_1 z_n|$. Putting everything together we have:

$$|x'_1 x'_n| \,>\, |x'_1 z_n| \,\geq\, |y_1 x_n| \,\geq\, |x_1 x_n|\,,$$

as desired. $\square$

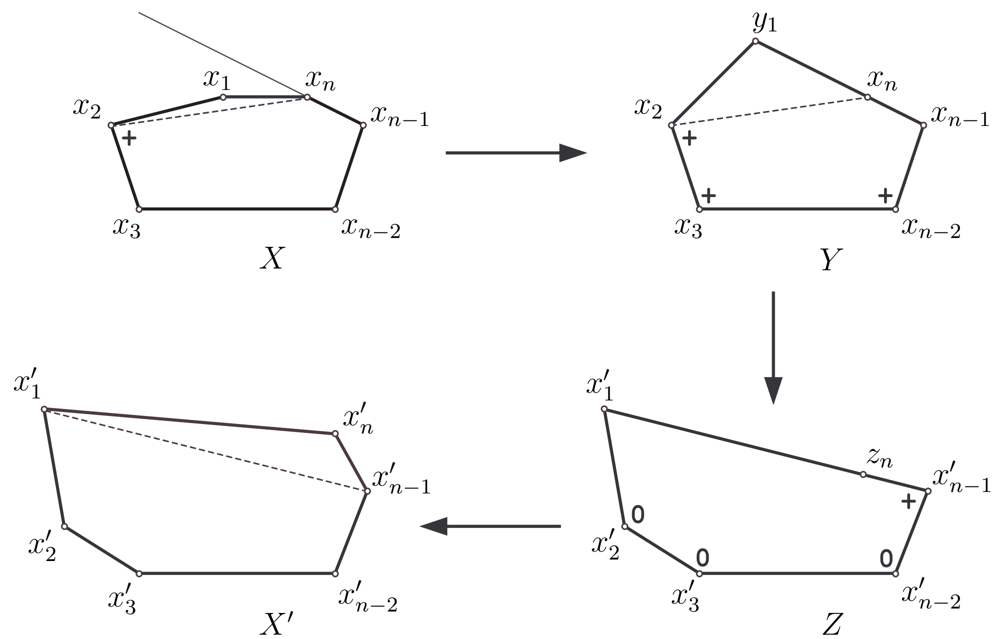

FIGURE 22.8. Transformation of polygons $X \to Y \to Z \to X'$.

In Figure 22.8 we show the process of moving from X to X', in the plane for simplicity. Note that the angle at x_{n-1} here decreases when going from Y to Z; the interesting feature of the proof is that this does not affect the conclusion.

22.6. **The Alexandrov lemma.** Just like one can view the Legendre–Cauchy lemma (Theorem 22.1) as the relative version of the discrete four vertex theorem (Theorem 21.4), one can ask about the relative version of the dual discrete four vertex theorem (Theorem 21.10). The following result is an unexpected generalization.

Recall that two convex polygons are called *parallel* if they have parallel edges. Of course, parallel polygons have equal corresponding angles.

Theorem 22.4 (Alexandrov lemma)**.** *Let $X = [x_1 \dots x_n]$ and $Y = [y_1 \dots y_n]$ be two parallel convex polygons in the plane. Denote by a_i and b_i the edge lengths in X and Y, respectively. Suppose neither X fits inside Y, nor Y fits inside X by a translation. Then either there are at least four sign changes in the cyclic sequence $(a_1-b_1, \dots, a_n-b_n)$, or the sequence is zero.*

The condition that neither polygon fits inside another may seem unusual, as there is no such condition in Theorem 22.1. In fact, without this condition the claim is false (see Figure 22.9).

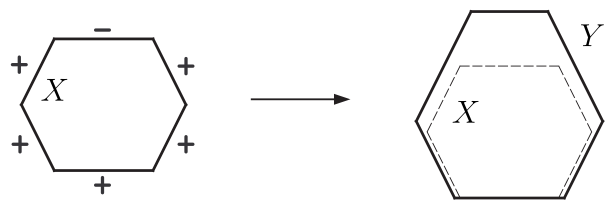

FIGURE 22.9. Parallel hexagons with exactly two sign changes.

The connection to the dual discrete four vertex theorem is similar to the case of the Legendre–Cauchy lemma. Consider an equiangular polygon $X = [x_1 \dots x_n]$ and a polygon Y obtained from X by a clockwise $2\pi/n$ rotation. If $X = Y$, the sequence in the theorem is a zero sequence. Otherwise, since $\text{area}(X) = \text{area}(Y)$, neither polygon fits inside another, and the cyclic sequence $(a_1 - a_2, \dots, a_n - a_1)$ has at least four sign changes. Thus we obtain Corollary 21.12. Here is a more general corollary from Alexandrov's lemma.

Corollary 22.5. *Let $X = [x_1 \dots x_n]$ and $Y = [y_1 \dots y_n]$ be two parallel convex polygons in the plane. Denote by a_i and b_i the edge lengths in X and Y, respectively. Suppose either of the following conditions holds:*

(i) $\text{area}(X) = \text{area}(Y)$,
(ii) $\text{perimeter}(X) = \text{perimeter}(Y)$.

Then either there are at least four sign changes in the cyclic sequence $(a_1-b_1, \dots, a_n-b_n)$, or the sequence is zero.

The follows from the Alexandrov lemma since under either of these conditions neither polygon fits inside another (for the perimeter see part a of Exercise 7.11). Of course, one can replace the area and perimeter in $(i), (ii)$ with any other function f of the polygons which satisfies: $f(X) \le f(Y)$ for all $X \subseteq Y$, where the equality holds only if $X = Y$. For example, take any polynomial of $\text{area}(X)$ and $\text{perimeter}(X)$ with positive coefficients. Alternatively, one can use two asymmetric conditions, such as, e.g., $\text{diam}(X) < \text{diam}(Y)$ and $\text{width}(X) > \text{width}(Y)$.

The proof of the Alexandrov lemma is elementary but tedious. Instead, we prove only the second part of the corollary.

Proof of part (ii) *of Corollary 22.5.* Let $\boldsymbol{e}_i = \overrightarrow{x_i x_{i+1}}$ be an edge vector in X, and let $\boldsymbol{u}_i = \boldsymbol{e}_i / a_i$ be a unit vector in direction $\boldsymbol{e}_i$, $1 \le i \le n$. Since these unit vectors are the same for Y, we have

$$\sum_{i=1}^{n} a_i\,\boldsymbol{u}_i \;=\; \sum_{i=1}^{n} b_i\,\boldsymbol{u}_i \;=\; \boldsymbol{0}.$$

Therefore, for every vector $\boldsymbol{w}$, we obtain

$$\sum_{i=1}^{n} (a_i - b_i)(\boldsymbol{u}_i - \boldsymbol{w}) \;=\; \sum_{i=1}^{n} (a_i - b_i)\,\boldsymbol{u}_i \;-\; \sum_{i=1}^{n} (a_i - b_i)\,\boldsymbol{w} \;=\; \boldsymbol{0},$$

where the second sum is zero since X and Y have equal perimeters. The rest of the proof follows the same argument as in the proof of Theorem 21.15. Clearly, the cyclic sequence $(a_1 - b_1, \ldots, a_n - b_n)$ is either zero or has at least two sign changes, since the sum of its elements is zero. Suppose there are exactly two sign changes. Choose $\boldsymbol{w}$ so that vectors $\boldsymbol{v}_i = \boldsymbol{e}_i - \boldsymbol{w}$ with positive and negative coefficients $(a_i - b_i)$ lie on different sides of the line. But then the sum on the l.h.s. in the equation above cannot be zero, a contradiction. □

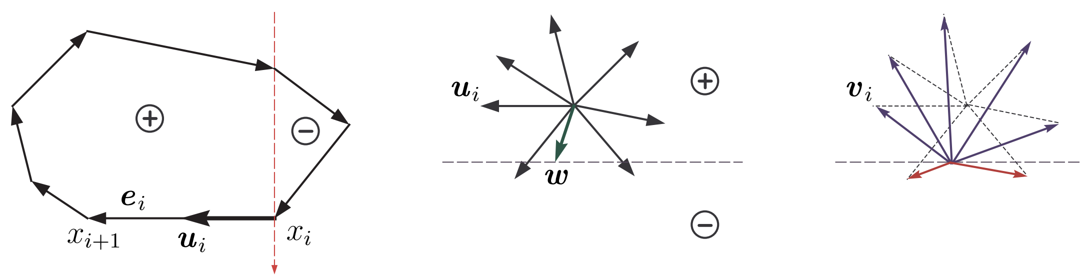

FIGURE 22.10. Edge vectors of polygon X (dotted lines separate vectors with positive and negative signs).

22.7. **Exercises.**

Exercise 22.1. ◊ (*Alexandrov's local lemma*) ◊ For a polygon $P \subset \mathbb{R}^3$, denote by ℓ_i the edge lengths and by $\boldsymbol{u}_i$ the unit outer normals to edges, where $i \in [n]$. Write P as the intersection of halfplanes $\langle \boldsymbol{x}, \boldsymbol{u}_i \rangle \le h_i$, for some $h_i \in \mathbb{R}$.
a) [1-] Prove that $\ell_1\,\boldsymbol{u}_1 + \ldots + \ell_n\,\boldsymbol{u}_n = 0$.
b) [1-] Prove that $2\,\mathrm{area}(P) = \ell_1 h_1 + \ldots + \ell_n h_n$.
c) [1] Suppose $\{P_t\}$ is a continuous deformation of P which preserves the area. Prove that all edge lengths ℓ_i cannot be increasing.
d) [1] Furthermore, prove that the sequence $(\ell_1', \ldots, \ell_n')$ of derivatives is either zero or has at least four sign changes. Note that this is an immediate consequence of Corollary 22.5, part (*i*).

Exercise 22.2. ◊ [1+] Prove Alexandrov's lemma (Theorem 22.4).

Exercise 22.3. (*Extended arm lemma*) *a*) [2-] Let $X = [x_1 x_2 \ldots x_n] \subset \mathbb{R}^2$ be a convex polygon, and let $X' = [x_1' x_2' \ldots x_n']$ be a (possibly non-convex) polygon such that

$$|x_1x_2| = |x_1'x_2'|, \;\;\ldots\;\; , |x_{n-1}x_n| = |x_{n-1}'x_n'|$$

and

$$|\pi - \angle x_1x_2x_3| \geq |\pi - \angle x'_1x'_2x'_3|, \ \ldots \ ,|\pi - \angle x_{n-2}x_{n-1}x_n| \geq |\pi - \angle x'_{n-2}x'_{n-1}x'_n|.$$

Then $|x_1x_n| \leq |x'_{n-1}x'_n|$.
b) [1+] What happens for $X \subset \mathbb{R}^3$?

Exercise 22.4. (*Line development*) *a*) [2-] Let $P \subset \mathbb{R}^3$ be a convex polytope and let $A \subset \partial P$ be a convex polygon on the surface of P, defined as a polygonal region with all interior angles satisfying $0 \leq \alpha_i \leq \pi$. Let $Q = \partial A$. Unfold the faces of F containing Q onto a plane, starting with any one of them, in the order of intersection with Q (note that, generally speaking, Q might go though the same facet more than once). Prove the unfolding of Q is not self-intersecting.[55]
b) [1] Let $P \subset \mathbb{R}^3$ be a convex polytope and let L be a plane which does not contain vertices of P. Consider the curve $C = L \cap \partial P$. Prove the unfolding of C is not self-intersecting.

22.8. **Final remarks.** While the arm lemma (Lemma 22.3) seem to be due to Legendre, it is rarely attributed to him [Sab6]. It became famous in Cauchy's original proof of the rigidity of convex polytopes (see Section 26). In any case, the mistake was discovered and corrected only by Steinitz in the 1920's (see Steinitz's proof in [Lyu]). The mistake is so subtle and occurs in such an 'obvious' claim, a number of textbooks continue to repeat it until this day. Since Steinitz, a number of correct proofs of the arm lemma have been proposed, some simpler and more elegant than others (compare the proofs in [A2, Ber1, Hada, Lyu, Sab6, SchoZ, Sto] and try not to be awed). The concise (and correct) proof of the arm lemma given in Subsection 22.5 was discovered in [SchoZ] (see also [AigZ, §11]). By now there are a number of papers completely dedicated to the arm lemma and its generalization (see [Sab6, Schl4] and references in [Con5]). We should mention here that the Pogorelov lemma (Lemma 28.5) can be viewed as another variation on the arm lemma.

Let us note that the arm lemma is more transparent and easier to prove in the plane than on a sphere (see e.g. the first proof in [SchoZ]), thus several authors gave a correct proof of a plane version, and then mislead the reader by saying that the proof in the spherical case is "almost the same". The reader should be careful with such claims.

The proof of the Alexandrov lemma (Theorem 22.4) is elementary and can be found in [A2, §6.1] and [Lyu].[56] The lemma is used in Section 36 in the elementary proof of the Minkowski theorem (Theorem 36.2) in $\mathbb{R}^3$ (see also Exercise 36.4). The elegant proof of the infinitesimal analogue of part (*i*) is given in Exercise 22.1 (see [A2, §9.1] and a concise presentation in Alexandrov's original 1937 article). The proof of part (*ii*) of Corollary 22.5 is new.

[55]If one imagines rolling a die of shape P on a plane, the unfolding of Q is the trace left by Q. See Section 40 for more on unfoldings.

[56]A simpler proof was recently found by Günter Rote (personal communication).

23. Global invariants of curves

In this section we concentrate on the "local move connectivity" method which we explored earlier in a very different context (see Section 17). We show how one can deform polygons and prove certain relations based on their invariance under transitions. Among other things, we sketch how this can be used to prove one of the *four vertex theorem* extensions (see Section 21) and outline our own proof of the *square peg problem* (see Section 5). We also give an unexpected application to *halving lines*, an important subject, which we do not explore elsewhere in the book.

23.1. **Counting the intersections.** The following result is elementary and can be proved in a number of ways. We present three different short proofs which give a good illustration of three flavors of proofs in polyhedral geometry.

Let $X = [x_1 \dots x_n] \subset \mathbb{R}^2$ be a plane, possibly self-intersecting polygon, and let ℓ_i be the line spanned by the edge $e_i = (x_i, x_{i+1})$, $1 \le i \le n$. We say that X is *generic* if no vertex x_r lies on a line ℓ_i, for all $i \ne r-1, r$. Denote by $a(X)$ the number of intersections of lines ℓ_i and edges e_j, $i \ne j, j \pm 1$, where $1 \le i, j \le n$. Note that when X is not simple, the intersections of edges e_i and e_j are counted twice: once as $\ell_i \cap e_j$ and once as $e_i \cap \ell_j$.

Theorem 23.1. *Let $X = [x_1 \dots x_n] \subset \mathbb{R}^2$ be a generic plane polygon. Then the number $a(X)$ of intersections of lines ℓ_i and edges e_j is even.*

For convex polygons we have $a(X) = 0$ and the theorem is trivial. It is less trivial in the general case. We first present three different proofs of the theorem and then continue in the next subsections with two generalizations of the result.

First proof. Orient X counterclockwise. An edge e_i of X is called *convex* if the edges e_{i-1} and e_{i+1} lie on the left of line ℓ_i. Similarly, edge e_i is *concave* if e_{i-1} and e_{i+1} lie on the right of ℓ_i. Alternatively, we say that e_i is a *rightward inflection edge* if e_{i-1} lies on the left and e_{i+1} lies on the right of ℓ_i. Finally, edge e_i is a *leftward inflection edge* if e_{i-1} lies on the right and e_{i+1} lies on the left of ℓ_i. Clearly, the leftward and rightward inflection edges alternate. Thus the number of inflection edges is always even.

Observe that every line ℓ_i intersects edges in X an even number of times when e_i is convex or concave, and an odd number of times when e_i is an inflection edge (see Figure 23.1). Thus the total number of intersections $a(X)$ is even. $\square$

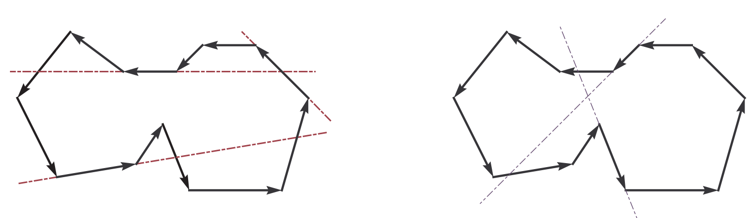

Figure 23.1. Lines through convex, concave and inflection edges.

Second proof. Since the number of intersections of two edges of X is counted twice in $a(X)$, it suffices to count the remaining intersections. Orient X counterclockwise and denote by ℓ_i' and ℓ_i'' two rays spanned by edge e_i, along and against orientation, starting at x_i and x_{i+1}, respectively. Observe that rays ℓ_i'' and ℓ_{i-1}' start at the same vertex x_i and separate the plane into two parts. Therefore, for every i the total number of intersections of ℓ_i'' and ℓ_{i-1}' and edges in Q is always even (see Figure 23.2). Summing these numbers, we conclude that $a(X)$ is also even. □

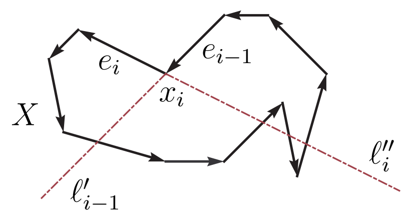

FIGURE 23.2. Rays ℓ_i'' and ℓ_{i-1}' intersect an even number of edges in X.

Third proof. Let us deform $X = [x_1 \ldots x_n]$ into a convex polygon $Y = [y_1 \ldots y_n]$, by moving one vertex a time: first move linearly x_1 into y_1, then move linearly x_2 into y_2, etc. By choosing y_i in general position the resulting continuous family of polygons is generic except at a finite number of *elementary transitions*. There are four types of these transitions as shown in Figure 23.3. In transitions of the first and fourth type the number of intersections $a(X)$ changes by two, and in transitions of the second and third type $a(X)$ is unchanged. This immediately implies the result. □

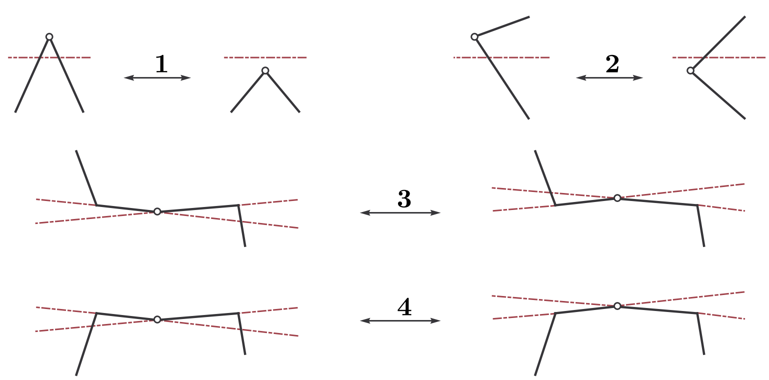

FIGURE 23.3. Four types of transitions.

23.2. **Post-proof analysis.** The best things about having several proofs of the same result is the ability to make a relative comparison of their strengths and weaknesses when it comes to generalizations. This can often shed some light on the nature of the result.

(1) The first proof is a global counting argument, suggesting that the key to the theorem is the even number of inflection edges. In fact, this kind of argument works in other interesting cases, and has analogues in higher dimensions as well (see below).

(2) The second proof is the shortest and the most ingenious of the three. It is fundamentally a local counting argument, which suggests that the key to the theorem is a local structure of lines around vertices. Of course, in other cases there is little hope of having this kind of argument.

(3) The third proof is the most general approach of all. As we shall see later in this section, it is widely applicable whenever one tries to prove a "global" enumerative statement about general curves. On the other hand, the "local moves" approach gives the least insight into the nature of the result. It is essentially a verification technique which gives no hints to potential generalizations.

To underscore the point in (1), let us present the following simple 3-dimensional result motivated by the proof. A generalization to higher dimensions can be obtained in a similar way.

Let $X = [x_1 \dots x_n] \subset \mathbb{R}^3$ be a space polygon. We say that X is *generic* if no three vertices lie on the same line and no four vertices lie on the same plane. Denote by L_i the plane spanned by edges e_{i-1} and e_i, where $e_i = (x_i, x_{i+1})$. We say that L_i is an *inflection plane* if edges e_{i-2} and e_{i+1} lie on the different sides of L_i.

Proposition 23.2. *The number of inflection planes of a generic space n-gon in $\mathbb{R}^3$ has same parity as n, for all $n \geq 4$.*

In particular, when n is odd, the proposition implies that there is at least one inflection plane (see Figure 23.4).

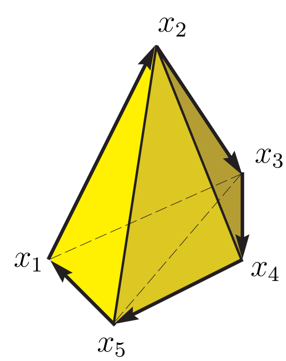

FIGURE 23.4. A space pentagon (drawn on the surface of a hexahedron) with a unique inflection plane $L_5 = (x_4 x_5 x_1)$.

Proof. Let $X = [x_1 \dots x_n] \subset \mathbb{R}^3$ be a generic space polygon. Denote by $\boldsymbol{e}_i = \overrightarrow{x_i x_{i+1}}$ the edge vectors of the polygon, $1 \leq i \leq n$, and let $\varepsilon_i \in \{\pm 1\}$ be the sign of the determinant $\det(\boldsymbol{e}_{i-1}, \boldsymbol{e}_i, \boldsymbol{e}_{i+1})$. Let $\nu_i = \varepsilon_{i-1}\varepsilon_i$, and observe that L_i is an inflection plane if and only if $\nu_i = 1$. Clearly,

$$\prod_{i=1}^{n} \nu_i \, = \, \prod_{i=1}^{n} \varepsilon_{i-1}\varepsilon_i \, = \, \left(\prod_{i=1}^{n} \varepsilon_i\right)^2 \, = \, 1,$$

which implies that the number of planes L_i that are not inflection planes, is even. $\square$

23.3. **Counting double supporting lines.** Let $X = [x_1 \ldots x_n] \subset \mathbb{R}^2$ be a plane polygon. Denote by e_i the edge (x_i, x_{i+1}) and by ℓ_{ij} the line through x_i and x_j. We say that X is *generic* if no three vertices lie on a line and no three edges intersect. Recall that e_i is called an *inflection edge* if edges e_{i-1} and e_{i+1} lie on different sides of line $\ell_{i,i+1}$. It is easy to see that the number of inflection edges is always even.

Line $\ell \subset \mathbb{R}^2$ is called *supporting* at x_i if $x_i \in \ell$ and both edges e_i and e_{i+1} lie on the same sides of ℓ. Line ℓ_{ij} through non-adjacent vertices x_i and x_j is called *double supporting* if it supporting at x_i and x_j. Observe that there are two types of double supporting lines depending on the direction of the edges adjacent to x_i and x_j. We say that ℓ_{ij} is *exterior* if all four edges e_{i-1}, e_i, e_{j-1} and e_j lie on the same side of ℓ_{ij}. Otherwise, if edges e_{i-1}, e_i and e_{j-1}, e_j lie on different sides of ℓ_{ij}, we say that ℓ_{ij} is *interior* (see Figure 23.5). Finally, a *crossing* is a pair of intersecting edges $e_i, e_j \in X$.

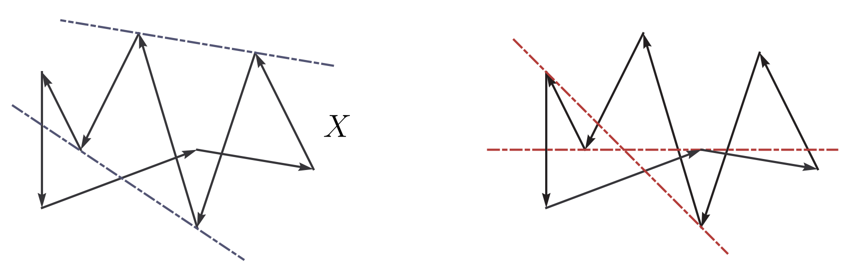

FIGURE 23.5. An example of two exterior and two interior lines in a polygon.

Theorem 23.3 (Fabricius-Bjerre's formula)**.** *Let $X \subset \mathbb{R}^2$ be a generic plane polygon. Let $a(X)$ be the number of inflection edges, let $c(X)$ be the number of crossings, and let $t_0(X)$ and $t_1(X)$ the number of exterior and interior double supporting lines, respectively. Then*

$$t_0(X) \, - \, t_1(X) \, = \, c(X) \, + \, \frac{a(X)}{2}\,.$$

For example, when X is convex, all numbers in the theorem are equal to zero. For a polygon X in Figure 23.5 we have: $a(X) = 6$, $c(X) = 2$, $t_0(X) = 8$, $t_1(X) = 3$, and the theorems states that $8 - 3 = 2 + 6/2$. The proof below is similar to the third proof of Theorem 23.1, based on deformation of polygons.

Proof. Deform $X = [x_1 \ldots x_n]$ into a convex polygon $Y = [y_1 \ldots y_n]$, by moving one vertex a time (see Subsection 23.1). By choosing y_i in general position the resulting continuous family of polygons is generic except for a finite number of elementary transitions when either three points lie on a line or three lines intersect at a point. The number of such transition is clearly finite, even if rather large. We summarize all transitions in Figure 23.6 and Figure 23.7. Now it suffices to check the relation in the theorem is invariant under these transitions. Since the relation holds for a convex polygon Y, it also holds for X. This completes the proof. $\square$

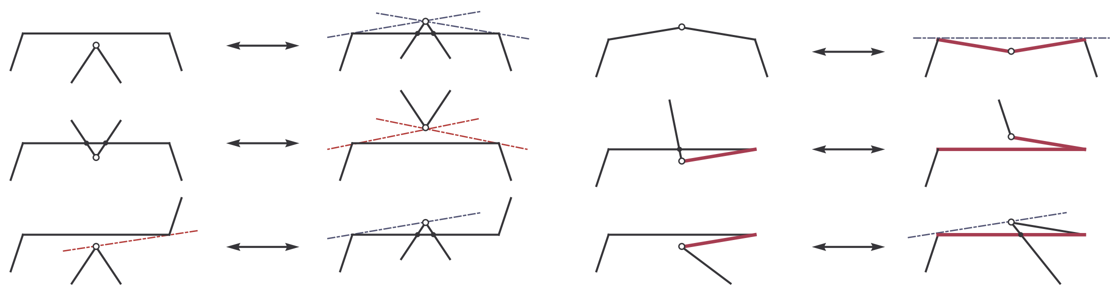

FIGURE 23.6. Transitions where some $a(X), c(X), t_1(X)$ and $t_2(X)$ change.

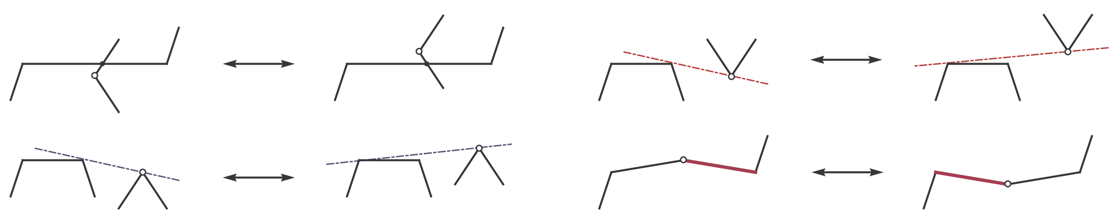

FIGURE 23.7. Transitions where $a(X), c(X), t_1(X)$ and $t_2(X)$ do not change.

One advantage of the "local move" approach is its flexibility. Let us now present a straightforward extension of Theorem 23.3 to unions of polygons.

Let Y be a finite union of polygons in the plane. We still assume that no three vertices lie on a line (and thus no two coincide), but do allow multiple intersections of edges. Denote by $c_i(Y)$ the number of points where i edges intersect. Define the number $t_0(Y)$ and $t_1(Y)$ of exterior and interior double supporting lines of Y in the same way.

Theorem 23.4 (Extended Fabricius-Bjerre's formula)**.** *For a union of polygons Y as above, we have:*

$$t_0(Y) - t_1(Y) = \sum_{i>1} \binom{i}{2} c_i(Y) + \frac{a(Y)}{2}.$$

To prove this extension, start with a nested union of convex polygons, when all parameters are zero and the formula trivially holds. Applying transformations as in the proof of Theorem 23.3, we obtain the formula for generic unions of polygons. When multiple crossings are allowed, slightly perturb the vertices of Y. Each i-intersection now creates $\binom{i}{2}$ intersections, which are all counted as in the formula.

Before we move on, let us note that Theorem 23.4 is non-trivial already when Y is a union of convex polygons. In this case there are no inflection edges, i.e., $a(Y) = 0$. Note that the interior double supporting lines separate the polygons they are supporting, while the exterior double supporting lines have them on the same side. As before, we say that Y is *generic* if no three vertices lie on a line and no three edges intersect. We obtain the following.

Corollary 23.5. *For a generic union Y of convex polygons, we have:*

$$t_0(Y) \, - \, t_1(Y) \, = \, c(Y).$$

The corollary can also be proved directly (see Exercise 23.1).

Remark 23.6. While the checking of elementary transitions in the proof above might seem overly tedious, there are certain advantages in the proof of this type. First, it is *automatic* in a sense that the number of transitions is obviously finite and they involve a finite number of edges. Thus, given a relation one can in principle write a program which verifies whether the relation is invariant under all transitions.

In a different direction, one can use elementary transitions to determine (the vector space of) all relations for any given set of parameters. For that, simply take a quotient of the space of relations by the vectors spanned by the transition vectors.

23.4. **An application: counting k-lines.** Let $X = \{x_1, \ldots, x_n\} \subset \mathbb{R}^2$ be a set of n points in general position, i.e., such that no three points x_i lie on a line and no three lines through different pairs of points intersect. Consider the set of directed lines ℓ_{ij} through vertices x_i and x_j, which are directed from x_i to x_j. Line ℓ_{ij} is called a *k-line* if it contains exactly k vertices of X on the left. Denote by $m_k = m_k(X)$ the number of k-lines in X.

Fix $k \geq 0$ and consider an oriented graph Γ_k on X, where edges (x_i, x_j) corresponding to k-lines ℓ_{ij} (see Figure 23.8). For example, Γ_0 consists of the boundary of the convex hull of X, oriented clockwise. Denote by $d_i = d_i(\Gamma_k)$ the number of vertices of out-degree i. Finally, denote by $c = c(\Gamma_k)$ the number of intersections of edges in Γ_k.

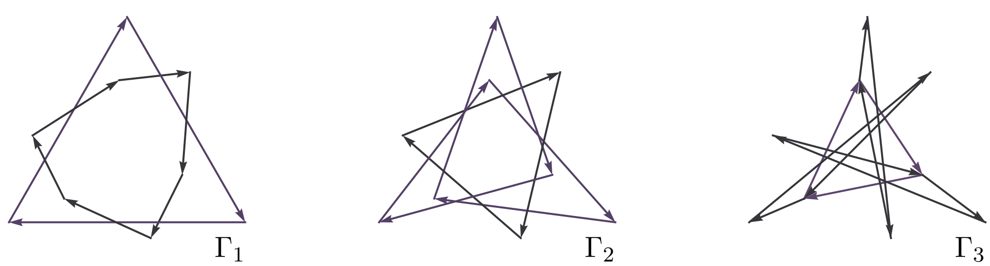

FIGURE 23.8. Graphs Γ_1, Γ_2 and Γ_3 on the same set of nine points.

Theorem 23.7. *Let X be a set of n points in general position, and let $k \leq n/2 - 1$. Then*

$$(\maltese) \qquad m_0 + m_1 + \ldots + m_{k-1} \, = \, c \, + \, \sum_{i>1} d_i \binom{i}{2}.$$

Proof. First, observe that in the graph Γ_k the in-degree is equal to the out-degree at every vertex x_j. To see that, start rotating clockwise line ℓ_{ij} around x_j. The number of edges to the left of the line will initially drop to $k-1$, but eventually will increase up to $(n-k-2)$ when the line turns into ℓ_{ji}. Clearly, when the line passes vertices on the left of ℓ_{ij}, this number decreases, and when those on the right it increases. Thus, the first time this number is equal to k happens when the line passes through x_j and the vertex on the right, which gives a correspondence between ingoing and outgoing edges at x_j. This splits edges of Γ_k into a union Y of polygons (which may have common vertices and be self-intersecting).

We need two more observations which follow by the same argument, i.e., by rotating a line around a vertex. One can check that at every vertex the order of ingoing edges is the same as the order of corresponding outgoing edges (see Figure 23.9). Further, at no point x_j can there be two pairs of corresponding (ingoing and outgoing) edges which lie on one side of a line ℓ_{ij}.

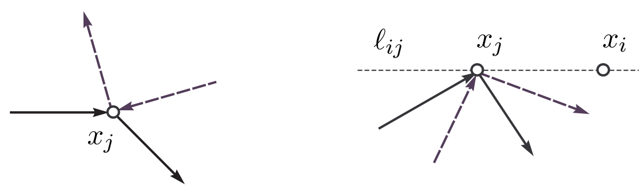

FIGURE 23.9. Impossible configurations of edges in Γ_k.

We are now ready to apply Theorem 23.4. Since x_i are in general position, no three different lines ℓ_{ij} can intersect. Further, by construction of Y the edges always turn right, so there are no inflection edges. This implies that the r.h.s. of ($\maltese$) is equal to $t_0(Y) - t_1(Y)$. Moreover, by the observation above, all these double supporting lines must be different.

Now observe that a line ℓ_{ij} is a supporting line at i if two edges of the same polygon in Y lie on the same side of ℓ_{ij}. This can happen only if on the other side of ℓ_{ij} there are fewer than k vertices. This implies that there are no interior double supported lines: $t_1(Y) = 0$. In the opposite direction, by rotating the line again one can check that for every r-line ℓ_{ij}, $r < k$, there exists two pairs of edges in Y at x_i and x_j, such that ℓ_{ij} is an exterior double supporting line. Thus, the number $t_0(Y)$ of exterior double supporting lines is equal to the number of r-lines, $0 \le r < k$. $\square$

Corollary 23.8. *For the number of k-lines, we have $m_k = O(n\sqrt[3]{k})$.*

The proof of the corollary is outlined in Exercise 23.3.

23.5. **Rigid deformations.** Let $X = [x_1 \dots x_n]$ and $Y = [y_1 \dots y_n]$ be two polygons in the plane with equal edge lengths:

$$|x_i x_{i+1}| = |y_i y_{i+1}| = a_i, \quad \text{where } 1 \le i \le n.$$

We say that X can be *rigidly deformed* into Y if there is a continuous family of polygons $X_t = [x_1(t) \dots x_n(t)]$, $t \in [0,1]$, such that $|x_i(t)x_{i+1}(t)| = a_i$ for all $1 \le i \le n$, and $X_0 = X$, $X_1 = Y$. The family $\{X_t, t \in [0,1]\}$ is called a *rigid deformation.*

Theorem 23.9. *Every two polygons with edge lengths* $(a_1, \dots, a_n)$ *can be rigidly deformed into each other, unless* $a_i + a_j > p/2$, $a_i + a_k > p/2$, *and* $a_j + a_k > p/2$, *for some* $i < j < k$, *where* $p = a_1 + \ldots + a_n$.

The inequalities in the theorem are necessary, since, for example, a polygon in Figure 23.10 cannot be deformed into its inverse. More generally, consider a polygon linkage $\mathcal{L}$ with given edges length (see Section 13), and let $\mathcal{M}_{\mathcal{L}}$ be its realization space. The theorem says that $\mathcal{M}_{\mathcal{L}}$ is connected unless these inequalities hold. In fact, in the latter case, it is known that the space $\mathcal{M}_{\mathcal{L}}$ has two connected components separating polygons and their inverses.

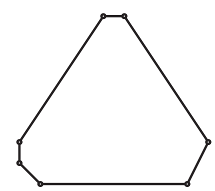

FIGURE 23.10. A polygon which cannot be deformed into its inverse.

We define a *simple rigid deformation* to be a rigid deformation $\{X_t\}$, where all X_t are simple. Clearly, a simple deformation preserves orientation of a polygon. The following result is the analogue of Theorem 23.9 for simple polygons.

Theorem 23.10 (Carpenter's rule problem)**.** *Every two simple polygons with the same corresponding edge lengths and the same orientation can be rigidly deformed into each other by a family of simple polygons.*

Just like in the previous two sections, this result can be used to prove a version of the four vertex theorem (Theorem 21.1) for all equilateral (but not necessarily convex) polygons.

Sketch of proof of Corollary 21.19. Denote by $N = s - t$ the difference in the number of neighboring and disjoint cycles. If we can prove that $N = 4$ for a specific simple polygon and that N does not change under the rigid deformation, Theorem 23.10 implies the result. Observe that N does not change while the polygon is generic.

Check that we can always deform one simple polygon into another without creating more than one degeneracy at a time: line with three vertices or circle with four vertices. Call a transition through one degeneracy a *local move.* Going over all possible local moves as in Figure 23.11, check that $s - t$ is indeed invariant under rigid deformations. It remains to check that $N = 4$ for a specific polygon. $\square$

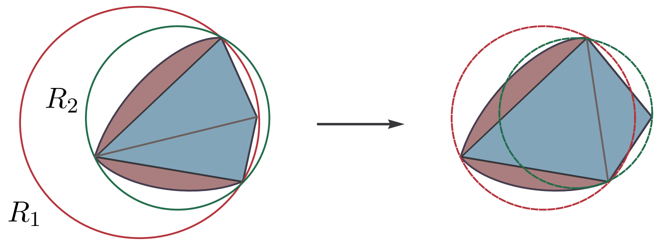

FIGURE 23.11. Rigid deformation of a polygon: a neighboring circle R_1 and a disjoint circle R_2 disappear after a local move.

23.6. **Back to inscribed squares.** We conclude this section with another proof of Theorem 5.12, which says that every simple polygon in the plane has an inscribed square.[57] In the spirit of this section, we prove this result by deforming the polygon, but first we need to find a global relation which holds for (almost all) simple polygons. This relation is simple: every generic simple polygon has an odd number of inscribed squares. Now that we have the relation, we can try to prove that it is invariant under certain elementary transitions.

Theorem 23.11. *Every generic simple polygon has an odd number of inscribed squares.*[58]

Theorem 5.12 now follows by a straightforward limit argument, as we repeatedly did in Section 5. Note also that the theorem is false for *all* simple polygons; for example every right triangle has exactly two inscribed squares. We begin the proof with the following simple statement, which will prove crucial (cf. Exercise 5.17).

Lemma 23.12. *Let ℓ_1, ℓ_2, ℓ_3 and ℓ_4 be four lines in $\mathbb{R}^2$ in general position. Then there exists a unique square $A = [a_1a_2a_3a_4]$ such that $x_i \in \ell_i$ and A is oriented clockwise. Moreover, the map $(\ell_1, \ell_2, \ell_3, \ell_4) \to (a_1, a_2, a_3, a_4)$ is continuously differentiable, where defined.*

Proof. Fix $z_1 \in \ell_1$. Rotate ℓ_4 around z_1 by $\pi/2$, and denote by ℓ_4' the resulting line, and by $z_2 = \ell_2 \cap \ell_4'$ the intersection point. Except when $\ell_2 \perp \ell_4$, such z_2 is unique. Denote by $z_4 \in \ell_4$ the inverse rotation of z_2 around z_1. We obtain the right isosceles triangle $\Delta = (z_2z_1z_4)$ oriented clockwise in the plane. The fourth vertex z_3 of a square is uniquely determined. Start moving z_1 along ℓ_1 and observe that the locus of z_3 is a line, which we denote by ℓ_3'. Since line ℓ_3 is in general position with respect to ℓ_3', these two line intersect at a unique point x_3, i.e., determines uniquely the square $[a_1a_2a_3a_4]$ as in the theorem. The second part follows immediately from the above construction. $\square$

[57] Of course, the result extends to general (self-intersecting) polygons, but for technical reasons it is easier to work with simple polygons.

[58] It takes some effort to clarify what we mean by a generic polygon (see the proof). For now, the reader can read this as saying that in the space $\mathbb{R}^{2n}$ of n-gons, almost all are generic.

Sketch of proof of Theorem 23.11. We begin with the following restatement of the second part of the lemma. Let $X = [x_1 \ldots x_n]$ be a generic simple polygon and let $\{X_t, t \in [0,1]\}$ be its continuous piecewise linear deformation. Suppose $A = [a_1a_2a_3a_4]$ is an inscribed square with vertices a_i at different edges of X, and none at the vertices of X, i.e., $a_i \neq x_j$. Then, for sufficiently small t, there exists a continuous deformation $\{A_t\}$ of inscribed squares, i.e., squares A_t inscribed into X_t. Moreover, for sufficiently small t, the vertices a_i of A_t move monotonically along the edges of X_t.

Consider what can happen to inscribed squares A_t as t increases. First, we may have some non-generic polygon X_s, where such a square in non-unique or undefined. Note that the latter case is impossible, since by compactness we can always define a limiting square A_s. If the piecewise linear deformation $\{X_t\}$ is chosen generically, it is linear at time s, and we can extend the deformation of A_t beyond A_s.

The second obstacle is more delicate and occurs when the vertex a_i of square A_s is at a vertex $v = x_j$ of X_s. Clearly, we can no longer deform A_s beyond this point. Denote by e_1 the edge of X which contains vertices a_i of A_t for $t < s$. Clearly, $e_1 = (x_{j-1}, x_j)$ or $e_1 = (x_j, x_{j+1})$. Denote by e_1' the other edge adjacent to v. Denote by e_2, e_3 and e_4 the other three edges of X containing vertices of A_t (see Figure 23.12).

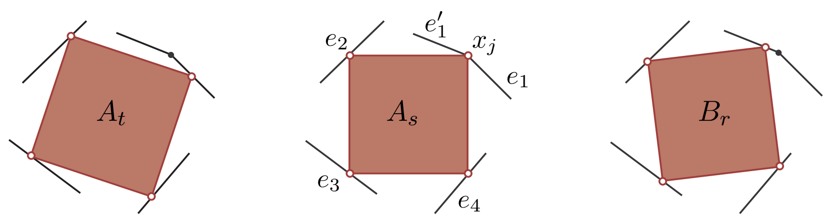

FIGURE 23.12. Inscribed squares A_t, $A_s = B_s$ and B_r, where $t < s < r$. Here e_2, e_3 and e_4 are fixed, while e_1 and e_1' move away from the squares.

Now consider a family $\{B_t\}$ of squares inscribed into lines spanned by edges e_1', e_2, e_3 and e_4. By construction, $A_s = B_s$. There are two possibilities: either the corresponding vertex b_i approaches x_j from inside e_1' or from the outside, when $t \to s$ and $t < s$. In the former case, we conclude that the number of inscribed squares decreases by 2 as t passes through s. In the latter case, one square appears and one disappears, so the parity of the number of squares remains the same. In summary, the parity of the number of squares inscribed into X_t with vertices at different edges is invariant under the deformation.

It remains to show that one can always deform the polygon X in such a way that at no point in the deformation do there exist inscribed squares with more than one vertex at the same edge, and such that the resulting polygon has an odd number of inscribed squares.

Fix a triangulation T of X. Find a triangle Δ in T with two edges the edges of X and one edge a diagonal in X. Subdivide the edges of X into small edges, so that neither of the new vertices is a vertex of an inscribed square. If the edge length is now small enough, we can guarantee that no square with two vertices at the same edge is inscribed into X. Now move the edges along two sides of the triangle Δ toward the diagonal as shown in Figure 23.13. Repeat the procedure. At the end we obtain a polygon Z with edges close to an interval. Observe that Z has a unique inscribed square (see Figure 23.13). This finishes the proof. $\square$

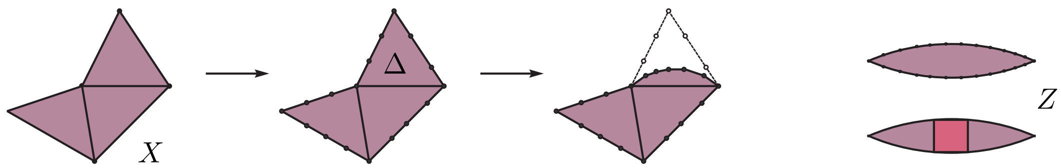

FIGURE 23.13. The first step of the polygon deformation which preserves the parity of the number of inscribed squares; the final polygon Z.

23.7. **Exercises.**

Exercise 23.1. $\lozenge$ [1] Prove Corollary 23.5 by a direct argument.

Exercise 23.2. [1] Suppose a closed curve $C \subset \mathbb{R}^2$ has no triple intersections and a winding number n around the origin. Prove that C has at least $n-1$ intersections.

Exercise 23.3. $\lozenge$ *a*) [1+] Let G be a graph with n vertices and $m \geq 4n$ edges drawn on a plane with c crossings. For a fixed probability p, compute the expected number of edges in a random p-subgraph of G. Use Euler's formula to show that there exists on average at least one crossing when p is sufficiently large. Compare this to the average number of crossing in a p-subgraph. Optimize for p to prove that $c \geq m^3/64n^2$.
b) [1+] In the notation of Theorem 23.7, prove that $m_0 + \ldots + m_k = O(kn)$. Conclude from here that $c = O(kn)$ and use part *a*) to obtain $m_k = O(n\sqrt[3]{k})$ (see Corollary 23.8).

Exercise 23.4. $\lozenge$ [1+] Find all local moves described in Subsection 23.5 and complete the proof of Corollary 21.19.

Exercise 23.5. $\lozenge$ *a*) [1-] For the case of numbers $a(X), c(X)$, $t_0(X)$ and $t_1(X)$ as above, show that the relation in Theorem 23.3 is the only possible relation with these parameters.
b) [1] Prove that every nonnegative integer 4-tuple $(t_0, t_1, c, a) \in \mathbb{Z}_+^4$ which satisfies $a \geq 2$ and the Fabricius-Bjerre formula can be realized by a polygon $X \subset \mathbb{R}^2$.
c) [2-] In notations of Theorem 23.3, prove that if $a(X) = 0$, then $t_1(X)$ is even and satisfies $t_1(X) \leq c(X)^2 - c(X)$.
d) [1+] Prove that every nonnegative integer 4-tuple $(t_0, t_1, c, 0) \in \mathbb{Z}_+^4$ which satisfies the Fabricius-Bjerre formula and the inequality in part *c*), can be realized by a polygon $X \subset \mathbb{R}^2$.

Exercise 23.6. [2-] Let $Q = [x_1 \ldots x_n] \subset \mathbb{R}^3$ be a simple space polygon such that no four vertices lie on the same plane. Denote by L_{ijk} the plane plane containing vertices x_i, x_j and x_k, $1 \leq i < j < k \leq n$. Note that L can be oriented according to the orientation of the triangle $(x_i, x_j, x_k) \in L$. We say that L is *tritangent* if two edges adjacent to each of the vertices x_i, x_j and x_k lie either on L side of L. Classify all planes L_{ijk} by the number of edges on it and by the side (positive or negative) on which adjacent edges lie. Find all linear relations for the resulting numbers.[59]

[59]For the weakly convex polygons some of these numbers were studied in Subsection 21.7. However, in the full generality not all relations remain true.

Exercise 23.7. [2-] Use the stereographic projection to obtain the analogues of results in the previous exercise for the numbers of neighboring and disjoint circumscribed circles (see Subsection 21.7).

Exercise 23.8. *a*) [1+] Find a space polygon in $\mathbb{R}^3$ isotopic to the trefoil knot, and without tritangent planes.
b) [2-] Let Q be space polygon in $\mathbb{R}^3$ isotopic to the trefoil knot, and such that Q projects onto a plane with the usual diagram with three crossings. Prove that Q has a tritangent plane.

Exercise 23.9. [2-] Let $X = [x_1 \ldots x_n] \subset \mathbb{R}^2$ be a polygon with vertices in general position, and let $e_i = (x_i, x_{i+1})$ be the edges of X. An interval $u = (x_i, y)$, $y \in X$, is called *quasi-normal* at x_i if both angles of u and e_{i-1}, e_i are acute or both are obtuse. Define a *sign* $\varepsilon(u) \in \{\pm 1\}$ of a normal u to be 1 if the angles are acute and -1 if the angles are obtuse. A diagonal (x_i, x_j) is called a *double quasi-normal* if it is quasi-normal at both vertices (see Subsection 9.4). It is called *positive* if $\varepsilon(x_i, x_j)\varepsilon(x_j, x_i) = 1$; otherwise, it is called *negative*. Denote by $n_+(X)$ and $n_-(X)$ the number of positive and negative double quasi-normals. Define a *perpendicular* to be an interval $u = (x_i, y)$ such that $y \in e_j$ for some e_j and $(x_i, y) \perp e_j$. Denote by $p_+(X)$ and $p_-(X)$ the number of perpendiculars u such that $\varepsilon(u) = 1$ and $\varepsilon(u) = -1$, respectively. Use deformations of polygons to prove that

$$n_+(X) \,-\, n_-(X) \,+\, p_+(X) \,-\, p_-(X) \,+\, c(X) \,=\, 0,$$

where $c(X)$ is the number of crossings of X.

Exercise 23.10. Let $C \subset \mathbb{R}^2$ be an oriented closed piecewise linear curve with n double intersections and no triple intersections. A *loop* is a closed portion of curve without double points.
a) [1] Note that C has at most $2n$ loops. Prove that if C has exactly $2n$ loops, then between every two points on C corresponding to a double crossing there are exactly n curve arcs. Such curves are called *maximally looped*.
b) [1+] Suppose C has at least $n + 1$ loops. Prove that n is odd.
c) [1+] Denote by $r(C)$ the sum of $(\pi - \alpha_i)$, over all angles α_i on the left of the curve. Prove that $r(C) = 2\pi(n_+ - n_- \pm 1)$, where n_+ and n_- is the number of positive and negative double points, respectively.
d) [1] Use this to prove that if C is maximally looped, then $r(C) \in \{-4\pi, 0, 4\pi\}$.
d) [1] Use part *c*) of Exercise 23.5 to show that if C is maximally looped and has no inflection edges, then C has no interior double supporting lines.
e) [1-] Use Theorem 23.3 to conclude that every curve C as in part *d*) has exactly n double supporting lines.

Exercise 23.11. [2-] Find a simple unknotted space polygon $Q \subset \mathbb{R}^3$ which cannot be rigidly deformed into a flat convex polygon. In other words, prove that Theorem 23.10 does not extend to $\mathbb{R}^3$.

Exercise 23.12. [1+] Prove that every two simple polygons in $\mathbb{R}^4$, can be rigidly deformed into each other. In other words, prove that Theorem 23.10 extends to $\mathbb{R}^4$.

Exercise 23.13. (*Erdős flip problem*) [2] Let $Q \subset \mathbb{R}^2$ be a non-convex polygon and let C be the convex hull of Q. Choose edge e of C that is not in Q and reflect the portion of Q across e, as in Figure 23.14. Repeat such *flip transformations* until a convex polygon is obtained. Prove that the process stops after finitely many steps.

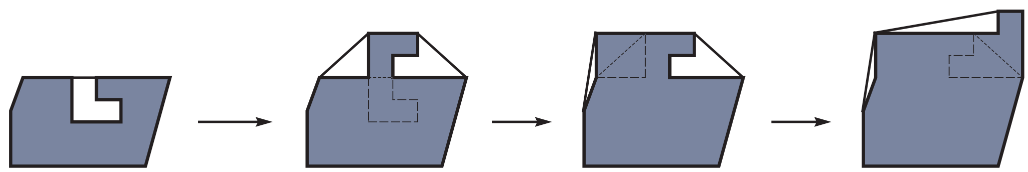

FIGURE 23.14. First few flip transformations of a polygon.

Exercise 23.14. [1+] Let $Q \subset \mathbb{R}^2$ be a self-intersecting polygon. Define a *reflection move* to be a reflection of an arc $[xy]$ of Q along the line (x, y), where $x, y \in Q$ are any points on the polygon (see Figure 23.15). Prove that every polygon can be made simple (not self-intersecting) by a finite sequence of reflection moves.

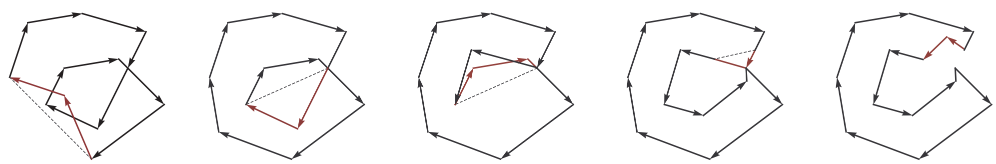

FIGURE 23.15. A sequence of reflection moves of a polygon.

23.8. **Final remarks.** Theorem 23.1 and the first two proofs are given in [VasE], Problem 235. The original smooth analogue of Theorem 23.3 is due to Fabricius-Bjerre; this version and the proof follows [Ban2] (see also [Fab]). A complete description of the values in the Fabricius-Bjerre formula is given in Exercise 23.5. For a precise statement in Remark 23.6 see [Pak3] which discusses various local move connectivity arguments for finite tilings.

The bound in Corollary 23.8 on the number of k-lines is due to Dey (1998). Theorem 23.7 implying it (see Exercise 23.3) was proved in [And+]. A different, more combinatorial proof of the theorem was given in [AroW].[60] The proof of Theorem 23.7 we present here is due to Uli Wagner (unpublished), who graciously allowed us to use it here (see also a related proof in [And+]). Let us note that all these proofs use basic properties of graphs Γ_k, discovered by Lovász in [Lov].

Theorem 23.9 was proved independently in [KM1] and [LenW]. For other versions of this results in the literature see e.g., [Whit]. Theorem 23.10 is known as "carpenter's rule problem" and was proved in [CDR] (see also [Stre]).

The deformation idea in the proof of Theorem 23.1 is due to Shnirelman [Shn] (see also [Gug2]), although the details are largely different. Our presentation follows [Pak9] (see Subsection 5.8 for further results on inscribed squares).

[60]This proof can be viewed as a combination of the argument we give and the Fabricius-Bjerre's proof in [Fab] applied to the special case of polygons with no inflection edges and interior double supporting lines.

24. Geometry of space curves

In this mostly stand-alone section we finish our exploration of the geometry of space curves. Much of this section is aimed at the proof of the Fáry–Milnor theorem and related results. Although these results are standard in differential geometry, their proofs are completely discrete and use important technical tools. Let us single out the *averaging technique* in *Crofton's formula* (see also Exercises 24.5–24.7), and *knot decomposition* in the *second hull theorem*.

24.1. **Curve covers.** Let $C \subset \mathbb{R}^d$ be a curve of length L. It is obvious that C can be covered by a ball of radius $L/2$. The following result shows that this bound can be improved to a sharp bound of $L/4$.

Theorem 24.1. *Every space polygon $Q \subset \mathbb{R}^d$ of length L can be covered by a ball of radius $L/4$.*

The theorem is a d-dimensional generalization of Exercise 1.3. By the Helly theorem (Theorem 1.2) it suffices to check this for all $(d+1)$-gons in $\mathbb{R}^d$. Here is a simple direct proof.

Proof. Let v, w be two points on Q which divide the length into equal halves. For the midpoint z of the interval (v, w) and any point $x \in Q$ we have:

$$|zx| \,=\, \frac{1}{2}\,\big\|\overrightarrow{vx} + \overrightarrow{wx}\big\| \,\le\, \frac{1}{2}\big(|vx| + |wx|\big) \,\le\, \frac{1}{2}\big(L/2\big) \,=\, \frac{L}{4}\,.$$

Thus, the ball of radius $L/4$ centered at z covers Q. $\square$

The following is a spherical analogue of Theorem 24.1. The proof above can also be modified to work in this case.

Theorem 24.2. *Every spherical polygon $Q \subset \mathbb{S}^d$, $d \ge 2$, of length $L \le 2\pi$ can be covered by a spherical ball of radius $L/4$.*

Proof. As in the proof above, let v, w be two points on Q which divide Q into paths of equal length $L/2$, and let z be the midpoint of the arc of a great circle from v to w. For every point $x \in Q$, denote by $x' \in \mathbb{S}^d$ a point on a sphere such that z is the midpoint of the arc of a great circle from x to x'. By the symmetry, we have:

$$|xz|_{\mathbb{S}^d} \,=\, \frac{1}{2}|xx'|_{\mathbb{S}^d} \,\le\, \frac{1}{2}\big(|xv|_{\mathbb{S}^d} + |vx'|_{\mathbb{S}^d}\big) \,=\, \frac{1}{2}\big(|xv|_{\mathbb{S}^d} + |xw|_{\mathbb{S}^d}\big).$$

Since the geodesic distance $|xv|_{\mathbb{S}^d} + |xw|_{\mathbb{S}^d} \le L/2$, we obtain the result.[61] $\square$

We will need the following special case of the theorem.

Corollary 24.3. *Every spherical polygon $Q \subset \mathbb{S}^d$, $d \ge 2$, of length $L < 2\pi$ can be covered by an open hemisphere. Similarly, every spherical polygon $Q \subset \mathbb{S}^d$, $d \ge 2$, of length $L = 2\pi$ can be covered by a closed hemisphere.*

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[61] Note that we never used condition $L \le 2\pi$, which is added for simplicity. The midpoints of intervals longer than π are ambiguous and the spherical balls of radius more than $\pi/4$ are less natural.

24.2. **Total curvature.** For a triangle (xyz), the *exterior angle* $\eta(xyz)$ at a vertex y is defined as

$$\eta(xyz) \, = \, \pi - \angle xyz \, .$$

Let $Q \subset \mathbb{R}^d$ be a simple (non-self-intersecting) space polygon. The *total curvature* $\varkappa(Q)$ is the sum of the exterior angles of Q. This definition is robust enough so that when polygons approximate a smooth closed curve $C \subset \mathbb{R}^d$, the total curvature of the polygons converges to the total curvature of the curve $\varkappa(C)$.

The following beautiful result is the first step in the understanding of total curvature.

Theorem 24.4 (Fenchel)**.** *For every simple space polygon $Q \subset \mathbb{R}^3$, the total curvature $\varkappa(Q) \geq 2\pi$. The equality holds if and only Q is flat and convex.*

Proof. Let $Q = [v_1 \ldots v_n] \subset \mathbb{R}^3$. For every edge vector $e_1 = (v_1 v_2), \ldots, e_{d-1} = (v_{n-1} v_n)$, $e_n = (v_n v_1)$, consider a point on a unit sphere $x_i \in \mathbb{S}^2$ so that (Ox_i) is a unit vector along e_i. Connect points x_i and x_{i+1} with the shortest path, so that $R = [x_1 \ldots x_n]$ is a spherical polygon. By definition, the exterior angle $\eta(v_{i-1} v_i v_{i+1})$ of Q is equal to the length $|x_i x_{i+1}|_{\mathbb{S}^2}$ on a sphere. Thus $\varkappa(Q) = |R|_{\mathbb{S}^2}$, the length of the spherical polygon.

Now observe that R cannot lie in an (open) hemisphere. Indeed, otherwise all edge vectors e_i are pointing into the same half-space, and since the sum of e_i is equal to O, this is impossible. On the other hand, by Corollary 24.3, any curve on a sphere of length $< 2\pi$ can be covered by a hemisphere. Therefore $|R|_{\mathbb{S}^2} \geq 2\pi$, which proves the first part of the theorem.

For the second part, from above and the second part of Corollary 24.3, the equality can occur only when Q lies in a plane. Finally, for the plane convex polygons the result follows from a direct calculation of the interior angles. $\square$

24.3. **The Fáry–Milnor theorem.** In the previous section we proved Fenchel's theorem that every curve in $\mathbb{R}^3$ has total curvature at least 2π. The main result of this section is a lower bound of 4π for knotted curves.

We say that a space polygon $Q \subset \mathbb{R}^3$ is *knotted* if it is simple and knotted as a closed curve.

Theorem 24.5 (Fáry–Milnor)**.** *For every knotted space polygon $Q \subset \mathbb{R}^3$, the total curvature $\varkappa(Q) \geq 4\pi$.*

The proof is again based on spherical geometry. Let $C \subset \mathbb{S}^2$ be a spherical polygonal curve on a unit sphere. For every point $x \in \mathbb{S}^2$, denote by $n_C(x)$ the number of points of intersection of C and a plane H_x orthogonal to (Ox). The values of n_C subdivide the sphere into spherical polygons $A_1, \ldots, A_k$ with constant value ν_i on A_i. Define

$$N_C \, = \, \sum_{i=1}^{k} \nu_i \cdot \operatorname{area}(A_i) \, = \, \int_{\mathbb{S}^2} n_C(x) \, dx \, .$$

Lemma 24.6 (Crofton's formula)**.** *The length $|C|$ of a spherical polygonal curve $C \subset \mathbb{S}^2$ satisfies*

$$|C| \, = \, \frac{1}{4} \, N_C$$

Proof. Both sides are clearly additive, so it suffices to prove the lemma only for the intervals on a sphere. Suppose $C \subset \mathbb{S}^2$ is an interval, $|C| = \alpha \le \pi$. Then $n_C(x) = 1$ when $x \in \mathbb{S}^2$ lies on a great circle with normal (Oz) for some $z \in C$, and $n_C(x) = 0$ otherwise. The set A of points x with $n_C(x) = 1$ is a union of two sectors with angle α. We have:

$$N_C \, = \, \operatorname{area}(A) \, = \, \frac{2\alpha}{2\pi} \cdot 4\pi \, = \, 4\alpha \, ,$$

which completes the proof. □

We are now ready to prove the Fáry–Milnor theorem.

Proof of Theorem 24.5. For a closed knotted space polygon $Q \subset \mathbb{R}^3$, fix an orientation and consider the spherical polygon $R \subset \mathbb{S}^2$ as in the proof of Theorem 24.4. Then $\varkappa(Q) = |R|$. For a spherical polygon R, consider a subdivision $A_1, \ldots, A_k$ of a sphere and the corresponding values $\nu_1, \ldots, \nu_k$. If $\nu_i \ge 4$ for all i, we have $|R| = N_Q/4 \ge 4\pi$, as desired.

Suppose now that $\nu_i < 4$. Choose a point $x \in A_i$ in general position, so no edge of Q is parallel to H_x. Recall that ν_i is equal to the number of intersections of R and H_x, which in turn is equal to the number of changes in direction of edges of Q with respect to H_x. Therefore, $\nu_i = 2$. The result now follows from the assumption that Q is unknotted and the following Milnor's lemma. □

Lemma 24.7 (Milnor's lemma)**.** *Let $Q \subset \mathbb{R}^3$ be a simple space polygon such that the distance function to a plane L has exactly two critical points. Then Q is unknotted.*

Proof. Denote by x and y the points of Q with the minimum and maximal distance, respectively. Then x, y divide Q into two paths where the distance from L is monotone increasing. Consider a projection of Q onto a generic plane $H \perp L$ as in Figure 24.1. Undo all crossings, from x to y to obtain a projection without crossings. This implies that Q is an unknot. □

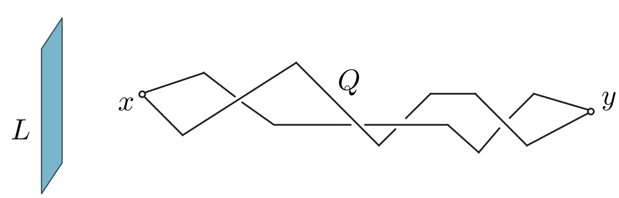

FIGURE 24.1. Projection of Q onto a plane orthogonal to L.

24.4. **Linked tubes.** Consider two linked simple space curves C_1 and C_2 in $\mathbb{R}^3$ of length 2π. The question is how big a tube can be made around the curves so that these tubes do not intersect. It is intuitively obvious that the tubes cannot have radius bigger than $1/2$. This radius is achieved when two curves are circles which lie in the orthogonal planes, and such that each circle passes through the center of the other (see Figure 24.2).

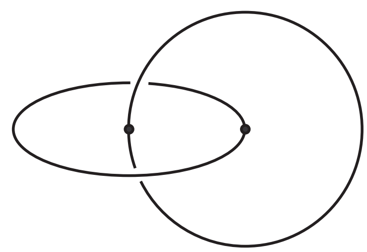

FIGURE 24.2. Two linked circles with maximal distance between them.

The following result is a discrete version of this observation. For two space polygons $Q_1, Q_2 \subset \mathbb{R}^3$ denote by

$$d(Q_1, Q_2) \,=\, \min\bigl\{|xy|,\, x \in Q_1,\, y \in Q_2\bigr\}$$

the distance between them.

Theorem 24.8. *For every two linked space polygons $Q_1, Q_2 \subset \mathbb{R}^3$ the distance $d(Q_1, Q_2)$ between satisfies:*

$$d(Q_1, Q_2) \,\le\, \frac{1}{2\pi}\, \min\bigl\{|Q_1|, |Q_2|\bigr\}.$$

Proof. Fix a point $x \in Q_1$ and a surface A spanned by intervals (x, x'), where $x' \in Q_1$. Since polygons are linked, Q_2 intersects A. Fix a point $y \in Q_2 \cap A$ and consider the largest ball B_r of radius r around y, such that Q_1 is outside B_r. Then the projection R of Q_1 onto the sphere $S = \partial B_r$ has length $|R| \le |Q_1|$. Since $y \in A$, the projection R is a spherical polygon which contains two opposite points. Since the spherical distance between the opposite points is equal to π, we have $|R| \ge 2\pi r$.[62] Therefore, for the radius r we have $r \le \frac{1}{2\pi}|Q_1|$, which implies the result. □

24.5. **Inscribed space polygons.** The main goal of this subsection is to give a combinatorial proof of the Fáry–Milnor theorem (Theorem 24.5). We begin with an important technical lemma.

We say that a space polygon X is *inscribed* into a space polygon Q, where $X, Q \subset \mathbb{R}^3$, if the vertices of X lie on the edges of Q.

Lemma 24.9 (Monotonicity of the total curvature). *Let $Q, X \subset \mathbb{R}^3$ be space polygons, such that X is inscribed into Q. Then $\varkappa(X) \le \varkappa(Q)$.*

[62]This also follows immediately from Theorem 24.2.

Proof. Add vertices of X to Q and observe that the curvature $\varkappa(Q)$ does not change. One by one, remove the vertices of Q that are not in X and check that the curvature is nonincreasing.

Formally, let $Q = [v_1 \ldots v_n]$. Remove vertex v_i and replace edges (v_{i-1}, v_i) and (v_i, v_{i+1}) with (v_{i-1}, v_{i+1}). Denote by Q' the resulting space polygon. Let us show that $\varkappa(Q') \leq \varkappa(Q)$. We have:

$$\begin{aligned}\varkappa(Q) - \varkappa(Q') &= (\pi - \angle v_{i-2}v_{i-1}v_i) + (\pi - \angle v_{i-1}v_iv_{i+1}) + (\pi - \angle v_iv_{i+1}v_{i+2}) \\ &\qquad - (\pi - \angle v_{i-2}v_{i-1}v_{i+1}) - (\pi - \angle v_{i-1}v_{i+1}v_{i+2}) \\ &= (\angle v_iv_{i-1}v_{i+1} + \angle v_{i-2}v_{i-1}v_{i+1} - \angle v_{i-2}v_{i-1}v_i) \\ &\qquad + (\angle v_iv_{i+1}v_{i-1} + \angle v_{i-1}v_{i+1}v_{i+2} - \angle v_iv_{i+1}v_{i+2}) \geq 0,\end{aligned}$$

where the last inequality is the spherical triangle inequality applied to each term. $\square$

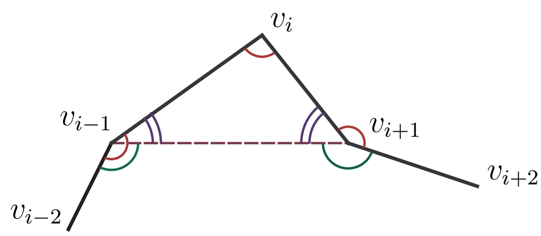

FIGURE 24.3. Removing vertex v_i from space polygon $Q = [v_1 \ldots v_n]$.

Here is a simple idea of another proof of the Fáry–Milnor theorem. Starting with a knotted polygon $Q \subset \mathbb{R}^3$, consider a sequence of space polygons inscribed into each other. If we can show that the last polygon in a sequence has total curvature 4π, we obtain the desired lower bound. In fact, we will prove that one can take the last inscribed polygon to be a degenerated quadrilateral $D = [xyxy]$ which has $\varkappa(D) = 4\pi$.

Combinatorial proof of Theorem 24.5. We prove the theorem by induction on the number n of vertices of a knotted space polygon $Q = [v_1 \ldots v_n] \subset \mathbb{R}^3$. When $n = 3$, polygon Q is a triangle. Then Q is unknotted and the claim trivially holds.

By the continuity of the total curvature, we can always assume that the vertices of Q are in general position. We start with the following observation. Suppose there exists a vertex v_i such that triangle $(v_{i-1}v_iv_{i+1})$ does not intersect any other edge of Q. We can then remove vertex v_i as in the proof above. Since the resulting polygon is still knotted, we obtain the step of induction.

We now begin to change polygon Q as follows. Starting at v_1, consider all points $z \in (v_2, v_3)$, $z \neq v_2$, such that (v_1, z) intersects an edge in Q. If this set of such points is empty, we can remove the triangle as above. Otherwise, let z_2 be the closest such point to v_2. Since Q is generic, we have $z_2 \neq v_3$. Denote by y_1 the corresponding point of intersection of (v_1, z_2) and Q (see Figure 24.4).

Move z_2 slightly towards v_2. Denote by x_1 the closest to y_1 point on a "new" (v_1, z_2). Now replace vertex v_2 of Q with z_2, i.e., replace edge (v_1, v_2) and interval

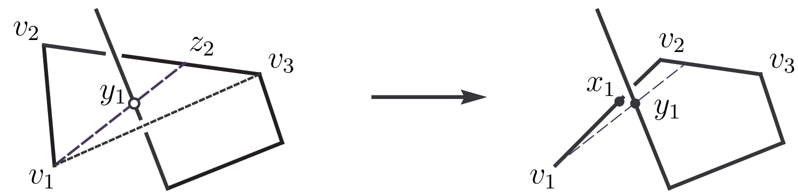

FIGURE 24.4. Transforming space polygon $Q = [v_1 \ldots v_n]$.

(v_2, z_2) with (v_1, z_2). The resulting polygon Q remains knotted, but has points x_1 and y_1 very close to each other.

Repeat the procedure for the (new) vertex v_2. Again, replace vertex v_3 with $z_3 \in (v_3, v_4)$, and obtain two points $x_2 \in (v_2, v_3)$ and $y_2 \in Q$ very close to each other. There are two possibilities for y_2 relative to the points $x_1, y_1 \in Q$. If y_2 is on the arc $[y_1 \ldots x_1]$ of the polygon, then the inscribed quadrilateral $[x_1 x_2 y_1 y_2]$ is very close to a degenerated quadrilateral D as above, and can be made as close as desired.

Alternatively, if y_2 is on the arc $[x_1 \ldots y_1]$ of the polygon, then the chords (x_1, y_1) and (x_2, y_2) as in Figure 24.5 do not intersect. Repeat the procedure until two intersecting chords are found. That will always happen since by construction the points (x_i, y_i) cannot lie on adjacent edges. This completes the induction step and proves the theorem. $\square$

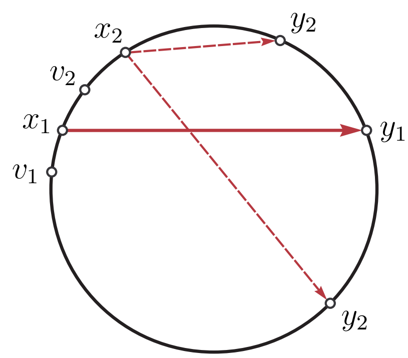

FIGURE 24.5. Two possibilities for y_2.

24.6. **Second hull of a polygon.** The goal of this section is to present the following unusual combinatorial generalization of Milnor's lemma (Lemma 24.7).

Let $Q \subset \mathbb{R}^3$ be a simple space polygon and let $P = \mathrm{conv}(Q)$ be its convex hull. Observe that every plane H through an interior point $x \in P$ intersects Q in at least two points. We say that a plane is *generic* if it contains no vertices of Q. Define the *second hull* $\mathrm{sh}(Q)$ to be the set of points $x \in P$ such that every generic plane H through x intersects Q in at least four points. The second hull $\mathrm{sh}(Q)$ can be empty, for example when Q projects onto a plane without intersections. Thus it is natural to ask what happens for knotted polygons.

Theorem 24.10 (Second hull theorem)**.** *For every knotted space polygon $Q \subset \mathbb{R}^3$, the second hull* $\mathrm{sh}(Q)$ *is nonempty.*

FIGURE 24.6. Second hull of the trefoil knot.

To see that the second hull theorem generalizes Milnor's lemma, let us restate them in the same language. Milnor's lemma says that if for a space polygon Q there exists a family of parallel planes through every point in $\mathbb{R}^3$ which intersects Q at most twice, then Q is unknotted. The second hull theorem makes the same conclusion without the restriction on all planes to be parallel.

Let us mention that $\mathrm{sh}(Q)$ can be disconnected, but is always a disjoint union of convex polytopes, except at the boundary (see Exercise 24.9). Also, the second hull theorem has no natural inverse, i.e., there exist unknots Q with $\mathrm{sh}(Q) \neq \varnothing$ (see Exercise 24.10).

Proof of Theorem 24.10. For two points $x, y \in Q$ consider a polygon Q' obtained by replacing the portion of Q between x and y by a straight interval (x, y). Observe that $\mathrm{sh}(Q') \subset \mathrm{sh}(Q)$. Indeed, if a generic plane H through a point in $\mathrm{sh}(Q')$ intersects Q' four times, it contains at least three of them in $Q' - (x, y) \subset Q$. Since the number of intersections of H and Q is even, it must be at least four.

From now on we assume that Q is *geometrically prime*, i.e., that there are no planes which cut Q into two knotted arcs. Otherwise, we can consider a knotted space polygon Q' instead and reduce the problem to a polygon with fewer edges.

Let $F \subset \mathbb{R}^3$ be a (closed) half-space on one side of plane H. We say that F *essentially contains* Q if it either contains Q, or intersects Q at exactly two points x, y such that the arc outside of F union (x, y) is unknotted. In this case, define the *clipping of Q by F* to be $\mathrm{clip}_F(Q) = (F \cap Q) \cup (x, y)$ the isotopic polygon inside F.

Let F, G be two half-spaces which essentially contain Q. We shall prove that G essentially contains the clipping $\mathrm{clip}_F(Q)$. Note that there are three possible relative positions of points $x, y \in Q \cap \partial F$ and points $x', y' \in Q \cap \partial G$ (see Figure 24.7). In the first case, we have $x', y' \notin F$. Then $\mathrm{clip}_F(Q) \subset G$ and the claim holds by definition. In the second case, we have $x', y' \in F$. Since the portion between x' and y' in Q is unknotted, so is the arc between x' and y' in $\mathrm{clip}_F(Q)$ and the claim holds by definition again. Finally, in the third case, we have $x' \notin F$ and $y' \notin G$. Then planes F

and G divide Q into four arcs, such that the sum of two outside of F and the sum of two outside of G are unknotted. Since the sum of two non-trivial knots cannot be unknotted (see Exercise 24.3), the arc A outside $F \cap G$ is unknotted. Therefore, $\mathrm{clip}_F(Q)$ consists of A inside G and an unknotted arc outside G, which proves that G essentially contains $\mathrm{clip}_F(Q)$ in this case as well.

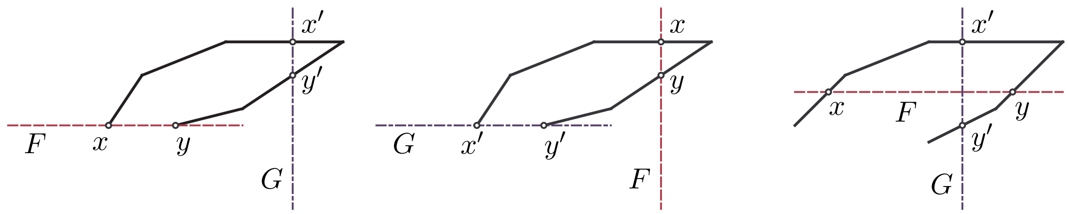

FIGURE 24.7. Clipping of knots: three cases.

We are now ready to finish the proof. Denote by $\mathcal{F} = \mathcal{F}(Q)$ the set of all closed half-spaces which essentially contain Q, and let $Z = \cap_{F \in \mathcal{F}} F$ be the intersection of all such half-spaces. We claim that Z is a nonempty convex set, such that $Z \subset \mathrm{sh}(Q)$. For the second part, consider a generic plane H through any point $z \in Z$ and assume that H intersects Q at two points. Since Q is geometrically prime, the arcs on both sides cannot be knotted. On the other hand, if either arc is unknotted, take the half-space F which lies in the other side of H, such that the plane ∂F is at distance ε from H. For $\varepsilon > 0$ small enough, the half-space F essentially contains Q, but $z \notin F$, a contradiction with the choice of $z \in Z$. Thus H intersects Q in at least three points. Since H is generic, it has at least four intersection points with Q then.

By definition, the set Z is the intersection of half-spaces $F \in \mathcal{F}$. Since the half-spaces are convex and closed, by the Helly theorem (Theorem 1.2 and Exercise 1.1), to prove that Z is nonempty it suffices to show that the intersection of every four half-spaces $F_1, \ldots, F_4 \in \mathcal{F}$ is nonempty. Consider a sequence of knotted polygons $Q_0, \ldots, Q_4$, defined by $Q_0 = Q$, and $Q_i = \mathrm{clip}_{F_i}(Q_{i-1})$, $1 \le i \le 4$. From above, by induction, plane F_1 essentially contains Q_0, plane F_2 essentially contains Q_1, etc. Therefore, Q_i are well defined knotted space polygon. Since $Q_4 \subset F_1 \cap F_2 \cap F_3 \cap F_4$, we conclude that this intersection is nonempty, which finishes the proof. $\square$

24.7. **Exercises.**

Exercise 24.1. *a*) [1-] Let $X \subset \mathbb{R}^2$ be a set of n points in general position. Prove that there exist a simple (not self-intersecting) polygon with vertices in X.
b) [1-] Let $X \subset \mathbb{R}^3$ be a set of n points in general position. Prove that there exists an unknotted polygon with vertices in X.
c) [1-] Let $X \subset \mathbb{R}^3$ be a set of $3n$ points in general position. Prove that there exist n disjoint triangles with vertices in X, such that no two triangles are linked.
d) [1+] Let $X \subset \mathbb{R}^3$ be a set of 6 points in general position. Prove that there exist 2 linked disjoint triangles with vertices in X.

Exercise 24.2. (*Local convexity of polygons*) $\Diamond$ [1] Prove that every simple polygon $Q \subset \mathbb{R}^2$ with interior angles $< \pi$ is convex. Prove a spherical analogue.

Exercise 24.3. (*Sum of knots*) [2-] Prove that the sum of two non-trivial knots cannot be an unknot.

Exercise 24.4. (*Borsuk*) [1] Generalize Fenchel's Theorem 24.4 to curves in $\mathbb{R}^d$.

Exercise 24.5. (*Crofton's formula for surfaces*) $\Diamond$ [1] Let $S \subset \mathbb{R}^3$ be a non-convex 2-dimensional polyhedral surface with the set of vertices $V = \{v_1, \ldots, v_n\}$. Define the *total absolute curvature* $\varkappa(S) = \sum_i |\omega_i|$ the sum of absolute values of curvatures $\omega_i = \omega(v_i)$ of vertices (see Section 25). Generalize Crofton's formula (Lemma 24.6) to show that $\varkappa(S)$ is 2π times the average number of critical points of linear functions on S.

Exercise 24.6. (*Buffon's needle*) $\Diamond$ Consider an infinite family $\mathcal{L}$ of horizontal lines in the plane which stand apart at distance 1 from each other. Let C be a rectifiable curve in $\mathbb{R}^2$. Denote by $N(C)$ the number of intersections of C and lines in $\mathcal{L}$. Define $\rho = \rho(\alpha, b)$ to be a rigid motion of a plane obtained as a rotation by angle α and then a translation by vector b. Denote by B an axis parallel unit square. Finally, let

$$N(C) \, = \, \mathbb{E}\big[N(\rho C)\big],$$

where $\rho = \rho(\alpha, b)$, $\alpha \in [0, 2\pi]$ is a random angle, and $b \in B$ is a random vector. Heuristically, think of $N(C)$ as the average number of intersections of C and a random rigid motion of C.
a) [1] Prove that the average $N(C)$ is invariant under rigid motions. Prove that $N(C)$ is additive when a union of two curves is taken. Check that $N(C)$ is continuous as curve C is deformed.
b) [1-] Observe that $N(C) = 2$ when C is a circle of diameter 1. Conclude from here that $N(C) = 1/\pi$ when C is a unit interval.[63] More generally, prove that $N(C) = |C|/\pi$ for every C.
c) [1-] Observe that $N(C) = 2$ when C is a curve of constant width 1 (see Exercise 3.6). Conclude that $|C| = \pi$.

Exercise 24.7. (*Random intersections*) Let $\mathfrak{R}$ be the set of rigid motions $\rho(\alpha, b)$ as above. Define the *kinematic density* $d\rho = |\sin\alpha|\, d\alpha \wedge db$. For two convex sets $X, Y \subset \mathbb{R}^2$, denote by $N(X, Y)$ the number of points in the intersection $\partial X \cap \partial Y$. Fix a convex set X and let m_i be the measure of the set of $\rho \in \mathcal{A}$ such that $N(X, \rho X) = i$. Finally, denote $\ell = \text{perimeter}(X)$ and $a = \text{area}(X)$.
a) [2-] Prove that

$$\ell^2 \, - \, 4\pi a \, = \, m_4 \, + \, 2m_6 \, + \, 3m_8 \, + \, \ldots$$

Conclude from here the isoperimetric inequality in the plane (Theorem 7.1).
b) [1] Consider a subset $\mathcal{A}(X) \subset \mathfrak{R}$ of rigid motions $\rho(\alpha, b)$ such that $N(X, \rho X) \geq 4$. Use part *a*) to show that if X is a convex set not equal to a circle, then the subset $\mathcal{A}(X)$ has positive kinematic measure.
c) [1] Give an independent proof of part *b*) using Lemma 9.6.

Exercise 24.8. (*Wienholtz*) *a*) [1+] Prove that every polygon $Q \subset \mathbb{R}^2$ of length $|Q|$ has a projection onto a line of length at most $|Q|/\pi$.
b) [2-] Prove that every space polygon $Q \subset \mathbb{R}^3$ of length $|Q|$ has a projection onto a plane which has diameter $|Q|/\pi$.

[63]This result, well known under the name *Buffon's needle*, gives a Monte Carlo method for approximating the number π.

c) [2] Prove that every space polygon $Q \subset \mathbb{R}^3$ of length $|Q|$ has a projection onto a plane which lies in a circle or diameter $|Q|/\pi$.

Exercise 24.9. $\diamondsuit$ *a*) [1] Prove that the closure of every connected component of $\mathrm{sh}(Q)$ is a convex polytope.
b) [1-] Let $L \subset \mathbb{R}^3$ be a disjoint union of space polygons which form a nontrivial link. Extend the definition of the second hull to links. Prove that $\mathrm{sh}(L) \neq \varnothing$.

Exercise 24.10. $\diamondsuit$ *a*) [1-] Find a simple space polygon in $\mathbb{R}^3$ with a disjoint second hull.
b) [1-] Find an unknotted space polygon in $\mathbb{R}^3$ with a nonempty second hull.
c) [1] Find an unknotted space polygon in $\mathbb{R}^3$ such that its second hull contains a unit ball.

Exercise 24.11. [2+] In the conditions of the second hull theorem (Theorem 24.10), prove that the closure of $\mathrm{sh}(Q)$ intersects Q.

Exercise 24.12. [2] Let L be a link with n components such that every two of them are linked. Prove that there exists a point $x \in \mathrm{conv}(L)$ such that every plane through x intersects at least $n/2$ components of L.

Exercise 24.13. (*Alexandrov*) [1+] Let C be a polygonal curve in $\mathbb{R}^2$ with endpoints x, y. Define the *total curvature* $\varkappa = \varkappa(C)$ to be the sum of exterior angles in the vertices of C. Suppose $\varkappa(C) < \pi$. Prove that

$$|C| \le \frac{|xy|}{\cos \varkappa},$$

and the equality holds only when C has two edges of equal length and angle $(\pi - \varkappa)$ between them. Extend the result to $\mathbb{R}^d$.

Exercise 24.14. $\diamondsuit$ *a*) [1+] Let C be a polygonal curve in $\mathbb{R}^2$ with endpoints x, y. Suppose C does not contain the origin O. Define the *visibility angle* $\varphi(C)$ to be the length of a projection of C onto a unit circle centered at O. Denote by α, β the angles between (Ox), (Oy) and C. Prove that

$$(*) \qquad \varphi(C) \le \varkappa(C) + \pi - \alpha - \beta.$$

b) [1+] Prove the analogue of *a*) for paths in $\mathbb{R}^3$. Formally, let C be a polygonal curve in $\mathbb{R}^3$ with endpoints x, y, and such that $O \notin C$. Let $\varphi(C)$ be the length of a projection of C onto a unit sphere centered at O, and let α, β be the angles between (Ox), (Oy) and C. Prove that $(*)$ holds. Conclude that $\varphi(Q) \le \varkappa(Q)$ for every (closed, simple) space polygon.
c) [1] Use the second hull theorem (Theorem 24.10) to prove that for every knotted space polygon $Q \subset \mathbb{R}^3$ there exists a translation Q' of Q such that $\varphi(Q') \ge 4\pi$. Use part *b*) to obtain the Fáry–Milnor theorem (Theorem 24.5).

Exercise 24.15. [1+] Let $Q \subset \mathbb{R}^3$ be a knotted space polygon. Prove that there exist three distinct points $x, y, z \in Q$ such that y is a midpoint of (x, z).

Exercise 24.16. (*Quadrisecants of knots*) Let $Q \subset \mathbb{R}^3$ be a knotted space polygon.
a) [1-] Prove that every point in Q lies on a line which intersects Q in at least two other points.
b) [2] Prove that there exists a line which intersects Q in at least four points lying on different edges. Such quadruples of points are called *quadrisecants*.
c) [2+] Prove that there there exists a quadrisecant with the order of points on a line alternating with the order of points on Q. These are called *alternating quadrisecants*. Conclude from here the Fáry–Milnor theorem (Theorem 24.5).

Exercise 24.17. (*Fáry theorem and generalizations*) *a*) [1+] Let $C \subset B_r$ be a general (possibly non-convex and self-intersecting) polygon in the plane, where B_r is a disk of radius $r > 0$. Denote by $|C|$ the length of C and by $\varkappa(C)$ the total curvature of C. Prove that $|C| < r\,\varkappa(C)$.
b) [2-] Let $C \subset B_r$ be a polygon in $\mathbb{R}^d$, where B_r is a ball of radius $r > 0$. As before, let $\varkappa(C)$ be the total curvature of C. Prove that $|C| < r\,\varkappa(C)$.
c) [2] Let $C, Q \subset \mathbb{R}^2$ be two polygons in the plane such that Q is convex and C is inside Q. Prove that the *average total curvature* is nonincreasing:

$$\frac{\varkappa(C)}{|C|} \geq \frac{\varkappa(Q)}{|Q|}.$$

Moreover, the equality holds only when C is a multiple of Q.

Exercise 24.18. Let $S_1, S_2 \subset \mathbb{R}^d$ be two PL-homeomorphic PL-surfaces. Define the *Fréchet distance*

$$\operatorname{dist}_F(C_1, C_2) \, = \, \inf_{\phi} \max_{x \in C_1} |x\,\phi(x)|,$$

where the infimum is over all piecewise linear homeomorphisms $\phi : S_1 \to S_2$.
a) [1+] Let $C_1, C_2 \subset \mathbb{R}^2$ be two convex polygons in the plane. Prove that

$$|C_1| - |C_2| \, \leq \, 2\pi \operatorname{dist}_F(C_1, C_2).$$

b) [2-] Let $C_1, C_2 \subset \mathbb{R}^2$ be two simple polygons in the plane. Prove that

$$|C_1| - |C_2| \, \leq \, \bigl[\varkappa(C_1) + \varkappa(C_2) - 2\pi\bigr] \operatorname{dist}_F(C_1, C_2).$$

c) [2] Let $C_1, C_2 \subset \mathbb{R}^3$ be two simple space polygons. Prove that

$$|C_1| - |C_2| \, \leq \, \pi\bigl[\varkappa(C_1) + \varkappa(C_2) - 2\pi\bigr] \operatorname{dist}_F(C_1, C_2).$$

d) [1-] Check that part *a*) of Exercise 24.17 follows from here, while part *b*) does not.

Exercise 24.19. (*Curvature of graphs in* $\mathbb{R}^3$) For a graph G denote by $\widehat{G}$ the graph on the same set of vertices, and with twice as many edges between every two vertices. For an embedding $\gamma : G \to \Gamma$, where $\Gamma \subset \mathbb{R}^3$, and an Eulerian circuit C in $\widehat{G}$, define

$$\varkappa_C(\Gamma) \, = \, \frac{1}{2}\,\varkappa\bigl(\gamma(C)\bigr).$$

This notion is called the *net total curvature* of Γ.
a) [2-] Find the analogue of Crofton's formula (Lemma 24.6) for the net total curvature
b) [1+] Define the *Kotzig transformation* on Eulerian circuits by changing the direction on a closed segment of a circuit. Prove that every two Eulerian circuits C_1, C_2 in G can be obtained from each other by a finite sequence of Kotzig transformations.
c) [1+] Use parts *a*) and *b*) to prove that the net total curvature is independent of the Eulerian circuit.

Exercise 24.20. Let $\Gamma \subset \mathbb{R}^3$ be a union of three non-intersecting space polygonal curves between two points x, y. Let $\varkappa(\Gamma)$ be the sum of all exterior angles between pairs of adjacent edges, where the exterior angles in x and y are weighted with $1/2$.
a) [1] Check that $\varkappa(\Gamma)$ is equal to the net total curvature of Γ.
b) [1+] Prove that $\varkappa(\Gamma) \geq 3\pi$.
c) [2-] Let G be a graph with two vertices and three edges between them, and let Γ be an embedding of G in $\mathbb{R}^3$. Prove that if $\varkappa(\Gamma) < 4\pi$, then the Γ is unknotted.

Exercise 24.21. *a*) [2-] Prove that the total length of a net (of inextensible strings) which holds a sphere is at least 3π.[64]
b) [1] Show that the bound in *a*) is tight.
c) [2] Suppose a net holds a sphere even if any k links are broken. Prove that the total length of such net is at least $(2k+3)\pi$.

24.8. **Final remarks.** Theorem 24.1 goes back to Segre (1937) and was subsequently rediscovered several times. This proof is given in [ChaK]. Theorem 24.4 was proved by Fenchel in 1929, and was extended by Borsuk in 1947 (see Exercise 24.4). A continuous version of the proof above was given by Liebmann and later rediscovered by Horn (see [Horn] for a simple proof and references).

We refer to [Ada2, Sos] for an accessible introduction to knot theory. Theorem 24.5 was discovered independently by Fáry and Milnor. In our proof we follow the sketch in [Cher], which is close to the original proof in [Mil1]. The linked tube problem in Subsection 24.4 was proposed by F. Gehring in 1973. The proof we present is due to M. Ortel [Can+] (see also [EdeS]). We refer to [Can+] for various extensions and further references.

The combinatorial proof of the Fáry–Milnor theorem presented in Subsection 24.5 is due to Alexander and Bishop [ABi]. They give additional arguments extending the result to general curves and making the inequality in the theorem strict. The result on alternating quadrisecants in the Exercise 24.16 is due to Denne [Den] and can be viewed as yet another proof of the Fáry–Milnor theorem.

The second hull theorem (Theorem 24.10) is proved in [CKKS], where various extensions and applications are also presented. These include the extension to nontrivial links (see Exercise 24.9) and another connection to the Fáry–Milnor theorem (see Exercise 24.14).

[64]The precise definition is a non-trivial part of the exercise. For a simple example of a net which holds a sphere, take vertices of an equilateral tetrahedron inscribed into $\mathbb{S}^2$ and connect them by geodesics. The total length of these geodesics is equal to $12\arcsin\sqrt{\frac{2}{3}} \approx 3.65\pi$.

25. Geometry of convex polyhedra: basic results

We begin the study of polyhedral surfaces with two most basic results: *Euler's formula* and the *Gauss–Bonnet theorem*. We use bits and pieces of spherical geometry and apply the results to closed geodesics on convex polyhedra (see Section 10).

25.1. **Euler's formula.** Let $P \subset \mathbb{R}^3$ be a convex polytope, and let $V, E, \mathcal{F}$ denote the set of vertices, edges, and faces of P, respectively. The classical Euler's formula in this case is the following result.

Theorem 25.1 (Euler's formula). $|V| - |E| + |\mathcal{F}| = 2$.

Even though there is a straightforward inductive proof, the spherical geometry proof below will be a helpful introduction to the method. Note also that in Section 8 we already proved the much more general Dehn-Sommerville equations for all simple polytopes (Theorem 8.1).

Proof. Move P so that it contains the origin O and consider the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ with the center at O. Consider a *radial projection* of the surface ∂P onto $\mathbb{S}^2$. In other words, for every edge $(v,w) \in E$ draw an intersection of $\mathbb{S}^2$ and a (planar) cone over (v,w) starting from O. This gives an arc $(v',w') \subset \mathbb{S}^2$ of a great circle. Clearly, every k-gonal face in P becomes a spherical k-gon.

Let $F \in \mathcal{F}$ be an i-gonal face of the polytope, let F' denotes the corresponding spherical i-gon, and let $\alpha(F')$ be the sum of the angles of F'. By Girard's formula for polygons (see Theorem 41.2) we have:

$$\alpha(F') \,=\, \operatorname{area}(F') \,+\, (i-2)\pi\,.$$

Denote by f_i the number of i-gonal faces in $\mathcal{F}$ and sum the above equation over all faces $F \in \mathcal{F}$. Since the sum of the angles on a sphere in the vertex image v' is 2π, for all $v \in V$, we have:

$$2\pi \cdot |V| \,=\, \operatorname{area}(\mathbb{S}^2) \,+\, \sum_i f_i \cdot (i-2)\pi \,=\, 4\pi - 2\pi \cdot |\mathcal{F}| \,+\, \pi \cdot \sum_i i\, f_i\,.$$

On the other hand, $\sum_i i\, f_i \,=\, 2\,|E|$, since every edge belongs to exactly two faces. Substituting this into above equation gives Euler's formula. $\square$

Of course, Euler's formula holds in much greater generality, e.g., for all planar maps, or in higher dimensions (see Section 8). Here is a simple corollary which will prove useful.

Corollary 25.2. *Every simplicial polytope with n vertices has $3n-6$ edges and $2n-4$ faces. More generally, this holds for every triangulated surface homeomorphic to a sphere $\mathbb{S}^2$.*

Proof. Suppose P has k faces. Then the number of edges is $\frac{3}{2}k$, and by Euler's formula $k - \frac{3}{2}k + n = 2$. Solving this for k gives the result. $\square$

25.2. **Gauss–Bonnet theorem.** Let $P \in \mathbb{R}^3$ be a convex polytope with the set of vertices $V = \{v_1, \ldots, v_n\}$. Denote by $\alpha_i = \alpha(v_i)$ the sum of the face angles around v_i and let $\omega_i = 2\pi - \alpha_i$ be the (*Gaussian*) *curvature* of v_i. The sum of curvatures at all vertices is called the *total curvature* of the polytope.

Theorem 25.3 (Gauss–Bonnet)**.** *Let $\omega_1, \ldots, \omega_n$ be the curvatures of vertices of a convex polytope $P \subset \mathbb{R}^3$. Then $\omega_1 + \ldots + \omega_n = 4\pi$.*

Observe that ω_i is an intrinsic rather than extrinsic parameter of vertices of a closed convex surface $S = \partial P$. In view of the theorem, one can think of $\omega_i/4\pi$ as a point concentrated measure on S. Here is a straightforward proof of the theorem based on a counting argument. A more enlightening proof based on spherical geometry is given in the next subsection.

Proof. Triangulate the faces of P. By Corollary 25.2, the resulting triangulation has $2n - 4$ triangles. Thus, the total sum of face angles $A = (2n - 4)\pi$. We conclude $\omega_1 + \ldots + \omega_n = 2\pi n - A = 4\pi$. $\square$

25.3. **Spherical geometry proof.** Now we present a different proof of the Gauss–Bonnet theorem (Theorem 25.3), based on spherical geometry. Not only this proof is more insightful and "from the book", it also leads to some useful extensions.

For every vertex $v_i \in V$ of polytope P, consider all planes H containing v_i, such that the whole polytope lies on one side of H. Such planes are said to be *supporting* vertex v_i. Taking normals $\boldsymbol{u}_H$ (unit vectors $\perp H$) these planes define a region R_i on a unit sphere $\mathbb{S}^2$. Clearly, these regions are disjoint (except for the boundary) and they cover the whole sphere: $\mathbb{S}^2 = \cup_i R_i$. Indeed, taking any plane H and moving it from infinity towards the polytope must hit some (perhaps more than one) vertex first (see Figure 25.1).

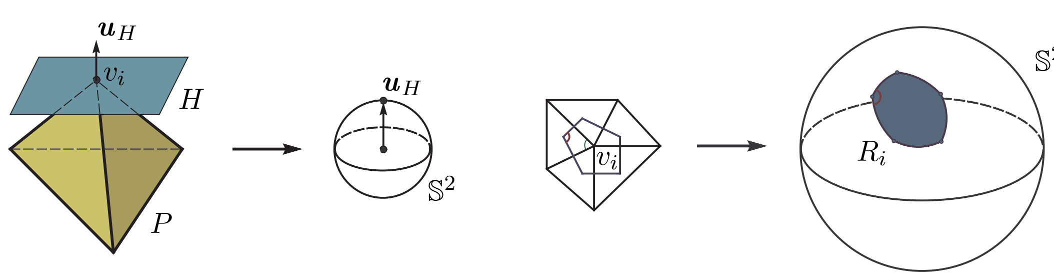

FIGURE 25.1. Normal $\boldsymbol{u}_H$ to plane H. Spherical k-gon $R_i \subset \mathbb{S}^2$ corresponding to vertex v_i of the polytope.

Suppose v_i is a vertex of degree k (i.e., has k adjacent edges). Denote by $\beta_1, \ldots, \beta_k$ the corresponding face angles. Observe that R_i is a spherical k-gon with sides orthogonal to projections of edges onto $\mathbb{S}^2$ (see Figure 25.1). Therefore, polygon R_i has angles $\pi - \beta_1, \ldots, \pi - \beta_k$ and by Girard's formula for spherical k-gons we have:

$$\operatorname{area}(R_i) \, = \, \sum_{j=1}^{k} (\pi - \beta_j) \, - \, (k-2) \cdot \pi \, = \, 2\pi - \alpha_i \, = \, \omega_i \, .$$

From here, for the total area of a sphere we have:

$$4\pi \, = \, \operatorname{area}(\mathbb{S}^2) \, = \, \sum_{i=1}^{n} \operatorname{area}(R_i) \, = \, \sum_{i=1}^{n} \omega_i \, ,$$

which finishes the proof of the Gauss–Bonnet theorem.

25.4. **Gauss–Bonnet theorem in higher dimensions.** One useful thing about a good proof is its robustness and ability to work in generalizations. It is even better if the proof leads to the "right" definitions and then becomes a complete triviality. This is what happens with the generalization of the Gauss–Bonnet theorem to higher dimensions.

Let $C \subset \mathbb{R}^d$ be a cone starting at the origin O, and let $S = \mathbb{S}^{d-1} \subset \mathbb{R}^d$ be a unit sphere centered at O. Define the *solid angle* $\sigma(C)$ to be $(d-1)$-dimensional volume of the intersection:

$$\sigma(C) \, := \, \operatorname{area}(C \cap S) \, = \, \operatorname{vol}_{d-1}(C \cap S).$$

Now, let $P \subset \mathbb{R}^d$ be a convex polytope with the set of vertices $V = \{v_1, \ldots, v_n\}$. Consider cones C_i starting in the vertices $v_i \in V$ of the polytope and spanned by the edges. As the proof above shows, the "right measure" of C_i is not $\sigma(C_i)$, but rather $\sigma(C_i^*)$, the solid angle of the dual cone defined as follows.

The *dual cone* C_i^* is the cone of the normals $\boldsymbol{u}_H$ to the hyperplanes $H \subset \mathbb{R}^d$ supporting v_i, i.e., hyperplanes H containing v_i and having P in one half-space of H (as in Figure 25.1 above). Define the *curvature* of a cone $\omega(C) = \sigma(C^*)$, and let $\omega_i = \omega(C_i) = \sigma(C_i^*)$. Finally, denote by Σ_d the $(d-1)$-dimensional volume of the unit sphere: $\Sigma_d = \operatorname{area}(\mathbb{S}^{d-1})$.

Theorem 25.4 (Gauss–Bonnet in $\mathbb{R}^d$)**.** *Let $\omega_1, \ldots, \omega_n$ be the curvatures of vertices of a convex polytope $P \subset \mathbb{R}^d$. Then: $\omega_1 + \ldots + \omega_n = \Sigma_d$.*

The proof is completely straightforward and follows the steps in the proof above. Consider the regions $R_i = C_i^* \cap S$, and check that

$$\sum_{i=1}^{n} \omega_i \, = \, \sum_{i=1}^{n} \operatorname{area}(R_i) \, = \, \operatorname{area}(S) \, = \, \Sigma_d \, .$$

We omit the details.

25.5. **Infinite polyhedra.** Let $P \subset \mathbb{R}^d$ be a (possibly unbounded) *convex polyhedron* defined as an intersection of a finite number of half-spaces. Convex polytopes and cones are examples of convex polyhedra. Below we show how to extend Gauss–Bonnet theorem to convex polyhedra in every dimension.

We assume that convex polyhedron P contains the origin O. Consider all *rays* ℓ (half-lines) starting at O and going to infinity and contained inside the polytope: $\ell \in P$. Denote by C_P the cone spanned by these rays. It is easy to see that C_P is unchanged under translations of P, still containing O. Cone C_P is called the *base cone of* P. Define the curvatures ω_i of vertices v_i of P as for the polytopes. Similarly, let $\omega(C_P) = \sigma(C_P^*)$ be the curvature of the base cone.

Theorem 25.5 (Total curvature of convex polyhedra). *Let $\omega_1, \ldots, \omega_n$ be the curvatures of vertices of a convex polyhedron $P \subset \mathbb{R}^3$. Then: $\omega_1 + \ldots + \omega_n = \omega(C_P)$.*

In particular, the theorem implies that $\omega(C_P)$ depends only on P and is invariant under the rigid motions. The proof is straightforward once again. Note that the spherical region $R_P = C_P^* \cap S$ corresponds to the set normals of hyperplanes supporting vertices of the polyhedron P.

Note that $\omega(C_P)$ is well defined even when P is *degenerate*, i.e., has dimension $< d$. For example, suppose polyhedron $P \subset \mathbb{R}^3$ has a base cone C_P which consists of a single ray. Such convex polyhedra are called *convex caps*. From the theorem, for the total curvature of convex caps we have $\omega(C_P) = \text{area}(\mathbb{S}^2)/2 = 2\pi$. More generally, for all infinite convex polyhedra $P \subset \mathbb{R}^3$, we have $\omega(C_P) \le 2\pi$, where the equality holds only for convex caps.

25.6. **Back to closed geodesics.** Let S be the surface of a convex polytope $P \subset \mathbb{R}^3$. A *surface polygon* $M = [w_1, \ldots, w_k]$ is a collection of shortest paths (w_i, w_{i+1}), $i \in [k-1]$, on the surface S. By Proposition 10.1, the shortest paths may cross edges but not vertices, except when $w_i \in V$. In the intrinsic geometry of S the edges of P are "invisible," in a sense that the metric is flat along them.

As before, polygon M is *simple* if it is not self-intersecting, and is *closed* if $w_1 = w_k$. Denote by A the region on one side of M. Denote by $\omega(A)$ the sum of curvatures of vertices $v_i \in A$. Recall that M is a closed geodesic on S if only if M does not contain vertices of P and all surface angles of A at w_i are equal to π (see Section 10.2). The following result formalizes and extends the results on closed geodesics obtained earlier in Section 9.

Theorem 25.6 (Total curvature of simple closed geodesics). *Let M be a simple closed geodesic on the surface $S = \partial P$ of a convex polytope. Then the total curvature on each side is equal to 2π.*

Proof. Suppose $M \subset S$ is a surface k-gon and a region A on one side has ℓ interior vertices. A triangulation of A has

$$m = \big(2(k+\ell) - 4\big) - (k-2) = k + 2\ell - 2 \quad \text{triangles.}$$

Thus, the total sum $\alpha(A)$ of the face and boundary angles in A satisfies $\alpha(A) = m\pi$. On the other hand, summing over all vertices, we have $\alpha(A) = 2\pi\ell - \omega(A) - k\pi$. Together these equations imply the result. $\square$

The proposition implies that if there is no way to split the total curvature of all vertices into two equal parts, then there are no simple closed geodesics (see Section 9). Here is another result in this direction.

Corollary 25.7. *An infinite convex polyhedron $P \subset \mathbb{R}^3$ has a simple closed geodesic if and only if P is a convex cap.*

Proof. The corollary follows immediately from Theorem 25.5 and the observations above: $\omega_1 + \ldots + \omega_n = \omega(C_P) < 2\pi$ unless P is a convex cap. To prove the second part, suppose P is a convex cap. Consider a hyperplane H orthogonal to a ray C_P, and such that all vertices of P lie on one side of H. Then the intersection $M = H \cap \partial P$ is a closed geodesic. $\square$

The following is a somewhat unexpected application of the proposition.

Theorem 25.8 (Cohn-Vossen)**.** *Two simple closed geodesics on a surface of a convex polytope either intersect or have equal lengths.*

For example, a hexagon and a square shown in Figure 10.4 are two closed geodesics on a cube of different length, and they do, in fact, intersect. The theorem also allows an easy calculation of the lengths of geodesics. For example, in a regular tetrahedron in the same figure, the rectangular geodesics can be sled down towards an edge. Thus its length is twice the edge length.

Proof. Let P be a convex polytope, and let M_1, M_2 be two simple closed geodesics on the surface S of P. As before, let V be the set of vertices of P. Assume that M_1 and M_2 do not intersect. Then polygons M_1 and M_2 divide the surface S into three regions: B_1, B_2 and C, such that $\partial B_1 = M_1$, $\partial B_2 = M_2$, and $\partial C = M_1 \cup M_2$. By proposition, the sum of curvatures inside B_1 and B_2 is equal to 2π, so by the Gauss–Bonnet theorem region C does not have interior points.

In the language of the graph $G = (V, E)$ of P, observe that the sets of interior vertices $V_1 = V(B_1)$ and $V_2 = V(B_2)$ are disjoint, nonempty and contain all graph vertices: $V = V_1 \cup V_2$. Thus, there exists an edge $(x, y) \in E$, such that $x \in V_1$, $y \in V_2$ Since both M_1 and M_2 must intersect (x, y), denote by z_1, z_2 points of intersection. Now cut the region C with an interval (z_1, z_2), and unfold C on a plane (see Figure 25.2). Each geodesic becomes a straight interval (z_1, z_1') and (z_2, z_2'), where (z_1', z_2') used to be identified with (z_1, z_2). Thus, the unfolding is a plane 4-gon $Z = [z_1 z_1' z_2' z_2]$ with $|z_1 z_2| = |z_1' z_2'|$. Since M_1 and M_2 are both geodesics, we have:

$$\angle\, z_1' z_1 z_2 + \angle\, z_1 z_1' z_2' = \angle\, z_1 z_2 z_2' + \angle\, z_1' z_2' z_2 = \pi\,.$$

Therefore, Z is a parallelogram and geodesics M_1 and M_2 have equal length. $\square$

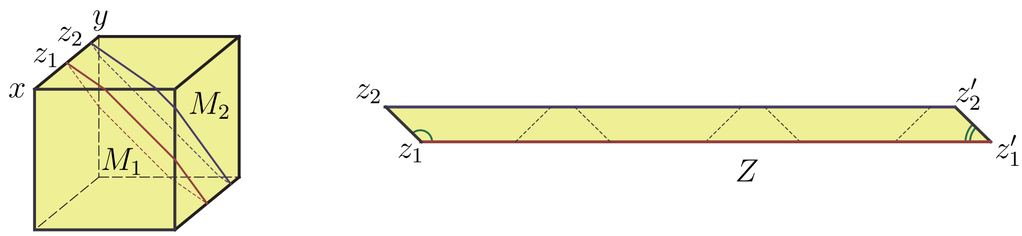

Figure 25.2. Two non-intersecting closed geodesics M_1 and M_2 on a cube, and an unfolding Z of the region between them.

25.7. **Exercises.**

Exercise 25.1. $\Diamond$ [1] Extend Euler's formula and the Gauss–Bonnet theorem to general closed polyhedral surfaces (of any genus).

Exercise 25.2. $\Diamond$ *a* [1] Let $S = \partial P$ be the surface of a convex polytope P, and let $A \subset S$ be a polygonal region. Denote by $\alpha(A)$ the sum of the interior surface angles at the boundary vertices. Suppose the boundary $M = \partial A$ contains r surface polygons, and k vertices. Then:

$$\alpha(A) \, = \, (n + 2r - 4) \cdot \pi \, + \sum_{v_i \in A} \omega_i \, .$$

b) [1] Deduce the Gauss–Bonnet theorem from here.

Exercise 25.3. (*From Gauss–Bonnet back to Euler*) $\Diamond$ [1] Consider a polygonal region $A \subset \partial P$ obtained as an ε-neighborhood of all edges of the polytope (see Figure 25.3), for $\varepsilon > 0$ small enough. Apply the previous exercise to A and derive Euler's formula.

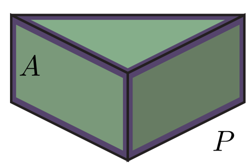

Figure 25.3. Region A obtained as an ε-neighborhood of edges of P.

Exercise 25.4. $\Diamond$ [1-] Let $C_v \subset \mathbb{R}^d$ be a convex cone with vertex v. Check that the dual cone satisfies: $C_v^* = \{w \in \mathbb{R}^3 \mid \langle w, v\rangle \le 0 \text{ for all } v \in C\}$.

Exercise 25.5. (*Monotonicity of cone curvature*) $\Diamond$ *a*) [1-] Suppose $C, D \subset \mathbb{R}^d$ are convex cones such that $C \subset D$. Prove that $\omega(C) \ge \omega(D)$.
b) [1] Moreover, $\omega(C) = \omega(D)$ only if the cone C is a translation of D.
c) [1-] Suppose $P, Q \subset \mathbb{R}^d$ are convex polyhedra such that $P \subset Q$. Prove that $\omega(P) \ge \omega(Q)$.

Exercise 25.6. *a*) [1] Suppose all faces of a convex polytope $P \subset \mathbb{R}^3$ are centrally symmetric. Prove that P has at least eight simple vertices (vertices adjacent to exactly three edges). Note that this is tight since the cube has exactly eight simple vertices.
b) [1-] Prove that there is no upper bound of the number of faces.

Exercise 25.7. [1] Let $P \subset \mathbb{R}^3$ be a convex polytope whose vertices have degrees ≥ 4. Prove that it has at least eight triangular faces.

Exercise 25.8. Let $P \subset \mathbb{R}^3$ be a convex polytope. Denote by $c(P)$ the sum of the number of triangular faces and the number of vertices of degree 3.
a) [1-] Check that the graph-theoretic proof in Subsection 11.2 implies that $c(P) \geq 1$.
b) [1] Use Euler's formula to prove that $c(P) \geq 8$.

Exercise 25.9. *a*) [1] Suppose the surface S of a convex polytope $P \subset \mathbb{R}^3$ can be decomposed into a finite number of rectangles. Prove that P has at most eight vertices.
b) [1] Suppose S can be decomposed into a finite number of rectangles. Prove that P has at most twelve vertices.

Exercise 25.10. (*de Gua's formula*) [1] Let $P \subset \mathbb{R}^3$ be a convex polytope. Denote by A and B the sums of all solid and dihedral angles, respectively. Prove that

$$2B - A = 2\pi(m-2),$$

where m is the number of faces in P.

Exercise 25.11. Let $P \subset \mathbb{R}^3$ be a convex polytope containing the origin O. For a facet F of P, denote by $\alpha(F)$ the sum of the angles of F and by $\beta(F)$ the sum of the angles of the projection of F onto a unit sphere centered at O. Finally, let $\omega(F) = \beta(F) - \alpha(F)$. Prove that

$$\sum_{F \subset P} \omega(F) = 4\pi.$$

Exercise 25.12. (*Equihedral tetrahedra*) $\Diamond$ [1+] Let $\Delta \subset \mathbb{R}^3$ be a tetrahedron. Prove that the following conditions are equivalent:
(*i*) all faces of Δ are congruent triangles,
(*ii*) all faces have equal perimeter,
(*iii*) all vertices have equal curvature,
(*iv*) the opposite edges have equal dihedral angles,
(*v*) all solid angles are equal.
Such Δ are called *equihedral tetrahedra*.

Exercise 25.13. [1+] The edges of a convex polytope $P \subset \mathbb{R}^3$ are colored with two colors. Prove that there exists a vertex in P such that the adjacent edges either have the same color or have two color intervals (in cyclic order).

Exercise 25.14. [1] Let Γ be the graph of a convex polytope $P \subset \mathbb{R}^3$ and let $\mathcal{O}$ be an orientation of edges. As always, let V and $\mathcal{F}$ denote the set of vertices and faces of P. For a vertex $v \in V$, denote by $\operatorname{ind}(v)$ to be $1 - c(v)/2$, where $c(v)$ is the number of changes in the orientation of edges adjacent to v (in cyclic order). Similarly, for a face $F \in \mathcal{F}$, denote by $\operatorname{ind}(F)$ to be $1 - c(F)/2$, where $c(F)$ is the number of changes in the orientation (clockwise vs. counterclockwise) of edges of F. Prove that

$$\sum_{v \in V} \operatorname{ind}(v) + \sum_{F \in \mathcal{F}} \operatorname{ind}(F) = 2.$$

Exercise 25.15. (*Discrete Poincaré–Hopf index theorem*) *a*) [1] Let $P \subset \mathbb{R}^3$ be a convex polytope. We say that an orientation of edges of P is *balanced* if every vertex has at least one ingoing and one outgoing edge (i.e., the oriented graph of P has no sinks and no sources). Prove that the edges of at least two faces of P form oriented cycles.
b) [1] Generalize part *a*) to orientable 2-dimensional polyhedral surfaces of higher genus.

Exercise 25.16. [1] Prove that the inverse to Theorem 25.6 does not hold, i.e., construct a convex polytope P such that a subset of vertices has sum of curvatures equal to 2π, but which has no simple closed geodesics.

Exercise 25.17. [1+] Let $P \subset \mathbb{R}^3$ be an unbounded polyhedron with $\sigma(C_P) > 0$. Prove that between every two points in $S = \partial P$ there is only a finite number of geodesics.

Exercise 25.18. Let γ denote the largest dihedral angle in a convex polytope $P \subset \mathbb{R}^3$ with n vertices.
a) [1] Suppose $\gamma < \pi/2$. Prove that P is a tetrahedron.
b) [1] Suppose $\gamma \leq \pi/2$. Prove that $n \leq 8$.
c) [1+] Suppose $\gamma \leq \pi - \varepsilon$, for some $\varepsilon > 0$. Disprove: n is bounded as a function of ε.

Exercise 25.19. [1+] Suppose integers $n, m, k > 0$ satisfy $n - m + k = 2$, $n \geq 2k - 4$, and $k \geq 2n - 4$. Give a direct construction of a convex polytope in $\mathbb{R}^3$ with n vertices, m edges and k faces.

Exercise 25.20. (*Eberhard's theorem*) *a*) [1] Use Euler's formula to show that for every simple polytope $P \subset \mathbb{R}^3$, we have:

$$\sum_{i \geq 3} (6 - i) \cdot f_i(P) \, = \, 12,$$

where $f_i(P)$ is the number of faces with k sides.
b) [2] Suppose $(f_3, f_4, \ldots)$ is an integer sequence which satisfies the above equation. Use the Steinitz theorem (Theorem 11.1) to prove that there exists a simple polytope $P \subset \mathbb{R}^3$, such that $f_i(P) = f_i$ for all $i \neq 6$.

25.8. **Final remarks.** Our presentation of Euler's and Gauss–Bonnet theorems is a variation on a standard theme (see [SCY, §57], [Hop2] and [Ber1, §12.7]). The curvature ω_i is the discrete version of the classical *Gaussian curvature*, and is usually called the *angle defect*.

The Gauss–Bonnet theorem for 3-dimensional convex polyhedra (Theorem 25.3) is due to Descartes [Fed], so it precedes Euler's formula by over a century. In fact, the spherical geometry proof of Euler's theorem is due to Legendre, and is probably the oldest rigorous proof (see [Bla2, Crom] for more on the history of the subject). The extension of the Gauss–Bonnet theorem to general polyhedra (Theorem 25.5) will be used at a technical point in the proof of the main result in Section 35. See [A2, §1.5] for 3-dimensional treatment. The higher dimensional generalization (Theorem 25.4) is due to Shephard [Grü4].

The idea of taking an ε-neighborhood in Exercise 25.3 is standard in geometry. In a similar context it leads to an advanced generalization of the Gauss–Bonnet theorem to all even-dimensional Riemannian manifolds [Gray, §5.5]. We refer to [Gray, §5.6] for the history of the Gauss–Bonnet theorem, various extensions and further references. Theorems 25.6 and 25.8 are essentially due to Cohn-Vossen [CV]. Although he does not consider convex polyhedral surfaces, his study of geodesics on Riemann surfaces have polyhedral analogues (see [Bus] for further results and references).

The formula in Exercise 25.10 is due to de Gua (1783) and can be generalized to higher dimensions by scaling the angles and taking the alternating sum (see [Grü4]). For an interesting discussion on the nature of Euler's formula see also [Lak].

26. Cauchy theorem: the statement, the proof, and the story

In this section we give the classical proof of the Cauchy theorem, of Alexandrov's extension, and of Stoker's converse. Much of the rest of the book is based on this section.

26.1. **Polytopes are surprisingly rigid.** Let P and P' be two combinatorially equivalent polytopes in $\mathbb{R}^3$ (see Section 12), with vertices $v_1, \ldots, v_n$ and $v'_1, \ldots, v'_n$, and faces $F_1, \ldots, F_m$ and $F'_1, \ldots, F'_m$, respectively. From here on, we will always assume that the equivalence maps the corresponding vertices and the corresponding faces, i.e., $v_i \to v'_i$ and $F_j \to F'_j$, all i and j. Finally, suppose the corresponding vertices belong to the corresponding faces: $v_i \in F_j$ if and only if $v'_i \in F'_j$. We say that two faces F_j and F'_j are *equal* if they are equal as polygons, i.e., the corresponding edges have equal lengths and the corresponding face angles are equal.

Theorem 26.1 (Cauchy theorem)**.** *Let P and $P' \subset \mathbb{R}^3$ be two convex polytopes as above, with all faces equal polygons: $F_j \simeq F'_j$, for all $1 \le j \le m$. Then P and P' are equal polytopes: $P \simeq P'$, i.e., P can be moved into P' by a rigid motion.*

There is a way to think of the Cauchy theorem in terms of graph realizability. Suppose P is simplicial, i.e., all faces are triangles. Then faces are determined by the edge lengths. Formally, let G be the graph of P, and let $L : E \to \mathbb{R}_+$ be the length function on edges of P. The Cauchy theorem implies that if convex polytope P' has the same pair (G, L), then $P \simeq P'$. In other words, polytope P with (G, L) is *unique* up to a rigid motion. This *uniqueness property* is an important way of thinking of Theorem 26.1.

More generally, let T be a graph of some triangulation of faces of P, i.e., a graph on the set V of vertices of P with the set of edges $H \supset E$ being edges of the polytope and some diagonals in T. Denote by $L : H \to \mathbb{R}_+$ be the corresponding length function. Now the full power of the Cauchy theorem implies that if convex polytope P' has the same pair (T, L), then $P \simeq P'$.

A more traditional way to think of the Cauchy theorem is in terms of (*continuous*) *rigidity*. A continuous family of polytopes $\{P_t : t \in [0,1]\}$ is called a *continuous deformation* of P_0 into P_1 if under deformation the faces remain equal. The Cauchy theorem now implies the following result.

Corollary 26.2 (Rigidity of convex polytopes)**.** *Let $\{P_t : t \in [0,1]\}$ be a continuous deformation of convex polytopes. Then P_t is a rigid transformation from P_0 to P_1.*

While the corollary follows immediately from the Cauchy theorem, it is easier to prove, has many applications and generalizations, some of which will be mentioned below.

Remark 26.3. Let us calculate the degrees of freedom of convex polytopes to see if the Cauchy theorem makes sense. A simplicial polytope $P \subset \mathbb{R}^3$ is determined by its $3n$ coordinates. The space of rigid motions has $\dim \mathrm{O}(3, \mathbb{R}) = 6$ dimensions (three rotations around pairs of axes and three translations along the axes). Thus, up to a rigid motion, polytope P has $3n - 6$ degrees of freedom. Since the number of edges is $|E| = 3n - 6$, this makes sense (see Corollary 25.2). If we had $|E| < 3n - 6$, we would have a hard time showing that they have a unique real solution.[65] While this is not a formal argument, the 'magic number' $(3n - 6)$ is good to keep in mind.

26.2. **What to expect when you are expecting a proof.** Surprise! There are many interesting proofs of the Cauchy theorem and its relatives. The classical proof (essentially due to Cauchy) will occupy the rest of this section. As the reader shall see in the next section, the Cauchy theorem can be extended and generalized in a number of ways. We outline a few of these directions and prove some extensions.

In fact, much of the rest of the book is also connected to the Cauchy theorem, one way or another, but the following seven sections are directly related. In Sections 28, 29 we present two other proofs of the Cauchy theorem, one of a 'local', while another of 'global' nature. In Section 30 we present several examples of non-unique and flexible polyhedra. By the Cauchy theorem these polyhedra must be non-convex, but they are interesting and enlightening nonetheless.

Later, in Section 31, we introduce an algebraic approach to rigidity of polyhedra. This approach leads us to two proofs of Dehn's rigidity theorem (in Sections 32 and 33) and eventually culminates in Section 34 with the proof of the bellows conjecture (Theorem 31.2). Let us note that the final remarks subsections in these sections contain references not listed in other sections, so the reader is advised to read all of them to get the "big picture".

26.3. **The traditional proof of the Cauchy theorem.** Let P and P' be two polytopes as in the Cauchy theorem (Theorem 26.1). Denote by $G = (V, E)$ the graph of P (and of P'), and by α_e and α'_e the dihedral angles in polytopes P and P' of the corresponding edge $e \in E$. Let us label every edge $e \in E$ with $(+)$ if $\alpha_e < \alpha'_e$, with $(-)$ if $\alpha_e > \alpha'_e$, and with (0) if $\alpha_e = \alpha'_e$. Now let us analyze this labeled graph G.

For a vertex v in P, consider the labels of the edges containing v, written in cyclical order, and ignoring the zero labels. Denote by m_v the number of *sign changes*, i.e., pairs of adjacent edges (cyclically, skipping all 0's) with labels of different signs (see e.g., Figure 26.1). Clearly, m_v is always even.

Lemma 26.4 (Sign changes lemma)**.** *For every vertex $v \in V$, the number of sign changes m_v among pairs of edges containing v is at least four, unless all labels of edges containing v are zero.*

Proof. Let C_v be the cone spanned by edges of the polytope P containing v. Think of C_v as an infinite cone starting at v and containing P. Now let Q_v be the spherical convex polygon obtained by intersection of C_v with a unit sphere $\mathbb{S}^2$ centered at v. The sides of Q_v are equal to the face angles, while angles are equal to dihedral angles in edges containing v. Note also that Q_v lies in a hemisphere $\mathbb{S}^2_+$. Similarly, let Q'_v be the corresponding polygon obtained from P'. From above, Q_v and Q'_v have equal

[65] The system would have infinitely many complex realizations of (G, L), but that make very little sense in the context of convex polytopes over the real numbers.

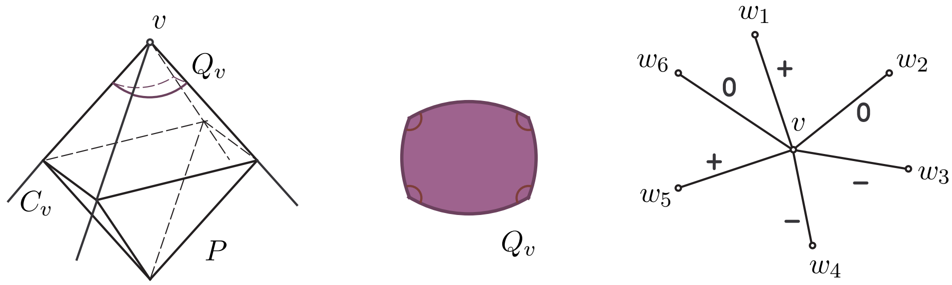

FIGURE 26.1. Spherical polygon Q_v corresponding to vertex v. A vertex v in graph G with $m_v = 2$ sign changes.

edge lengths but may have different angles. The result now follows immediately from the spherical Legendre–Cauchy lemma (Theorem 22.2). □

We are now ready to prove the Cauchy theorem.

Proof of Theorem 26.1. Assume for now that the labeling has no zeroes. Denote by $M = \sum_{v \in V} m_v$ the total number of pairs of edges with a sign change. By the lemma, we have a lower bound on the number of sign changes: $M \geq 4 \cdot |V|$.

Now let us get an upper bound on M by counting the number of sign changes summing over all faces $F \in \mathcal{F}$. As before (see Section 25), let f_k denotes the number of faces with k sides. We have:

$$\begin{aligned} |\mathcal{F}| &= f_3 + f_4 + f_5 + f_6 + f_7 + f_8 + \ldots \quad \text{and} \\ 2\,|E| &= 3f_3 + 4f_4 + 5f_5 + 6f_6 + 7f_7 + 8f_8 + \ldots \end{aligned}$$

Therefore,

$$4\,|E| - 4\,|\mathcal{F}| = 2\,f_3 + 4\,f_4 + 6\,f_5 + 8\,f_6 + 10\,f_7 + 12\,f_8 + \ldots$$

Observe that the number of sign changes in a $(2r+1)$-gon face is at most $2r$, the same as in a $2r$-gon (see Figure 26.2). From here we have:

$$\begin{aligned} M &\leq 2\,f_3 + 4\,f_4 + 4\,f_5 + 6\,f_6 + 6\,f_7 + 8\,f_8 + \ldots \\ &\leq 2\,f_3 + 4\,f_4 + 6\,f_5 + 8\,f_6 + 10\,f_7 + 12\,f_8 + \ldots \\ &\leq 4\,|E| - 4\,|\mathcal{F}| = 4\,|V| - 8\,, \end{aligned}$$

where the last equality follows from Euler's formula. Since $M \geq 4\,|V|$, we get a contradiction.

Now suppose some edges in G are labeled zero, but not all of them. Remove zero labeled edges from G and let H be a connected component in the resulting graph (see Figure 26.2). Note that for the graph H we also have Euler's formula, since the removal of non-bridge edges (i.e., those edges whose removal does not make H disconnected) decreases only the number of faces and edges, so it does not change $|V| - |E| + |\mathcal{F}|$. Then the argument above gives a contradiction once again.

From above, we conclude that H is empty and all edges are labeled zero. In other words, polytopes P and P' have equal dihedral angles between the corresponding faces. This immediately implies the polytopes are equal, i.e., can be moved by a rigid motion. □

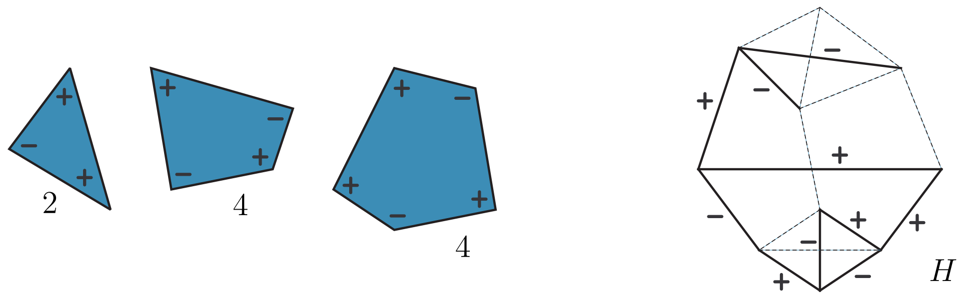

FIGURE 26.2. Maximum number of sign changes in different faces, and graph H obtained after removal of zero labeled edges in G.

The proof above is important enough to be stated as a separate result, so that we can refer to it later on. Since the proof is essentially a graph-theoretic statement, let us phrase it as such:

Lemma 26.5 (Sign counting lemma)**.** *Suppose the edges of a plane simple graph are labeled with* 0*,* $(+)$ *and* $(-)$ *such that around each vertex either all labels are* 0 *or have at least four sign changes. Then all signs are* 0*.*

Here by *plane graph* we mean a graph which is already drawn on a plane, since otherwise, for graphs with low connectivity, the faces are not well defined. Note that the faces include the outside face as well.

26.4. **Better language for the Cauchy theorem.** As the reader can see, we use a rather clumsy way of saying that two polytopes are "assembled in the same way from the same faces". Let us straighten the language and restate the result.

Theorem 26.6 (Cauchy, restated)**.** *Let* P *and* $Q \subset \mathbb{R}^3$ *be two combinatorially equivalent convex polytopes whose corresponding faces are isometric. Then* P *and* Q *are isometric.*

Here by *isometry* between polytopes we mean that one can be mapped into another such that the pairwise distances between the corresponding points are always equal. This map is called the *isometry map*. Same for the isometry between faces, but here we are assuming that the isometry respects combinatorial equivalence.

Note that in $\mathbb{R}^3$ two polytopes are isometric if and only if they are equal, i.e., can be mapped into each other by a rigid motion. Beside sounding more scientific, the isometry is a useful concept in several generalizations of the Cauchy theorem.

Note also that the underlying theme in this version is the 'local $\Rightarrow$ global' property of isometry, since we are saying that isometry of the faces (all lying on a surface of polytopes) implies global isometry. As we will see in Section 30, this principle does not hold for non-convex polytopes. However it does apply in many other convex situations.

Now let us redefine rigidity of convex polytopes and restate Corollary 26.2. We say that $\{P_t : t \in [0,1]\}$ is a *continuous deformation* of polytope P if the faces of P_t remain isometric, for all $t \in [0,1]$. We say that P is *(continuously) rigid* if in every continuous deformation $\{P_t\}$ of polytope P the polytopes P_t are isometric.

Corollary 26.7 (Rigidity of convex polytopes, revisited)**.** *Every convex polytope $P \subset \mathbb{R}^3$ is continuously rigid.*

This definition of rigidity is also robust enough to allow advanced generalizations. For simplicity, we drop 'continuous' and use 'rigidity' for the rest of the lectures[66].

26.5. **Parallel polytopes.** We start a short series of extensions of the Cauchy theorem (Theorem 26.1) with the following side result. While easily equivalent to the Cauchy theorem, it brings to light some properties of the proof given in Section 26.3.

Theorem 26.8 (Alexandrov)**.** *Let $P, Q \subset \mathbb{R}^3$ be two combinatorially equivalent convex polytopes with equal corresponding face angles. Then they have equal corresponding dihedral angles.*

This result easily implies the Cauchy theorem: faces and dihedral angles determine the whole polytope. In fact, we used this observation in the proof of the Cauchy theorem. On the other hand, the result is applicable to distinct polytopes, such as bricks $[a \times b \times c]$.

The proof of Theorem 26.8 is essentially an observation. Note that in the proof of the Cauchy theorem we never used the geometry of faces, except for the face angles. In addition to the (intrinsic) convexity of polyhedra, we use only only one global parameter: Euler's theorem, that is a consequence from the fact that the surfaces of convex polytopes are homeomorphic to a sphere. Thus, basically, in the proof of the Cauchy theorem, we first establish Theorem 26.8, and only then prove the result.

We say that two polytopes $P, Q \in \mathbb{R}^d$ are *parallel* if they are combinatorially equivalent and the corresponding facets are parallel. Clearly, every two combinatorially equivalent polytopes with equal corresponding dihedral angles can be made parallel by a rigid motion. Basically, one can view Theorem 26.8 as a local condition for being parallel, up to a global rotation.

Let us mention that the inverse to Theorem 26.8 is also true. Given two polytopes which are combinatorially equivalent and have equal corresponding dihedral angles we conclude that they are parallel, i.e., the corresponding faces lie on parallel planes. Therefore, the corresponding edges lie on parallel lines. Finally, the corresponding face angles are equal as angles between pairwise parallel lines.

26.6. **Face angles are not as good as you think they are.** Before we move to further generalizations of the Cauchy theorem, let us mention that from the point of view of convex polytopes, face angles are not a good way to define a polytope. In fact, they overdetermine the polytope, even up to translation of faces. Indeed, let us

[66]There are other kinds of rigidity, such as *static rigidity*, *infinitesimal rigidity*, *k-th order rigidity*, *global rigidity*, etc. (see [Con5]). We will define the first two in Sections 31 and 33.

compare the degrees of freedom for a simplicial polytope P with n vertices. First, there are

$$\sum_i i f_i \,=\, 2\,|E| \,=\, 2\,(3n-6)$$

face angles. These angles satisfy $|\mathcal{F}| = (2n-4)$ linear relations on the sums along faces[67]. In total, angles give $2\,|E| - |\mathcal{F}| = 4n-8$ degrees of freedom.

On the other hand, n vertices of a polytope $P \subset \mathbb{R}^3$ have $(3n-6)$ degrees of freedom (see Remark 26.3). There are $|\mathcal{F}| = (2n-4)$ degrees of freedom of translation of faces, $|\mathcal{F}| - 3 = 2n-7$ up to translations of the whole space $\mathbb{R}^3$. Thus there are $(3n-6)-(2n-7) = n+1$ remaining degrees of freedom. Comparing $n+1$ and $4n-8$ we obtain a number of (nonlinear) relations on face angles of polytopes, making face angles an unattractive way to define a polytope.

Let us illustrate this in a tetrahedron: 12 face angles in this case determine a tetrahedron up to rigid motions and expansion, a 5-dimensional space of realizations. Taking into account 4 linear relations on face angles, we conclude that there are three extra nonlinear equations on the angles. Let us compute one of these two equations (the others are similar).

In a tetrahedron as in Figure 26.3, denote the vertices by $i \in [4]$. Recall the *law of sines*:

$$\frac{|12|}{|23|} \,=\, \frac{\sin(132)}{\sin(213)}\,.$$

Applying this equation to all four faces, we have:

$$1 \,=\, \frac{|12|}{|23|}\cdot\frac{|23|}{|34|}\cdot\frac{|34|}{|14|}\cdot\frac{|14|}{|12|} \,=\, \frac{\sin(132)}{\sin(213)}\cdot\frac{\sin(243)}{\sin(324)}\cdot\frac{\sin(314)}{\sin(134)}\cdot\frac{\sin(124)}{\sin(142)}\,.$$

This is a non-linear equation on 8 angles, all linearly independent otherwise.

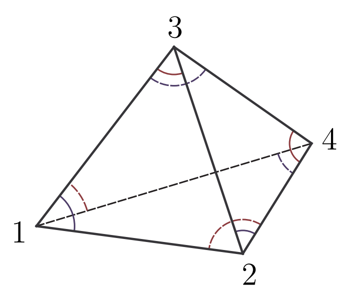

FIGURE 26.3. Getting a nonlinear relation on face angles of a tetrahedron.

[67]The Gauss–Bonnet theorem (Theorem 25.3) gives another linear relation, but it follows from the linear relations on faces (as made explicitly in the proof of the theorem).

26.7. **The converse of the Cauchy theorem.** From the discussion above, one can think of the Cauchy theorem as a statement that in all convex polytopes in $\mathbb{R}^3$ the edge length and face angles determine the dihedral angles, and with them the whole polytope. Thus, it is reasonable to ask if the converse is true as well: is it true that the edge lengths and dihedral angles determine the face angles as well? The following result confirms the suspicion:

Theorem 26.9 (Stoker)**.** *Let $P, Q \subset \mathbb{R}^3$ be two combinatorially equivalent convex polytopes with equal corresponding edge lengths and dihedral angles. Then P and Q are isometric.*

Note that the theorem is obvious for simplicial polytopes: in that case the edge lengths alone determine face angles (consider separately each triangular face), and with them the whole polytope. Of course, for simple polytopes knowing edge lengths is insufficient. On the other hand, dihedral angles alone can determine the face angle (consider separately each vertex cone), and with them the whole polytope once again. Therefore, this result is actually easy for extreme cases and the main difficulty is with 'intermediate' polytopes. The main idea of the proof below is to combine these two different approaches into one argument.

Proof. Using the approach in the proof of the Cauchy theorem, compare face angles in P and Q and label them with $(+)$, $(-)$ and 0 accordingly. Note that around every face either all labels are zero or there are at least four sign changes. This is the analogue of the arm lemma (Lemma 22.3) for planar polygons and the proof is verbatim. Similarly, around every vertex all labels are zero or there are at least four sign changes. Indeed, consider a vertex v of P and the dual cone C_v^* (see Section 25). The dihedral angles β_i equal to $\pi - \alpha_i$, where α_i are face angles in v. Now, as in the proof of the sign changes lemma (Lemma 26.4), by the arm lemma for spherical polygons we have the claim.

Consider the *medial graph* H with vertices corresponding to edges of P and edges connecting two adjacent edges in P lying in the same face. Label the edges of H with the same label as the corresponding face angle in P. Note that the faces of H

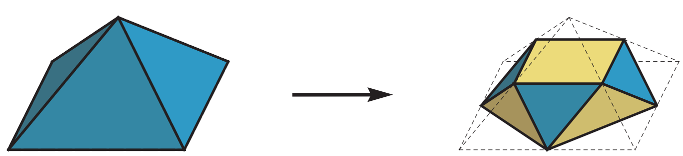

Figure 26.4. Medial graph H of the graph of a square prism is the graph of a square antiprism.

correspond to vertices and faces of P, and have all zero labels, or at least four sign changes. Applying the sign counting lemma (Lemma 26.5) to the dual graph H^*, we conclude that all labels in H are zero. In other words, the corresponding face angles in P and Q are equal. Now the isometry of faces and the equality of dihedral angles implies the result. $\square$

26.8. **Exercises.**

Exercise 26.1. (*Stoker's conjecture*) $\diamondsuit$ The conjecture states that every two combinatorially equivalent convex polytopes in $\mathbb{R}^3$ with equal corresponding dihedral angles have equal corresponding face angles.
a) [1-] Check that the conjecture implies Theorem 26.9.
b) [1] Prove that the conjecture holds for simple polytopes.
c) [1+] Prove that the conjecture holds for circumscribed polytopes.
d) [$*$] Prove the conjecture for simplicial polytopes.

Exercise 26.2. [1+] State the spherical analogue of Stoker's conjecture. Find a counterexample.

26.9. **Final remarks.** The Cauchy theorem was proposed to Cauchy by Legendre, who established it in several special cases (see [Bla2, Mi3]). Legendre wrote a standard geometry course, and was led to the problem by trying to formalize (and prove) Definition 11.10 from Euclid's "Elements" [Euc]:

Equal and similar solid figures are those contained by
similar planes equal in multitude and in magnitude.

Here by "planes" Euclid means the faces of the polytope, which he required to be equal polygons. For centuries this definition was viewed as a claim that polytopes with equal faces are equal. However, the notion of equality as congruence of polytopes is absent in Euclid, and the statement really concerns the volumes of polytopes. As was noted in [Mi3, §2.5], it seems Euclid meant to apply the definition only to volumes of "basic" polytopes, such as triangular prisms and pyramids. In any event, Legendre proved the Cauchy theorem in several special cases, and gave the general problem to his student Cauchy.[68]

The proof of the Cauchy theorem more or less follows Cauchy's original approach. The Alexandrov theorem (Theorem 27.7) can be found in [A2]. Let us mention that similar polytopes play an important role in the study of mixed volumes of convex bodies. We use them in the proof of the Minkowski theorem (Theorem 36.2).

Our proof of Theorem 26.9 follows the original proof by Stoker, who further conjectured that convex polytopes with equal corresponding dihedral angles have equal face angles [Sto, §6] (see Exercise 26.1). Let us mention that Theorem 26.8 extends verbatim to $\mathbb{S}^3$ and $\mathbb{H}^3$ (see Subsection 27.1), and so does Stoker's theorem (Theorem 26.9). Although the natural analogue of Stoker's conjecture is open in full generality, the spherical analogue in false (see Exercise 26.2), and the infinitesimal version was recently established in [MazM] (see also [Ale6, §5], [Schl2] and Exercise 32.2).

In the next several sections we present further results on rigidity of polyhedral surfaces. In Sections 28, 29 we present two other (more involved) proofs of the Cauchy theorem, and in Sections 32, 33 we present about four different (and somewhat less involved) proofs of the rigidity of convex polytopes (Corollary 26.2).

[68]The works of Legendre were studied by I.Kh. Sabitov (personal communication).

27. Cauchy theorem: extensions and generalizations

In this short section we discuss a number of extensions of the Cauchy theorem, some easy and straightforward, some important (the *Alexandrov uniqueness theorem* and its converse in Section 37) and some beyond the scope of this book (the *Pogorelov uniqueness theorem*).

27.1. **Spherical polyhedra.** As it turns out, in order to extend the Cauchy theorem to higher dimensions, we first need a spherical analogue.

Note first that the proof of the Cauchy theorem is robust enough to work in other geometries, e.g., on a sphere $\mathbb{S}^3$, Lobachevsky space $\mathbb{H}^3$, as well as other metric spaces with bounded curvature.

We begin with the definitions. Let $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ be a d-dimensional unit sphere. A convex *spherical d-dimensional polyhedron* $P \subset \mathbb{S}^d$ is an intersection of a finite number of hemispheres $\mathbb{S}^d_+$, i.e., half-spheres lying on one side of a hyperplane in $\mathbb{R}^4$ containing the origin O. Note that the boundaries of these hemispheres are the $(d-1)$-dimensional spheres, which we denote by $C_1, \ldots, C_m$. The *vertices*, *edges* and k-dimensional *faces* of a convex spherical polytope $P \subset \mathbb{S}^d$ are the points of the boundary ∂P which lie in the intersections of d, $d-1$, and $d-k$ spheres C_i, respectively. We can now define *combinatorially equivalent* spherical polyhedra the same way as we do in the Euclidian space.

Define the *spherical distance* between points $x, y \in \mathbb{S}^d$ as the length of the shortest path between them. Equivalently, this is the length along the unique great circle which contains x, y (such circle is unique unless the points are equal or opposite). Define now *isometric* spherical polyhedra, write $P \simeq Q$, the same way as before. Note that spherical polyhedra $P, Q \subset \mathbb{S}^d$ are (globally) isometric if and only if there is an orthogonal transformation $M \in \mathrm{O}(d+1, \mathbb{R})$ which maps P into Q, i.e. $M(P) = Q$. We will need the isometry only for $d = 2$ (faces), and $d = 3$ (polytopes).

Theorem 27.1 (Cauchy theorem for spherical polyhedra)**.** *Let $P, Q \subset \mathbb{S}^3_+$ be two combinatorially equivalent convex spherical polyhedra in the upper hemisphere, whose corresponding faces are isometric. Then P and Q are isometric.*

The proof follows verbatim the proof of the Cauchy theorem, with appropriate substitutions of spherical geometry concepts, such as dihedral angles in edges of P and Q. We skip the details.

27.2. **Cauchy theorem in higher dimensions.** We are now ready to present the following important extension.

Theorem 27.2 (Cauchy theorem in higher dimensions)**.** *Let $P, Q \subset \mathbb{R}^d$ (or $P, Q \subset \mathbb{S}^d_+$), $d \geq 3$ be two combinatorially equivalent (spherical) convex polyhedra whose corresponding facets are isometric. Then P and Q are isometric.*

Here by (*global*) *isometry* between polyhedra we mean that one can be mapped into another such that the pairwise distances between the corresponding points are always equal. By $\mathbb{S}^d_+$ we denote a fixed (upper) hemisphere.

Now, to prove Theorem 27.2, we need the corresponding result for convex spherical polytopes first. Such a result is easy to prove by a straightforward induction. For $d = 3$ this is Theorem 27.1. Suppose the result is established for some $d \geq 3$. Take a $(d+1)$-dimensional spherical polyhedron P and let Q_v be an intersection of P and a d-dimensional sphere centered at a vertex v of small radius. By inductive assumption applied to the d-dimensional convex spherical polytope Q_v, we conclude that Q_v and thus the spherical pyramid $P_v = \operatorname{conv}\{v, Q_v\} \subset \mathbb{S}^{d+1}$ are completely determined. This gives the Cauchy theorem in any dimension for spherical polytopes.

Now, for a Euclidean polytope $P \subset \mathbb{R}^d$, again consider an intersection Q_v with a small $(d-1)$-dimensional sphere. Use the result for spherical polyhedra to conclude that the pyramid P_v over Q_v is completely determined, and thus gives all dihedral angles between facets. We omit the details.

Before we conclude, let us state the rigidity of higher-dimensional polytopes as well.

Corollary 27.3 (Rigidity in $\mathbb{R}^d$). *Every convex polytope in $\mathbb{R}^d$, $d \geq 3$, is rigid.*

Remark 27.4. There is a case to be made that in higher dimensions the polytopes are largely overdetermined, i.e., conditions in the corollary can be weakened in most cases. Without making this precise, let us make a counting argument calculating the degrees of freedom (cf. Remark 26.3).

Consider a convex simplicial polytope $P \in \mathbb{R}^d$, in which case the facets are determined by the edge lengths. Suppose P has n vertices and k edges. The space of realizations of the graph of P up to rigid motions is $dn - \binom{d+1}{2}$, since the dimension of the group of rigid motions (translations and rotations) is $d + \dim \mathrm{O}(d, \mathbb{R}) = d(d+1)/2$. Now the rigidity of P implies[69] that this is the lower bound for the number of edges:

$$(\triangledown) \qquad k \geq dn - \binom{d+1}{2}.$$

Interestingly, we already have seen this inequality. Denote by $P^* \subset \mathbb{R}^d$ a dual polytope to P. Clearly, P^* is simple, and in the notation of Section 8 we have $f_{d-1} = n$ and $f_{d-2} = k$. Thus, the inequality $g_2 \geq g_1$ presented in Remark 8.2 becomes inequality $(\triangledown)$ given above.

Let us show that inequality $(\triangledown)$ can be quite weak. This means that a polytope can have too many edges, more than necessary to justify the rigidity. The easiest example is a *cross-polytope* $C_d \subset \mathbb{R}^d$, defined as a regular polytope dual to a cube. For example, C_3 is a regular octahedron. Clearly, C_d has $n = 2d$ vertices and $k = \binom{2d}{2} - d$ edges, making $(\triangledown)$ into strict inequality for $d \geq 4$.

27.3. **Unbounded convex polyhedra.** Recall the notion of (unbounded) convex polyhedra considered in Section 25. Namely, let $P \subset \mathbb{R}^3$ be an intersection of a finite number of half-spaces. The question is: can we extend the Cauchy theorem to this case? Phrased differently, do there exist non-isometric convex polyhedra which are combinatorially equivalent and have isometric corresponding faces?

The answer is easily "no", and examples include a cone and a product of an unbounded convex polygon and a line (see Figure 27.1). Somewhat surprisingly, there is an easy way to enforce rigidity and generalize the Cauchy theorem to this case.

[69]Formally speaking, this conclusion can be made for realizations only over $\mathbb{C}$, rather than over $\mathbb{R}$. The reader can think of this as a heuristic argument which can in fact be formalized.

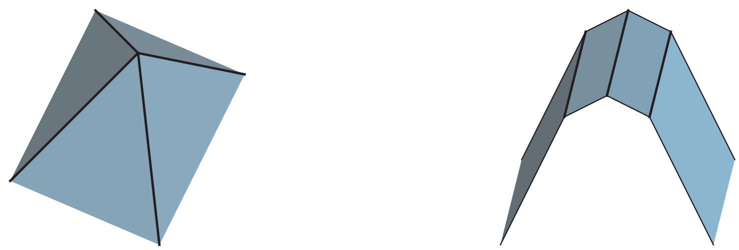

Figure 27.1. Non-rigid unbounded convex polyhedra.

Theorem 27.5 (Cauchy theorem for unbounded polyhedra)**.** *Let P and $Q \subset \mathbb{R}^3$ be two combinatorially equivalent unbounded convex polyhedra which do not contain straight lines and whose corresponding faces are isometric. Suppose further that P and Q contain the origin O and have equal base cones: $C_P = C_Q$. Then polyhedra P and Q are isometric.*

One can think of the theorem as of a limiting case of the Cauchy theorem, when vertices of a face are sent to infinity along the rays of the base cone. Unfortunately, we do not know a rigorous limit argument in this case. In fact, there is a 'global' proof along the same lines as that of the Cauchy theorem. Again, consider dihedral angles and the sign changes. The infinite edges of P and Q have equal dihedral angles to that of $C_P = C_Q$. Now proceed as in the proof of the Cauchy theorem, noting that Euler's formula also applies in this case.

27.4. **Non-strictly convex polytopes.** Consider a sequence of convex polytopes $P_i \subset \mathbb{R}^3$ converging to a convex polytope Q. Note that the geometric convergence does not necessarily translate into a combinatorial convergence, i.e., the limit graph $G = \lim_{i\to\infty} G_i$ will, of course, contain graph H of Q, but may be much larger than H as many edges now can lie on the surface ∂Q (see Figure 27.2). One can think of such polyhedra as *non-strictly convex polytopes*, defined not to have vertices in the relative interior of faces. We will use a different terminology.

Figure 27.2. Examples of non-strictly convex polyhedral surfaces.

Let $S = \cup F_i \subset \mathbb{R}^3$ be a *polyhedral surface* defined as a union of convex polygons F_i embedded into $\mathbb{R}^3$. We say that S is a *convex polyhedral surface* if $S = \partial P$ for some convex polytope $P \subset \mathbb{R}^3$, where the equality is in term of sets of points. Of course, different polyhedral surfaces may be equal (as sets of points) to the boundary of the same convex polytope.

We say that convex polyhedral surfaces S and S' in $\mathbb{R}^3$ are *globally isometric*,[70] write $S \simeq S'$, if there is a map between them (as sets of points in $\mathbb{R}^3$) which preserves distances in $\mathbb{R}^3$. In other words, $S \simeq S'$ if there exists a rigid motion which moves S into S'.

Theorem 27.6 (Cauchy–Alexandrov theorem for polyhedral surfaces)**.** *Let $S = \cup_{i=1}^m F_i$ and $S' = \cup_{i=1}^m F_i'$ be two convex polyhedral surfaces in $\mathbb{R}^3$, which are combinatorially equivalent and the corresponding faces are isometric: $F_i \simeq F_i'$. Then surfaces S and S' are globally isometric.*

Proof. This is again essentially an observation on the proof of the Cauchy theorem.

We say that a point v in S is *flat* if the sum of the face angles in it is equal to 2π (the point v may actually lie on an edge of P). Since the sum of the face angles around vertices is always $\le 2\pi$, every flat vertex v of polygon F_i is mapped into a flat vertex v' of polygon F_i'. There are three possibilities: either both v and v' lie in the interior of faces, or one of them lies

on the proof of the Cauchy theorem.

Suppose $S = \partial P$ and $S' = \partial P'$. From above, the union of all polygons F_i which form a polytope face $P_j \subset P$ is isometrically mapped into the union of all F_i' which form a face $P_j' \subset P'$. Now use the same argument as in the Cauchy theorem to obtain the result. ☐

27.5. **Alexandrov's uniqueness theorem for isometric convex surfaces.** Let $S \subset \mathbb{R}^d$ be a (convex) surface embedded into a space. For two points $x, y \in S$ define the *surface distance* $|x, y|_S$ to be the length $\ell(\gamma)$ of the shortest path $\gamma \subset S$ between x and y. Thus we obtain the *surface metric* on S.

Let $S = \cup_{i=1}^m F_i$ and $S' = \cup_{j=1}^{m'} F_j'$ be two (convex) polyhedral surfaces in $\mathbb{R}^3$. We say that S and S' are *intrinsically isometric*, write $S \doteq S'$, if there exists a map $\varphi : S \to S'$ (between sets of points), which maps the surface metric on S into the surface metric on S':

$$|x, y|_S = |\varphi(x), \varphi(y)|_{S'}, \quad \text{for all } x, y \in S.$$

Of course, global isometry implies intrinsic isometry, but the converse is not true for non-convex surfaces. For example, two polyhedral surfaces that lie on the boundary of the same polytope are intrinsically and globally isometric. Alternatively, the surfaces of two realizations of the same polytope are intrinsically, but not globally, isometric (see, e.g., Figure 30.1). By the Cauchy theorem (Theorem 26.1), one of these polytopes must be non-convex. The following result shows that one cannot obtain two intrinsically isometric surfaces from two different convex polytopes.

Theorem 27.7 (Alexandrov uniqueness theorem)**.** *Let $S, S' \subset \mathbb{R}^3$ be two convex polyhedral surfaces which are intrinsically isometric. Then they are globally isometric.*

[70]We use the term "globally isometry" in place of the usual "isometry" to emphasize that we are working with surfaces rather than polytopes. The "intrinsic isometry" is defined below.

Once again, let us emphasize that one can view Theorem 27.7 as of a uniqueness result. It says that up to a rigid motion there exists only one convex polyhedral surface with a given metric.

Proof. Let $\varphi : S \to S'$ be a map in the definition of intrinsic isometry, $S = \partial P$, and $S' = \partial P'$. Consider the regions $G_i = \varphi(F_i)$. Since all interior points in F_i are flat in S, so are the points in G_i; otherwise shortest paths in F_i will not be mapped into shortest paths in G_i by Proposition 10.1. Similarly, since sides of the polygon F_i are the shortest path in S, the sides of G_i are also shortest paths. We conclude that G_i are polygons in S' not containing vertices of P' in their relative interior. Similarly, regions $H_j := \varphi^{-1}(F_j)$ are polygons in S not containing vertices of P in their relative interior. Finally, note that all G_i and H_j are intrinsically convex in the surface metric of S' and S.

Let $G_{ij} = G_i \cap F'_j$, and $H_{ij} = H_i \cap F_j$, for all $1 \le i \le m$, $i \le j \le m'$. Clearly, each G_{ij} and H_{ij} is a convex polygon and the map $\varphi : H_{ij} \to G_{ij}$ is an isometry between them. Finally, combinatorial equivalence of convex polyhedral surfaces $\widetilde{S} = \cup_{ij} H_{ij}$ and $\widetilde{S}' = \cup_{ij} G_{ij}$ follows from adjacency of the corresponding edges as zero distance. By Theorem 27.6, surfaces $\widetilde{S}$ and $\widetilde{S}'$ are globally isometric. Since $S = \widetilde{S}$ and $S' = \widetilde{S}'$ (as sets of points), we obtain the result. □

An example of an isometric embedding of a polygon into a surface is given in Figure 27.3. The edges subdivide it into smaller convex polygons, as in the proof above.

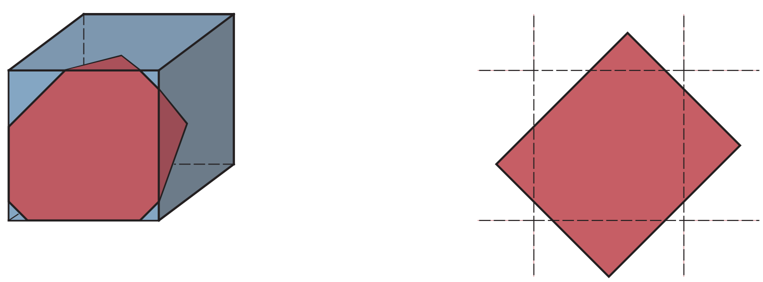

FIGURE 27.3. An isometric embedding of a rectangle into a surface of a cube.

27.6. **General convex surfaces.** Let $S = \partial P$ be the surface of a compact convex body. Define a metric on S again via the *geodesic distance*, i.e., let $|xy|_S$ be the length of a shortest path (it may not be unique). As before, we say that surfaces S and S' are *intrinsically isometric* if there exists a map $\varphi : S \to S'$, which maps the surface metric on S into the surface metric on S' (see above). The following result is an advanced extension of the Cauchy theorem and the Alexandrov uniqueness theorem.

Theorem 27.8 (Pogorelov uniqueness theorem)**.** *Let $S, S' \subset \mathbb{R}^3$ be two convex surfaces which are intrinsically isometric. Then they are globally isometric.*

The proof is a delicate and at times technical argument based on approximation of convex surfaces by convex polytopes. We will not present it here.

27.7. **Exercises.**

Exercise 27.1. [1+] Let $S \subset \mathbb{R}^3$ be a (non-convex) 2-dimensional polyhedral surface homeomorphic to a sphere, such that every face of S is a unit square. Prove that all dihedral angles of S are right angles.

Exercise 27.2. *a*) [1-] Prove or disprove: $\Delta \subset \mathbb{R}^3$ is a regular tetrahedron if and only if it has five equal dihedral angles.
b) [1-] The same for all six equal dihedral angles.

Exercise 27.3. (*Locally convex polyhedra*) $\diamondsuit$ [1] A polyhedron is called *locally convex* if the cone at every vertex is a convex cone. Prove that every locally convex 2-dimensional polyhedral surface homeomorphic to a sphere and embedded in $\mathbb{R}^3$ is convex.[71]

Exercise 27.4. [1+] Show that every two spherical tetrahedra in $\mathbb{S}^3$ with equal corresponding dihedral angles are isometric.

Exercise 27.5. $\diamondsuit$ [1] Complete the proof of Theorem 27.1.

Exercise 27.6. $\diamondsuit$ [1+] Complete the proof of Theorem 27.2.

Exercise 27.7. $\diamondsuit$ [1+] Complete the proof of Theorem 27.5.

Exercise 27.8. $\diamondsuit$ [1] Complete the proof of Theorem 27.6.

Exercise 27.9. (*Olovianishnikoff*) [2] Let $P \subset \mathbb{R}^3$ be a convex polytope and let $Q \subset \mathbb{R}^3$ be a convex body. Suppose the surface ∂P is isometric to ∂Q. Prove that Q is a convex polytope. Conclude from here that P and Q are globally isometric.

27.8. **Final remarks.** There are a number of extensions of the Cauchy theorem to polyhedra in other metric spaces, some of them going back to Alexandrov, while some are more recent and quite advanced. In particular, extensions of the Cauchy theorem to spherical polytopes and polytopes in a Lobachevsky space (hyperbolic polytopes) are due to Alexandrov. We refer to [A1, A2, Pog3] for classical results, to [Con5, IKS] for the surveys, and to papers [Mi6, Mi4, Schl1] for more recent advances. Theorem 27.6 is also due to Alexandrov, as are most of the results in this section.

The study of unbounded polyhedra and Theorem 27.5 are due to Olovianishnikoff [Olo1] (sometimes spelled Olovjanišnikov). He also established an important intermediate result between the Alexandrov and Pogorelov uniqueness theorems in [Olo2] (see Exercise 27.9). For Pogorelov's theorem see [Pog3].

Also, a number of further extensions of the uniqueness theorems (to convex spherical surfaces, unbounded convex surfaces, surfaces with bounded curvature, etc.) were obtained by Alexandrov and his students. We refer to an extensive survey [Sab1] (see also [Sen2]) for rigidity of general surfaces.

[71] This is unfortunate since, of course, the Cauchy rigidity theorem immediately extends to locally convex polyhedra homeomorphic to a sphere.

28. Mean curvature and Pogorelov's lemma

We introduce the *mean curvature* of polyhedral surfaces and prove that it is invariant under isometric deformations. We then present *Pogorelov's lemma*, which implies both the *Cauchy theorem* and the isoperimetry of spherical polygons (cf. Section 7).

28.1. Mean curvature.

The proof below requires a few preliminary results on geometry of convex polyhedra. Since we need these results for other purposes, we separate them from the main part of the proof.

Let $P \subset \mathbb{R}^3$ be a convex polytope with the set of vertices V and the set of edges E. For every $e = (v, w) \in E$ denote by $\ell_e = |vw|$ the length of edge e. Let $\boldsymbol{u}_{v,e} = (v, w)/\ell_e$ be a unit vector starting at $v \in V$ and pointing along edge $e = (v, w) \in E$. Fix a real function $f : e \to f_e$, where $f_e \in \mathbb{R}$, $e \in E$. Define vectors $\boldsymbol{a}_v(f) \in \mathbb{R}^3$, $v \in V$, by the following summation formula:

$$\boldsymbol{a}_v(f) := \sum_{e=(v,w)\in E} f_e \cdot \boldsymbol{u}_{v,e}\,.$$

Finally, let

$$\mathrm{M}(P, f) := \sum_{e\in E} f_e \cdot \ell_e\,.$$

A special case of $\mathrm{M}(P, f)$ is of particular importance. Denote by θ_e the dihedral angle in e, and let $\mathrm{M}(P) := \mathrm{M}(P, \pi - \theta)/2$ be the *mean curvature* of P:

$$\mathrm{M}(P) := \frac{1}{2} \sum_{e\in E} (\pi - \theta_e) \cdot \ell_e\,.$$

This function comes from differential and integral geometry and has a special invariance property which we prove later in this section.

Lemma 28.1 (Edge summation lemma)**.** *For every point $z \in \mathbb{R}^3$, vectors $\boldsymbol{r}_v = \overrightarrow{vz}$ corresponding to vertices $v \in V$, and function $f : E \to \mathbb{R}$ as above, we have:*

$$\mathrm{M}(P, f) = \sum_{v\in V} \big\langle \boldsymbol{a}_v(f), \boldsymbol{r}_v \big\rangle.$$

Proof. Observe that

$$(\triangle) \qquad \langle \boldsymbol{u}_{v,e}, \boldsymbol{r}_v \rangle + \langle \boldsymbol{u}_{w,e}, \boldsymbol{r}_w \rangle = \ell_e\,,$$

for every edge $e = (v, w) \in E$ with length ℓ_e. Indeed, the l.h.s. in $(\triangle)$ is the sum of the (signed) lengths of projections p_v and p_w of vectors $\boldsymbol{r}_v$ and $\boldsymbol{r}_w$ onto line (vw) (see Figure 28.1). Changing the order of summation from vertices to edges we obtain:

$$\sum_{v\in V} \big\langle \boldsymbol{a}_v(f), \boldsymbol{r}_v \big\rangle = \sum_{v\in V} \Big\langle \sum_{e=(v,w)\in E} f_e\, \boldsymbol{u}_{v,e}\,,\, \boldsymbol{r}_v \Big\rangle$$

$$= \sum_{e=(v,w)\in E} f_e \cdot \Big[\langle \boldsymbol{u}_{v,e}, \boldsymbol{r}_v \rangle + \langle \boldsymbol{u}_{w,e}, \boldsymbol{r}_w \rangle \Big] = \sum_{e\in E} f_e\, \ell_e = \mathrm{M}(P, f),$$

as desired. $\square$

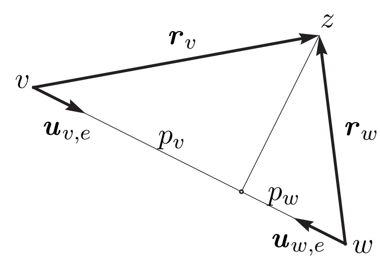

FIGURE 28.1. The equality $p_v + p_w = \ell_e$, where $p_v = \langle \boldsymbol{u}_{v,e}, \boldsymbol{r}_v \rangle$ and $p_w = \langle \boldsymbol{u}_{w,e}, \boldsymbol{r}_w \rangle$ are projections onto edge $e = (v, w)$.

28.2. **Some things never change.** Let us study the mean curvature in greater detail, by observing what happens when we continuously change the polytope. Start with the continuous deformations of a cone, defined as follows.

Let $C \subset \mathbb{R}^3$ be a polyhedral cone, not necessarily convex, defined as follows. Let $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k$ be unit vectors in $\mathbb{R}^3$, and fix a point $v \in \mathbb{R}^3$. Consider cone faces starting at v and spanned by consecutive pairs of vectors $(\boldsymbol{u}_1, \boldsymbol{u}_2), \ldots, (\boldsymbol{u}_{k-1}, \boldsymbol{u}_k)$, and $(\boldsymbol{u}_k, \boldsymbol{u}_1)$. Denote by θ_i, $i \in [k]$, the (signed) dihedral angle of C at $\boldsymbol{u}_i$.

Now, suppose we have a continuous family of cones $\{C_t : t \in [0,1]\}$, such that $C_0 = C$, spanned by continuously changing unit vectors $\boldsymbol{u}_i(t)$, where $\boldsymbol{u}_i = \boldsymbol{u}_i(0)$. We call $\{C_t\}$ a *continuous deformation* of the cone C.

Lemma 28.2 (Angular velocity equation)**.** *Let $\{C_t : t \in [0,1]\}$ be a continuous deformation of the cone C preserving face angles, and let $\theta_i(t)$ be the dihedral angles in the cone C_t. Then*

$$\sum_{i=1}^{k} \boldsymbol{u}_i(t) \cdot \theta_i'(t) \;=\; \mathbf{0} \quad \textit{for all} \quad t \in [0,1].$$

Here in the lemma we are assuming that the dihedral angles $\theta_e(t)$ on $[0,1]$ have derivatives $\theta_e'(t)$ with respect to t. If not, by continuity of $\theta_e(t)$, the claim can be replaced with the left or right derivative at any point $t = t_0$ with no change in the proof. We find the former version more aesthetically pleasing, while the second more useful.

Proof. The lemma follows immediately from the following observation. Note that $\theta_i'(t)\,\boldsymbol{u}_i(t)$ are angular velocity vectors of the i-th face around $(i-1)$-st face of C_t. Thus the sum is the rotational velocity of the 1-st face over itself, a zero.

Formally, consider a vector $\boldsymbol{w} \in \mathbb{R}^3$ in the first face of C. Suppose $\boldsymbol{w}$ is the composition of rotations around $\boldsymbol{u}_i$ by an angle $\theta_i(t) - \theta_i(0)$, for all $i \in [k]$. We have:

$$\boldsymbol{w}' \;=\; \sum_{i=1}^{k} \theta_i' \boldsymbol{u}_i \times \boldsymbol{w} \;=\; \left[\sum_{i=1}^{k} \theta_i' \boldsymbol{u}_i\right] \times \boldsymbol{w} \;=\; \boldsymbol{a} \times \boldsymbol{w}\,.$$

By the argument above, $\boldsymbol{w}' = 0$, and we conclude that $\boldsymbol{a} = 0$. $\square$

Before we turn to the proof of the Cauchy theorem, let us make an interesting corollary from the lemma. Let $\{P_t : t \in [0,1]\}$ be a continuous family of combinatorial equivalent polyhedra in $\mathbb{R}^3$, not necessarily convex or even embedded (see Section 26.2), such that vertex coordinates change continuously. Then, with the same disclaimer as after Lemma 28.2, we obtain the following result.

Theorem 28.3 (Schläfli formula). *For a continuous family of polyhedra $\{P_t : t \in [0,1]\}$ preserving the faces, we have:*

$$\sum_{e \in E} \ell_e(t) \cdot \theta_e'(t) \, = \, 0, \quad \textit{for all} \quad t \in [0,1],$$

where $\ell_e(t)$ are edge lengths and $\theta_e(t)$ are dihedral angles.

Proof. In the notation above, let $f_e = \theta_e'(t)$ be derivatives of the dihedral angles. By Lemma 28.2 we have $\boldsymbol{a}_e(\theta'(t)) = \mathbf{0}$, for every edge $e \in E$. On the other hand, $\mathrm{M}(P_t, \theta'(t)) = \sum_{e \in E} \ell_e(t)\theta_e'(t)$, as in the theorem. Now the edge summation lemma (Lemma 28.1) gives:

$$\mathrm{M}(P_t, \theta'(t)) \, = \, \sum_{v \in V} \big\langle \boldsymbol{a}_v(\theta'(t)), \boldsymbol{r}_v \big\rangle \, = \, 0,$$

as desired. $\square$

The following corollary follows easily from the Schläfli formula, but is important enough to single out. Recall that a continuous deformation $\{P_t : t \in [0,1]\}$ is a continuous family of polyhedra with all edge lengths constant under deformation: $\ell_e(t) = c_e$ for some $c_e > 0$ and all $t \in [0,1]$.

Corollary 28.4 (Invariance of the mean curvature). *Let $\{P_t : t \in [0,1]\}$ be a continuous isometric deformation of a (possibly non-convex) polyhedron. Then the mean curvature $\mathrm{M}(P_t)$ is a constant independent of t.*

For convex polyhedra this follows immediately from Corollary 26.7. On the other hand, for non-convex polyhedra this is a new result, a special property of *flexible polyhedra*. We present their examples and further properties in Sections 30, 31.

Proof. Suppose first that all dihedral angles $\theta_e(t)$ are differentiable on $[0,1]$. Then

$$\mathrm{M}'(P_t) \, = \, \left[\frac{1}{2} \sum_{e \in E} (\pi - \theta_e) \cdot \ell_e \right]' \, = \, -\frac{1}{2} \sum_{e \in E} \ell_e(t) \cdot \theta_e'(t) \, = \, 0,$$

so the mean curvature $\mathrm{M}(P_t)$ is a constant independent of t. Suppose now that not all $\theta_e(t)$ are differentiable. By continuity, at every point $t_0 \in (0,1)$ we can compute the left and right derivatives of $\theta_e(t)$, and then of $\mathrm{M}(P_t)$. From above, the left and right derivatives of $\mathrm{M}(P_t)$ are equal to 0 at all $t \in (0,1)$. Thus, $\mathrm{M}(P_t)$ is differentiable with zero derivative in $(0,1)$. This again implies that $\mathrm{M}(P_t)$ is independent of t. $\square$

28.3. **The ultimate local proof.** Observe that the structure of the proof of the Cauchy theorem splits naturally into a local and a global part: the sign changes lemma (Lemma 26.4) which easily reduces to the arm lemma (Lemma 22.3) and the sign counting lemma (Lemma 26.5) based on the use of a counting argument and Euler's formula. There are several other proofs of the Cauchy theorem which follow the same pattern, but have varying levels of complexity of local and global parts. The idea of this proof is to make the 'local claim' strong enough, so that the contradiction follows easily, by a geometric and a straightforward double counting argument, rather than by a calculation and Euler's formula. After all, Euler's formula can also be deduced by a 'global argument' (see Section 25), so it makes sense that the Cauchy theorem can be proved along these lines.

Let $C, D \subset \mathbb{R}^3$ be two convex cones starting at the origin O and with the same sequence of side angles (in cyclic order). In this case we say that cones C and D have *isometric sides*. The cones are called *equal*, write $C \simeq D$ if there exists a rotation mapping one into the other. Finally, define the *dual cone* C^* as in Subsection 25.4, to be the cone of all normal vectors: $C^* = \{w \in \mathbb{R}^3 \mid (w, v) \le 0 \text{ for all } v \in C\}$.

Lemma 28.5 (Pogorelov). *Let $C, D \subset \mathbb{R}^3$ be two convex cones with isometric sides and dihedral angles $\theta_1, \ldots, \theta_k$ and $\vartheta_1, \ldots, \vartheta_k$, respectively. Denote by $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k$ the unit vectors along edges of C. Let $\boldsymbol{w} = \boldsymbol{w}(C, D)$ be a vector defined as follows:*

$$\boldsymbol{w} := \sum_{i=1}^{k} (\vartheta_i - \theta_i)\, \boldsymbol{u}_i \,.$$

Then $\boldsymbol{w} \in C^*$. Furthermore, $\boldsymbol{w} = \boldsymbol{0}$ if and only if $C \simeq D$.

We postpone the proof of Pogorelov's lemma and start with the following original proof of the Cauchy theorem based on the lemma.

Proof of the Cauchy theorem modulo Pogorelov's lemma. Let P and Q be combinatorially equivalent convex polytopes with isometric faces. Denote by V, E the set of vertices and edges of P, respectively, and let θ_e denote the dihedral angles in P. To simplify the notation we use the same notation E for the sets of edges in Q and let ϑ_e, $e \in E$, be the dihedral angles in Q.

Denote by C_v the cone spanned by the edges of P containing the vertex $v \in V$. Let $\boldsymbol{w}_v$ be the vector as in the lemma, corresponding to the cone C_v. Fix a point $z \in P$ and let $\boldsymbol{r}_v = (v, z)$, $v \in V$, be as above. Since $\boldsymbol{r}_v \in C_v$ for all $v \in V$, by Pogorelov's lemma we have $(\boldsymbol{w}_v, \boldsymbol{r}_v) \le 0$. Summing this over all vertices, we get:

$$A := \sum_{v \in V} \langle \boldsymbol{w}_v, \boldsymbol{r}_v \rangle \,\le\, 0\,.$$

In the notation of the edge summation lemma (Lemma 28.1), let $f_e = (\vartheta_e - \theta_e)$ and note that $\boldsymbol{w}_v = \boldsymbol{a}_v(f)$. By the lemma, we have:

$$A \,=\, \sum_{e \in E} (\vartheta_e - \theta_e)\, \ell_e \,=\, \sum_{e \in E} (\pi - \theta_e)\, \ell_e \,-\, \sum_{e \in E} (\pi - \vartheta_e)\, \ell_e \,=\, 2\mathrm{M}(P) \,-\, 2\mathrm{M}(Q),$$

where the last equality follows from the isometry between the corresponding edges of polytopes P and Q. From here, $\mathrm{M}(Q) \le \mathrm{M}(P)$.

Now switch the roles of P and Q. We similarly get $\mathrm{M}(P) \le \mathrm{M}(Q)$. Therefore, $\mathrm{M}(P) = \mathrm{M}(Q)$, $A = 0$, and every inequality above is an equality: $\langle \boldsymbol{w}_v, \boldsymbol{r}_v \rangle = 0$, for all $v \in V$. By convexity and the assumption that $z \in P$, this implies that $\boldsymbol{w}_v = \mathbf{0}$, for all $v \in V$. By the second part of Pogorelov's lemma, we conclude that the corresponding cones have isometric sides, and thus the corresponding dihedral angles are equal: $\theta_e = \vartheta_e$, for all $e \in E$. Thus, the polytopes P and Q are also equal. $\square$

28.4. **Proof of Pogorelov's lemma.** While Pogorelov's lemma may seem like a quantitative version of the sign changes lemma (Lemma 26.4), the proof we present below reduces the former to the latter. Unfortunately, as of now, there is no independent proof of Pogorelov's lemma. So while the search for such a proof is ongoing, let me present the proof we have.

Proof of Pogorelov's lemma modulo the sign changes lemma. Clearly, if $C \simeq D$, then the vector $\boldsymbol{w} = \mathbf{0}$. Now suppose C and D are not equal. Since cone C is convex, it suffices to show that $\langle \boldsymbol{w}, \boldsymbol{u}_i \rangle < 0$, for all $i \in [k]$. In other words, we need to prove that

$$\langle \boldsymbol{w}, \boldsymbol{u}_i \rangle \, = \, \Big\langle \sum_{j=1}^{k} (\vartheta_j - \theta_j)\, \boldsymbol{u}_j \, , \boldsymbol{u}_i \Big\rangle \, = \, \sum_{j=1}^{k} (\vartheta_j - \theta_j) \cos \alpha_{ij} \, < \, 0, \quad \text{for all } \ j \in [k],$$

where α_{ij} is the angle between $\boldsymbol{u}_i$ and $\boldsymbol{u}_j$. Fix $i \in [k]$ and denote by Φ_i the summation we need to bound:

$$\Phi_i \, := \, \sum_{j=1}^{k} (\vartheta_j - \theta_j) \cos \alpha_{ij} \, .$$

We fix cone D and vary cone C. Let us show that $\Phi_i = \Phi_i(C)$ maximizes only when $C \simeq D$. This would imply that $\Phi_i < 0$, and prove the result.

First, note that Φ_i is bounded, i.e., by definition it is at most πk. Thus, by compactness of the space of k-cones (equivalently, spherical k-gons), the maximum of Φ_i is reached at some cone, which can be degenerate and not strictly convex. Either way, let C be such a cone, and assume that $\Phi_i(C) > 0$.

Think of the cone C and the corresponding spherical polygon. Place labels $(+)$, $(-)$ and 0 on the edges of C as in the proof of the Cauchy theorem, i.e., according to the signs of $(\vartheta_j - \theta_j)$, $j \in [k]$. By the assumption above, C and D are not equal, and by the sign changes lemma (Lemma 26.4), there are at least four sign changes.

Let us assume that the i-th edge has label $(+)$. Then, there exists $r \in [k]$ such that r-th edge also has label $(+)$ and together the i-th and r-th edges divide all k edges into two parts (in cyclic order) so that each of them has positive and negative labels. Suppose edges p and q are the closest on each side with labels $(-)$. Consider a continuous deformation $\{C_t\}$ of the cone C, which increases dihedral angles at the i-th and r-th edges, decreases dihedral angles at the p-th and q-th, and leaves unchanged angles at the remaining edges. We claim that such a deformation is possible, with the cones C_t convex on a sufficiently small interval $t \in [0, \epsilon]$. Indeed, we need to worry only about increasing angles at the i-th and r-th edges beyond π, and decreasing angles at the i-th and r-th edges below 0. But since the corresponding angles at D are greater or less, respectively, we conclude that $\theta_i, \theta_r < \pi$, and $\theta_p, \theta_q > 0$, which implies the claim.

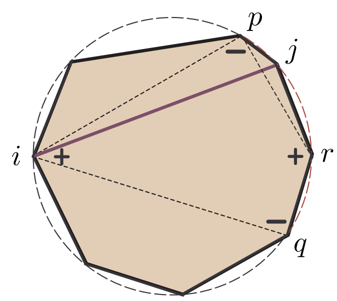

FIGURE 28.2. Change of dihedral angles in a continuous deformation $\{C_t\}$ of the cone C. Here the angle $\alpha_{ij}(t)$ decreases.

Observe that by the arm lemma (Lemma 22.3), the angles $\alpha_{ij}(t)$ are decreasing for j on an interval between p and q containing r, and are unchanged elsewhere (see Figure 28.2). Differentiating directly, we have:[72]

$$\begin{aligned}\Phi_i'(t) &= \Big(\sum_{j=1}^{k} (\vartheta_j - \theta_j(t)) \cos \alpha_{ij}(t)\Big)' \\ &= \sum_{j=1}^{k} (\theta_j(t) - \vartheta_j) \sin \alpha_{ij}(t)\, \alpha_{ij}'(t) \;-\; \sum_{j=1}^{k} \theta_j'(t) \cos \alpha_{ij}(t).\end{aligned}$$

Note that the second sum is equal to 0. This follows from the angular velocity equation (Lemma 28.2) by taking the scalar product with $\boldsymbol{u}_i$ on both sides. Note also that each term in the first sum is ≥ 0, with the r-th term > 0. We conclude that $\Phi_i'(t) > 0$, which contradicts our assumption that Φ_i maximizes at C.

We have two more cases to consider: when the label of the i-th edge is 0 and $(-)$. In the latter case we proceed similar fashion, by taking a deformation with the dihedral angles increasing at p-th and q-th edges, while decreasing at i-th and r-th edges. When the label is 0 we can treat it either as $(+)$ or as $(-)$, since the i-th term in the first summation is always zero. One has to be careful with the extremal cases: if $\theta_i = 0$ we should increase the angle, and if $\theta_i = \pi$, then decrease it. Either way we get a contradiction with the maximality assumption on Φ_i. $\square$

[72]It is easy to make the differentiation rigorous. Let us parameterize the deformation, say, by having θ_i increase linearly: $\theta_i(t) = \theta_i + \varepsilon t$, for some $\varepsilon > 0$. Then the remaining angles $\theta_j(t)$ and $\alpha_{ij}(t)$ are completely determined, and can be computed by the spherical law of cosines (see Appendix 41.2). Clearly, the resulting functions are all analytic in t.

28.5. **Isoperimetry of spherical polygons.** The following result is classical and has no place in this section, and fits well other isoperimetric problems in Section 7, except that the proof is a straightforward application of Pogorelov's lemma (Lemma 28.5).

We say that two spherical polygons $P, Q \subset \mathbb{S}^2_+$ are *isometric* if they have equal corresponding side lengths. By restricting them to the upper hemisphere $\mathbb{S}^2_+$ we can define the area of the polygons as the area of the corresponding polygonal regions on a sphere.

Theorem 28.6. *Let $P, Q \subset \mathbb{S}^2_+$ be two isometric convex spherical polygons, such that P is inscribed into a circle in $\mathbb{S}^2$ and contains its center. Then* $\operatorname{area}(P) \geq \operatorname{area}(Q)$.

Proof. Denote by α_i and β_i the corresponding angles of polygons P and Q, $i \in [k]$. By Girard's formula (see Section 41.3) we have:

$$\operatorname{area}(P) \, = \, \sum_{i=1}^{k} \alpha_i \, - \, (k-2)\pi \, , \quad \operatorname{area}(Q) \, = \, \sum_{i=1}^{k} \beta_i \, - \, (k-2)\pi \, .$$

Denote by O_1 the center of the sphere containing convex polygons P and Q. Denote by C and D the cones over P and Q, respectively, with center at O_1. By the assumption, the polygon $P = [x_1 \ldots x_k]$ is inscribed into a circle with center $O_2 \in \mathbb{S}^2$. Finally, let $\boldsymbol{r} = \overrightarrow{O_1 O_2}$ and $\boldsymbol{u}_i = \overrightarrow{O_1 x_i}$, for all $i \in [k]$. We have:

$$\langle \boldsymbol{u}_1, \boldsymbol{r} \rangle \, = \, \ldots \, = \, \langle \boldsymbol{u}_k, \boldsymbol{r} \rangle \, = \, c, \quad \text{for some } \ c > 0.$$

Now, let $\boldsymbol{w} = \boldsymbol{w}(C, D)$ be as in Pogorelov's lemma:

$$\boldsymbol{w} = \sum_{i=1}^{k} (\beta_i - \alpha_i) \, \boldsymbol{u}_i \, .$$

By the lemma, $\boldsymbol{w} \in C^*$. By the assumption in the theorem, $\boldsymbol{r} \in C$, and, therefore, $\langle \boldsymbol{w}, \boldsymbol{r} \rangle \leq 0$. We conclude:

$$\begin{aligned} \operatorname{area}(Q) \, - \, \operatorname{area}(P) \, &= \, \sum_{i=1}^{k} (\beta_i - \alpha_i) \, = \, \sum_{i=1}^{k} (\beta_i - \alpha_i) \cdot \frac{\langle \boldsymbol{u}_i, \boldsymbol{r} \rangle}{c} \\ &= \, \frac{1}{c} \left\langle \sum_{i=1}^{k} (\beta_i - \alpha_i) \, \boldsymbol{u}_i, \boldsymbol{r} \right\rangle \, = \, \frac{1}{c} \, \langle \boldsymbol{w}, \boldsymbol{r} \rangle \, \leq \, 0, \end{aligned}$$

as desired. □

28.6. **Exercises.**

Exercise 28.1. $\diamondsuit$ [1-] Prove the result of Theorem 28.6 without the convexity assumption on Q, and the assumption that P contains the center O_2 of the circle it is inscribed into. Deduce from here the isoperimetric inequality in the plane (Theorem 7.1).

Exercise 28.2. (*Monotonicity of the mean curvature*) $\diamondsuit$ *a*) [1+] Let $P \subset \mathbb{R}^3$ be a convex polytope inside a unit ball B. Denote by ℓ_e the length of edge e, and let γ_e be the dihedral angle at e. Prove that the *mean curvature* of P satisfies: $M(P) \leq 4\pi$.
b) [2-] Let $P, Q \subset \mathbb{R}^3$ be two convex polytopes and $P \subset Q$. Then $M(P) \leq M(Q)$.

c) [1] Define $M(X) = \sup_{P \subset X} M(P)$ to be the *mean curvature* of a convex body X. Prove that $M(B) = 4\pi$.

Exercise 28.3. *a*) [1+] Let $C \subset \mathbb{R}^3$ be a convex cone with a vertex at the origin. Denote by $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k$ the unit vectors along the cone edges, and let $\theta_1, \ldots, \theta_k$ be the corresponding dihedral angles. Prove the following inequality:

$$\frac{1}{2}\big\|\theta_1\,\boldsymbol{u}_1 + \ldots + \theta_k\,\boldsymbol{u}_k\big\| \,\le\, \omega(C),$$

where $\omega(C)$ is the curvature of C.
b) [1-] Use the edge summation lemma (Lemma 28.1) and the Gauss–Bonnet theorem (Theorem 25.3) to obtain part *a*) of Exercise 28.2.
c) [$*$] Generalize *a*) to higher dimensions.

Exercise 28.4. Let $S_1, S_2 \subset \mathbb{R}^3$ be two 2-dimensional polyhedral surfaces. Define the Fréchet distance $\operatorname{dist}_F(S_1, S_2)$ as in Exercise 24.18.
a) [2-] When S_1, S_2 are convex, prove that

$$\big|M(S_1) - M(S_2)\big| \,\le\, 4\pi \operatorname{dist}_F(C_1, C_2).$$

Check that this implies part *a*) of Exercise 28.2.
b) [2] Suppose S_1 and S_2 are homeomorphic to a sphere. The *total absolute curvature* $\varkappa(S)$ is the sum of absolute values of curvatures of vertices in S (see Exercise 24.5). Prove that:

$$\big|M(S_1) - M(S_2)\big| \,\le\, \big(\varkappa(S_1) + \varkappa(S_2) + 4\pi\big) \operatorname{dist}_F(C_1, C_2).$$

Exercise 28.5. (*Minkowski formula*) There is a classical differential geometry approach to the mean curvature of surfaces in $\mathbb{R}^d$, but that would lead us away from the subject. Instead, we present the classical *Minkowski formula* which can be viewed as an alternative definition:

$$\mathrm{M}_d(P) \,=\, \int_{\mathbb{S}^{d-1}} H(u)\, d\sigma(u),$$

where $P \subset \mathbb{R}^d$ is a convex polytope containing the origin O, and $H(u)$ is the support function defined by $H(u) = \max\{\langle x, u\rangle \mid x \in P\}$.

a) [1] Prove that $\mathrm{M}_2(P) = \operatorname{perimeter}(P)$.

Now, let us compute $\mathrm{M}_3(P)$ by rewriting the integral as a sum over vertices. In the notation above, using $R_v = C_v^* \cap \mathbb{S}^2$, we have:

$$\mathrm{M}_3(P) \,=\, \sum_{v \in V} \int_{R_v} -\langle u, \boldsymbol{r}_v\rangle\, d\sigma(u) \,=\, -\sum_{v \in V} \Big\langle \int_{R_v} u\, d\sigma(u), \boldsymbol{r}_v \Big\rangle.$$

b) [1+] For a simple cone $C \subset \mathbb{R}^3$ calculate the integral $\int_R u\, d\sigma(u)$, where $R = C \cap \mathbb{S}^2$.

c) [1+] Use additivity to compute the integral above for general cones. Write the answer for each vertex:

$$\int_{R_v} u\, d\sigma(u) \,=\, -\sum_{e=(v,w)\in E} \theta_e\, \boldsymbol{u}_{v,e}\,.$$

d) [1] In the edge summation lemma, let $f_e = \theta_e$ and compute $\mathrm{M}_3(P)$ explicitly.

Exercise 28.6. (*General Schläfli formula*) *a*) [1+] Prove the Schläfli formula for *every* (not necessarily isometric) deformation $\{T_t, t \in [0,1]\}$ of a tetrahedron $T_0 \subset \mathbb{R}^3$, i.e.,

$$\sum_{e \in E} \ell_e(t) \cdot \theta_e'(t) \, = \, 0, \quad \text{for all} \;\; t \in [0,1].$$

b) [1] Deduce from here the Schläfli formula for deformations of all polyhedral surfaces.

Exercise 28.7. (*Spherical Schläfli formula*) *a*) [2-] Let $T \subset \mathbb{S}^3$ be a spherical tetrahedron with edge lengths ℓ_e and dihedral angles θ_e, as above. Suppose $\{T_t, t \in [0,1]\}$ is a deformation of T. Prove:

$$\frac{1}{2} \sum_{e \in E} \ell_e(t) \cdot \theta_e'(t) \, = \, \operatorname{vol}(T_t)'.$$

b) [1] Extend this to deformations of general spherical polyhedral surfaces in $\mathbb{S}^3$.

Exercise 28.8. [1+] Let $\Delta \subset \mathbb{R}^3$ be a fixed tetrahedron and let $a, b \subset \Delta$ denote its two edges. When we change the length ℓ_a of a while keeping other edge lengths fixed, the dihedral angle θ_b at b also changes. Let $L_{ab} = \partial\theta_b / \partial\ell_a$. Prove that $L_{ab} = L_{ba}$, for all a and b.

Exercise 28.9. *a*) [1] Let $x_1, x_2, x_3, x_4 \in \mathbb{R}^2$ be four points in general position, and let $\ell_{ij} = |x_i x_j|$ be the pairwise distances between them. Suppose all distances except for ℓ_{12} and ℓ_{34} are fixed. Since $(x_1x_2x_3x_4)$ has five degrees of freedom, change of ℓ_{12} inflicts a change of ℓ_{34}. Prove that

$$\frac{\ell_{12}\, d\ell_{12}}{\operatorname{area}(x_1x_2x_3) \cdot \operatorname{area}(x_1x_2x_4)} \, = \, \frac{-\,\ell_{34}\, d\ell_{34}}{\operatorname{area}(x_1x_3x_4) \cdot \operatorname{area}(x_2x_3x_4)}\,.$$

b) [1+] Let $x_1, x_2, x_3, x_4, x_5 \in \mathbb{R}^3$ denote five points in general position, and let $\ell_{ij} = |x_i x_j|$ be as above. Suppose all ℓ_{ij} except for ℓ_{12} and ℓ_{34} are fixed. Prove that

$$\frac{\ell_{12}\, d\ell_{12}}{\operatorname{vol}(x_1x_2x_3x_5) \cdot \operatorname{vol}(x_1x_2x_4x_5)} \, = \, \frac{\ell_{34}\, d\ell_{34}}{\operatorname{vol}(x_1x_3x_4x_5) \cdot \operatorname{vol}(x_2x_3x_4x_5)}\,.$$

28.7. **Final remarks.** The angular velocity equation (Lemma 28.2) is standard, see e.g., [A2, §10.1]. The edge summation lemma (Lemma 28.1) is a straightforward extension of the argument in [Pog2]. Corollary 28.4 is due to Alexander [Ale] who presented two proofs. The first proof is based on simplicial subdivisions of polyhedra and the additivity property of the mean curvature. Our proof of the (special case of) Schläfli formula (Theorem 28.3) is in fact similar to the second proof of Alexander (see [Ale, §9]).

The 'local proof' presented above is due to Pogorelov [Pog2], who never seems to have published the proof of his lemma (Lemma 28.5). Pogorelov's lemma was proved in a companion paper [Vol1] by Volkov, which we follow. The combined Pogorelov–Volkov proof of the Cauchy theorem is obviously harder than Cauchy's proof, since it basically replaces the straightforward sign counting lemma (Lemma 26.5) with two non-trivial proofs (make it four, if one counts two supporting lemmas). On the other hand, this approach has the advantage of being connected to other results, and also gives ground for further investigations.

Most recently, Schlenker found an infinitesimal analogue of Pogorelov's lemma, and a similar in spirit (double counting) proof of the infinitesimal rigidity [Schl4] (see Section 32). His proof of the infinitesimal analogue of Pogorelov's lemma is independent of the sign changes lemma, and is based on some explicit calculation for spherical 4-gons. It would be natural to look for a similar argument to obtain a new proof of Pogorelov's lemma, independent of the sign changes lemma.

As we mentioned above, the fact that the mean curvature of polyhedral surfaces are invariant under the flexing was shown in [Ale]. This result was rediscovered and extended in [AlmR], where further connections to the Schläfli formula were found (see also [SchlS]). For the history of the Schläfli formula, references and related results see [AVS, Mil2]. We refer to [Sant, Ch. 13] for more on mean curvature in higher dimensions, its connections to differential and integral geometry, and the proof of the Minkowski formula (Exercise 28.5).

Our proof of Theorem 28.6 follows [Mi2] (see also [Mi3, §1.5]). Note that isoperimetric inequalities were studied earlier in Section 7 and in fact one can view Theorem 28.6 as a generalization of the isoperimetric inequality in the plane (see Exercise 28.1).

29. Sen'kin-Zalgaller's proof of the Cauchy theorem

29.1. **Avoiding counting at all cost.** Below we present yet another proof of the Cauchy theorem (Theorem 26.1) with the opposite philosophy compared to the proof in the previous section. Previously, of the Cauchy theorem, we replaced a simple double counting 'global argument' with a delicate and well crafted 'local argument' (Pogorelov's Lemma 28.5). Now, instead, we give an elaborate geometric 'global argument', reducing the problem to a simple looking 'local' lemma, which is a weak version of the arm lemma (Lemma 22.3). The latter will be transformed in such a way as to allow an advanced but straightforward 'global' proof. This makes the following proof fundamentally global, at least as much as such proofs ever are. We begin by stating the local result.

Lemma 29.1 (Weak arm lemma)**.** *Let $X = [x_1 x_2 \ldots x_n]$ and $X' = [x'_1 x'_2 \ldots x'_n]$ be two convex spherical polygons in the upper hemisphere, such that*

$$\sphericalangle\, x_1x_2x_3 < \sphericalangle\, x'_1x'_2x'_3\,,\ \ldots\ ,\sphericalangle\, x_{n-1}x_nx_1 < \sphericalangle\, x'_{n-1}x'_nx'_1\,,\ \sphericalangle\, x_nx_1x_2 < \sphericalangle\, x'_nx'_1x'_2\,.$$

Then X and X' are not isometric*, i.e., the following equations cannot hold:*

$$|x_1x_2| = |x'_1x'_2|,\ \ldots\ ,\ |x_{n-1}x_n| = |x'_{n-1}x'_n|\ ,\ |x_nx_1| = |x'_nx'_1|\,.$$

In the notation of Section 26, the lemma says that the number of sign changes cannot be zero unless all labels are zero. This case appears at the very end of the proof of the sign changes lemma (Lemma 26.4), and is an immediate consequence of the arm lemma (Lemma 22.3). Following the pattern of the previous proofs, we continue with the proof of the Cauchy theorem, and then derive the lemma.

29.2. **Step back and take a look from the outside.** The idea of this proof is to use the global convexity of convex polytopes by comparing the distances from the same vertex to the corresponding points on the surfaces. There are certain restrictions on this distance function which can be utilized by the following geometric argument.

Proof of the Cauchy theorem modulo the weak arm lemma. Fix any vertex v of the polytope P, and the corresponding vertex v' of P'. Denote by $S = \partial P$ and $S' = \partial P'$ the polyhedral surfaces of the polytopes, and let $\zeta : S \to S'$ be the isometry map as in the Cauchy theorem (Theorem 26.6). We will use primes in the notation $\zeta : x \to x'$ between points of the surfaces throughout the proof.

Consider the dual cone C_v^* (see Section 25) and fix a point $w \in C_v^*$ (see Figure 29.1). Denote by $f = f_v : S \to \mathbb{R}_+$ and $f' = f'_w : S' \to \mathbb{R}_+$ the distance function (in $\mathbb{R}^3$) between v, w and points of the surfaces: $f(x) = |vx|$, and $f'(x') = |wx'|$. Consider a subset $G = G_w \subset S$ of points where function f is greater than f' (on the corresponding points):

$$G := \{x \in S \mid f(x) > f'(x'),\ \text{where}\, x' = \zeta(x)\}\,.$$

Similarly, let $F = F_w \subset S$ be the set of points where the functions are equal:

$$F := \{x \in S \mid f(x) = f'(x'),\ \text{where}\, x' = \zeta(x)\}\,.$$

To visualize set F, consider a rigid motion of $\mathbb{R}^3$ which moves face A into A'. Denote by $\widetilde{w}$ the image of the point w. The set of points at equal distance from v and $\widetilde{w}$ is the plane T which intersects face A either by a face or by an interval with endpoints at the edges of A, or T does not intersect A at all (see Figure 29.1). In other words, the set F is a union of faces and straight intervals between the edges.

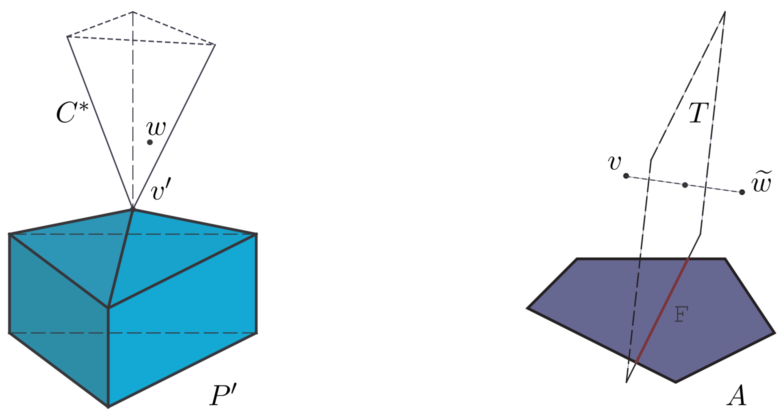

FIGURE 29.1. Point $w \in C^*$ and the construction of F'.

In the construction above we have some flexibility in the choice of a point w, an advantage we can exploit. When $w = v'$, we can assume without loss of generality that G is nonempty. From the contrary, if $G = \varnothing$ switch the role of P and P'. If we still have $G = \varnothing$, then $F = S$ and the polytopes are equal. Indeed, simply note that the distances between any vertex and the corresponding points on faces determine the polytope up to a rigid motion.

Now, start moving w away from v' in direction of the cone C^*. Then the set $G = G_w$ is decreasing since all the distances between w and points in S' are increasing. Since $G_{v'}$ is nonempty, when w is close enough to v' the set G_w will remain nonempty. Now, in order for a vertex u of the polytope P to be in F, the point w must be on a sphere with radius $|uv|$ centered at $u' = \zeta(u')$. For points w in general position this cannot happen.

To summarize, we can choose w is such a way that G_w is nonempty and F_w does not contain any vertices of P. The set $F = F_w$ is a union of straight intervals between the edges (see Figure 29.2).

Denote by H a connected component of G_w. From above, the boundary ∂H is a union of polygons on the surface of S, with polygon vertices on the edges of P, but not in the vertices of P. Denote by $Z = [z_1 z_2 \ldots z_n]$ a polygon corresponding to the exterior boundary of H, defined as a boundary of the only connected component of $S \smallsetminus H$ containing v. Similarly, define $Z' = [z'_1 z'_2 \ldots z'_n] = \zeta(Z)$, where $z'_i = \zeta(z_i)$, the corresponding polygon in S'.

By construction, polygons Z and Z' have equal sides and the distance from v and w to the corresponding points is the same: $|vz_i| = |wz'_i|$, for all $1 \le i \le n$. Consider a polygon vertex z_i, and let $y \in H$ be a point on the same edge as z_i. Let $a = z_{i-1}$,

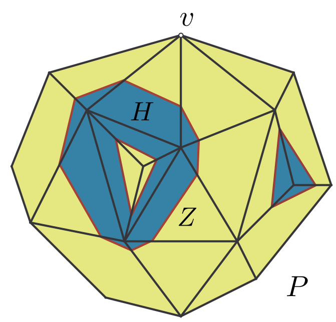

FIGURE 29.2. Connected component H of the set G (shaded) on the surface S of the polytope P. Polygon Z is the exterior boundary of H.

and $b = z_{i+1}$ be the previous and next vertices of the polygon P (from here on in this section all indices in subscripts are taken mod n). Define $a' = \zeta(a)$, $b' = \zeta(b)$, $y' = \zeta(y)$ to be the corresponding points on S' (see Figure 29.3).

We can now consider a four-sided cone D starting at z_i and spanned by the vectors (z_i, y), (z_i, a), (z_i, b), and (z_i, v). Similarly, consider a four-sided cone D' starting at z_i and spanned by the vectors (z_i', y'), (z_i', a'), (z_i', b'), and (z_i', w). Comparing D and D' we see that these are cones with isometric sides (see above) and by definition of the set $G \ni y$, the distance $|vy| > |wy'|$. This implies that $|ab| < |a'b'|$, and that the dihedral angle in D at (vz_i) is *strictly smaller* than the corresponding dihedral angle in D' at (wz_i'), for every $i \in [n]$ (see Figure 29.3). This crucial observation allows us to obtain a contradiction with the weak arm lemma (Lemma 29.1).

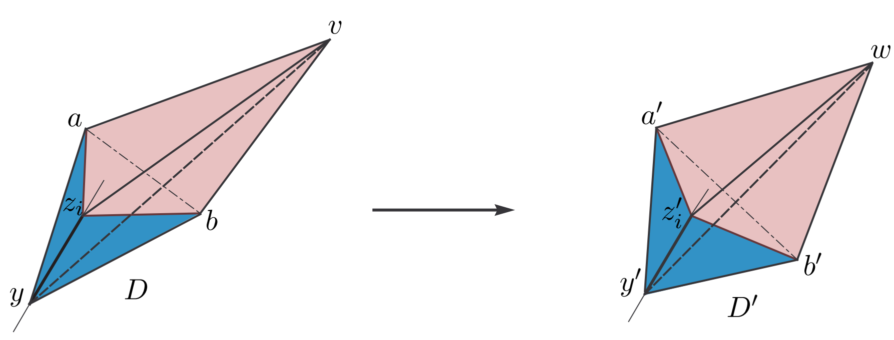

FIGURE 29.3. Two corresponding four-sided cones D and D'.

Let $\mathbb{S}$ and $\mathbb{S}'$ be the unit spheres centered at v and w. Consider $X \subset \mathbb{S}$ and $X' \subset \mathbb{S}'$ the projections of Z and Z' from v and w on $\mathbb{S}$ and $\mathbb{S}'$, respectively. By the equality of triangles (vz_iz_{i+1}) and $(vz_i'z_{i+1}')$, we have $\angle x_ivx_{i+1} = \angle x_i'wx_{i+1}'$, and thus the spherical polygons X and X' have equal corresponding sides. Observe that the spherical angles $\sphericalangle x_{i-1}x_ix_{i+1}$ and $\sphericalangle x_{i-1}'x_i'x_{i+1}'$ in X and X' are equal to the dihedral angles in (vz_i) and (wz_i'), defined as above. Thus, $\sphericalangle x_{i-1}x_ix_{i+1} < \sphericalangle x_{i-1}'x_i'x_{i+1}'$ for all $i \in [n]$.

Finally, note that X lies in the cone C_v and thus can be moved into an upper hemisphere. Similarly, X' lies in the cone $C_{v'}$. Since the cone over X' with a vertex at w is strictly smaller than $C_{v'}$, the polygon X' also can be moved into an upper hemisphere. By the weak arm lemma (Lemma 29.1), we conclude that such polygons X and X' do not exist, a contradiction. This proves that $P \simeq Q$. □

29.3. **Killing the monsters.** It would be a shame to have a "global" but rather intricate proof as above, and a rather unexciting inductive proof of the weak arm lemma (Lemma 29.1). Fortunately, the fact that the lemma is qualitative rather than quantitative allows us to create from two polygons a nice imaginary 'monster', i.e., an impossible construction whose nonexistence is easy to establish.

We need a few preliminary definitions and remarks. Let M be a compact metric space homeomorphic to a sphere. The example to keep in mind is a surface $S = \partial P$ of a convex body with the distance defined as the length of the shortest path in S between the points. The advantage of the abstract metric spaces is that it can be studied without an explicit embedding into Euclidean space. Below we use only the abstract *spherical polyhedral surfaces* (s.p.s.) which are defined by gluing along the edges a finite number of spherical polygons (on a unit sphere $\mathbb{S}^2$), such that the resulting manifolds are homeomorphic (but not necessarily isometric) to a sphere. These are analogues of polyhedral surfaces for spherical polyhedra (we will add an analogue of convexity below). An example is a 'spherical tetrahedron' which can be obtained by gluing together four equilateral spherical triangles with the same sidelength.

Given a s.p.s. M as above, we can define lines, circles, and angles accordingly. At a point $x \in M$, define the *curvature* $\omega(x) = 2\pi - \alpha(x)$ where $\alpha(x)$ is the total angle around x. A point $x \in S$ is called *flat* if $\omega(x) = 0$, and *non-flat* otherwise. For a k-gon $Q \subset M$ define

$$\omega(Q) := \operatorname{area}(Q) + \sum_{x \in \mathrm{NF}(Q)} \omega(x),$$

where $\mathrm{NF}(Q)$ is the set of non-flat interior points $x \in Q$. Using these definitions one can extend the inductive proof of the Gauss–Bonnet theorem (Theorem 25.3) to obtain that the total curvature of every s.p.s. satisfies $\omega(M) = 4\pi$.

Finally, we will always assume that our s.p.s. is nonnegatively curved: $\omega(x) \geq 0$ for all $x \in M$. Observe that for every two points $x, y \in M$ we have $|xy| \leq \pi$. Indeed, the shortest path cannot go through points of positive curvature by Proposition 10.1. Thus, every shortest path $\gamma \subset M$ between x and y must lie in a flat neighborhood $R \subset M$. Since R is isometric to a region in $\mathbb{S}^2$ (possibly overlapping), the path γ of length $|\gamma| > \pi$ can be shortened. This condition will be important in the proof.

Proof of the weak arm lemma. Let X, X' be two spherical polygons as in the lemma. Remove X' from a unit sphere $\mathbb{S}^2$ and attach X in its place. Denote by M the resulting s.p.s. The only non-flat points in M are the vertices of $X = [x_1 x_2 \ldots x_n]$. Using notation $\alpha_i = \sphericalangle x'_{i-1} x'_i x'_{i+1} - \sphericalangle x_{i-1} x_i x_{i+1}$, we have $\omega(x_i) = \alpha_i > 0$. Since X' is in the upper hemisphere, the s.p.s. M contains a hemisphere $B \subset \mathbb{S}^2$, and $x_i \in M \smallsetminus B$ for all $i \in [n]$.

Consider a shortest path γ between x_1 and x_2 and cut M along γ. Let us prove that the length $|\gamma| = |x_1x_2|_M < \pi$. Indeed, from above $|x_1x_2| \le \pi$. Now, if $|x_1x_2| = \pi$, we can consider all shortest paths from x_1 to x_2, and show that they either go through other points x_i, or $n = 2$, both leading to a contradiction.

Construct two symmetric spherical triangles $T_1 = [a_1a_2a_3]$ and $T_2 = [b_1b_2b_3]$ on a unit sphere U $|a_1a_2| = |b_1b_2| = |x_1x_2|$, $\sphericalangle\, a_2a_1a_3 = \sphericalangle\, b_2b_1b_3 = \alpha_1/2$ and $\sphericalangle\, a_1a_2a_3 = \sphericalangle\, b_1b_2b_3 = \alpha_2/2$. From $|x_1x_2|_M < \pi$, for the third angle we have $\sphericalangle\, a_1a_3a_2 = \sphericalangle\, b_1b_3b_2 < \pi$. Glue T_1 and T_2 to each other: (a_1a_3) with (b_1b_3), (a_2a_3) with (b_2b_3). Now glue sides (a_1a_2) and (b_1b_2) of the triangles to the sides of the cut γ. This gives the nonnegatively curved s.p.s. M_1 with only $n-1$ non-flat points.

Continue this process to obtain M_2, M_3, etc., until we obtain M_{n-1} with only one non-flat point y. The complement $M_{n-1} - y$ is isometric to the complement to a point of a unit sphere, and has area 4π. Therefore, $\omega(M_{n-1}) = 4\pi + \omega(y) > 4\pi$, which contradicts the Gauss–Bonnet theorem for s.p.s. (see Exercise 29.1). $\square$

Remark 29.2. The condition that the polygons (or at least X') are in the upper hemisphere is critical in the weak arm lemma by the same argument as in Section 22.4. Let us see how this affects the proof above. Consider a sector R on a unit sphere which can be viewed as a 2-gon with vertices x, y at the North and South Pole, and some angle $\beta > 0$ between two meridians. Define M to be a s.p.s. obtained by gluing two sides of A. Now the curvature in each vertex is $\omega(x) = \omega(y) = 2\pi - \beta$, while area$(R) = 2\beta$. The triangles $T_1 = T_2$ have two angles $(\pi - \beta/2)$ and the third angle is π. Therefore, when we glue R and two triangles together we obtain back the sphere $\mathbb{S}^2$. As expected, no contradiction in this case.

29.4. **Exercises.**

Exercise 29.1. (*Gauss–Bonnet theorem for s.p.s.*) $\diamondsuit$ [1+] State and prove the analogue of the Gauss–Bonnet theorem (Theorem 25.3) for spherical polyhedral surfaces.

Exercise 29.2. [1] Check that the proof above extends verbatim to the proof of the Alexandrov uniqueness theorem (Theorem 27.7).

Exercise 29.3 (*Alexandrov-Sen'kin*)**.** [1] Let $S, S' \subset \mathbb{R}^3$ be two intrinsically isometric polyhedral surfaces with the same boundary $Q = \partial S = \partial S'$. Suppose the origin O is separated from S, S' by a hyperplane, and that both S, S' are seen from O from inside (outside). Then $S = S'$.

Exercise 29.4. [2-] Modify the proof above to obtain the Pogorelov uniqueness theorem (Theorem 27.8) in the piecewise smooth case.

29.5. **Final remarks.** The 'global proof' of the Cauchy theorem is due to Sen'kin [Sen1] and is not alike any other proof we know. The proof of the weak arm lemma in this section follows a companion paper by Zalgaller [Zal1]. While not as simple as the arm lemma (even in the corrected version), it is very insightful and brings to light some useful ideas. We refer to [BVK] for a slightly expanded presentation of the original proof. The extension of the Cauchy theorem in Exercise 29.3 is given in [AS].

Interestingly, there seems to be a bit of a tradeoff: the easier the global part is, the harder the local part is, and vice versa. Pogorelov's proof in the previous section has a strong local part, and an easy global part. Similarly, the proof in this section has a reverse emphasis.

30. Flexible polyhedra

30.1. **The importance of being flexible.** To better understand and appreciate the Cauchy theorem consider what happens with non-convex polyhedra. At this point there are several alternative definitions that can be accepted, some of them more interesting than others. We will go over these definitions in this section and see where they lead us.

30.2. **Rigidity and uniqueness of realization.** Let S be a 2-dimensional polyhedral surface, defined as a closed compact metric space obtained by gluing a finite number of triangles. Sometimes it is convenient to use polygons F_i in place of triangles. Of course, this is an equivalent definition since one can always triangulate polygons F_i. A *realization* of S is an isometric immersion $Q \subset \mathbb{R}^3$ given by a map $f : S \to \mathbb{R}^3$, which maps polygons F_i into equal polygons in $Q = f(S)$. In other words, polygons F_i lie on 'faces' of Q, but they are allowed to intersect as sets in $\mathbb{R}^3$.

Of course, one cannot hope to have a unique realization even when S is convex. For example, in Figure 30.1 we present two realizations of the same bipyramid, one convex and one non-convex. Further, for a *cyclic polytope* Z_k as in the same figure, the number of realizations is exponential in the number of vertices (polytope Z_k has $k+3$ vertices and 2^{k-1} realizations). On the other hand, it is easy to show that the number of different realizations of Z_k is always finite, thus it does not have any continuous deformations (cf. Corollary 26.2).

In summary, one cannot hope for uniqueness of realizations, just the lack of continuous families or realizations, corresponding to continuous deformations of polyhedra.

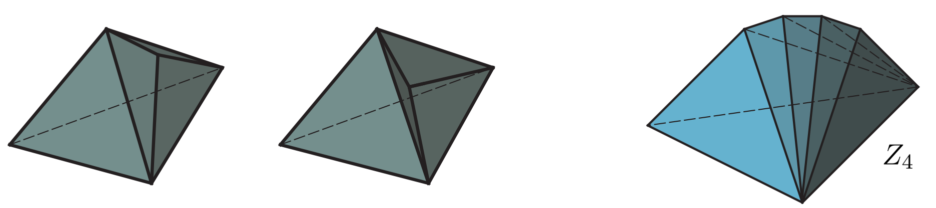

Figure 30.1. Two realizations of the same bipyramid and a cyclic polytope Z_4.

30.3. **Tight polyhedra.** A polyhedron $P \subset \mathbb{R}^3$ (a 2-dimensional polyhedral surface) is called *tight* if every plane divides P into at most two connected components. The definition is equivalent to the following weak convexity condition: every edge of $\mathrm{conv}(P)$ must lie in P and every vertex $v \in P$ which is a local minimum of some linear function $\varphi : \mathbb{R}^3 \to \mathbb{R}$, must also be a vertex in $\mathrm{conv}(P)$ (see Exercise 30.1).

The tightness condition is so restrictive, the reader might find it difficult to find any non-convex examples of tight polyhedra. As it happens, all tight polyhedra homeomorphic to a sphere are convex (see Exercise 30.1), but there are various natural examples of tight polyhedra of higher genus (see Figure 30.2).

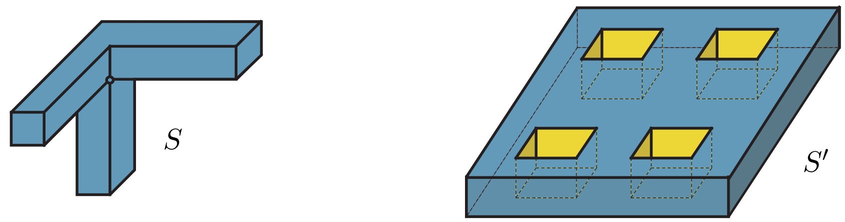

FIGURE 30.2. Surfaces $S, S' \subset \mathbb{R}^3$, where S' is tight and S is not.

Let us show that the Cauchy theorem does not extend to tight polyhedra. We present two non-congruent embedded toric polyhedra which are both tight and have congruent corresponding faces. In other words, we show that being embedded and tight is not sufficient for the uniqueness of realization.

Consider a convex polygon $Q = [v_1 \dots v_n]$ in the right half-plane of the xz-plane. Rotate Q around the z axis by $\pi/2$, π and $3\pi/2$ to obtain four copies of Q, two in the xz-plane and two in the yz-plane. Connect the corresponding edges in the orthogonal copies of Q by trapezoid faces to obtain a polyhedron P_Q homeomorphic to a torus. Let $A = [a_1 \dots a_4]$, $B = [b_1 \dots b_4]$ be the following two quadrilaterals:

$$a_1 = (1,2), \qquad a_2 = (2,4), \qquad a_3 = (4,3), \qquad a_4 = (3,1),$$
$$b_1 = (1,1), \qquad b_2 = (2,3), \qquad b_3 = (4,4), \qquad b_4 = (3,2),$$

(see Figure 30.3). Consider polyhedra P_A and P_B defined as above. It is easy to check that the corresponding faces of P_A and P_B are congruent, while the polyhedra themselves are not.

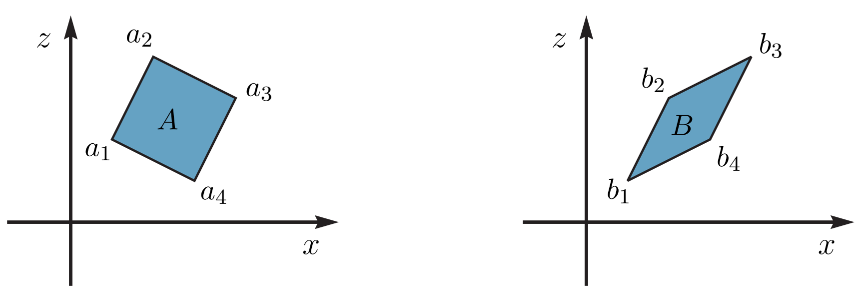

FIGURE 30.3. Polygons A and B corresponding to toric polyhedra with congruent faces.

30.4. **Flexible polyhedra.** We say that a polyhedral surface S is *flexible* if there exists a continuous family $\{Q_t : t \in [0,1]\}$ of realizations of S which are (globally) pairwise non-isometric. In other words, no two realizations $Q_t, Q_{t'}$ can be moved into each by a rigid motion. Such realizations are called *flexible polyhedra.* Below we give an example of a flexible polyhedron[73] called the *Bricard octahedron*, which is a non-convex realization of a (non-regular) convex octahedron in $\mathbb{R}^3$.

Consider all plane self-intersecting realizations of the polygon with sides a, b, a, b, where $0 < a < b$. By symmetry, such polygon realizations Q can be inscribed into a circle. Take a bipyramid with the main diagonal orthogonal to the plane and side edge length $c \geq (a + b)$ (see Figure 30.4). By the argument above, the resulting polyhedral surfaces are intrinsically isometric and form a continuous family of globally non-isometric realizations.

Note that the polyhedral surface S of the above realizations is that of a convex octahedron with four triangles with sides a, c, c, and four triangles with sides b, c, c. Therefore, S is homeomorphic to a sphere.

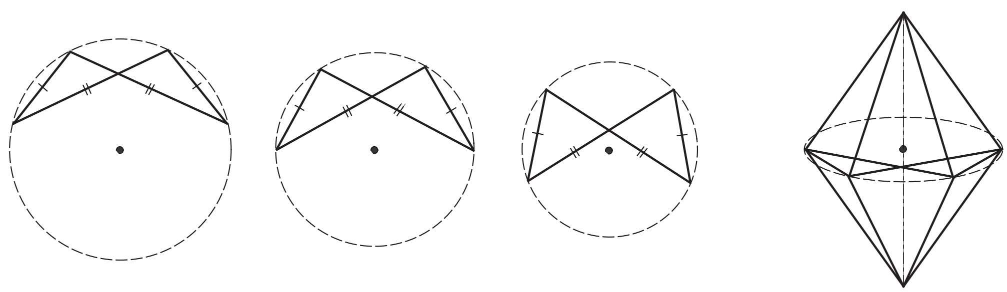

FIGURE 30.4. Three self-intersecting realizations of the same parallelogram, and the Bricard octahedron.

Since the surface S is triangulated, one can think of realizations of S as realizations of the corresponding graph. Formally, let G be a graph of an octahedron and let L be a length function taking values a, b, c as in the figure. From above, there is a continuous family of realizations of (G, L) which cannot be obtained one from another by rigid motions.

30.5. **Flexors.** While the Bricard octahedra can be viewed as an obstacle to an extension of the Cauchy theorem (Theorem 26.1) to non-convex polytopes, one can restrict the set of examples further, by requiring a realization to be an embedding (i.e., with no self-intersections). Such polyhedra are called *flexors* and were first constructed by Connelly. This is the most restrictive definition of non-convex polyhedra, suggesting that the Cauchy theorem really cannot be extended in this direction.

Unfortunately very few constructions of flexors are known, all of them somehow related to a trick used by Connelly. Being a flexible polyhedron carries already too many restrictions and flexors do not seem to have any additional properties separating them from the flexible polyhedra. We skip their constructions.

[73]In this section we will use the term 'polyhedron' quite loosely, applying it to all objects at hand.

30.6. **Flexible polyhedron of higher genus.** Let us give an example of a flexible polyhedron which is not homeomorphic to a sphere. Consider a 'Renault style' polyhedron T shown in Figure 30.5. This polyhedron consists of four symmetric 'tubes' with equal parallelogram sides, glued together along rhombi which are parallel in $\mathbb{R}^3$. Observe that T is homeomorphic to a torus and can be continuously deformed by 'flattening'. We omit the details.

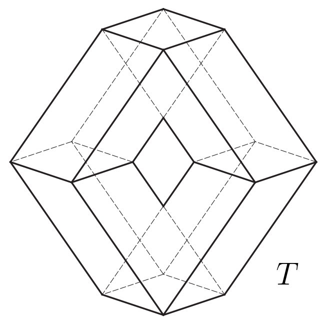

FIGURE 30.5. Flexible polyhedron T homeomorphic to a torus.

30.7. **Flexible spherical polyhedron.** As we explain in Section 27.2, if we want to have any chance of making flexible polyhedra in higher dimensions, we should be able to make flexible spherical polyhedra in $\mathbb{S}^3$. As it turns out, flexible spherical polyhedra are *easier* to construct than (the usual) flexible polyhedra in $\mathbb{R}^3$.

We start with a simple construction of a convex spherical polyhedron $P \subset \mathbb{S}^3$. There are two ways to think about it. First, one can take a spherical rhombus $R = [abcd]$, i.e., a polygon with all sides of length α on a unit sphere $S_1 \simeq \mathbb{S}^2$. Think of S_1 as the equator on the unit sphere $\mathbb{S}^3$ and consider a bipyramid over R with additional vertices x, y in the North and South Pole. Another way to think about it is to take four equal sectors $[xayb]$ with angle α on a unit sphere $S_2 \simeq \mathbb{S}^2$. Then glue them along the sides, as schematically illustrated in Figure 30.6. Clearly, the rhombus R is non-rigid, and thus so is P. In fact, at the extremes the polytope becomes flat; more precisely, we get a doubly-covered spherical sector, a polyhedron of zero volume.

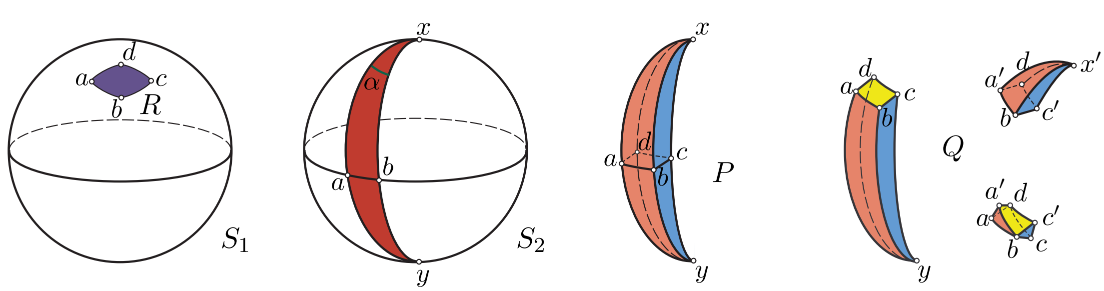

FIGURE 30.6. Construction of flexible spherical polyhedra P and Q

It is instructive to compare the construction of P with Theorem 27.1 which states that all spherical convex polyhedra are rigid. At first glance the two may seem contradictory until one goes into the details. Note a condition in the theorem that the polyhedron must lie in the upper hemisphere. This was used in the proof of the arm lemma (Lemma 22.3 and Lemma 29.1), a key ingredient in the proofs of the Cauchy theorem and its relatives. By construction, polyhedron P is 'almost' there: it can be rotated so that only points x, y are not in the upper hemisphere. This shows that the upper hemisphere condition is in fact necessary in Theorem 27.1.[74]

To obtain a spherical (non-convex) polyhedron in the upper hemisphere, let us modify the above construction as follows. Note that $[abcd]$ is really unnecessary to define P; in fact, these are not the real edges of the polyhedron. To simplify the construction, choose the rhombus $[abcd]$ now close to the North Pole. It divides the surface of the polyhedron P into two parts. Rotate the part containing x around diagonal (bd) to obtain a new 'top': pyramid with vertex x' over the rhombus $[a'bc'd]$. Keep the 'bottom' part, containing y unchanged. Now attach the 'top' part and the bottom part to intermediate triangles $(aa'b)$, $(aa'd)$, $(cc'b)$ and $(cc'd)$ (see Figure 30.6). The resulting polyhedron Q is flexible for the same reason as P. On the other hand, the direction of rotation around (bd) can be chosen in such way so that Q lies completely in a hemisphere.

30.8. **Flexible polyhedra in higher dimension.** Unfortunately, very little is known about flexible polyhedra in higher dimensions. The following construction of a 4-dimensional cross-polytope C_4 is an interesting variation on the Bricard octahedron theme.

Think of C_4 as dual to a 4-dimensional cube. Denote by $V = \{a_1, a_2, b_1, \ldots, d_2\}$ the set of 8 vertices of C_4. All pairs of vertices in the graph of C_4 are connected, except for the opposite pairs: (a_1, a_2), (b_1, b_2), (c_1, c_2), and (d_1, d_2). We think of a 4-dimensional realization of C_4 as of a map $f : V \to \mathbb{R}^2 \times \mathbb{R}^2$, where $f(a_1) = (a_1', a_1''), \ldots, f(d_2) = (d_2', d_2'')$. Visualize the first coordinates on one plane $\mathbb{R}^2$ and the second coordinate on another $\mathbb{R}^2$. We call these *plane coordinates*.

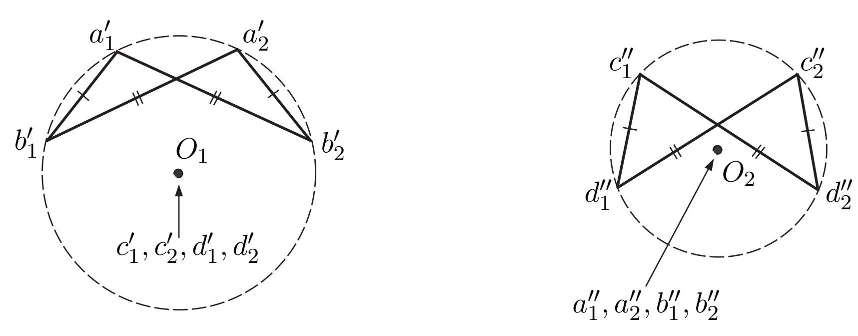

FIGURE 30.7. Walz's flexible 4-dimensional cross-polyhedron.

[74]We already mentioned in Remark 29.2 what happens to the (weak) arm lemma without this condition.

Now let the plane coordinates of C_4 be as shown in Figure 30.7. Here all plane coordinates either lie on circles or in their centers O_1 and O_2. Thus, the pairwise distances between points are either equal to the interval lengths as in the figure, or equal to $\ell = \sqrt{\rho_1^2 + \rho_2^2}$, where ρ_1, ρ_2 are the radii of the circles. Note that both inscribed 4-gons are flexible, as the circle radii change (compare this with the Bricard octahedron). Thus, we can keep the distance ℓ constant while changing the radii of the circles. Therefore, the above construction gives a flexible polyhedron.

Finally, note that by the argument in Subsection 27.2 every construction of a flexible d-dimensional polyhedron gives several constructions of flexible $(d-1)$-dimensional spherical polyhedra, one per vertex. Of course, in this case all such polyhedra are the spherical analogues of the Bricard octahedron.

30.9. **Polyhedral surfaces with boundary.** Now that we started expanding the class of non-convex polyhedra, there is no reason to stop on closed surfaces. One can (and, in fact, some people do) consider polyhedral surfaces with boundary. Unfortunately, in this case flexibility is not an exception but a rule. Basically, if one removes two adjacent faces from a surface of a simplicial polytope one obtains a flexible surface:

Theorem 30.1. *Let $P \subset \mathbb{R}^3$ be a simplicial convex polytope, let e be an edge of P, and let F_1, F_2 be two faces containing e. Then $S := \partial P \smallsetminus (F_1 \cup F_2)$ is a flexible polyhedral surface.*

Of course, the result cannot be extended to all polytopes. For example, if we remove two adjacent faces of a cube, the resulting surface is still rigid. The proof of the theorem is an easy application of the Alexandrov existence theorem (Theorem 37.1), and is presented in Subsection 37.3.

30.10. **Exercises.**

Exercise 30.1. (*Tight polyhedra*) $\diamondsuit$ *a*) [1+] Prove that a polyhedron $P \subset \mathbb{R}^3$ is tight if and only if every edge of $\operatorname{conv}(P)$ must lie in P and every vertex $v \in P$ which is a (strict) local minimum of some linear function $\varphi : \mathbb{R}^3 \to \mathbb{R}$ must be a vertex of $\operatorname{conv}(P)$.
b) [1-] Show that neither of the two conditions in *a*) suffice for the tightness.
c) [1-] Check that the two toric polyhedra constructed in Subsection 30.3 are tight.
d) [1] Prove that every tight polyhedron embedded into $\mathbb{R}^3$ and homeomorphic to a sphere is convex.[75]

Exercise 30.2. [2-] Generalize to higher dimensions the construction of a toric polyhedron in Subsection 30.6. Check whether they are flexible or rigid.

Exercise 30.3. [1] Consider the *Hooker polyhedron* defined in the US Patent 3894352.[76] Is it flexible or rigid?

Exercise 30.4. *a*) [$*$] Find examples of flexible polyhedra in $\mathbb{R}^d$, for all $d \geq 5$.
b) [$*$] Prove or disprove: every realization of a cross-polytope in $\mathbb{R}^5$ is rigid.

[75] One can view this result as a variation on the convexity criterion given in Exercise 1.25.

[76] See `http://www.google.com/patents?vid=USPAT3894352`.

30.11. **Final remarks.** The construction of tight toric polyhedra with isometric faces is due to Banchoff [Ban1]. See [Kuh] for more on tight polyhedra, generalizations and references.

We refer to [Con2, Con4, FucT] for an introduction to the subject of flexible polyhedra and easy-to-construct examples (see also [Ale4, Dol, Sab5]). Let us note also that Bricard completely characterized all flexible octahedra in 1897; there are in fact two additional families different from the construction we present in this section [Bri2] (see also [Leb3]). We refer to a well written survey [Ale6, §7] for detailed constructions and the references.

Flexible polyhedron homeomorphic to a torus in Subsection 30.6 is one of a large family of flexible polyhedra introduced by Goldberg [Gol2] (see also [Ale2]). Flexible spherical polyhedron in Subsection 30.7 is due to V. A. Alexandrov [Ale3]. Let us mention also that flexible polyhedra exist in the hyperbolic space $\mathbb{H}^3$, where the analogues of the Bricard octahedra can be proved to be flexible [St2].

The construction in Subsection 30.8 is due to A. Walz (unpublished). Our presentation follows [St1], where the author gives a more general construction of flexible cross-polytopes. While no examples of flexible polytopes in higher dimensions are known at the moment, we do expect the multitude of examples. On the other hand, Stachel conjectured that in dimension $d \geq 5$ there are no flexible cross-polytopes [St1].

31. The algebraic approach

The algebraization of realization spaces introduced earlier in the context of linkages (see Section 13) will prove critical to understand the rigidity of non-convex polyhedra. In this section we prove *Gluck's theorem* and set up a general approach for infinitesimal rigidity in the next two sections, as well as the bellows conjecture.

31.1. **Rigidity vs. flexibility: what is more likely?** Let P be a convex polytope and $S = \partial P$ be its polyhedral surface. The Cauchy theorem states that S is rigid. On the other hand, as the Bricard octahedron and the Connelly polytope show, this is no longer true for immersions and even for embeddings of S (see the previous section). The question is what is more likely: that a surface is rigid or flexible? Perhaps unsurprisingly, the answer is unambiguous: a polyhedral (not necessarily convex) surface in $\mathbb{R}^3$ is almost surely rigid.

We start with the definitions. Let $G = (V, E)$ be a *plane triangulation*, i.e., a connected planar graph with all faces (including exterior face) triangles. Let $|V| = n$. By Euler's formula we have $m = |E| = 3n - 6$ (see Corollary 25.2). Suppose we are given a length function $L : E \to \mathbb{R}_+$. For each triangle in the plane triangulation G, make a metric triangle with edge lengths given by the length function L. Now glue these triangles along the corresponding sides to obtain a simplicial surface S homeomorphic to a sphere.

Clearly, not all length functions define a metric space as the triangle inequality must be satisfied. Denote by $\mathcal{L}(G)$ the set of all length functions leading to a metric space. The set $\mathcal{L}(G)$ forms a convex cone in $\mathbb{R}^m$ with facets corresponding to triangle inequalities for each triangle in G. Denote by $\mathcal{L}_1(G)$ the set of all length functions $L \in \mathcal{L}(G)$ such that $\sum_{e \in E} L(e) \le 1$. From above, $\mathcal{L}_1(G)$ is a convex polytope in $\mathbb{R}^m$.

As before, a *realization* of (G, L) is a map $f : V \to \mathbb{R}^3$, such that the actual distance $\bigl|f(v)f(w)\bigr| = L(e)$ for every edge $e = (v, w) \in E$. Given $L \in \mathcal{L}(G)$, such a realization defines an (intrinsically) isometric immersion of S into $\mathbb{R}^3$. In the notation above, realizations of (G, L) now correspond to realizations of the surface S.

Finally, rigid motions in $\mathbb{R}^3$ act naturally on realizations of (G, L) by acting on sets $\{f(v) \mid v \in V\}$. We are now ready to state the main result.

Theorem 31.1 (Gluck, Poznjak)**.** *Let G be a planar triangulation and let L be a random length function in $\mathcal{L}_1(G)$. Then, almost surely, (G, L) has only a finite number of realizations, up to rigid motions.*

In other words, the set of non-rigid realizations has measure zero in $\mathcal{L}(G)$. We prove the result by an algebraic argument later in this section.

31.2. **The reason why metallic bellows do not exist.** The bellows, as any dictionary will explain, are mechanical devises which by expansion and contraction pump air. Basically, these are surfaces which change the volume under bending. Of course, in theory it would be nice to make these surfaces polyhedral, with all faces made out of some kind of metal. In practice this does not work. Here is why (a theoretical version).

We start with a celebrated *bellows conjecture* recently resolved by Sabitov. It states that the volume of flexible polyhedra is invariant under continuous deformations:

Theorem 31.2 (Former bellows conjecture). *Let $\{Q_t : t \in [0,1]\}$ be a continuous deformation of a closed polyhedral surface S homeomorphic to a sphere. Then the volume* $\mathrm{vol}(Q_t)$ *is independent of t.*

We prove this theorem in Section 34. In fact, the result holds for all closed orientable surface immersed in $\mathbb{R}^2$ (see Subsection 34.7). To illustrate the theorem, note that the (signed) volume of the Bricard octahedron and the toric polyhedron T defined in Subsection 30.6, is equal to zero by symmetry. Of course, one can attach a rigid convex polyhedron to the surface of the either of the two, to obtain a flexible polyhedron of positive volume, still constant under flexing.

31.3. **Constructing polytopes from scratch.** Let $P \subset \mathbb{R}^3$ be a simplicial convex polytope with n vertices: $V = \{v_1, \ldots, v_n\}$. For simplicity, let us assume that $(v_1 v_2 v_3)$ is a face of P. Observe that there always exists a rigid motion which maps vertex v_1 into the origin $O = (0,0,0)$, v_2 into $(a,0,0)$, and v_3 into $(b,c,0)$, for some $a,b,c \in \mathbb{R}_+$. Note also that up to reflection such rigid motion is unique. We call the resulting polytope *planted*.

Now, in order to construct a planted polytope from its graph and edge length we need to set up a system of algebraic (in fact, quadratic) equations. By the Cauchy theorem, there exists only one convex solution, so we are done. Of course, the first part of the plan is easy to set up as follows.

Formally, let $G = (V,E)$ be a plane triangulation and let $L : E \to \mathbb{R}_+$ be a length function, $L \in \mathcal{L}(G)$. Denote by $f : v_i \to (x_i, y_i, z_i)$, $i \in [n]$, realizations of (G,L). For every pair of vertices (v_i, v_j) consider a polynomial

$$F_{ij} := (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 .$$

We consider only planted solutions, i.e., values $(x_1, \ldots, z_n) \in \mathbb{R}^{3n}$ where

$$(\dagger) \qquad x_1 = y_1 = z_1 = y_2 = z_2 = z_3 = 0.$$

Denote by $\mathrm{R} = \mathbb{C}[x_2, x_3, \ldots, x_n, y_3, \ldots, y_n, z_4, \ldots, z_n]$ the ring of polynomials on the remaining variables. Now, in the ring R we have the system of equations:

$$(\ddagger) \quad \bigl\{F_{ij} = (\ell_{ij})^2, \ \text{for all } \ e = (v_i, v_j) \in E \ \text{ and } \ \ell_{ij} = L(e)\bigr\} .$$

As we mentioned before, $(\ddagger)$ is a system of $m = 3n - 6$ equations with $3n - 6$ variables. Therefore, if the equations are algebraically independent, we have a finite set of solutions. The following result positively resolves the problem:

Theorem 31.3 (Gluck). *Let $G = (V,E)$ be a plane triangulation. Then polynomials F_{ij} corresponding to edges $(v_i, v_j) \in E$, $v_i, v_j \in V$, are algebraically independent.*

We prove the theorem later in this section. First, let us mention the following corollary, which follows easily from the theorem.

Corollary 31.4. *Under conditions of Theorem 31.3, every polynomial* $g \in \mathrm{R}$ *is a root of a nontrivial algebraic equation:*

$$(\diamond) \qquad c_N t^N + \ldots + c_1 t + c_0 = 0,$$

where $c_r \in \mathbb{C}[F_{ij}, (v_i, v_j) \in E]$, $0 \le r \le N$.

We call polynomial relations $(\diamond)$ in the corollary the *Sabitov polynomial relations*. To obtain the corollary simply note that transcendency degree of R is $m = 3n - 6$, equal to the number $|E|$ of algebraically independent polynomials F_{ij} in the theorem.

Remark 31.5. (*Does this give an algorithm?*) It was shown in [FedP] that the degree of Sabitov polynomials $(\diamond)$ is at most 2^m and can be exponential even in the most simple cases (see Exercise 31.5). Starting with edge polynomials, one can use standard computer algebra tools to determine (numerically) the coordinates of all vertices.

31.4. **Why polynomials are always better than numbers.** Let us continue our construction of the polytope. For each coordinate x_i, y_i, and z_i we need to compute the Sabitov polynomial relations $(\diamond)$. Evaluating polynomials F_{ij} at ℓ_{ij}^2 makes $(\diamond)$ into polynomials of the desired values. Compute their roots and try one by one all resulting combinations until a convex one is found.

Alternatively, and more invariantly, we can compute Sabitov polynomials for all diagonals F_{1i}, F_{2i} and F_{3i}. Again, after their lengths are determined to belong to a certain finite set of solutions, we have a finite number of possibilities to consider.

Now, the above argument clearly contradicts the existence of flexible polyhedra (say, with self-intersections) since it implies that there is *always* only a finite number of solutions. The mistake in this argument is very important and may not be obvious at first sight.

. . . . . . [Think about it for a few minutes!] . . . . . .

The mistake is that all polynomial coefficients c_i in the Sabitov polynomial relations $(\diamond)$ may become zero when evaluated at $\{\ell_{ij}\}$. When this happens, we cannot determine the corresponding diagonal lengths, thus allowing for the flexible polyhedra. On the other hand, if at least one $c_i \ne 0$ in all Sabitov polynomials for diagonals as above, then there exists only a finite number of planted realizations. In particular, all realizations are (continuously) rigid.

We are now ready to prove Theorem 31.1. First, observe that the polytope $\mathcal{L}_1$ of length functions has full dimension $m = 3n - 6$ (see next subsection). On the other hand, for every diagonal (v_i, v_j), the set of roots of the equation $c_r = 0$ (over $\mathbb{R}$) has codimension at least 1, where c_r is a coefficient of $(\diamond)$ corresponding to F_{ij}. Thus, all relations $(\diamond)$ are nonzero almost surely, and there exists only a finite number of realizations of (G, L), as desired.

Let us think about what is needed to prove Theorem 31.2, the former bellows conjecture. First, one has to check that $\mathrm{vol}(P)$ is a polynomial in the ring R, i.e., depends polynomially on the vertex coordinates. Then, by Corollary 31.4 it satisfies a Sabitov polynomial relation $(\diamond)$. If one can prove that the coefficient c_N is nonzero, then the volume takes only a finite number of values, and, therefore, remains constant under continuous deformations. While proving polynomiality of the volume is an easy exercise, checking that $c_N \neq 0$ is a major task; we prove this in Section 34.

31.5. **Proof by an algebraic manipulation.** Let us first show that Theorem 31.3 follows easily from the Cauchy theorem and the Steinitz theorems. The proof is trivial in essence, but requires a good absorbtion of the results and definitions involved.

Proof of Theorem 31.3. Let $G = (V, E)$ be a plane triangulation as in the theorem. Recall that by the Steinitz theorem (Theorem 11.1) there exists a convex polytope P with graph G. We can assume that P is planted; otherwise use a rigid motion to make it so. Perturbing the vertex coordinates of P by $< \epsilon$, while keeping P planted, does not change convexity and the graph of the polytope, for sufficiently small $\epsilon > 0$. By the Cauchy theorem, all perturbations give different length functions $L : E \to \mathbb{R}_+$ of these realizations of P. In other words, we obtain an open subset $X \subset \mathbb{R}^m$ of perturbations of coordinates, each giving a length function $L \in \mathcal{L}(G) \subset \mathbb{R}^m$.

Now consider a *characteristic map* $\mathbf{F} = (\ldots, F_{ij}, \ldots) : \mathbb{R}^m \to \mathbb{R}^m$, where F_{ij} will always correspond to graph edges $(v_i, v_j) \in E$. The Jacobian $J(\cdot)$ in this case is a determinant of a $m \times m$ matrix of partial derivatives of F_{ij} by the coordinates. Since F_{ij} are polynomials, the Jacobian is also a polynomial. From above, $J(\cdot)$ does not vanish on an open subset $X \subset \mathbb{R}^m$ of the coordinates. Therefore, the Jacobian $J(\cdot)$ is a nontrivial polynomial, and polynomials F_{ij} are algebraically independent. $\square$

Remark 31.6. The proof above is based on the Cauchy theorem, but uses a 'local' rather than a 'global' argument. On the other hand, the (continuous) rigidity is insufficient to make the argument work, since in general the Jacobian can vanish at a point while a polyhedral surface remains continuously rigid. The examples are given in the previous section. This will also lead us to a notion of static and infinitesimal rigidity which are stronger than continuous rigidity, but somewhat easier to prove than the Cauchy theorem.

31.6. **Exercises.**

Exercise 31.1. [1+] Let $P \subset \mathbb{R}^3$ be a bipyramid with side length c, over an inscribed n-gon in the plane, with side lengths a. Find a minimal polynomial for the volume of P with coefficients in $\mathbb{C}[a^2, c^2]$.

Exercise 31.2. Denote by $\mathrm{Q}_n(a, b)$ the radius of a convex polygon inscribed into a circle with $n-1$ sides a and one side b. Let $r_n(a, b)$ be the radius of the this circle, i.e., the radius of the circle circumscribed around $\mathrm{Q}_n(a, b)$.
a) [1+] Compute a formula for $r_n(a, b)$ as a root of a polynomial equation in a^2 and b^2 in terms of the Chebyshev polynomials.
b) [1+] Compute the minimal degree of such polynomial relation for $r_n(a, b)$.

Exercise 31.3. (*Robbins's problem*) Denote by $\mathrm{A}(a_1, \ldots, a_n)$ and $\mathrm{R}(a_1, \ldots, a_n)$ the area and the circumradius of an inscribed convex polygon with sides a_i (see Example 34.6 and Exercises 34.1, 34.6).
a) [1] Prove that $\mathrm{A}^2(\cdot)$ and $\mathrm{R}^2(\cdot)$ are algebraic over $\mathbb{C}[a_1^2, \ldots, a_n^2]$.

b) [1+] Prove that in both cases the degrees α_n and ρ_n of the minimal polynomial is at least T_k when $n = 2k+1$, and $2T_k$ when $n = 2k+2$, where T_k is the number of (possibly self-intersecting) polygons inscribed into a circle.
c) [1+] Prove an explicit formula for T_k:

$$T_k \, = \, \frac{2k+1}{2}\binom{2k}{k} \, - \, 2^{2k-1}\,.$$

d) [2] Give an explicit formula for ρ_n. Check that it proves the upper bound matching the lower bound in *b*).
e) [2] Prove that $\alpha_n = \rho_n$.

Exercise 31.4. (*Connelly's suspensions*) Define a *suspension* to be a bipyramid over an inscribed polygon (as in Bricard's octahedron).
a) [2] Prove that in every flexible suspension, the inscribed (self-intersecting) polygon must have each edge length repeated twice, once in each direction.
b) [1] Check that such polygons have zero area (cf. Exercise 34.6). Conclude that every flexible suspension has a zero volume.

Exercise 31.5. (*Degrees of Sabitov polynomials*) $\Diamond$ *a*) [1] Let Q_n denote the bipyramid over an n-gon and let β_n be the degree of the Sabitov polynomial of the "main diagonal". Prove that $\beta_n = \rho_n$ (see Exercise 31.3). Conclude that β_n is exponential.
b) [1] Let ϱ_n denote the degree of Sabitov polynomial of the "main diagonal" in the cyclic polytope Z_n (see Figure 30.1). Prove that ϱ_n is exponential.
c) [2] Use a Bézout type result to show that degrees of Sabitov polynomials ($\diamond$) is at most 2^m.

Exercise 31.6. Let $\Delta = (v_0v_1v_2v_3) \subset \mathbb{R}^3$ be a tetrahedron, and let ℓ_{ij} be its edge lengths, $0 \le i < j \le 3$.
a) [1−] Find positive values $\{\ell_{ij}, 0 \le i < j \le 3\}$ which satisfy the triangle inequalities

$$\ell_{ij} + \ell_{jk} > \ell_{ik}\,,$$

and such that no tetrahedra with these edge lengths exist.
b) [1] Consider the set of all possible edge lengths of tetrahedra in $\mathbb{R}^3$:

$$L \, = \, \{(\ell_{01}, \ldots, \ell_{34})\} \, \subset \, \mathbb{R}^6.$$

Prove that L is not convex.
c) [2−] Consider the set of all possible square edge lengths of tetrahedra in $\mathbb{R}^3$:

$$S \, = \, \{(\ell_{01}^2, \ldots, \ell_{34}^2)\} \, \subset \, \mathbb{R}^6.$$

Prove that S is convex.

Exercise 31.7. Let $\Delta = (v_1v_2v_3v_4) \subset \mathbb{R}^3$ be a tetrahedron, and let γ_{ij} denote the dihedral angle at the edge (v_i, v_j), for all $1 \le i < j \le 4$.
a) [1] Prove that $\gamma_{12} + \gamma_{23} + \gamma_{34} + \gamma_{14} \le 2\pi$.
b) [1−] Prove that $2\pi \le \gamma_{12} + \gamma_{13} + \gamma_{14} + \gamma_{23} + \gamma_{24} + \gamma_{34} \le 3\pi$.
c) [1] Prove that $0 \le \cos\gamma_{12} + \cos\gamma_{13} + \cos\gamma_{14} + \cos\gamma_{23} + \cos\gamma_{24} + \cos\gamma_{34} \le 2$.
d) [1−] Prove that the second inequality in *c*) is an equality only if Δ is equihedral.
e) [1+] Show that part *a*) holds also in the hyperbolic space $\mathbb{H}^3$.

31.7. **Final remarks.** Versions of Theorem 31.1 have been proved in much greater generality, e.g., for closed surfaces of every genus. This result is usually attributed to Gluck [Glu], but has also appeared in an earlier paper by Poznjak [Poz].

We prove the bellows conjecture (Theorem 31.2) and give references in Subsection 34.7. To further appreciate Sabitov's theorem, compare it with the following *Connelly's conjecture*: In the conditions of Theorem 31.2, polytopes enclosed by Q_t are scissor congruent (see [Con2, Con4]). A version of this conjecture was recently disproved in [AleC].

Finally, let us mention that the algebraic approach to the subject and the method of places recently found unexpected applications to the Robbins' conjectures (see Exercise 31.3 and 34.6). These conjectures are concerned with properties of polynomial relations on areas of inscribed convex polygons, as functions of squares of its sides. We refer to [FedP] for connections between two problems and an algebraic background, and to [Pak4] for a short survey and further references (see also Exercise 12.3).

32. Static rigidity and Dehn's theorem

This is the first of two sections where we prove *Dehn's theorem*, an infinitesimal version of the *Cauchy rigidity theorem*. We include two proofs: a variation on Cauchy's original proof in Section 26 and Dehn's original proof.

32.1. **Who needs rigidity?** In the next two sections we introduce two new concepts, the *static* and the *infinitesimal rigidity* of convex polyhedra, which turn out to be equivalent to each other and imply continuous rigidity. These ideas are crucial in modern rigidity theory; their natural extensions to general frameworks (of bars, cables and struts) were born out of these considerations and have a number of related properties. While we spend no time at all on these extensions, we find these ideas useful in discussions on Cauchy's and Gluck's theorems.

To summarize the results in the next two sections, we show that Gluck's Theorem 31.3 follows from *Dehn's lemma* on the determinant of a *rigidity matrix*, which we also introduce. We also show that the continuous rigidity also follows from Dehn's lemma. We then present three new proofs of Dehn's lemma, all without the use of the Cauchy theorem, as well as one extra proof of continuous rigidity of convex polyhedra. As the reader shall see, all this is done and motivated by the two rigidity concepts.

32.2. **Loading the edges.** To define the static rigidity, we need to extract the key ingredient in the proof of Gluck's theorem we presented in the previous section.

Let $V = \{v_1, \ldots, v_n\}$ be the set of vertices of a plane triangulation $\Gamma = (V, E)$, and denote by $f : V \to \mathbb{R}^3$ its planted realization. Let E be a set of ordered pairs: if $(v_i, v_j) \in E$, then $(v_j, v_i) \in E$ as well. Now, for every edge $e = (v_i, v_j) \in E$, denote by

$$\boldsymbol{e}_{ij} \, = \, \overrightarrow{f(v_i)f(v_j)} \, = \, (x_j - x_i, y_j - y_i, z_j - z_i) \in \mathbb{R}^3$$

the corresponding edge vector in the realization. In this notation, $\boldsymbol{e}_{ij} = -\boldsymbol{e}_{ji}$, for all $(v_i, v_j) \in E$. The set of scalars $\{\lambda_{ij} \in \mathbb{R}, \, (v_i, v_j) \in E\}$ is said to be an *edge load* if $\lambda_{ij} = -\lambda_{ji}$, $\lambda_{12} = \lambda_{13} = \lambda_{23} = 0$, and

$$\sum_{j \, : \, (v_i, v_j) \in E} \lambda_{ij} \boldsymbol{e}_{ij} = 0, \quad \text{for all } \, i \in [n].$$

We say that a planted realization $f : V \to \mathbb{R}^3$ of (V, E) defining the polytope is *statically rigid* if there is no nonzero static load $\{\lambda_{ij}\}$. Finally, a simplicial convex polytope P with graph $\Gamma = (V, E)$ is *statically rigid* if so is the planted realization of Γ obtained by a rigid motion of P. The following result is the key result of this section.

Theorem 32.1 (Dehn's theorem; static rigidity of convex polytopes)**.** *Every simplicial convex polytope in* $\mathbb{R}^3$ *is statically rigid.*

We already proved this result in a different language. To see this, consider a matrix $\mathcal{R}_\Gamma$ with rows $\mathcal{R}_\Gamma^{(ij)}$ corresponding to edges $(v_i, v_j) \in E$, written in lexicographical order:

$$\mathcal{R}_\Gamma^{(ij)} \, = \, (\ldots, x_i - x_j, y_i - y_j, z_i - z_j, \ldots, x_j - x_i, y_j - y_i, z_j - z_i, \ldots).$$

The matrix $\mathcal{R}_\Gamma$ is called the *rigidity matrix*. Now observe, the Jacobian $J(\cdot)$ is a determinant of the matrix with the following rows:

$$dF_{ij} \, = \, \left(\ldots, \frac{\partial F_{ij}}{\partial x_r}, \frac{\partial F_{ij}}{\partial y_r}, \frac{\partial F_{ij}}{\partial z_r}, \ldots\right) \, = \, 2\,\mathcal{R}_\Gamma^{(ij)}.$$

We showed that the Jacobian $J(\cdot) = 2^m \det \mathcal{R}_\Gamma \neq 0$, when evaluated at a planted realization $f : \Gamma \to \mathbb{R}^3$ defined by a convex polytope P. Thus, there is no nonzero linear combination of the rows of the rigidity matrix, with coefficients λ_{ij} as above. Interpreting the set of coefficients $\{\lambda_{ij}\}$ as the edge load, we obtain the statement of Dehn's theorem.

In the opposite direction, given Theorem 32.1, we obtain that $\det \mathcal{R}_\Gamma \neq 0$. Therefore, the Jacobian is nonzero, which in turn implies Gluck's theorem without the use of the Cauchy theorem. To conclude this discussion, the static rigidity of convex polytopes is equivalent to the following technical statement.

Lemma 32.2 (Dehn)**.** *Let $P \subset \mathbb{R}^3$ be a simplicial convex polytope with a graph $\Gamma = (V, E)$. Then the rigidity matrix $\mathcal{R}_\Gamma$ is nonsingular.*

In the following two subsections we present three independent proofs of Dehn's lemma, all (hopefully) easier and more elegant than any of the previous proofs of the Cauchy theorem. Until then, let us make few more comments.

First, let us show that Dehn's lemma implies the (continuous) rigidity of convex polytopes (Corollary 26.7). Indeed, consider the m-dimensional space W of planted realizations of $\Gamma = (V, E)$. The space W is mapped onto the m-dimensional space of all length functions, and the determinant $J(\cdot) = 2^m \det \mathcal{R}_\Gamma$ is nonzero at convex realizations. Therefore, in a small neighborhood of a convex realization the edge lengths are different, and thus a simplicial convex polytope is always rigid.

Our second observation is that the Cauchy theorem is more powerful than Theorem 32.1. To see this, recall the Cauchy–Alexandrov theorem on uniqueness of convex polyhedral surfaces (Theorem 27.6). This immediately implies the rigidity of these surfaces[77]. On the other hand, such realizations are not necessarily statically rigid, as the example in Figure 32.1 shows. Here we make arrows in the directions with positive coefficients which are written next to the corresponding edges.

32.3. **Proof of Dehn's lemma via sign changes.** This proof goes along the very same lines as the traditional proof of the Cauchy theorem (see Section 26.3). We first show that the edge load $\{\lambda_{ij}\}$ gives a certain assignment of signs on edges, then prove the analogue of the sign changes lemma (Lemma 26.4), and conclude by using the sign counting lemma (Lemma 26.5).

Proof of Dehn's lemma. Consider an edge load $\{\lambda_{ij} \mid (v_i, v_j) \in E\}$ on the edges in P. To remove ubiquity, consider only coefficients λ_{ij} with $i < j$. Let us label the edge $(v_i, v_j) \in E$, $i < j$, with $(+)$ if $\lambda_{ij} > 0$, with $(-)$ if $\lambda_{ij} < 0$, and with (0) if $\lambda_{ij} = 0$.

[77] One has to be careful here: this only proves rigidity in the space of *convex realizations*. In fact, the continuous rigidity holds for *all* non-strictly convex realizations; this is a stronger result due to Connelly (see [Con5])

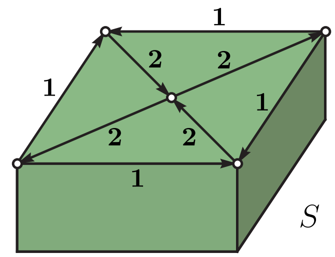

FIGURE 32.1. Stresses on a non-strictly convex polyhedral surface S show that it is not statically rigid.

Lemma 32.3 (Static analogue of the sign changes lemma)**.** *Unless all labels around a vertex are zero, there are at least four sign changes.*

By Lemma 26.5, we conclude that all labels must be zero. This proves Dehn's lemma modulo Lemma 32.3. □

Proof of Lemma 32.3. Denote by $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_k$ the edge vectors of edges leaving vertex w of a convex polytope P. We assume that w is at the origin and the edge vectors are written in cyclic order. Suppose we have a nonzero linear combination

$$\boldsymbol{u} := \lambda_1 \boldsymbol{e}_1 + \ldots + \lambda_k \boldsymbol{e}_k = 0.$$

Denote by H any generic hyperplane supporting P at w, i.e., a hyperplane containing w, such that all vectors $\boldsymbol{e}_i$ lie in the same half-space. If there are no sign changes, i.e., $\lambda_1, \ldots, \lambda_k \geq 0$ or $\lambda_1, \ldots, \lambda_k \leq 0$. Then their linear combination $\boldsymbol{u}$ is also in the same half-space unless all $\lambda_i = 0$, a contradiction.

Suppose now that there are two sign changes, for simplicity $\lambda_1, \ldots, \lambda_i \geq 0$ and $\lambda_{i+1}, \ldots, \lambda_k \leq 0$. Denote by H a hyperplane which contains vectors $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_i$ in a half-space H_+, and $\boldsymbol{e}_{i+1}, \ldots, \boldsymbol{e}_k$ in the other half-space H_-. Then the linear combi-

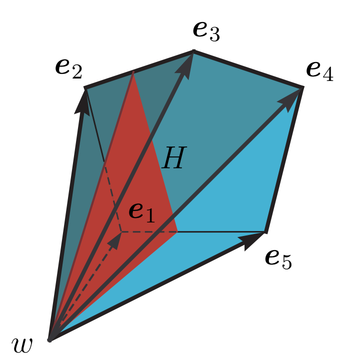

FIGURE 32.2. Hyperplane H separating edges in a polytope P.

nation $\boldsymbol{u}$ is in H_+ unless all $\lambda_i = 0$, another contradiction (see Figure 32.2). Thus, there are at least four sign changes unless all labels are zero. $\square$

32.4. **Proof of continuous rigidity from the angular velocity equation.** Before we continue with other proofs of Dehn's lemma, let us show how continuous rigidity (Corollary 26.7) follows from the angular velocity equation (Lemma 28.2), which was proved in Section 29 by a simple independent argument. The proof will be almost completely the same as the above proof of Dehn's lemma.

Proof of Corollary 26.7 modulo Lemma 28.2. Consider the angular velocity equation for each vertex and assign labels to all edges according to the signs of derivatives $\theta'_e(t)$. By the proof of Lemma 32.5, either all labels around a vertex are zero, or there are at least four sign changes. Now use the sign counting lemma (Lemma 26.5) to conclude that all labels bust be zero, i.e., all derivatives are zero (more precisely, all left and right derivatives at each point are zero, which is equivalent). Thus the dihedral angles remain constant under the continuous deformation, and the deformation itself is a rigid motion. $\square$

32.5. **Graph-theoretic proof of Dehn's lemma.** Let $\Gamma = (V, E)$ be a plane triangulation, and let $\mathcal{R}_\Gamma(\ldots, x_r, y_r, z_r, \ldots)$ be the rigidity matrix defined above. To prove Dehn's lemma we compute $\mathfrak{D} = \det(\mathcal{R}_\Gamma)$ and show that it is $\neq 0$ for convex realizations.

Let us use the fact that most entries in $\mathcal{R}_\Gamma$ are zero. Observe that every 3×3 minor of $\mathcal{R}_\Gamma$ either contains a zero row or column, or two columns which add up to zero, or looks like

$$M(a \mid b, c, d) \;=\; \begin{pmatrix} x_a - x_b & y_a - y_b & z_a - z_b \\ x_a - x_c & y_a - y_c & z_a - z_c \\ x_a - x_d & y_a - y_d & z_a - z_d \end{pmatrix},$$

where a, b, c, d represent distinct integers in $[n]$. Here we assume that $b < c < d$ and the ordering on rows corresponding to edges $(v_i, v_j) \in E$ is lexicographic. In addition to these minors, there is one special non-degenerate 3×3 minor of $\mathcal{R}_\Gamma$, with rows corresponding to the edges (v_1, v_2), (v_1, v_3), and (v_2, v_3), and the columns to x_2, x_3, and y_3:

$$M(1, 2, 3) \;=\; \begin{pmatrix} x_2 & 0 & 0 \\ 0 & x_3 & y_3 \\ x_2 - x_3 & x_3 - x_2 & y_3 \end{pmatrix}.$$

Now, using the Laplace expansion for $\det \mathcal{R}_\Gamma$ over triples of rows we conclude that the determinant $\mathfrak{D}$ is the product of determinants of the 3×3 minors as above, each given up to a sign. Since we need these signs let us formalize this as follows.

We say that vertices v_1, v_2, v_3 are *base vertices* and the edges between them are *base edges*. A *claw* in Γ is a subgraph H of Γ isomorphic to $K_{1,3}$, i.e., a subgraph $K(a \mid b, c, d)$ consisting of four distinct vertices v_a, v_b, v_c, v_d and three edges: (v_a, v_b), (v_a, v_c), and (v_a, v_d). We call vertex v_a the *root* of the claw $K(a \mid b, c, d)$. Recall that Γ

contains $(3n-6)-3 = 3(n-3)$ non-base edges, exactly three per non-base vertex (cf. Corollary 25.2). One can ask whether non-base edges of Γ can be partitioned into claws, with a root at every non-base vertex. Denote such *claw partitions* by π, and the set of claw partitions by Π_Γ. We have:

$$\mathfrak{D} \,=\, \det \mathcal{R}_\Gamma \,=\, \det M(1,2,3) \cdot \sum_{\pi\in\Pi_\Gamma} \varepsilon(\pi) \prod_{H(a|b,c,d)\in\pi} \det M(a \mid b,c,d),$$

where $\varepsilon(\pi) \in \{\pm 1\}$ and is given by the sign of the corresponding permutation $\sigma(\pi) \in S_m$ of the set of edges E. Now, observe that each determinant in the minor as above can be written as a volume of a parallelepiped spanned by the edge vectors:

$$\det M(a \mid b,c,d) \,=\, \frac{1}{6}\,\mathrm{vol}\langle \boldsymbol{e}_{a,b},\, \boldsymbol{e}_{a,c},\, \boldsymbol{e}_{a,d}\rangle\,.$$

Fix an orientation of the surface $S = \partial P$ induced by $\mathbb{R}^3$. By convexity of P,[78] the determinant is positive if the order of edges $(v_a,v_b),(v_a,v_c),(v_a,v_d)$ coincides with the orientation of S, and negative if it is the opposite. We call such claws $K(a \mid b,c,d)$ *positive* and *negative*, respectively. For a claw partition $\pi \in \Pi_\Gamma$, denote by $s(\pi)$ the number of negative claws $H \in \pi$, and let $\delta(\pi) = (-1)^{s(\pi)}$. We need the following two lemmas:

Lemma 32.4 (Existence of claw partitions)**.** *For every plane triangulation Γ and every base triangle, the set of claw partitions Π_Γ is nonempty.*

Lemma 32.5 (Signs of claw partitions)**.** *Every two claw partitions $\pi,\pi' \in \Pi_\Gamma$ have the same products of signs: $\varepsilon(\pi)\cdot\delta(\pi) = \varepsilon(\pi')\cdot\delta(\pi')$.*

Now the claim follows easily from the lemmas: the determinant $\mathfrak{D}$ is a nonempty sum of products, all of the same sign. Thus, $\mathfrak{D} \neq 0$, which completes the proof of Lemma 32.2 modulo the above two lemmas. $\square$

Proof of Lemma 32.4. Let us orient the non-base edges in a claw away from the root. Every non-base vertex must have exactly three outgoing edges then, which determine the claw partition. Use induction on the number n of vertices in Γ. For $n=4$, when $\Gamma = K_4$ the result is obvious.

For $n>4$, let a be a vertex of degree at most 5. This vertex exists, since otherwise the number of edges $|E| = 3n-6 \geq (6\cdot n)/2 = 3n$. Remove vertex a with all adjacent edges, and denote by Γ' any triangulation of the remaining graph. Since Γ' has a claw partition by inductive assumption, return the removed vertex and modify the edge orientation as in Figure 32.3. There are five cases to consider: when $\deg(a)=3$, when $\deg(a)=4$, and three cases corresponding to different orientations of diagonals in a pentagon, when $\deg(a)=5$.

All the cases are straightforward except for the last one, with two diagonals leaving from the same pentagon vertex v. Since at most three edges can leave v, one of the two side edges of the pentagon, say (w,v), must go into v. Then make the change as in the figure. We should add that when triangulating the pentagon, we need to

[78] Incidently, this is the only use of convexity of P in the whole proof.

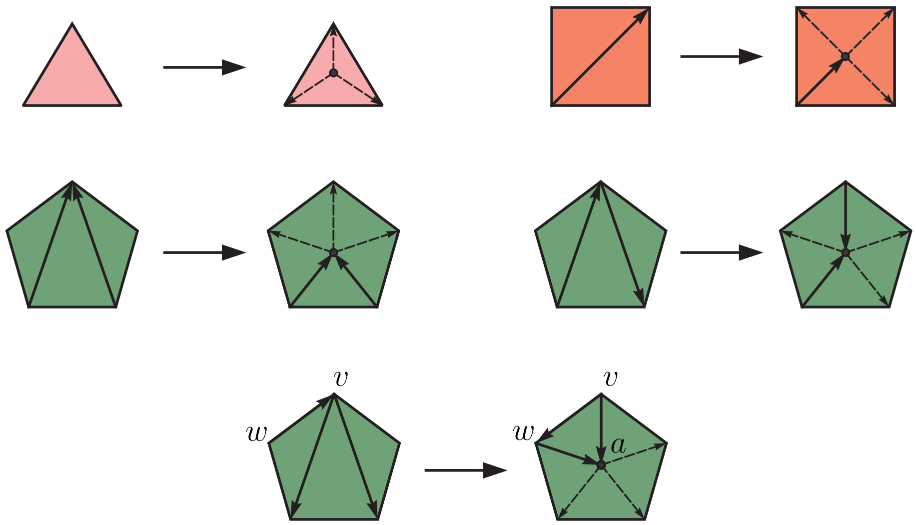

FIGURE 32.3. Inductive proof of existence of claw partition.

make sure that v is not a base vertex; this is easy to do since there are only three base vertices. □

Proof of Lemma 32.5 is motivated by the following example.

Example 32.6. It is easy to see that for every base triangle in the graph of an octahedron, there exist exactly two claw partitions. In Figure 32.4 we show both of them and how they are connected by reversal of a cycle of length 3. Here the base vertices are the vertices of the outside triangle. We also show a claw partition of the graph of an icosahedron and an oriented cycle of length 6. Reversal along this cycle gives a different claw partition.

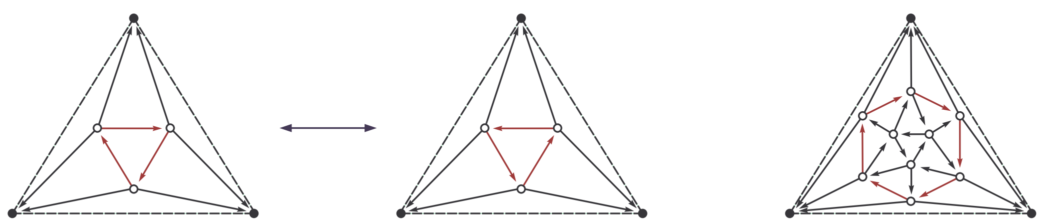

FIGURE 32.4. Two claw partitions of an octahedron and a claw partition of an icosahedron.

We say that two claw partitions s $\pi, \pi' \in \Pi_\Gamma$ are connected by a 3-*cycle reversal* if they coincide everywhere except for three edges which belong to different claws and form a 3-cycle as in the example. We need the following lemma.

Lemma 32.7. *Every two claw partitions* $\pi, \pi' \in \Pi_\Gamma$ *can be connected by a sequence of claw partitions:*

$$\pi = \pi_0 \to \pi_1 \to \ldots \to \pi_{\ell-1} \to \pi_\ell = \pi',$$

such that $\pi_i \in \Pi_\Gamma$, $i \in [\ell]$, and every π_i is obtained from π_{i-1} by the reversal of edges in a 3-cycle.

Note that the number of number $s(\pi)$ of negative claws in π remains invariant under a 3-move reversal, so the sign $\delta(\pi)$ is constant on all $\pi \in \Pi_\Gamma$. Similarly, because 3-cycle is an even permutation, the sign $\varepsilon(\pi)$ is also constant. Therefore, Lemma 32.7 immediately implies Lemma 32.5.

Proof of Lemma 32.7. Let $e_1 = (v_i, v_j) \in E$ be an edge in Γ which belongs to a claw with root v_i in partition π and belongs to a claw with root v_j in partition π'. If there are no such edges, then $\pi = \pi'$, and there is nothing to prove. Think of e_1 as an oriented edge: it belongs to π, but not to π'. Clearly, there also exists an edge $e_2 = (v_j, v_r)$ which belongs to π, but not to π', etc. Continuing in this manner, we eventually return to the starting vertex v_i, and obtain an oriented cycle of edges $C = (e_1, e_2, \ldots, e_k)$. Note that if we reverse the direction of the edges, we obtain an oriented cycle in π' (see a cycle on the right in Figure 32.4).

Note that the reversal operation of edges in an oriented cycle does not move from π to π'. Instead, we obtain a new claw partition π_1. Now, comparing π_1 to π' we can find a new oriented cycle of edges in π but not in π', whose reversal will create a new partition π_2. Repeating this procedure we obtain a sequence of claw partition with more and more edges oriented as in π', and eventually reach π' :

$$(\mathsf{T}) \qquad \pi \to \pi_1 \to \pi_2 \to \ldots \to \pi' .$$

We can also assume that all cycles reversed at each step are not self-intersecting since otherwise we can split each such cycle at the intersection point (see Figure 32.5), and further refine sequence (T).

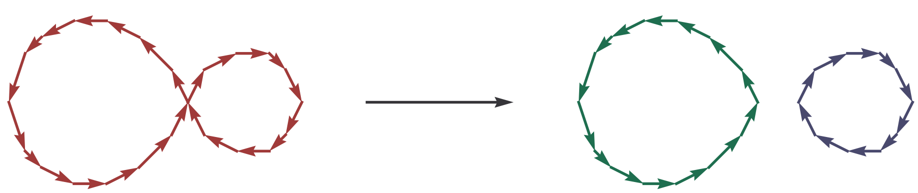

FIGURE 32.5. Splitting a self-intersecting cycle into two cycles.

In summary, we obtain a sequence (T) of claw partitions, each subsequent obtained by a reversal along some oriented not self-intersecting cycle C. Let further refine this sequence, by writing each reversal of edges of C as a composition of reversals along 3-cycles.

Clearly, if cycle C is removed from the surface $S = \partial P$ one part will contain the base triangle $(v_1 v_2 v_3)$ Denote by H an induced subgraph of Γ obtained by removal of all interior vertices in that part. Suppose that $H = (V', E')$ contains m interior vertices. Note that H is planar, and consists of triangles and one k-gonal face. We have:

$$|V'| = m + k, \quad 2\,|E'| = k + 3t, \quad \text{and} \quad |\mathcal{F}'| = t + 1 .$$

By Euler's formula $|V'|-|E'|+|\mathcal{F}'|=2$, graph H has exactly $t=2m+k-2$ triangles and

$$|E'| \, = \, \frac{1}{2}\,(k+3t) \, = \, 3m+2k-3$$

edges. Observe that there are $3m$ edges in claws rooted at the interior m vertices, and neither of these edges can belong to the cycle C. Thus, the remaining $(2k-3)$ edges consist of k edges in C, and of $s=(k-3)$ edges rooted at vertices of C and oriented inside of H. Therefore, every $k\geq 4$ there exist at least one such edge e (see Figure 32.6).

Consider a path γ starting with e, obtained as in the argument above. Clearly, we eventually reach C again. Two sides of path γ will divide C into two parts, and reversal along C is the composition or reversals along one of them, call it C_1, and then along the other, call it C_2 (see Figure 32.6). Since cycle C_1, C_2 have smaller area than C, we can continue splitting of the cycles until we have a composition of 3-cycles, refining (T). $\square$

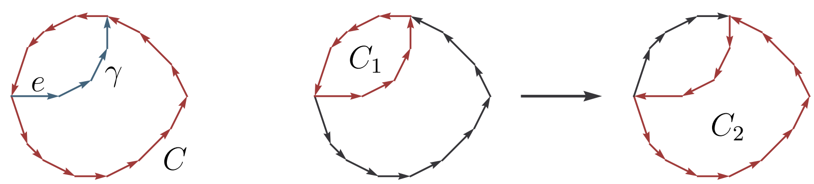

FIGURE 32.6. Reversal of a long cycle.

32.6. **Exercises.**

Exercise 32.1. (*Rigidity of non-convex polyhedra with saddle vertices*) *a*) [1] Let P be a non-convex polyhedron in $\mathbb{R}^3$ homeomorphic to a sphere, but possibly with self-intersections. We say that a vertex is a *saddle*, if it is adjacent to exactly 4 edges, and there exists a hyperplane separating two opposite edges from the other two. Check that Lemma 32.3 extends to saddle vertices.

[1] We say that a vertex v in P is *convex* if the corresponding cone C_v is convex. Check that the proof above extend verbatim to polyhedra homeomorphic to a sphere, with only convex and saddle vertices. An example is given in Figure 32.7.

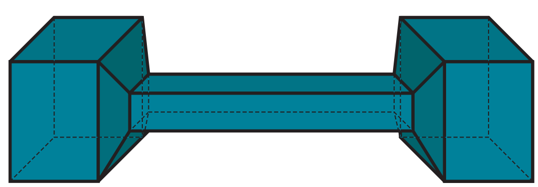

FIGURE 32.7. A barbell style polyhedron with only convex and saddle vertices.

Exercise 32.2. [2] Let $P \subset \mathbb{R}^3$ be a simplicial convex polytope and let $S = \partial P$. Suppose $\{S_t\}$ is a polyhedral deformation which preserves combinatorial structure and dihedral angles. Prove that $\{S_t\}$ is a rigid motion.[79]

Exercise 32.3. *a*) [2-] Let $P \subset \mathbb{R}^3$ be a convex polytope with even-sided faces (quadrilaterals, hexagons, etc.) with surface $S = \partial P$. Suppose $S' \subset \mathbb{R}^3$ is an embedded surface which is combinatorially equivalent to S and whose faces are isometric. Prove that S' is continuously rigid.
b) [2-] Extend part *a*) to polyhedra with at most 7 odd-sided faces.

Exercise 32.4. (*Alexandrov*) $\diamondsuit$ [1+] Let $S = \partial P$ be a triangulated surface of a convex polytope $P \subset \mathbb{R}^3$. Suppose there are no triangle vertices in the relative interior of faces (they are allowed only at vertices and on natural edges of P). Prove that S is statically rigid.

32.7. **Final remarks.** Dehn's lemma was used by Dehn [Dehn] to prove Theorem 32.1 and deduce from here the continuous rigidity of convex polyhedra (Corollary 26.7). This approach was repeatedly rediscovered and is now only the first basic step in the study of rigidity of general frameworks. We refer to [Con5, Whi2] for the introduction and a broad overview of general problems in rigidity theory and various applications.

The static (and thus continuous) rigidity of various families of non-convex polyhedra (see Exercise 32.1) is an interesting part of modern rigidity theory. We refer to surveys [Ale6, Con5] for various references on the subject.

The first proof of Dehn's lemma (Subsection 32.3) follows [FedP], but can also be found in the early works on rigidity of frameworks (see [Roth, Whi1]). The proof of continuous rigidity given in Subsection 32.4 is due to Alexandrov [A2, §10.3].

The first part of the graph theoretic proof (Subsection 32.5, until Example 32.6) follows the original paper of Dehn [Dehn] (see also [Ale6, §6]). The connectivity of all claw partitions by a sequence of 3-*cycle reversals* is probably new. The argument here is a variation on theme of several "local move connectivity" arguments (see Subsection 14.7) We refer to [KorP] for further discussion on claw partitions of graphs and connections to tilings, and to [Pak3] for a survey on local move connectivity arguments (cf. Section 23).

The extension of the Cauchy and Dehn theorems to polytopes with saddle vertices (as in Exercise 32.1) in fact goes through with many proofs. This result was was proved by Stoker [Sto] and then repeatedly rediscovered. Alexandrov's extension (Exercise 32.4) is the first step towards Connelly's theorem on the second order rigidity of all triangulated convex surfaces (Exercise 33.5).

Finally, let us note that for small enough angles the argument in the original (incorrect) proof of the arm lemma (Lemma 22.3) is completely valid. Thus the "incorrect proof" in fact implies the continuous rigidity of convex polytopes (Corollary 26.2). As we show in this and the next section, the rigidity of convex polyhedra is somewhat easier to prove than the uniqueness (the Cauchy theorem). Thus, it is quite tempting to ignore the full power of the Cauchy theorem, thinking that the continuous rigidity is the most important implication of uniqueness. From the "real life applications" point of view this idea is actually not without merits (you really need rigidity, not uniqueness, to prevent a bridge from falling apart). On the other hand, the proofs of a number of important results, such as the Alexandrov

[79] This is the infinitesimal version of Stoker's conjecture for simplicial polytopes.

existence theorem (Theorem 37.1), substantially rely on uniqueness. As a consequence, the full strength of the Cauchy theorem should be neither undervalued nor undermined.

33. Infinitesimal rigidity

In this short section we give another approach to the rigidity, and give our own new proof of *Dehn's theorem* in the previous section.

33.1. **You cannot change anything if you do not know how to start.** Consider a continuous deformation $\{P_t : t \in [0,1]\}$ of the simplicial polytope $P = P_0 \subset \mathbb{R}^3$. As before, let V and E be the set of vertices and edges of P. Denote by $\boldsymbol{v}_i(t) = \overrightarrow{Ov_i}$ the vector from the origin to the vertex $v_i = v_i(t)$ of P_t. Think of vectors $\boldsymbol{v}_i'(t)$ as *velocities* of vertices v_i.[80] For an edge length $|v_i v_j|$ to be constant under the deformation we need $\|\boldsymbol{v}_i(t) - \boldsymbol{v}_j(t)\|' = 0$, where $\|\boldsymbol{w}\| = (w,w) = |w|^2$. Thus, in particular, at $t = 0$ we must have:

$$\begin{aligned} 0 &= \frac{d}{dt}\,\big\|\boldsymbol{v}_i(t) - \boldsymbol{v}_j(t)\big\|_{t=0} = \frac{d}{dt}\,\big\|\big(\boldsymbol{v}_i(0) - \boldsymbol{v}_j(0)\big) + t\big(\boldsymbol{v}_i'(0) - \boldsymbol{v}_j'(0)\big)\big\|_{t=0} \\ &= 2\big(\boldsymbol{v}_i(0) - \boldsymbol{v}_j(0), \boldsymbol{v}_i'(0) - \boldsymbol{v}_j'(0)\big). \end{aligned}$$

This leads to the following natural definition.

Let $P \subset \mathbb{R}^3$ be a planted simplicial polytope. Suppose we are given a vector $\boldsymbol{a}_i$, for every vertex $v_i \in V$. We say that the set of vectors $\{\boldsymbol{a}_i\}$ defines an *infinitesimal rigid motion* if

$$(\ominus) \qquad (\boldsymbol{v}_i - \boldsymbol{v}_j, \boldsymbol{a}_i - \boldsymbol{a}_j) = 0, \quad \text{for every } (v_i, v_j) \in E.$$

An infinitesimal rigid motion is called *planted* if the velocities of base vertices are equal to zero: $\boldsymbol{a}_1 = \boldsymbol{a}_2 = \boldsymbol{a}_3 = \boldsymbol{0}$. Finally, we say that a simplicial polytope $P \subset \mathbb{R}^3$ is *infinitesimally rigid* if every planted infinitesimal rigid motion is trivial: $\boldsymbol{a}_i = \boldsymbol{0}$ for all $v_i \in V$.

Theorem 33.1 (Dehn's theorem; infinitesimal rigidity of convex polytopes)**.** *Every simplicial convex polytope in* $\mathbb{R}^3$ *is infinitesimally rigid.*

Of course, the restriction to planted infinitesimal rigid motions is necessary, as the usual rigid motions of P in $\mathbb{R}^3$ can define nontrivial infinitesimal rigid motions. Also, by definition, the infinitesimal rigidity implies the continuous rigidity.

To prove the theorem, simply observe that infinitesimal rigidity is in fact equivalent to static rigidity, but written from a dual point of view. Indeed, if the determinant $\mathfrak{D}$ of the rigidity matrix is nonzero that means that locally all edge lengths of planted realizations must be different. Therefore, the space of planted infinitesimal rigid motions is zero, and vice versa. The details are straightforward.

To summarize, both static and infinitesimal rigidity are equivalent to Dehn's lemma (Lemma 32.2). In fact, this equivalence extends to general bar frameworks (see [Con5, Whi2]) with no difference in the proof.

[80] As in Section 28, when the derivatives do not exist we can consider both left and right derivatives.

33.2. **A synthetic proof of infinitesimal rigidity.** Now, one may ask what is so great about yet another way to rephrase Dehn's lemma. Well, as it turns out, the infinitesimal rigidity notion is nice enough to allow a graph theoretic approach markedly different from the one used above.

The proof we present below yet again splits into two parts: global and local. The local part, while different, has roughly the same level of difficulty as the static analogue of the sign changes lemma (Lemma 32.3). The global part is a simple graph theoretic argument similar in style to the proof of the sign counting lemma (Lemma 26.5).

Proof of Theorem 33.1. First, note that equations ($\ominus$) above say that the difference in the velocity of vertices is orthogonal to the edges of the polytope. Think of velocity vectors as vector functions on vertices of P which are equal to $\mathbf{0}$ on base vertices v_1, v_2, v_3. The idea of the proof is to enlarge the set of such functions and prove a stronger result.

Let $V = \{v_1, \ldots, v_n\}$ be the set of vertices of a simplicial convex polytope $P \subset \mathbb{R}^3$, and let E be the set of edges. Consider the set of all vector sequences $(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n)$, $\boldsymbol{a}_i \in \mathbb{R}^3$, such that for every edge $(v_i, v_j) \in E$ we have one of the following three possibilities:

1. $(\boldsymbol{v}_i - \boldsymbol{v}_j, \boldsymbol{a}_i) = (\boldsymbol{v}_i - \boldsymbol{v}_j, \boldsymbol{a}_j) = 0$,
2. $(\boldsymbol{v}_i - \boldsymbol{v}_j, \boldsymbol{a}_i) < 0$ and $(\boldsymbol{v}_i - \boldsymbol{v}_j, \boldsymbol{a}_j) < 0$,
3. $(\boldsymbol{v}_i - \boldsymbol{v}_j, \boldsymbol{a}_i) > 0$ and $(\boldsymbol{v}_i - \boldsymbol{v}_j, \boldsymbol{a}_j) > 0$.

In other words, we require that projections of velocity vectors $\boldsymbol{a}_i$ and $\boldsymbol{a}_j$ onto edge $(\boldsymbol{v}_i, \boldsymbol{v}_j)$ have the same signs. We say that a vertex v_i is *dead* if $\boldsymbol{a}_i = \mathbf{0}$; it is *live* otherwise. We need to prove that for every vector sequence $(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n)$ as above, if base vertices are dead, then all vertices $v_i \in V$ are dead. By definition of infinitesimal rigidity, this would immediately imply the theorem.

Denote by $\Gamma = (V, E)$ the graph of P. Since P is simplicial, Γ is a plane triangulation. Consider an orientation edges of Γ in the direction of projections of the velocity vectors. More precisely, we orient the edge $v_i \to v_j$ in case **2**, we orient the edge $v_i \leftarrow v_j$ in case **3**, and leave it unoriented v_i — v_j in case **1**.

Consider two edges $e = (v_i, v_j)$ and $e' = (v_i, v_r)$, $e, e' \in E$, with a common vertex v_i, such that (v_i, v_j, v_r) is a face in P. We say that edges e and e'

- have *one inversion* if one of them is oriented into v_i, and the other out of v_i,
- have *zero inversions* if both of them are oriented into v_i or out of v_i,
- have a *half-inversion* if one of the edges is oriented and the other is unoriented,
- have *one inversion* if both of them are unoriented and v_i is a live vertex,
- have *zero inversions* if both of them are unoriented and v_i is a dead vertex.

We say that a triangle is *active* if at least one of its vertices is live; it is *inactive* otherwise. Now consider orientations of an active triangle $(v_i v_j v_r)$ where vertex v_i is live (see some of them in Figure 33.1). A simple enumeration of all possible cases gives the following result.

Lemma 33.2. *Every active triangle has at least one inversion.*

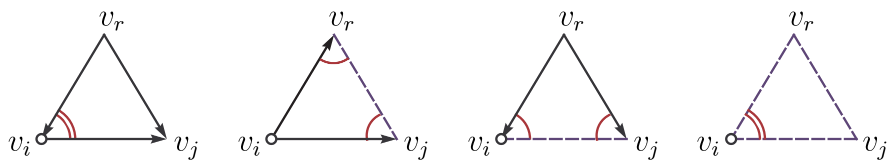

FIGURE 33.1. Different orientations of $(v_iv_jv_r)$, where vertex v_i is live.

This gives a lower bound on the number of inversions Γ. To get an upper bound, we use the following infinitesimal analogue of the sign changes lemma.

Lemma 33.3. *There are at most two inversions around every live vertex.*

We postpone the proof of the lemma until we finish the proof of the theorem. Consider what this gives us when the only unoriented edges in Γ are the base edges (v_1, v_2), (v_1, v_3), and (v_2, v_3). In this case we have $n-3$ live vertices and $(2n-4)-1 = 2n-5$ triangles with at least one live vertex. By Lemma 33.3, there are at most $2(n-3) = 2n-6$ inversions in Γ, while by Lemma 33.2 there are at least $2n-5$ inversions, a contradiction.

We use the same strategy in the general case. Remove from Γ all inactive triangles together with all edges and vertices which belong only to inactive triangles. Denote by $H = (V', E')$ a connected component of the remaining graph. Since Γ is planar, the induced subgraph H of Γ has a well defined boundary ∂H. Denote by k the number of vertices in ∂H (all of them dead), and by ℓ the number of connected components of ∂H. Finally, denote by m the number of vertices in $H \smallsetminus \partial H$ (some of them live and some possibly dead). By Lemma 33.3, there are at most $2m$ inversions in H.

Now let us estimate the number of inversions via the number t of triangles in H. Observe that the total number of vertices, edges, and faces in H is given by

$$|V'| = m + k, \quad 2|E'| = k + 3t, \quad \text{and} \quad |\mathcal{F}'| = \ell + t\,.$$

Thus, by Euler's formula, graph H has exactly $t = 2m + k + 2\ell - 4$ triangles. Since there is at least one inactive triangle $(v_1v_2v_3)$, we have $k \geq 3$ and $\ell \geq 1$. Therefore, by Lemma 33.2, there are at least

$$t = 2m + k + 2\ell - 4 \geq 2m + 3 + 2 - 4 = 2m + 1$$

inversions in H, a contradiction. $\square$

Proof of Lemma 33.3. Consider all possibilities one by one and check the claim in each case. Suppose a vertex v_i is adjacent to three or more unoriented edges. This means that $\boldsymbol{a}_i$ is orthogonal to at least three vectors spanning $\mathbb{R}^3$. Therefore, $\boldsymbol{a}_i = \mathbf{0}$ and v_i is a dead vertex with zero inversions.

Suppose now that v_i is adjacent to exactly two unoriented edges $\boldsymbol{e}, \boldsymbol{e}'$. This means that $\boldsymbol{a}_i \neq \mathbf{0}$ is orthogonal to a plane spanned by these edges. Observe that $\boldsymbol{e}, \boldsymbol{e}'$ separate the edges oriented into v_i from those oriented out of v_i. Thus, there are

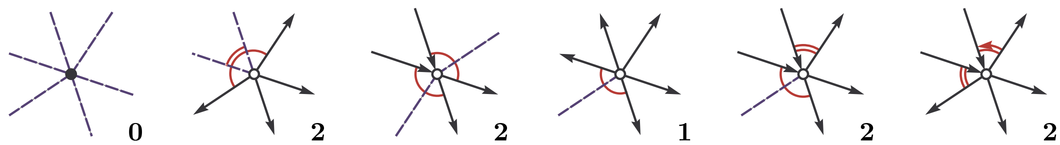

FIGURE 33.2. The number of inversions around a vertex in different cases.

either two half-inversions and one inversion if the edges $\boldsymbol{e}, \boldsymbol{e}'$ are adjacent, or four half-inversions if $\boldsymbol{e}, \boldsymbol{e}'$ are not adjacent (see Figure 33.2).

Next, suppose that v_i is adjacent to exactly one unoriented edge $\boldsymbol{e}$. Since $\boldsymbol{a}_i \neq \mathbf{0}$ in this case, consider a plane containing v_i and orthogonal to $\boldsymbol{a}_i$. This plane contains β and separates the edges oriented into v_i from those oriented out of v_i. Therefore, there are either two half-inversions if all other edges are oriented into v_i, or out of v_i, or two half-inversions and one inversion otherwise.

Finally, if v_i is a live vertex and not adjacent to unoriented edges, then the plane orthogonal to $\boldsymbol{a}_i$ separates the edges into two parts: those oriented into v_i from those oriented out of v_i. Thus, there are exactly two inversions in this case. $\square$

33.3. **Exercises.**

Exercise 33.1. [1+] Prove the infinitesimal analogue of Alexandrov's theorem (Exercise 32.4).

Exercise 33.2. [1] Prove that Jessen's orthogonal icosahedron (see Exercise 19.17) is continuously rigid, but not infinitesimally rigid.

Exercise 33.3. $\diamondsuit$ [1] Give a direct proof that the static and the infinitesimal rigidity are equivalent.

Exercise 33.4. [1+] Modify and prove the analogue of Dehn's theorem for unbounded polyhedra (cf. Subsection 27.3).

Exercise 33.5. (*Connelly's second order rigidity theorem*) $\diamondsuit$ *a*) [1] In notation as in the beginning of the section, take the second derivative of the edge length $|v_iv_j|$. Letting both the first and second derivatives to be zero, extend ($\ominus$) to define the *second order rigidity* in combinatorial terms. Conclude that this is a necessary condition for the continuous rigidity.
b) [1-] Prove that the non-strictly convex polyhedron in Figure 32.1 is second order rigid.
c) [1] Give a polynomial time algorithm for testing the second order rigidity.
d) [2] Prove that every triangulated convex polyhedral surface in $\mathbb{R}^3$ is second order rigid.

Exercise 33.6. [2] Prove that the infinitesimal rigidity is invariant under projective transformations.

Exercise 33.7. [1+] Let $X = [x_1 \dots x_n] \subset \mathbb{R}^2$ be a convex polygon in the plane. Suppose the edges of X are bars and all diagonals (x_i, x_{i+1}) are cables, where the index $i \in \{1, \dots, n\}$ is taken modulo n.[81] This framework is called the *tensegrity polygon*. Formally state and prove that it is infinitesimally rigid.

[81]Here *bars* lengths must remain the same, while *cables* can be contracted.

Exercise 33.8. [1+] Let $(x_{000}, x_{001}, \ldots, x_{111}) \subset \mathbb{R}^3$ be eight vertices corresponding to vertices of a cube. Suppose all cube edges, as well as the diagonals in the top and bottom face, are connected by cables. Also, suppose the diagonals in the four side edges are connected by bars. Prove that this framework is infinitesimally rigid.

Exercise 33.9. Let $\Delta = (x_1x_2x_3x_4)$ be a tetrahedron in $\mathbb{R}^3$, let $\ell_{ij} = |x_ix_j|$, and let γ_{ij} be the dihedral angle at edge (x_ix_j).
a) [1] Prove that angles γ_{ij} determine Δ uniquely, up to expansion.
b) [1] Show that lengths $\ell_{12}, \ell_{13}, \ell_{14}$ and angles $\alpha_{23}, \alpha_{24}, \alpha_{34}$ do not necessarily determine Δ.
c) [1+] Show that Δ cannot be continuously deformed so that the angles and lengths in *b*) remain invariant.

Exercise 33.10. (*Rivin, Luo*) *a*) [2+] Let S be an abstract 2-dimensional triangulated surface homeomorphic to a sphere, and defined by the edge lengths $\ell_{ij} = |e_{ij}|$. Denote by $\varphi_{ij} = \pi - \alpha - \beta$, where α and β are the angles opposite to edge e_{ij}. Prove that every continuous deformation of S which preserves all φ_{ij} is a homothety.
b) [2+] Same result for $\varphi_{ij} = \ln(\tan\alpha) + \ln(\tan\beta)$.
c) [2+] Same result for $\varphi_{ij} = \cot\alpha + \cot\beta$.

Exercise 33.11. (*Schramm*) [2] Suppose two combinatorially equivalent simplicial convex polyhedra $P_1, P_2 \subset \mathbb{R}^3$ have a midscribed sphere, i.e., all their edges touch a fixed unit sphere $\mathbb{S}^2$. Suppose further that a face in P_1 coincides with the corresponding face in P_2. Prove that $P_1 = P_2$.

33.4. **Final remarks.** The synthetic proof of infinitesimal rigidity in this section follows [Pak5] and is based on the proof idea in [Tru]. We refer to [Con5, §4.6] for further references and other proofs of Dehn's theorem.

Polyhedra that are continuously but not infinitesimally rigid are called *shaky polyhedra* [Gol4]. Jessen's orthogonal icosahedron is a classical example of a shaky polyhedron (see Exercises 19.17 and 33.2). A non-strictly convex polyhedron in Figure 32.1 is another, degenerate example.

While continuous rigidity is often hard to establish even for very specific frameworks, the infinitesimal rigidity is equivalent to a nonzero determinant of the rigidity matrix $\mathcal{R}_\Gamma$. Similarly, one can define the second and higher order rigidity, which give further necessary conditions on continuous rigidity (see Exercise 33.5). We refer to [ConS] for the introduction to higher order rigidity, and to survey papers [Con5, IKS] for further references.

34. Proof of the bellows conjecture

We continue the algebraic approach started in Section 31. Here we give a complete proof of the *bellows conjecture* stated there, modulo some basic algebraic results given in the Appendix (see Subsection 41.7).

34.1. **Polynomiality and integrality of the volume.** Let $S \subset \mathbb{R}^d$ be a polyhedral surface homeomorphic to a sphere. When $S = \partial P$ is a surface of a convex polytope, we can define the volume enclosed by S as the volume of the polytope P. In general, we can triangulate the surface and define

$$\operatorname{vol}(S) := \frac{1}{n!} \sum_{(v_1 \ldots v_d) \in S} \det[v_1, \ldots, v_d],$$

where $(v_1 \ldots v_d)$ is an oriented simplex in S and $\det[v_1, \ldots, v_d]$ is a determinant of a matrix where v_i are column-vectors. This implies that the volume of convex polytopes is polynomial in the vertex coordinates.

Now let $S = \partial P$ be the surface of a simplicial convex polytope $P \subset \mathbb{R}^3$. Its volume is polynomial in vertex coordinates, and by Corollary 31.4 this implies that as a polynomial this volume is a root of an equation with coefficients in $\mathbb{C}[F_{ij}]$, where F_{ij} are polynomials giving the squared distances between v_i and v_j. In other words, there exists a Sabitov polynomial relation ($\diamond$) for the volume polynomial (see Corollary 31.4).

More generally, consider a simplicial surface $S \subset \mathbb{R}^3$ homeomorphic to a sphere, i.e., a polyhedral surface with only triangular faces. From above, the volume is a root of a polynomial obtained by an evaluation of the Sabitov polynomial relation ($\diamond$) at the vertex coordinates. Now, as we showed in Section 31, the bellows conjecture (Theorem 31.2) follows immediately if we show that the leading coefficient is 1, or any other nonzero integer, since this implies that the above polynomial does not degenerate. Let us formalize this property as follows.

Let L be a field which contains a ring R. For our purposes one can always assume that L is either $\mathbb{C}$ or $\mathbb{R}$. We say that an element $x \in L$ is *integral* over R if

$$x^r + c_{r-1}x^{r-1} + \ldots + c_1 x + c_0 = 0.$$

Now comes the crucial result.

Lemma 34.1 (Integrality of the volume)**.** *Let $S \subset \mathbb{R}^3$ be a simplicial surface homeomorphic to a sphere and let R be a ring generated by the squares of the edge lengths ℓ_{ij}^2. Then $24\operatorname{vol}(S)$ is integral over R.*

From above, the lemma implies the bellows conjecture (Theorem 31.2). Its proof is based on a mixture of algebraic and geometric properties of volume and will occupy the rest of this section.

34.2. **Playing with infinities.** To prove the integrality of the volume we need to develop some algebraic tools. Below is a quick introduction to the *theory of places*, which will help resolve the problem.

Let F be a field, and let $\widehat{F} = F \cup \{\infty\}$. Elements $a \in F$ are called *finite* elements in $\widehat{F}$. As before, let L be a field. A map $\varphi : L \to \widehat{F}$ is called a *place* if $\varphi(1) = 1$ and it satisfies the following conditions:

$$(\circledast) \qquad \varphi(a \pm b) = \varphi(a) \pm \varphi(b), \quad \varphi(a \cdot b) = \varphi(a) \cdot \varphi(b), \quad \text{for all } \ a, b \in L.$$

Here it is understood that for all $a \in F$

$$a \pm \infty \,=\, \infty \cdot \infty \,=\, \frac{1}{0} \,=\, \infty, \qquad \frac{a}{\infty} \,=\, 0,$$

and $b \cdot \infty = \infty$ for all $b \neq 0$. Furthermore, the expressions

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad 0 \cdot \infty, \quad \text{and} \quad \infty \pm \infty$$

are not defined, and the conditions on φ do not have to hold in the case when the r.h.s. of $(\circledast)$ is undefined. Note that $\varphi(0) = \varphi(1-1) = \varphi(1) - \varphi(1) = 1 - 1 = 0$.

For example, take the field of rational Laurent polynomials $L = \mathbb{Q}(x)$ and define a place $\varphi : \mathbb{Q}(x) \to \widehat{\mathbb{Q}}$ by setting $\varphi(x) = \infty$, $\varphi(\frac{1}{x}) = 0$, and $\varphi(a) = a$, for all $a \in \mathbb{Q}$. Viewing the elements $f(x) \in \mathbb{Q}(x)$ as functions in x defined on $\mathbb{R}_+$, have then: $\varphi(f) = \lim_{x\to\infty} f(x)$.

A place $\varphi : L \to \widehat{F}$ is called *finite on* L (on R) if $\varphi(a) \in F$ for all $a \in L$ (for all $a \in R$). The following standard algebraic result is a key tool in our approach.

Theorem 34.2 (Integrality criterion)**.** *Let L be a field containing a ring R. An element $x \in L$ is integral over R if and only if every place that is finite on R, is also finite on x.*

Obviously, all elements $x \in R$ are integral over R. Note that by the theorem, if x and y are integral over R, then so are $-x$, $x + y$ and $x \cdot y$. On the other hand, even if $x \in L$, $x \neq 0$, is integral, then $1/x \in L$ is not necessarily integral. For example, if $L = \mathbb{C}$ and $R = \mathbb{Z}$, then -2, $\sqrt{2}$ and $i = \sqrt{-1}$ are integral over $\mathbb{Z}$, while $1/2$ is not.

34.3. **Proof of the bellows conjecture.** Just like in several proofs of the Cauchy and Dehn's theorems, the proof of the bellows conjecture (Theorem 31.2) neatly splits into the local and the global part. As before, we first present the global part of the proof, and only then prove the local part.

Proof of Lemma 34.1. Let $L = \mathbb{R}$ be the field of vertex coordinates. As in the lemma, denote by R the ring generated by the squared edge lengths. Let F be any fixed field. Consider a place $\varphi : L \to \widehat{F}$ which is finite on R.

Now, let $\Gamma = (V, E)$ be a graph of the vertices and edges in S. We prove the claim by induction on the number $n = |V|$ of vertices and the smallest degree of a vertex in Γ. Let us start with the base of induction, which is summarized in the following lemma, proved later in this section.

Lemma 34.3. *Let $\Delta \subset \mathbb{R}^3$ be a simplex and let R be a ring generated by the squares of the edge lengths ℓ_{ij}^2. Then $24\,\mathrm{vol}(\Delta)$ is integral over R.*

Suppose Γ has vertex w of degree 3. In this case we show that one can make the inductive step by reducing the number of vertices. Denote by v_1, v_2, v_3 the neighbors of w. Consider a new surface S' obtained from S by removing triangles (wv_1v_2), (wv_2v_3), (wv_3v_1), and adding triangle $(v_1v_2v_3)$ (see Figure 34.1). Clearly, S' has $|V| - 1$ vertices and

$$\mathrm{vol}(S) \,=\, \mathrm{vol}(S') \,+\, \mathrm{vol}(\Delta),$$

where Δ is a simplex $(wv_1v_2v_3)$. By the lemma and by the inductive assumption, $24\,\mathrm{vol}(\Delta)$ and $24\,\mathrm{vol}(S')$ are both integral over R. Thus, $24\,\mathrm{vol}(S)$ is also integral over R, and we proved the claim in this case.

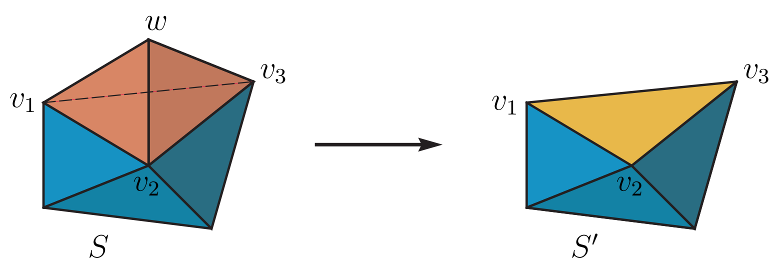

FIGURE 34.1. The inductive step $S \to S'$ when the degree of w is 3.

Suppose now that Γ has no vertices of degree 3. The inductive step in this case consists of reducing the minimal degree of Γ. Since Γ is a plane triangulation, there exists a vertex w of degree at most 5 (otherwise there are $\geq 6n/2 = 3n$ edges in Γ). We will show that φ is also finite on certain diagonals in S. Then, by the argument similar to the one above, we conclude that φ is finite on the volume.

Formally, we say that $g = (v, v')$ is a *diagonal* if $v, v' \in V$ and $g \notin E$. Let $d = |g|$ be the length of the diagonal g in S. The diagonal g is called *supportive* if $\varphi(d^2)$ is finite.

Let w be a vertex in S of the smallest degree k, $4 \leq k \leq 5$, and let $v_1, \ldots, v_k$ be its neighbors, in cyclic order. Denote by H_w the graph induced by the vertices $w, v_1, \ldots, v_k$. Define *small diagonals* in H_w to be pairs of vertices $g_1 = (v_1, v_3)$, ..., $g_{k-2} = (v_{k-2}, v_k)$, $g_{k-1} = (v_{k-1}, v_1)$, and $g_k = (v_k, v_2)$. The following lemma shows that at least one of the small diagonals in Γ is supportive.

Lemma 34.4 (Supportive small diagonals)**.** *Let w be a vertex in S of degree 4 or 5, and let $\varphi : L \to \widehat{F}$ be a place which is finite on squared edge lengths of H_w. Then one of the small diagonals g_i in H_w is supportive.*

In other words, the lemma says that at least one of the place values $\varphi(|g_1|^2)$, ..., $\varphi(|g_k|^2)$ is finite. The lemma represents the local part of the proof of the bellows conjecture and will be proved later in this section.

Suppose $\varphi(|g_i|^2)$ is finite, for the small diagonal $g_i = (v_i, v_{i+2})$ as above. Consider a new surface S' obtained from S by removing triangles (wv_iv_{i+1}), $(wv_{i+1}v_{i+2})$, and adding triangles (wv_iv_{i+2}) and $(v_iv_{i+1}v_{i+2})$ (see Figure 34.2). Clearly, the degree of w in S' is $k-1$, and

$$\mathrm{vol}(S) \,=\, \mathrm{vol}(S') \,\pm\, \mathrm{vol}(\Delta),$$

where Δ is a simplex $(wv_iv_{i+1}v_{i+2})$ with finite squared edge lengths. Here the orientation of the surface of Δ can be either positive or negative, which determines the sign. By the same argument as above, Lemma 34.3 and the inductive assumption imply

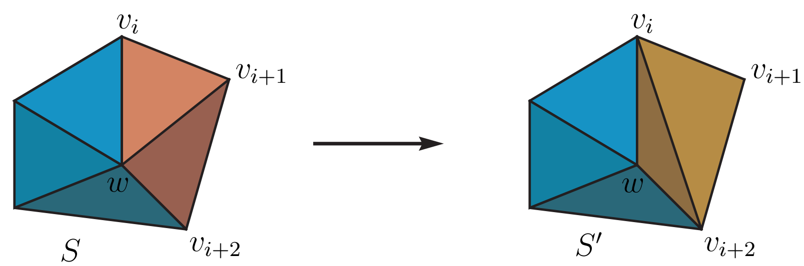

FIGURE 34.2. The inductive step $S \to S'$ when the degree of w is 5.

that $24\,\mathrm{vol}(\Delta)$ and $24\,\mathrm{vol}(S')$ are both integral over R. Thus, $24\,\mathrm{vol}(S)$ is also integral over R, and we proved inductive claim. This completes the proof of Lemma 34.1. $\square$

34.4. **Computing the volume of a simplex.** To prove Lemma 34.3, we present the following classical result which implies the lemma.

Denote by $\Delta = (v_0v_1\ldots v_d)$ the simplex in $\mathbb{R}^d$ with vertices v_i and edge lengths ℓ_{ij}. Define the *Cayley–Menger determinant* $\mathrm{CM}(\cdot)$ as follows:

$$\mathrm{CM}(v_0, v_1, \ldots, v_d) \,:=\, \det \begin{pmatrix} 0 & 1 & 1 & 1 & \ldots & 1 \\ 1 & 0 & \ell_{01}^2 & \ell_{02}^2 & \ldots & \ell_{0d}^2 \\ 1 & \ell_{01}^2 & 0 & \ell_{12}^2 & \ldots & \ell_{1d}^2 \\ 1 & \ell_{02}^2 & \ell_{12}^2 & 0 & \ldots & \ell_{2d}^2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \ell_{0d}^2 & \ell_{1d}^2 & \ell_{2d}^2 & \ldots & 0 \end{pmatrix}.$$

The following formula computes the volume of Δ via the Cayley–Menger determinant.

Theorem 34.5 (Cayley–Menger)**.** *For every simplex $\Delta = (v_0v_1\ldots v_d) \subset \mathbb{R}^d$, we have:*

$$\mathrm{vol}^2(\Delta) \,=\, \frac{(-1)^{d-1}}{2^d\, d!^2}\, \mathrm{CM}(v_0, v_1, \ldots, v_d).$$

The proof of the Cayley–Menger theorem is a straightforward calculation with determinants (see Appendix 41.6). We can now prove Lemma 34.3. When $d = 3$, we have from the theorem:

$$2 \cdot \mathrm{CM}(v_0, v_1, v_2, v_3) \,=\, 2 \cdot 288\, \mathrm{vol}^2(\Delta) \,=\, \big(24\, \mathrm{vol}(\Delta)\big)^2.$$

Since the l.h.s. in this equation is a polynomial in the squared edge lengths, we conclude that $24\,\mathrm{vol}(\Delta)$ is integral over R, as desired.

Example 34.6. (*Heron's formula*) Let P be a triangle in $\mathbb{R}^2$ with side lengths a, b, and c. Recall the classical *Heron's formula* for the area of a triangle:

$$\mathrm{area}(P) \,=\, \sqrt{p\,(p-a)\,(p-b)\,(p-c)}\,, \quad \text{where } \ p \,=\, \frac{1}{2}(a+b+c).$$

Expanding the product we obtain the same result as given by the Cayley–Menger determinant:

$$\mathrm{area}^2(P) = \frac{1}{16}\bigl(2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4\bigr) = \frac{1}{16}\,\det\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & a^2 & b^2 \\ 1 & a^2 & 0 & c^2 \\ 1 & b^2 & c^2 & 0 \end{pmatrix}.$$

This implies that $4\,\mathrm{area}(P)$ is integral over the ring generated by a^2, b^2 and c^2. Note also that the multiple 4 cannot be lowered here.

34.5. **Proof of Lemma 34.4.** We start with the following technical result. Let w, v_1, v_2, v_3, v_4 be distinct points in $\mathbb{R}^3$. Consider the squared distances $a_1 = |v_1v_2|^2$, $a_2 = |v_2v_3|^2$, $a_3 = |v_3v_4|^2$, and $b_i = |wv_i|^2$, $1 \le i \le 4$. Further, let $c = |v_1v_4|^2$, $s_1 = |v_1v_3|^2$, and $s_2 = |v_2v_4|^2$.

Lemma 34.7. *Let $\varphi : L \to \widehat{F}$ be a place such that $\varphi(a_i)$ and $\varphi(b_j)$ are finite, where $1 \le i \le 3$, and $1 \le j \le 4$. Suppose also that $\varphi(s_1) = \varphi(s_2) = \infty$. Then $\varphi(c/s_1) = \infty$.*

This lemma easily implies Lemma 34.4. For $k = 4$, suppose that $\varphi(s_1) = \varphi(s_2) = \infty$ (see Figure 34.3). Then the conditions of Lemma 34.7 are satisfied. In addition, we require that $\varphi(c)$ is finite, which implies that $\varphi(c/s_1) = 0$. This contradicts $\varphi(c/s_1) = \infty$ in the lemma. Therefore, either $\varphi(s_1)$ or $\varphi(s_2)$ is finite. In other words, at least one of the two small diagonals must be supportive.

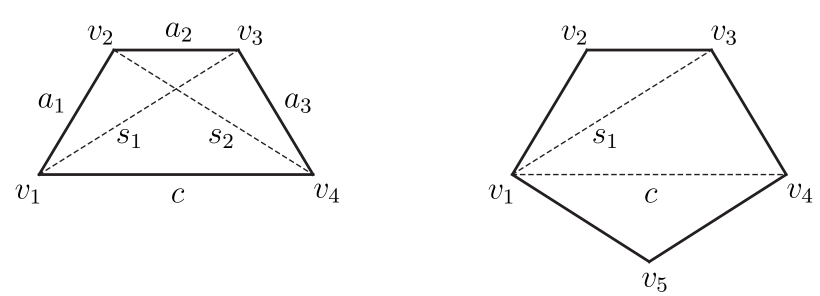

FIGURE 34.3. Two cases in Lemma 34.4: when $k = 4$ and when $k = 5$.

For $n = 5$, suppose neither of the small diagonals in a pentagon $[v_1v_2v_3v_4v_5]$ is supportive. From the lemma, $\varphi(c/s_1) = \infty$. Similarly, by the symmetry, $\varphi(s_1/c) = \infty$ (see Figure 34.3). But then

$$\varphi(1) \,=\, \varphi(c/s_1)\cdot\varphi(s_1/c) \,=\, \infty\cdot\infty \,=\, \infty,$$

a contradiction. This implies that at least one of the small diagonals is supportive, and completes the proof of Lemma 34.4 from Lemma 34.7.

Proof of Lemma 34.7. Think of five points $v_1,\ldots,v_4$ and w as being in $\mathbb{R}^4$. Then the Cayley–Menger determinant $\mathrm{CM}(v_1,v_2,v_3,v_4,w)=0$. In the notation of the lemma, we have:

$$\det\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & a_1 & s_1 & c & b_1 \\ 1 & a_1 & 0 & a_2 & s_2 & b_2 \\ 1 & s_1 & a_2 & 0 & a_3 & b_3 \\ 1 & c & s_2 & a_3 & 0 & b_4 \\ 1 & b_1 & b_2 & b_3 & b_4 & 0 \end{pmatrix} = 0.$$

Divide the second row and second column by s_1. Similarly, divide the fifth row and fifth column by s_2. We obtain:

$$D := \det\begin{pmatrix} 0 & \frac{1}{s_1} & 1 & 1 & \frac{1}{s_2} & 1 \\ \frac{1}{s_1} & 0 & \frac{a_1}{s_1} & 1 & \frac{c}{s_1s_2} & \frac{b_1}{s_1} \\ 1 & \frac{a_1}{s_1} & 0 & a_2 & 1 & b_2 \\ 1 & 1 & a_2 & 0 & \frac{a_3}{s_2} & b_3 \\ \frac{1}{s_2} & \frac{c}{s_1s_2} & 1 & \frac{a_3}{s_2} & 0 & \frac{b_4}{s_2} \\ 1 & \frac{b_1}{s_1} & b_2 & b_3 & \frac{b_4}{s_2} & 0 \end{pmatrix} = 0.$$

Suppose that $\varphi\bigl(\frac{c}{s_1s_2}\bigr)=0$. Using the assumptions on the values of φ and conditions ($\circledast$) in the definition of a place, we obtain:

$$\varphi(D) = \det\begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & \varphi(a_2) & 1 & \varphi(b_2) \\ 1 & 1 & \varphi(a_2) & 0 & 0 & \varphi(b_3) \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & \varphi(b_2) & \varphi(b_3) & 0 & 0 \end{pmatrix} = \det\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = -1,$$

which contradicts $\varphi(D)=\varphi(0)=0$. Therefore, $\varphi\bigl(\frac{c}{s_1s_2}\bigr)\neq 0$, and

$$\varphi\left(\frac{c}{s_1}\right) = \varphi\left(\frac{c}{s_1s_2}\right)\cdot\varphi(s_2) = \infty,$$

as desired. $\square$

34.6. **Exercises.**

Exercise 34.1. (*Brahmagupta's formula*) *a*) [1-] Prove an explicit formula for the area of a convex quadrilateral Q with side lengths a,b,c,d, and which is inscribed into a circle:

$$\mathrm{area}^2(Q) = (\rho-a)(\rho-b)(\rho-c)(\rho-d),$$

where $\rho=\frac{1}{2}(a+b+c+d)$ is half the perimeter of Q.
b) [1] Conclude that $4\,\mathrm{area}(Q)$ is integral over $\mathbb{C}[a^2,b^2,c^2,d^2]$.

Exercise 34.2. *a*) [1-] Show that a triangle in the plane cannot have odd sides and integral volume.
b) [1] Show that a tetrahedron in $\mathbb{R}^3$ cannot have odd edge lengths and integral volume.
c) [1-] Conclude from *a*) that no 4 points in the plane can have odd pairwise distances.

Exercise 34.3. Two sets A and B of n points are called *equivalent* if their have the same multiset of pairwise distances. Similarly, they are called *congruent* if one can be obtained from the other by a rigid motion.
a) [1] Find two equivalent but non-congruent sets of 4 points in the plane.
b) [1+] Show that almost all sets of $n \geq 4$ points in the plane have no sets equivalent but not congruent to them.

Exercise 34.4. $\Diamond$ [1] Extend Lemma 34.4 to vertices of degree > 5.

Exercise 34.5. [1+] Let $v_0, \ldots, v_d \in \mathbb{R}^{d-1}$ be any $d+1$ points. Denote by

$$\mathrm{CM}'(v_0, v_1, \ldots, v_d) \, = \, \det(\ell_{ij}^2)$$

the principal minor of the Cayley–Menger matrix, obtained by removing the first row and the first column. Prove that $\mathrm{CM}'(v_0, v_1, \ldots, v_d) = 0$ if and only if the points lie on a sphere or a hyperplane.

Exercise 34.6. (*Robbins problem*) Denote by $\mathrm{A}(a_1, \ldots, a_n)$ the area of a convex polygon inscribed into a circle with sides a_i (cf. Example 34.6, Exercises 13.1, 31.3, and 34.1).
a) [1-] Prove that $\mathrm{A}(\cdot)$ is a symmetric function.
b) [2-] Prove that $4\mathrm{A}(\cdot)$ is integral over $\mathbb{C}[a_1^2, \ldots, a_n^2]$.

Exercise 34.7. (*Extended bellows conjecture*) *a*) [1+] Extend the bellows conjecture to all orientable 2-dimensional polyhedral surfaces in $\mathbb{R}^3$. In other words, extend Lemma 34.1 to surfaces of any genus.
b) [$*$] Extend the bellows conjecture to higher dimensions.

Exercise 34.8. (*Equiareal triangulations*) A dissection of a polygon into triangles of equal area is called an *equiareal triangulation*.
a) [2] Prove that every equiareal triangulation of a unit square has an odd number of triangles.
b) [2] Suppose a unit square is dissected into triangles with areas $a_1, \ldots, a_m$. Prove that there exists a polynomial $f \in \mathbb{Z}[x_1, \ldots, x_m]$ such that $f(a_1, \ldots, a_m) = 1/2$.
c) [2+] Prove that every equiareal triangulation of a centrally symmetric polygon has an odd number of triangles.
d) [2+] Extend part *c*) to centrally symmetric polygons of area 1.
e) [2] Find a plane quadrilateral which has no equiareal triangulations.
f) [2] Find a convex polytope $R \subset \mathbb{R}^3$ which cannot be dissected into tetrahedra of equal volume.

34.7. **Final remarks.** The bellows conjecture is usually attributed to Connelly and Sullivan (see [Con2]). The idea of finding and studying polynomial relations for the volume (as in Corollary 31.4) seems to be due to Connelly and Sabitov and goes back to the late 80's (see [Sab1]). The bellows conjecture, including the case of surfaces of higher genus, was completely resolved by Sabitov in a series of papers of varying difficulty [Sab2, Sab3, Sab4]. Our presentation is based on [CSW], where the idea of using the theory of places has been introduced. This approach was later used by Connelly to give a simple proof of one of the Robbins conjectures [Con6] (see Exercise 34.6 and [Pak4]). We refer to [Schl3] for for more on the background of the conjecture and the proof.

The bellows conjecture remains open in higher dimensions (see Exercise 34.7). It is known to hold for the few available examples of flexible polyhedra (see Subsection 30.8). Interestingly, the bellows conjecture fails for spherical polyhedra, as exhibited by a flexible spherical polyhedron given in Subsection 30.7 (see [Ale3]). A number of counterexamples to the infinitesimal versions of the conjecture was given in [Ale1]. Finally, the conjecture remains open for hyperbolic polyhedra (see [Ale6] for the references).

The theory of places is often presented as part of a more general theory of valuations. In this context the integrality criterion (Theorem 34.2) is called the *Chevalley extension lemma* (see [Cohn, §9.1]) and is usually stated in terms of valuation rings (see [AtiM, Corollary 5.22] and [Mats, Theorem 10.4]).

35. The Alexandrov curvature theorem

This is the first of three sections giving a description of convex polyhedra in terms of certain metric invariants. In this section we consider polytopes whose vertices lie on given rays starting at the origin. The main result is Alexandrov's theorem giving their complete characterization. The tools include the Gauss–Bonnet theorem and its variations (see Section 25) and the mapping lemma in the Appendix (see Subsection 41.8). The proof ideas from this section are also used in the next two sections.

35.1. **First touch rule.** Let $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n \in \mathbb{S}^2$ be a fixed set of unit vectors. Denote by $R_1, \ldots, R_n \subset \mathbb{R}^3$ the *rays* starting at the origin $O \in \mathbb{R}^3$ and spanned by these vectors. Consider all convex polytopes P with vertices on the rays (one on each). In other words, let $P = \operatorname{conv}\{v_1, \ldots, v_n\}$ where $\overrightarrow{Ov_i} = r_i \boldsymbol{u}_i$, $r_i > 0$, and $i \in [n] = \{1, \ldots, n\}$. We say that polytope P *lies on rays* R_i, and call r_i the *ray coordinates* of P.

We assume that rays R_i do not lie in the same half-space, so that polytope P contains the origin O in its relative interior. As before, denote by $\omega_i = \omega(C_i)$ the curvature of vertex v_i (see Section 25).

Define the *expansion* of polytope P with ray coordinates $r_1, \ldots, r_n$ to be a polytope cP with ray coordinates $cr_1, \ldots, cr_n$, for some $c > 0$. The following results shows that the curvature uniquely determines the polytope up to expansion:

Lemma 35.1 (Curvature uniqueness). *Let $P, P' \subset \mathbb{R}^3$ be polytopes which lie on rays R_i containing the origin O, where $i \in [n]$. Suppose these polytopes have equal curvature at all vertices: $\omega_i = \omega_i'$, for all $i \in [n]$. Then P is an expansion of P'.*

Proof. Consider all expansions cP' which lie inside P. Clearly, for sufficiently small $c > 0$ such expansions exist, and for sufficiently large c they do not. Let $Q = \varepsilon P'$ be the largest such expansion, $Q \subset P$. Since no further expansion is possible, at least one vertex of Q lies on the boundary ∂P, and by construction of polytopes on rays this boundary point is a vertex of P (see Figure 35.1).

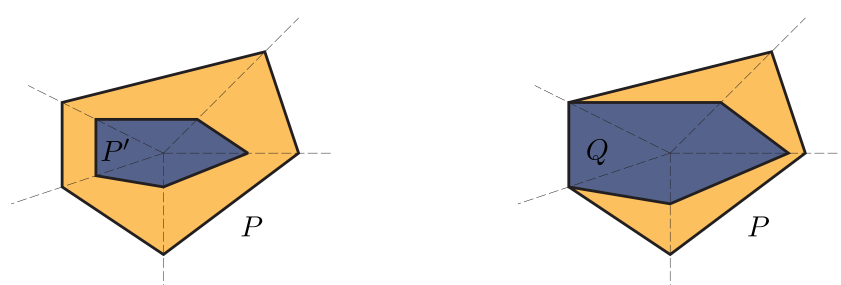

Figure 35.1. Polygons P, P' and the maximal expansion $Q = cP' \subset P$.

Consider all edges of Q which coincide with edges of P. There are two possibilities: either all of them do, or (by connectivity of the graph of Q there exists an edge

$e = (v_i, v_j)$ in Q, with one vertex v_i in P and the other vertex v_j in the interior of P. In the first case, all vertices of Q coincide with those in P and polytope P is an expansion of P'. In the second case, denote by C_i the cone spanned by edges of P starting at v_i, and by D_i the cone spanned by edges of Q starting at v_i'. Then, by construction, $C_i \supset D_i$ and $C_i \neq D_i$. Therefore, by the monotonicity of the cone curvature (Exercise 25.5), we have $\omega_i = \omega(C_i) < \omega(D_i) = \omega_i'$, a contradiction. □

35.2. **Inequalities on curvatures.** Before we can state the converse of Lemma 35.1, we need to give the necessary conditions on the curvatures.

As before, let P be a convex polytope which lies on rays R_i, which do not lie in the same half-space, and let $\omega_i > 0$ be the curvatures of the vertices, $i \in [n]$. First, recall the Gauss–Bonnet theorem (Theorem 25.3):

$$\omega_1 + \ldots + \omega_n = 4\pi.$$

In addition, there is a number of inequalities given by the geometry of rays.

Lemma 35.2. *For a subset $I \subset [n]$, denote by $C_I = \mathrm{conv}\{R_i : i \in I\}$ the convex cone spanned by the rays. Assume $C_I \neq \mathbb{R}^3$. Then $\sum_{k \notin I} \omega_k > \omega(C_I)$.*

Proof. Let P' be a convex polyhedron obtained as a convex hull of P and rays R_i, for all $i \in [n]$. Since $C_I \neq \mathbb{R}^3$, polyhedron P' is well defined. Note that the vertices of P' are given by $v_k \notin C_I$, so in particular, $k \notin I$. Since O is a point in the relative interior of P, the cones of P and P' at all such vertices v_k satisfy $C_k \subset C_k'$. By the monotonicity of the cone curvature (Exercise 25.5), we have $\omega_k' \leq \omega_k$, and the inequality is strict for vertices of unbounded faces of C_k'. Applying Theorem 25.5 to polyhedron P', we obtain:

$$\sum_{k \notin I} \omega_k > \sum_{k \notin I} \omega_k' \geq \sum_{k \in [n]\,:\, v_k \notin C_I} \omega_k' = \omega(C_I),$$

as desired. □

35.3. **Inequalities on ray coordinates.** Let us now describe the set of ray coordinates of all convex polytopes which lie on given rays $R_1, \ldots, R_n$. As it turns out, it is convenient to use the *inverse ray coordinates*. Formally, let $x_i = \frac{1}{r_i}$ and let $\mathcal{X}$ denote the set of all $(x_1, \ldots, x_n) \in \mathbb{R}^n$, such that $x_i > 0$ and $\left(\frac{1}{x_1}, \ldots, \frac{1}{x_n}\right)$ are ray coordinates of a polytope on rays R_i.

Lemma 35.3. *Suppose $R_1, \ldots, R_n \subset \mathbb{R}^3$ are rays which do not lie in the same half-space. Then set $\mathcal{X}$ of inverse ray coordinates is an open convex cone in $\mathbb{R}^n$.*

Proof. Since all polytopes P which lie on rays R_i must contain the origin, their combinatorics is completely determined as follows. For every three independent rays R_a, R_b and R_c, consider a cone $C = C_{\{a,b,c\}}$ spanned by these rays. Whenever ray R_i lies inside C, the vertex v_i of R_i must be separated from O by a plane H spanned by v_a, v_b and v_c.

As before, let $\boldsymbol{u}_a, \boldsymbol{u}_b, \boldsymbol{u}_c$ and $\boldsymbol{u}_i$ be the unit vectors along the rays. Suppose

$$\boldsymbol{u}_i \,=\, \alpha\, \boldsymbol{u}_a \,+\, \beta\, \boldsymbol{u}_b \,+\, \gamma\, \boldsymbol{u}_c .$$

We immediately have

$$r_i \boldsymbol{u}_i \,=\, r_i\, (\alpha \boldsymbol{u}_a + \beta \boldsymbol{u}_b + \gamma \boldsymbol{u}_c) \,=\, \left(r_i \frac{\alpha}{r_a} \right) r_a \boldsymbol{u}_a \,+\, \left(r_i \frac{\beta}{r_b} \right) r_b \boldsymbol{u}_b \,+\, \left(r_i \frac{\gamma}{r_c} \right) r_c \boldsymbol{u}_c .$$

Observe that the plane H consists of points $\lambda_a v_a + \lambda_b v_b + \lambda_c v_c$, for all $\lambda_a + \lambda_b + \lambda_c = 1$, and let $s > 0$ be defined by

$$s \cdot \left(\frac{\alpha}{r_a} + \frac{\beta}{r_b} + \frac{\gamma}{r_c} \right) \,=\, 1 .$$

Therefore, vertex v_i lies on H whenever $r_i = s$ and v_i is separated from O by H if $r_i > s$. Rewriting the last inequality in the inverse ray coordinates $x_i = \frac{1}{r_i}$ we obtain $x_i < \alpha x_a + \beta x_b + \gamma x_c$. Taking the intersection of these halfspaces for all cones C as above, with the halfspaces $x_i > 0$, we obtain convex cone $\mathcal{X}$ as in the lemma. $\square$

35.4. **Finding polytopes with given curvatures.** We are ready now to prove the main result of this section. For a subset $I \subset [n]$, denote by $C_I = \operatorname{conv}\{R_i : i \in I\}$ the convex cone spanned by the rays.

Theorem 35.4 (Alexandrov curvature theorem)**.** *Suppose $R_1, \ldots, R_n \subset \mathbb{R}^3$ are rays which do not lie in the same half-space, and suppose $\omega_1, \ldots, \omega_n \in \mathbb{R}$ satisfy the following conditions:*

1) $\omega_i > 0$, *for all* $i \in [n]$,
2) $\omega_1 + \ldots + \omega_n = 4\pi$,
3) $\sum_{k \notin I} \omega_k > \omega(C_I)$, *for every* $I \subset [n]$, *such that* $C_I \neq \mathbb{R}^3$.

Then, up to expansion, there exists a unique convex polytope $P \subset \mathbb{R}^3$ which lies on rays R_i and has curvatures ω_i at vertices $v_i \in R_i$, for all $i \in [n]$. Conversely, all curvatures ω_i of such polytopes P must satisfy conditions 1)–3).

One can think of this theorem as both the extension and the converse of Lemma 35.1. Of course, the second part follows from Lemma 35.2 and the Gauss-Bonnet theorem.

Remark 35.5. Let us first check that this result is plausible. There are $n-1$ degrees of freedom of the ray coordinates modulo expansion. This matches the $n-1$ degrees of freedom of the vertex curvatures modulo the Gauss–Bonnet formula. Therefore, by the curvature uniqueness theorem and the inverse function theorem, for small perturbations of ω_i, there is always a small perturbation of the ray coordinates r_i giving these curvatures. The following proof uses a variation on the same argument.

Proof of Theorem 35.4. Let $\mathcal{X} \subset \mathbb{R}^n$ be the cone of inverse ray coordinates x_i, and let $\mathcal{X}_\circ$ be the intersection of $\mathcal{X}$ with a hyperplane defined by $x_1 + \ldots + x_n = 1$. Clearly, $\mathcal{X}_\circ$ is an open $(n-1)$-dimensional convex polytope. Also, for every polytope P with inverse ray coordinates $\boldsymbol{x} \in \mathcal{X}$, there is a unique $c > 0$ and an expansion cP with inverse ray coordinates $c\boldsymbol{x} \in \mathcal{X}_\circ$.

Denote by $\Theta \subset \mathbb{R}^n$ the set of all $\boldsymbol{w} = (\omega_1, \ldots, \omega_n)$ as in the theorem. Denote by $\varphi : \mathcal{X} \to \Theta$ the map defined by $\varphi(\boldsymbol{x}) = \boldsymbol{w}$, where $\boldsymbol{x} = (x_1, \ldots, x_n)$. By Lemma 35.2, the curvatures satisfy three conditions in the theorem, so this map is well defined. We now deduce the result from the mapping lemma (Theorem 41.8). First, both $\mathcal{X}$ and Θ are open connected manifolds of the same dimension $n-1$. By Lemma 35.1, map φ is injective. Obviously, map φ is continuous. It remains to show that φ is proper.

Suppose $\boldsymbol{w}_k \to \boldsymbol{w}$ as $k \to \infty$, where $\boldsymbol{w}_k, \boldsymbol{w} \in \Theta$. Assume $\boldsymbol{w}_k = \varphi(\boldsymbol{x}_k)$ for some $\boldsymbol{x}_k \in \mathcal{X}$. Since $\mathcal{X}$ is a bounded convex polytope, the sequence $\{\boldsymbol{x}_k\}$ has a converging subsequence to some point $\boldsymbol{x} \in \mathbb{R}^n$. We need to show that $\boldsymbol{x} \in \mathcal{X}$. Denote by P_k the polytopes corresponding to $\boldsymbol{x}_k$. By contradiction, if $\boldsymbol{x} \notin \mathcal{X}$, then it lies on one of the facets, i.e., either $x_i = \alpha x_a + \beta x_b + \gamma x_c$ for some $i, a, b, c \in [n]$, or $x_i = 0$ for some $i \in [n]$, in notation of the proof of Lemma 35.3. In the first case, the limit of P_i is a non-strictly convex polytope P where vertex v_i lies in the plane spanned by $(v_a v_b v_c)$. But then the limit of the curvatures ω_i in $\boldsymbol{w}_k$ is 0, a contradiction with $\boldsymbol{w} \in \Theta$.

In the second case, if $x_i = \frac{1}{r_i} \to 0$, we have $r_i \to \infty$, and the limit polyhedron $P_k \to P$ is unbounded. Let us first show that P is well defined. Denote by I the set of all $i \in [n]$ such that $r_i \to \infty$. By convexity, this implied that the same holds for all $i \in [n]$ such that $R_i \subset C_I$. Thus, if $C_I = \mathbb{R}^3$ we have $x_i \to\to 0$ for all $i \in [n]$, a contradiction with $x_1 + \ldots + x_n = 1$. Therefore, polyhedron P is well defined and contains only those rays R_i, s.t. $i \in I$. In other words, the base cone C_P coincides with the cone C_I. Applying Theorem 25.5 to P, we have $\sum_{j \notin I} \omega_j = \omega(C_P) = \omega(C_I)$, a contradiction with $\boldsymbol{w} \in \Theta$. This finishes the proof that φ is proper. By the mapping lemma, φ is a homeomorphism, which proves the result. $\square$

35.5. **Convex cap curvatures.** Let $H \subset \mathbb{R}^3$ be a plane, which we always think as horizontal. Fix n points $a_1, \ldots, a_n \in H$ in general position, i.e., such that no three of them lie on a line. Consider lines $L_i \perp H$, such that $a_i \in L_i$, for all $i \in [n]$. For convenience, assume that the origin O is in the interior of $\operatorname{conv}\{a_1, \ldots, a_n\}$. Consider a convex cap $P \subset \mathbb{R}^3$ with vertices $v_i \in L_i$ and the base cone C_P a ray orthogonal to H and pointing down.

One can ask if there is a way to characterize the curvatures ω_i of vertices of convex caps. Of course, shifting the convex cap P up by a fixed constant does not change the curvatures. Also, by the Gauss–Bonnet theorem for convex caps (see Section 25.5), we have $\omega_1 + \ldots + \omega_n = 2\pi$. As the following theorem shows, there are no nontrivial inequalities on the curvatures.

Theorem 35.6 (Alexandrov curvature theorem for convex caps)**.** *Fix a plane H in $\mathbb{R}^3$ and lines $L_1, \ldots, L_n \subset \mathbb{R}^3$ orthogonal to H. Suppose $\omega_1, \ldots, \omega_n > 0$ satisfy $\omega_1 + \ldots + \omega_n = 2\pi$. Then, up to translation, there exists a unique convex cap P which lies on lines L_i and has curvatures ω_i at vertices $v_i \in R_i$, for all $i \in [n]$.*

There is a topological proof (using the mapping lemma) of this theorem which we leave to the reader (Exercise 35.2). Below we present a variational principle argument proving a stronger result (see the reduction in Exercise 35.5).

Suppose $a_1, \dots, a_k$ are the vertices of a polygon $A = \operatorname{conv}\{a_1, \dots, a_n\}$, so the remaining points $a_{k+1}, \dots, a_n$ lie in the relative interior of A. Consider a convex cap P with vertices $v_i \in L_i$, for all $i \in [n]$. Space polygon $Z = [v_1 \dots v_k]$ is called the *border* of P (see Figure 35.2).

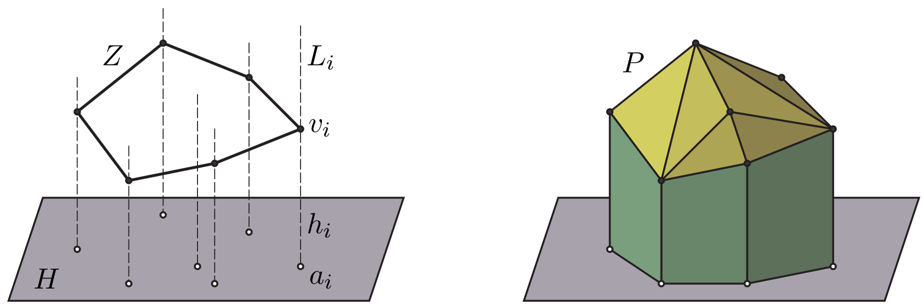

FIGURE 35.2. Space polygon Z and a convex cap P on lines L_i with border Z.

Theorem 35.7 (Pogorelov curvature theorem for convex caps)**.** *Fix a plane H in $\mathbb{R}^3$, lines $L_1, \dots, L_n \subset \mathbb{R}^3$ orthogonal to H, and space polygon $Z = [v_1 \dots v_k]$ as above. Suppose $\omega_{k+1}, \dots, \omega_n > 0$ satisfy $\omega_{k+1} + \ldots + \omega_n < 2\pi$. Then there exists a unique convex cap P which lies on lines L_i, has the border Z, and has curvatures ω_i at vertices $v_i \in L_i$, for all $k+1 \le i \le n$.*

Proof. We start with the uniqueness part. Suppose we have two convex caps P and P' as in the theorem. Shift P' down until it lies below P and start moving it up until the first time some two vertices v_i and v_i' coincide. As in the proof of Lemma 35.1, for the cones C_i and C_i' at these vertices we have $C_i' \subset C_i$ and $\omega(C_i) = \omega(C_i')$. By the monotonicity of the cone curvature, we have $C_i = C_i'$. Moving along the edges of the graph G of P, starting at v_i and that are not in Z, we conclude that $v_j = v_j'$ for all vertices in G.

For a convex cap P which lies of lines L_i, denote by h_i the height of points v_i, $i \in [n]$. Let $\mathcal{P}$ be the set of convex caps P which lie on lines L_i, have border Z and have curvatures $\nu_i = \omega(V_i) \le \omega_i$ for all $k+1 \le i \le n$. Of course, for $P \in \mathcal{P}$, the heights $h_1, \dots, h_k$ are fixed, while $h_{k+1}, \dots, h_n$ can vary. Let $\eta(P) = h_{k+1} + \ldots + h_n$ be the sum of the heights.

We will show that the cap $P_\circ \in \mathcal{P}$ with maximal value of η is the desired convex cap. First, let us show that the set $\mathcal{P}$ is bounded, i.e., that $h_i \le c$ for some constant c which depends on a_i, ω_i and Z. Let $w = \omega_{k+1} + \ldots + \omega_n < 2\pi$. Then there exists a vertex v_i, $i \in [k]$, with curvature $\nu_i > (2\pi - w)/k > 0$. Therefore, the maximal slope of the ray in the convex cone C_i starting at v_i is at most $(\pi - \nu_i)/2 < \pi/2$. Since every vertex v_j lies inside C_i, the height h_j is bounded from above (see Figure 35.3). In a different direction, by convexity, the heights h_j are bounded from below. This implies that $\mathcal{P}$ is a compact set and the sum of heights η maximizes on $\mathcal{P}$.

Suppose now that function η maximizes at some $P \in \mathcal{P}$, and that for the curvature ν_i of vertex v_i we have $\nu_i < \omega_i$. Increase the height of v_i by $\varepsilon > 0$ to obtain

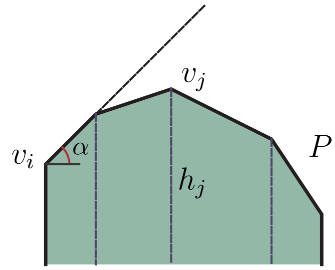

FIGURE 35.3. The heights h_j of convex caps $P \in \mathcal{P}$ are bounded.

a convex cap P'. By convexity, cone C_i' can be shifted down by ε to fit inside C_i. Thus, $\nu_i' = \omega(C_i') > \omega(C_i) = \nu_i$. If ε is small enough, the curvature $\nu_i' < \omega_i$. For the remaining vertices, we have $C_j' \supset C_j$ and thus $\nu_j' \le \nu_j \le \omega_i$. This proves that $P' \in \mathcal{P}$. By construction, $\eta(P') = \eta(P) + \varepsilon$, a contradiction. Therefore, η maximizes on the convex cap as in the theorem. $\square$

35.6. **Exercises.**

Exercise 35.1. Let $P, P' \subset \mathbb{R}^3$ be polytopes which lie on rays R, and have equal solid angles: $\sigma(C_i) = \sigma(C_i')$, for all $i \in [n]$.
a) [1-] Prove that P is an expansion of P'.
b) [1] Think of part *a*) as the uniqueness result (Lemma 35.1). Explain why there is no obvious counterpart of the existence theorem in this case.
c) [1+] Prove the analogue of Theorem 35.7 for solid angles.

Exercise 35.2. $\diamondsuit$ [1] Give a topological proof of Theorem 35.7.

Exercise 35.3. $\diamondsuit$ [1+] Generalize Theorem 35.4 to higher dimensions.

Exercise 35.4. [$*$] Give a variational principle proof of Theorem 35.4.

Exercise 35.5. $\diamondsuit$ [1] Show that Theorem 35.7 implies Theorem 35.6.

Exercise 35.6. [1-] Suppose n particles in space move at the same speed in different directions. Prove that all particles will eventually move into a convex position.

Exercise 35.7. [2] In notation of Lemma 35.1, suppose polytopes P and P' have curvatures which satisfy $|\omega_i - \omega_i'| < \varepsilon$. Prove that there exist a constant $c > 0$, such that $|r_i - cr_i| < O(\varepsilon^\alpha)$ for all $i \in [n]$, where $c > 0$ is a universal constant independent of the number n of rays.

Exercise 35.8. *a*) [1] Following the approach in Subsection 9.1, convert the variational proof of Theorem 35.7 into a greedy algorithm. Explain where the combinatorics of faces of the convex caps appears in the proof.
b) [$*$] For a greedy algorithm as in part *a*), consider triangulations of $A = \text{conv}\{a_1, \ldots, a_n\}$, corresponding to projections of convex caps P. Either construct an example which requires going through exponentially many such triangulations, or prove a polynomial upper bound. Check that the same triangulation can be repeated. Give the exponential upper bound on the number of triangulations in the greedy algorithm.

Exercise 35.9. Let $(\omega_1, \ldots, \omega_n)$, satisfy $n \geq 4$, $0 < \omega_i < 2\pi$, for all $1 \leq i \leq n$, and $\omega_1 + \ldots + \omega_n = 4\pi$.
a) [2-] Prove that there exists a convex polytope $P \subset \mathbb{R}^3$ with n vertices, which has curvatures ω_i, $1 \leq i \leq n$.
b) [2-] Suppose n is even. Prove that there exists a simple polytope $P \subset \mathbb{R}^3$ with n vertices, which has curvatures ω_i, $1 \leq i \leq n$.

Exercise 35.10. $\Diamond$ *a*) [1] Find a collection of rays R_i as in Theorem 35.4, such that the number of different cones C_I is exponential.
b) [$*$] Prove or disprove: deciding whether condition 3) is satisfied for given $R_1, \ldots, R_n \in \mathbb{R}^3$ and $\omega_1, \ldots, \omega_n > 0$, is NP-hard.

Exercise 35.11. $\Diamond$ [1] In notation of the mapping lemma (Theorem 41.8), suppose $\mathcal{A}$ and $\mathcal{B}$ are connected, but φ is only locally injective: for every $b \in \mathcal{B}$ we have $\varphi(a_1) = \varphi(a_2)$ implies $a_1 = a_2$ in a sufficiently small neighborhood of b. Prove the analogue of the mapping lemma in this case.

35.7. **Final remarks.** The curvature existence theorem (Theorem 35.4) is due to Alexandrov and our proof follows [A2]. Both Theorem 35.6 and 35.7 follow from Alexandrov's most general result. The version with the border Z and the variational principle proof in Subsection 35.5 is due to Pogorelov (see [Pog3, §2.4]).

To explain the motivation behind the results in this section, consider the history of the problem. Let $f : B \to \mathbb{R}$ be a convex function defined on a convex region $B \subset \mathbb{R}^2$. Think of points $s_b = (b, f(b))$, where $b \in B$ as forming a convex surface $S \subset \mathbb{R}^3$. Following the *Cauchy problem*, one can ask whether for given boundary values $f : \partial B \to \mathbb{R}$ and the curvature function $\sigma : b \to \sigma(s_b)$, one can *reconstruct* the surface S? Note here the logic in the question: the curvature is a fundamentally local parameter, so we are asking whether such local parameter can determine a global structure of the surface. This problem was resolved by Alexandrov in a much more general form. Theorem 35.7 is both a simple special case and an important lemma in the proof.

Now, one ask a similar question for the surfaces of general convex bodies in $\mathbb{R}^3$. Here the curvature is defined not on a plane but on a unit sphere, via spherical projections. This case is more technical because of the inequalities as in Lemma 35.2. Again, this was resolved by Alexandrov in full generality. Theorem 35.4 corresponds to the case of convex polytopes, when the curvature is concentrated in the vertices.

Recall that prior to this section, we proved a number of uniqueness and rigidity results on convex polyhedra, including the Cauchy and Dehn rigidity theorems, as well as their variations and generalizations. In the next two sections we give two more existence results. One can think of this section as a gate to the next two sections.

Namely, the Minkowski theorem (Theorem 36.2) on existence and uniqueness of convex polytopes with given normals and areas of the faces. Informally one can think of this result as dual to Theorem 36.2. As the reader shall see, we use an unusual variational type argument to prove this result, and a "first touch rule" analogue for unbounded polyhedra. Similarly, for the Alexandrov existence theorem (Theorem 37.1) we give a topological proof based on the mapping lemma. In fact, the proof uses Lemma 35.3.

Let us note also that while the existence is often harder and more delicate to establish (as in this section), the uniqueness results play an important role in classical Differential Geometry (see Subsection 38.5).

36. The Minkowski theorem

We continue the study of global geometry of convex polyhedra. The main result of this section is the Minkowski theorem, which characterizes convex polytopes by the facet areas and normals. The proof is based on the Brunn–Minkowski inequality (see Section 7). We also present Pogorelov's variation for the unbounded polyhedra and a curious Alexandrov's uniqueness theorem in $\mathbb{R}^3$, based on the Alexandrov lemma in Section 22.

36.1. **Normals to faces.** Let $P \subset \mathbb{R}^d$ be a convex polytope with facets $F_1, \ldots, F_n$. Denote by $\boldsymbol{u}_i$ the unit normal to the facet F_i, and by $a_i = \text{area}(F_i) = \text{vol}_{d-1}(F_i)$ its areas, $1 \le i \le n$. The following result gives a connection between them.

Proposition 36.1. *For every convex polytope $P \subset \mathbb{R}^d$ as above, we have:*

$$a_1 \boldsymbol{u}_1 + \ldots + a_n \boldsymbol{u}_n = \mathbf{0}.$$

The vector equation in the proposition is obvious in the plane (see Figure 36.1).

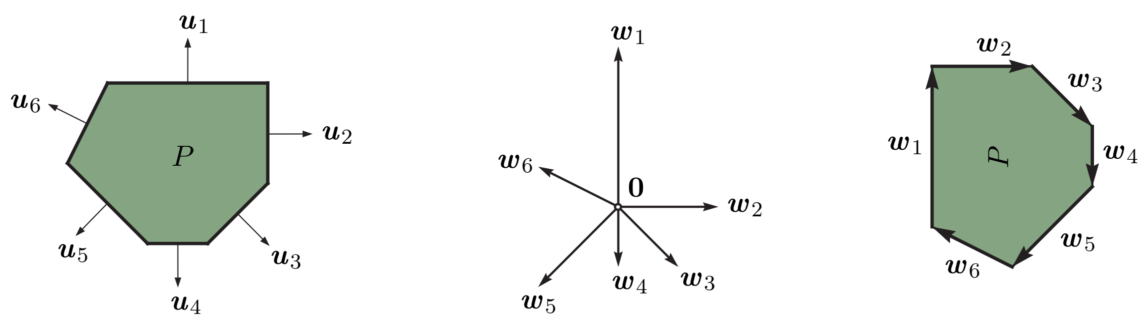

Figure 36.1. Polygon P, normals $\boldsymbol{u}_i$ and weighted normals $\boldsymbol{w}_i = a_i \boldsymbol{u}_i$.

Proof. Let $\mathbf{z} = \boldsymbol{u}_H$ be the normal to a hyperplane $H \subset \mathbb{R}^d$. Consider an orthogonal projection $Q \subset H$ of polytope P onto H. Similarly, consider projections A_i of facets F_i onto H. Recall that facets F_i are oriented according to the (fixed) orientation of the surface ∂P of the polytope. Observe that these projections cover Q twice: once with a positive and once with a negative orientation. Therefore, for the vector $\boldsymbol{x}$ defined by $\boldsymbol{x} = a_1 \boldsymbol{u}_1 + \ldots + a_n \boldsymbol{u}_n$ we have:

$$0 = \sum_{i=1}^{n} \text{area}(A_i) \cdot \langle \boldsymbol{u}_i, \mathbf{z} \rangle = \sum_{i=1}^{n} a_i \cdot \langle \boldsymbol{u}_i, \mathbf{z} \rangle = \left\langle \sum_{i=1}^{n} a_i \boldsymbol{u}_i \, , \mathbf{z} \right\rangle = \langle \boldsymbol{x}, \mathbf{z} \rangle .$$

In other words, the scalar product $\langle \boldsymbol{x}, \mathbf{z} \rangle = 0$ for every unit vector $\mathbf{z} \in \mathbb{R}^d$. This immediately implies that $\boldsymbol{x}$ is a zero vector, as desired. $\square$

We are now ready to state the main result of this section. We prove it in the next subsection.

Theorem 36.2 (Minkowski theorem)**.** *Let $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n \in \mathbb{R}^d$ be unit vectors which span $\mathbb{R}^d$, and let $a_1, \ldots, a_n > 0$ satisfy $a_1\boldsymbol{u}_1 + \ldots + a_n\boldsymbol{u}_n = \mathbf{0}$. Then there exists a unique convex polytope $P \subset \mathbb{R}^d$ with n facets $F_1, \ldots, F_n$, such that* $\operatorname{area}(F_i) = a_i$ *and the normal to F_i is $\boldsymbol{u}_i$, for all $i \in [n]$.*

36.2. **Proof of the Minkowski theorem.** Denote by $\mathcal{P}$ the set of all convex polytopes $P \in \mathbb{R}^d$, with normals $\boldsymbol{u}_i$ as in the theorem, up to a translation. Polytope P can be written as the intersection of halfspaces corresponding to hyperplanes H_i at distance z_i from the origin.

$$(\gimel) \qquad P = \bigcap_{i=1}^{n} \big\{ x \in \mathbb{R}^d \, : \, \langle \overrightarrow{Ox}, \boldsymbol{u}_i \rangle \le z_i \big\}$$

To simplify the notation, we assume that $z_1 = \ldots = z_d = 0$, and that $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_d$ are linearly independent, i.e., polytope P has a unique translation such that the facets $F_1, \ldots, F_d$ contain the origin O. Denote by W the space of all vectors $\boldsymbol{z} = (z_{d+1}, \ldots, z_n)$. Let $W_+ \subset W$ be the cone of vectors $\boldsymbol{z}$ with nonnegative coordinates, so $W_+ \simeq \mathbb{R}_+^{n-d}$. Of course, not all $\boldsymbol{z} \in W_+$ correspond to a polytope in $\mathcal{P}$, since some facets may become degenerate or disappear. We denote by $P_{\boldsymbol{z}}$ the polytope defined in $(\gimel)$, and by $\mathcal{Z}$ the set of $\boldsymbol{z} \in W_+$, such that $P_{\boldsymbol{z}} \in \mathcal{P}$. Observe also that

$$\operatorname{area}(F_i) \, = \, \frac{\partial \operatorname{vol}(P_{\boldsymbol{z}})}{\partial z_i}\,, \quad \text{for all } \; d+1 \le i \le n.$$

The following result is a key to both parts of the Minkowski theorem. It is an easy consequence of the Brunn–Minkowski inequality (see Section 7)

Lemma 36.3. *Let $\mathcal{K} \subset W$ be a set of all $\boldsymbol{z} \in \mathcal{Z}$ such that* $\operatorname{vol}(P_{\boldsymbol{z}}) \ge 1$. *Then $\mathcal{K}$ is a closed convex set with a unique supporting hyperplane through every boundary point $\boldsymbol{z} \in \partial\mathcal{K}$. Moreover, $\mathcal{K}$ is* strictly convex*, i.e., no straight interval can lie in the boundary of $\mathcal{K}$.*

Proof. Since the volume $\operatorname{vol}(P_{\boldsymbol{z}})$ is continuous in z_i, we obtain that $\mathcal{K}$ is closed. Similarly, since the supporting hyperplanes to $\mathcal{K}$ are given by the partial derivatives as above and the facet areas are continuous, we obtain uniqueness of the supporting hyperplanes.

For the convexity, take $\boldsymbol{z} = (1-\lambda)\boldsymbol{z}' + \lambda\boldsymbol{z}''$, where $\boldsymbol{z}', \boldsymbol{z}'' \in \mathcal{P}$ and $\lambda \in [0,1]$. Let $P = P_{\boldsymbol{z}'} + P_{\boldsymbol{z}''}$ be the Minkowski sum of polytopes. For every $x \in P$ we have $x = x' + x''$, where $x' \in (1-\lambda)P_{\boldsymbol{z}'}$ and $x'' \in \lambda P_{\boldsymbol{z}''}$. Therefore,

$$\langle \overrightarrow{Ox}, \boldsymbol{u}_i \rangle \, = \, \langle \overrightarrow{Ox'}, \boldsymbol{u}_i \rangle + \langle \overrightarrow{Ox''}, \boldsymbol{u}_i \rangle \, \le \, (1-\lambda)z_i' + \lambda z_i'' \, = \, z_i\,,$$

which implies that $x \in P_{\boldsymbol{z}}$ and $P \subset P_{\boldsymbol{z}}$. Thus, by Proposition 7.7, we have:

$$\begin{aligned}\operatorname{vol}(P_{\boldsymbol{z}}) \, &\ge \, \operatorname{vol}\big((1-\lambda)P_{\boldsymbol{z}'} + \lambda P_{\boldsymbol{z}''}\big) \, \ge \, \big((1-\lambda)\operatorname{vol}(P_{\boldsymbol{z}'})^d + \lambda \operatorname{vol}(P_{\boldsymbol{z}''})^d\big)^{1/d} \\ &\ge \, \big((1-\lambda) + \lambda\big)^{1/d} \, = \, 1.\end{aligned}$$

Hence, $\boldsymbol{z} \in \mathcal{K}$, which proves that $\mathcal{K}$ is convex. By the second part of Proposition 7.7, the above inequality is an equality if and only if the polytopes $P_{\boldsymbol{z}'}$ and $P_{\boldsymbol{z}''}$ are similar (equal up to homothety). Now, if $[\boldsymbol{z}', \boldsymbol{z}''] \subset \partial\mathcal{K}$ is an interval on the boundary, polytopes $P_{\boldsymbol{z}}$, $\boldsymbol{z} \in [\boldsymbol{z}', \boldsymbol{z}'']$ are similar and have volume 1. Thus, all polytopes $P_{\boldsymbol{z}}$ are congruent, and since they have the same cone at vertex $\mathbf{0}$, we conclude that all $P_{\boldsymbol{z}}$ coincide, a contradiction. Therefore, $\mathcal{K}$ is strictly convex. $\square$

Proof of Theorem 36.2. We prove the existence first. Consider a hyperplane $H \subset W$ defined by $a_{d+1} z_{d+1} + \ldots + a_n z_n = 0$. Since $a_i > 0$, the intersection $H \cap W_+$ contains only the origin O. By the lemma, set $\mathcal{K}$ is closed and strictly convex. Since $\mathcal{K} \subset W_+$, there exists a supporting hyperplane $H_{\boldsymbol{z}} \| H$, for some $\boldsymbol{z} \in \mathcal{K}$ (see Figure 36.2). Let us show that $P_{\boldsymbol{z}}$ is similar to the desired polytope.

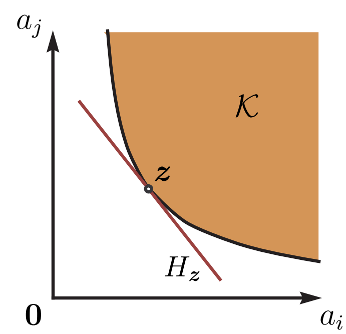

FIGURE 36.2. Hyperplane $H_{\boldsymbol{z}}$ tangent to $\mathcal{K}$ at $\boldsymbol{z}$.

Denote by $A_i = \text{area}(P_{\boldsymbol{z}})$ the facet areas of polytope $P_{\boldsymbol{z}}$, for all $i \in [n]$. Since the supporting hyperplane $H_{\boldsymbol{z}}$ is tangent to $\mathcal{K}$ at $\boldsymbol{z}$, we have

$$(\daleth) \quad \frac{A_i}{a_i} - \frac{A_j}{a_j} = \frac{1}{a_i} \cdot \frac{\partial \,\text{vol}(P_{\boldsymbol{z}})}{\partial z_i} - \frac{1}{a_j} \cdot \frac{\partial \,\text{vol}(P_{\boldsymbol{z}})}{\partial z_j} = \frac{\partial \,\text{vol}(P_{\boldsymbol{z}})}{\partial \xi_{ij}} = 0,$$

where $\xi_{ij} = a_i z_i - a_j z_j$ lies in H, and $d+1 \le i < j \le n$. We immediately have:

$$A_{d+1} = s a_{d+1}, \ \ldots, A_n = s a_n, \quad \text{for some } s > 0.$$

Since vectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_d \in \mathbb{R}^d$ are linearly independent by assumption, the vector relation in the theorem implies that $A_1, \ldots, A_d$ are uniquely determined by $A_{d+1}, \ldots, A_n$. We conclude that $A_1 = s a_1, \ldots, A_d = s a_d$, and $\frac{1}{\sqrt{s}} P_{\boldsymbol{z}}$ is the desired polytope.

For the uniqueness, reverse the above argument. For every $P \in \mathcal{P}$ as in the theorem, there exists a unique similar polytope $P_{\boldsymbol{z}} = cP$ of volume 1. The supporting hyperplane $H_{\boldsymbol{z}}$ is tangent at $\boldsymbol{z}$ and thus must satisfy $(\daleth)$. By the lemma, $H_{\boldsymbol{z}}$ can contain only one point in $\boldsymbol{z}$, which implies that $\boldsymbol{z}$ is uniquely determined. Thus, so is P, which completes the proof. $\square$

36.3. **Orthant shape variation.** Denote by $Q = \mathbb{R}^d_+$ the positive orthant. We say that a normal vector $\boldsymbol{u}$ is *positive* if it lies in the interior of Q. We say that an unbounded polyhedron $P \subset \mathbb{R}^d$ has *orthant shape* if it is an intersection of Q and and halfspaces with positive normals (see Figure 36.3).

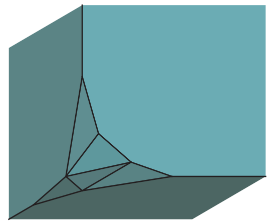

FIGURE 36.3. Orthant shape polyhedron $Q \subset \mathbb{R}^3$ with 5 boundary faces.

Theorem 36.4 (Pogorelov). *For every positive unit vectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n \in Q$, and $a_1, \ldots, a_n > 0$, there exists a unique orthant shape convex polyhedron $P \subset \mathbb{R}^d$ with n bounded facets $F_1, \ldots, F_n$, such that the facet F_i has normal $\boldsymbol{u}_i$ and area a_i, for all $i \in [n]$.*

Proof. Denote by $\mathcal{P} = \mathcal{P}(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_i)$ the set of all octant shape polyhedra P with normals $\boldsymbol{u}_i$ and face areas $\le a_i$. Let us prove that $\mathcal{P}$ is nonempty. Take halfspaces spanned by the hyperplane at unit distance from the point $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^d$ and with these normals. Denote the resulting polyhedron by P and observe that all facet areas > 0. Then, for sufficiently small $c > 0$, the polyhedron cP has facet areas $< a_i$, as desired.

Denote by H_i the planes spanned by F_i, and let $w_i \cdot \mathbf{1} = H_i \cap T$ be the intersection of H_i with the diagonal $T = \{t \cdot \mathbf{1} \,:\, t \ge 0\}$. A polyhedron $P \in \mathcal{P}$ is uniquely determined by $\boldsymbol{w} = (w_1, \ldots, w_n) \in \mathbb{R}^n$. Define $\varphi(\boldsymbol{w}) = w_1 + \ldots + w_n$ and let $\boldsymbol{w}_\circ$ be a maximum of φ. We first prove that polyhedron $P_\circ$ corresponding to $\boldsymbol{w}_\circ$ has face areas a_i, and then that such maximum is unique.

Suppose for the area of the facet F_i in $P_\circ$ we have $\text{area}(F_i) < a_i$. Increase the value of w_i in $\boldsymbol{w}_\circ$ by $\varepsilon > 0$ and let $\boldsymbol{w}_\triangleleft$ be the resulting vector. The area of F_i will increase, and the area of F_j, $j \ne i$, will decrease or remain unchanged. Thus, for $\varepsilon > 0$ sufficiently small, we obtain a polyhedron $P_\triangleleft \in \mathcal{P}$ with $\varphi(\boldsymbol{w}_\triangleleft) = \varphi(\boldsymbol{w}_\circ) + \varepsilon$, a contradiction. Therefore, for every maximum of φ, the corresponding polyhedron has facets a_i.

For the uniqueness, suppose two polyhedra $P, P' \in \mathcal{P}$ have facet areas a_i. Consider the corresponding vectors $\boldsymbol{w}, \boldsymbol{w}' \in \mathbb{R}^n$ and let $\delta = \max_i(w_i - w_i')$. We assume that $\delta \ge 0$; otherwise, relabel the polyhedra. Let $P^* = P' + \delta \cdot \mathbf{1}$ be a polyhedron with hyperplanes H_i^* at least as far away from the origin as H_i. By construction, $w_k = w_k' + \delta$ for some $k \in [n]$, which implies that $H_k^* = H_k$. Therefore, for the corresponding facets we have $F_k^* \subseteq F_k$, and $F_k^* = F_k$ only if $H_j^* = H_j$ for all facets F_j adjacent to F_k, where by adjacent facets we mean two facets which have a common $(d-2)$-dimensional face. But then $w_j = w_j' + \delta$ and we can repeat the argument for all such j. Since all facets of P are connected, we conclude that $w_i = w_i' + \delta$ for all $i \in [n]$. Applying the same argument to the unbounded facets of P, we conclude that the unbounded facets of P^* intersect T at the origin. Since by construction they are shifted by $\delta \cdot \mathbf{1}$, we immediately have $\delta = 0$ and $P = P^* = P'$, as desired. $\square$

36.4. **Alexandrov's approach.** Perhaps surprisingly, in $\mathbb{R}^3$ the Minkowski theorem can be proved in two steps: first the uniqueness by the argument as in the Cauchy theorem (Theorem 26.1), and then by the topological argument as in the previous section.

Theorem 36.5 (Alexandrov)**.** *Let $P, P' \subset \mathbb{R}^3$ be two convex polytopes with equal face normals and such that neither face F_i in P fits inside the corresponding face F_i' of P', nor vice versa. Then P and P' are congruent, i.e., equal up to a translation.*

In particular, when the areas of the corresponding faces are equal, we obtain the uniqueness part of the Minkowski theorem without using the BrunnMinkowski inequality. The existence part now can be done by a topological argument and is left to the reader (Exercise 36.3).

36.5. **Exercises.**

Exercise 36.1. $\Diamond$ [1-] Make a translation of all hyperplanes H_i and use equations $\text{area}(F_i) = \frac{\partial\, \text{vol}(P_z)}{\partial z_i}$ to obtain an alternative proof of Proposition 36.1.

Exercise 36.2. $\Diamond$ [1+] Use Alexandrov's lemma (Theorem 22.4) to prove Alexandrov's Theorem 36.5.

Exercise 36.3. $\Diamond$ [1+] Prove the Minkowski theorem in $\mathbb{R}^3$ using the Alexandrov theorem (Theorem 36.5) and the topological lemma (Theorem 41.8).

Exercise 36.4. $\Diamond$ [1+] Use Exercises 22.1 and 35.11 to obtain an alternative proof of the Minkowski theorem in $\mathbb{R}^3$, without the full power of the Alexandrov lemma (Theorem 22.4).

Exercise 36.5. $\Diamond$ *a*) [1-] In $\mathbb{R}^3$, prove the analogue of the Pogorelov theorem (Theorem 36.4) when perimeters of the faces are used in place of the areas.
b) [1-] In $\mathbb{R}^4$, prove the analogue of the Pogorelov theorem (Theorem 36.4) when the surface area of the facets, which are 3-dimensional convex polytopes, are used in place of their 3-dimensional volumes.
c) [1-] In $\mathbb{R}^4$, the same with mean curvatures.

Exercise 36.6. [$*$] Find the perimeter analogue of the Minkowski theorem in $\mathbb{R}^3$?

Exercise 36.7. $\Diamond$ [1] Extend Pogorelov's theorem to general convex cones in $\mathbb{R}^d$.

Exercise 36.8. (*Dihedral angles in simplices*) *a*) [1-] Use Proposition 36.1 to prove that every facet of a simplex in $\mathbb{R}^d$ has an acute dihedral angle with some other facet.
b) [1] Prove that every simplex in $\mathbb{R}^d$ has at least d acute dihedral angles.

Exercise 36.9. (*Equihedral tetrahedra*) $\Diamond$ [1+] Prove that every tetrahedron $\Delta \subset \mathbb{R}^3$ with equal face areas is equihedral (see Exercise 25.12).
b) [1-] Prove that for all $d \geq 4$, every simplex $\Delta \subset \mathbb{R}^d$ with equal areas of 2-dimensional faces is regular.

Exercise 36.10. (*Shephard*) A polytope $P \subset \mathbb{R}^d$ is called *decomposable* if $P = P_1 + P_2$ for some polytopes which are not similar to P. Otherwise, P is called *indecomposable*.
a) [1] Prove that except for a triangle, every convex polygon is decomposable.
b) [1-] Prove that every simplicial polytope $P \subset \mathbb{R}^d$ is indecomposable.
c) [1-] Prove that the truncated cube and the regular dodecahedron are decomposable.

d) [1+] Prove that except for a tetrahedron, every simple convex polytope in $\mathbb{R}^3$ is decomposable.
e) [1+] Find two combinatorially equivalent polytopes in $\mathbb{R}^3$, one of which is decomposable and the other is not.
f) [2-] Suppose every two edges of a polytope $P \subset \mathbb{R}^3$ are connected by a chain of adjacent triangles. Prove that P is indecomposable.
g) [2-] Prove that every convex polytope $P \subset \mathbb{R}^3$ with n faces is the Minkowski sum of at most n indecomposable polytopes.
h) [2-] Generalize *e*), *f*) and *g*) to higher dimensions.

Exercise 36.11. (*Wulff*) [1] Let P be a polytope defined by ($\daleth$) in the proof of Theorem 36.5. Then P maximizes the sum $z_1 + \ldots + z_n$ among all polytopes P' with the same volume and facet normals.

Exercise 36.12. (*Lindelöf*) Let $P, P' \subset \mathbb{R}^d$ be polytopes with the same face normals $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$ and facet areas $a_i = \operatorname{area}(F_i)$ and $a_i' = \operatorname{area}(F_i')$, where $i \in [n]$. Denote by z_i and z_i' be as in Subsection 36.2.
a) [1-] Define $\upsilon(P, P') = \frac{1}{d}(z_1 a_1' + \ldots + z_n a_n')$. Observe that $\upsilon(P, P) = d\operatorname{vol}(P)$. Check that $\upsilon(P, P')$ is translation invariant.
b) [1+] Fix P' and consider all P as above which satisfy $\upsilon(P, P') = d\operatorname{vol}(P')$. Use Lagrange multipliers to show that $\operatorname{vol}(P)$ is maximized if and only if P is an expansion of P'.
c) [1+] Conclude that $\upsilon^3(P, P') \geq \operatorname{vol}(P)\operatorname{vol}(P')^2$ and the equality holds only if P is an expansion of P'.
d) [1+] Let P' be a circumscribed polytope around a unit sphere. Observe that $\operatorname{vol}(P') = \frac{1}{d}\operatorname{area}(\partial P')$. Prove that of all polytopes in $\mathbb{R}^d$ with given volume and facet normals, the circumscribed polytope has the smallest surface area.
e) [1-] Deduce from here the isoperimetric inequality.

Exercise 36.13. Let $P, P' \subset \mathbb{R}^d$ be convex polytopes as in the previous exercise.
a) [1+] Define $f(t) = P + tP'$. Prove that $f'(0) = \upsilon(P, P')$.
b) [1+] Use the Brunn–Minkowski inequality (see Proposition 7.7) to deduce part *c*) in the previous exercise.
c) [1] Conversely, deduce the Brunn–Minkowski inequality from part *c*) in the previous exercise.

Exercise 36.14. (*Minkowski's symmetry criterion*) $\diamondsuit$ [1] Let $P \subset \mathbb{R}^d$ be a convex polytope such that for every facet F of P there exist a facet with the same area and opposite normal. Prove that P is centrally symmetric.

Exercise 36.15. (*Zonotopes in* $\mathbb{R}^d$) $\diamondsuit$ *a*) [1] Suppose convex polytope $P \subset \mathbb{R}^d$ and all its faces are centrally symmetric. Prove that P is also centrally symmetric.
b) [1+] Generalize Exercise 7.16 to prove that P is the Minkowski sum of intervals.

Exercise 36.16. Let $P \subset \mathbb{R}^d$ be a convex polytope such that translations of P tile the space.
a) [1] Prove that the opposite facets of P have equal areas.
b) [1] Use the Minkowski theorem to prove that P is centrally symmetric.
c) [1+] Extend this to show that all faces of P are centrally symmetric. Use the previous exercise to conclude that P is a zonotope.
d) [1] In $\mathbb{R}^3$, show that P has at most 14 faces and this bound is tight.

e) [1+] Classify completely the combinatorics of all such P in $\mathbb{R}^3$.
f) [1+] In $\mathbb{R}^d$, show that P has at most $2^{d+1} - 2$ facets.

Exercise 36.17. [$*$] Suppose a convex polytope $P \subset \mathbb{R}^3$ can tile the whole space (not necessarily periodically or even face-to-face). Prove that P has at most 10^6 vertices.

Exercise 36.18. *a*) [2-] Let $P, P' \subset \mathbb{R}^3$ be convex polytopes such that for every edge e of P there exists a parallel edge e' of P' such that $|e| \le |e'|$. Suppose further, that for every face F of P there exists a parallel face F' of P' such that a translation of F is contained in F'. Prove that there exists a translation of P which is contained inside P'.
b) [1] Check that neither of the two conditions suffice by themselves.

Exercise 36.19. [2+] Find two convex polytopes $P, Q \subset \mathbb{R}^3$ with parallel and non-equal faces which satisfy the following property: for every face F of P and the corresponding parallel face G of Q, there exists a unique translation which either fits F inside of G or fits G inside of F.[82]

36.6. **Final remarks.** Our proof of the Minkowski theorem is a reworking of Minkowski's original proof [Min], parts of which were later clarified in [McM1]. The main idea of this proof is implicitly based on Wulff's theorem in crystallography (see [Grub, §8.4]). Let us mention that most standard proofs the Minkowski theorem are based on the Brunn–Minkowski inequality (see [BonF, Schn2]), to the extent that they can be shown essentially equivalent [Kla].[83] Minkowski extended Theorem 36.2 to general surfaces (see [Pog4, Pog3] for further results and references). Let us mention also the algorithmic approach [GriH] and the important work on stability of solutions in the Minkowski theorem (see [Gro] and references therein).

The orthant shape variation given in Subsection 36.3 is due to Alexandrov, who proved it in [A2, §6.4] in the generality of all cones in $\mathbb{R}^3$ (cf. Exercise 36.7). It was extended to higher dimensions and other functionals (see Exercise 36.5) by Pogorelov [Pog3, §7.6] (see also [A2, §6.5] and [BárV]). The proof in Subsection 36.3 follows the original Pogorelov's proof.

For the complete proof of the Alexandrov theorem (Theorem 36.5) see [A2, §6.2] (see also [Lyu, §30] and [Mi3]). Let us mention that a series of Exercises 22.1, 35.11 and 36.4 outline a similar approach, which substitutes the tedious Alexandrov's lemma (Theorem 22.4) with an elegant argument in the Alexandrov's local lemma (Exercise 22.1).

[82]The corresponding faces must lie on the same side of the polytopes; otherwise any tetrahedron P and $Q = -2P$ will work.

[83]To quote [Kla], they have "equiprimordial relationship."

37. The Alexandrov existence theorem

In this section we prove the Alexandrov existence theorem, giving a complete characterization of convex polyhedral surfaces in $\mathbb{R}^3$. This is a celebrated result with has a number of important applications, including Theorem 30.1 (see later in this section) and Leibin's theorem (Theorem 38.2). We present a topological proof using the mapping lemma (Theorem 41.8) and the Alexandrov uniqueness theorem (Theorem 27.7).

37.1. **The statement.** Let S be an abstract 2-dimensional polyhedral surface with n vertices. We say that S is *intrinsically convex* if the curvatures $\omega_1, \ldots, \omega_n$ of all vertices satisfy $0 < \omega_i < 2\pi$, for all $i \in [n]$. Of course, the surface of every convex polytope in $\mathbb{R}^3$ is intrinsically convex. The following result shows that there are essentially no other examples.

Theorem 37.1 (Alexandrov existence theorem). *Every intrinsically convex 2-dimensional polyhedral surface homeomorphic to a sphere is isometric to the surface ∂P of a convex polytope $P \subset \mathbb{R}^3$, or to a doubly covered polygon.*

By the Alexandrov uniqueness theorem (Theorem 27.7), such a polytope is uniquely determined, up to a rigid motion. In other words, one can view Theorem 37.1 as the converse of Theorem 27.7.

To fully appreciate the Alexandrov existence theorem, let us note that the edges of the polytope are determined by the surface S, but there does not seem to be an efficient algorithm to construct it directly (see Exercise 37.11). If the edges are given and form a triangulation, one can obtain explicitly all realizations of S with these edges (see Remark 31.5). The Cauchy theorem says that for each triangulation there is at most one convex realization (up to rigid motions). In this language, the existence theorem implies that such convex realization exists for one or more triangulations of S.

Example 37.2. To see how the triangulation of the surface can change under a small deformation, consider the following two cases. Start with the surface of the cube and push the opposite vertices towards each other by a small ε. Take the convex hull. Alternatively, pull these vertices away from each other by a small ε. Take the convex hull. Now observe that although these are combinatorially different triangulations, intrinsically the surfaces are ε-close to each other.

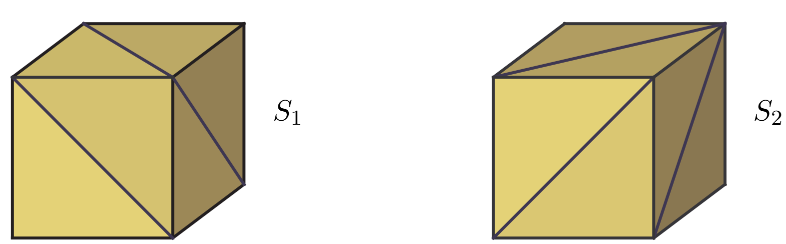

Figure 37.1. Two triangulated ε-close surfaces S_2 and S_2.

Example 37.3. (*Surfaces from unfoldings*) Following Alexandrov, let us restate the theorem in terms of unfoldings. Let $X \subset \mathbb{R}^2$ be a simple polygon in the plane with an even number of edges, and let $U \subset \mathbb{R}^2$ be the non-convex region bounded by X, i.e., $X = \partial U$. Orient the edges of X clockwise. Let $\mu : X \to X$ be an involution on edges of X describing the gluing of U. One can show that the surface U/μ is well defined and homeomorphic to a sphere if and only if the following conditions are satisfied:

(i) the attached edges must have equal lengths and the opposite orientation,

(ii) the involution on edges in this case can be drawn inside U (see Figure 37.2).

Note that the involution μ maps several different vertices of X into the same vertex v of the surface $S = U/\mu$. The 2-dimensional polyhedral surface S is intrinsically convex in the sum of angles around all preimages of v is $< 2\pi$. When all these conditions are satisfied, the Alexandrov existence theorem says that U is the unfolding of a convex polytope. Finally, let us mention that U is a general unfolding, not necessarily an edge unfolding (see Section 40).

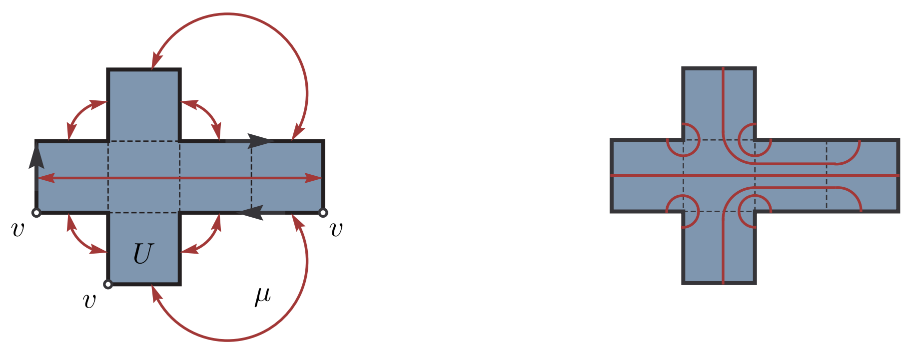

FIGURE 37.2. The standard unfolding U of a cube and the gluing map μ.

37.2. **Topological proof.** Denote by $\mathcal{P}_n$ the space of all convex polytopes in $\mathbb{R}^3$ with n (labeled) vertices. Here the distance between polytopes $P_1, P_2 \in \mathcal{P}_n$ can be defined in a number of different ways, e.g., as the maximal distance between the corresponding vertices. We include into $\mathcal{P}_n$ the doubly covered n-gons, which can be viewed as surfaces of flat polyhedra.

Lemma 37.4. *Space $\mathcal{P}_n$ is an open connected $3n$-dimensional manifold.*

It is tempting to deform a polytope so as to "simplify" it and then connect to a "special" polytope $P \in \mathcal{P}_n$. The main obstacle here is that the number of vertices of polytopes must remain the same at all times. Here is how we get around this problem.

Proof. For connectivity, fix a polytope $P \in \mathcal{P}_n$ and translate it so that the origin O lies in its relative interior. Consider rays R_i from $\mathbf{0}$ to all vertices v_i of P, where $i \in [n]$. Denote by P' the polytope which lies on rays R_i and has unit ray coordinates (see Section 35). By Lemma 35.3, polytope P is connected to P'. Observe that P' is inscribed into a unit sphere $\mathbb{S}^2$. Similarly, every doubly covered n-gon is connected to a doubly covered n-gon inscribed into $\mathbb{S}^2$. Therefore, it suffices to show that all polytopes in $\mathcal{P}_n$ inscribed into $\mathbb{S}^2$ are connected.

Let $v_i' \in \mathbb{S}^2$ denote the vertices of P'. By definition, $P' = \operatorname{conv}\{v_1', \ldots, v_n'\}$. In the other direction, for every $u_1, \ldots, u_n \in \mathbb{S}^2$ the polytope $Q = \operatorname{conv}\{u_1, \ldots, u_n\}$ lies in $\mathcal{P}_n$. Therefore, the space of polytopes $Q \in \mathcal{P}_n$ inscribed into $\mathbb{S}^2$ is homeomorphic to the space of distinct n-tuples of points in $\mathbb{S}^2$, and thus connected.

Now, take $P \in \mathcal{P}_n$ and perturb the vertices. When the perturbations are sufficiently small they give a convex polytope. The space of perturbations is $3n$-dimensional, which implies that $\mathcal{P}_n$ is an open $3n$-dimensional manifold. □

Denote by $\mathcal{M}_n$ the space of all 2-dimensional polyhedral surfaces homeomorphic to a sphere with n (labeled) vertices, defined as points with positive curvature. Again, the topology on $\mathcal{M}_n$ can be defined via the pairwise geodesic distance between the corresponding vertices.

Lemma 37.5. *Space $\mathcal{M}_n$ is an open connected $(3n-6)$-dimensional manifold.*

Proof. Take $S \in \mathcal{M}_n$ with vertices $v_1, \ldots, v_n$. Fix a shortest path γ_{ij} between v_i and v_j, for all $1 \le i < j \le n$. If there is more than one such path, choose either one. Since S is intrinsically convex, by Proposition 10.1 paths γ_{ij} do not contain vertices in its relative interior. The geodesic triangle $\Delta = \gamma_{12} \cup \gamma_{13} \cup \gamma_{23}$ subdivides S into two regions. Vertex v_4 lies in one of these regions and connected by γ_{14}, γ_{24} and γ_{34} to the vertices in its boundary. Vertex v_5 lies in in one of the resulting four regions, etc. Denote by T the resulting geodesic triangulation of S. Observe that T has $3n-6$ edges (see Corollary 25.2 or use induction in the construction).

By construction, every triangle in T is flat, i.e., has no vertices in its interior. Perturb the edge lengths of T. Since every vertex v_i has, by definition, strictly positive curvature, when the perturbations are sufficiently small they determine a surface $S' \in \mathcal{M}_n$. Thus, $\mathcal{M}_n$ is open and the space of such perturbations is $(3n-6)$-dimensional.

To prove that $\mathcal{M}_n$ is connected, let us show that triangulation T can be realized in $\mathbb{R}^3$ with the length function $\ell_{ij} = |\gamma_{ij}|$ (see Subsection 31.1). In other words, we claim that there exists a (possibly, self-intersecting) polyhedral surface $S_\circ \subset \mathbb{R}^3$ isometric to S, and such that $S_\circ$ is triangulated into flat triangles according to T, with the same edge lengths ℓ_{ij}.

Use the construction of T. Suppose vertex v_n is connected to a geodesic triangle $[v_i v_j v_k]$. The geodesic triangles $A_1 = [v_n v_i v_j]$, $A_2 = [v_n v_i v_k]$ and $A_3 = [v_n v_j v_k]$ are flat and their angles at v_n are together $< 2\pi$ and satisfy the triangle inequality. Thus, there exists a tetrahedron in $\mathbb{R}^3$ with faces A_1, A_2 and A_3. Change S into a surface S_1 by substituting A_1, A_2 and A_3 with a flat triangle $[v_i v_j v_k]$. Since the vertex curvatures do not increase, we conclude that $S_1 \in \mathcal{M}_{n-1}$.[84] Repeat the construction. At the end, we obtain a doubly covered triangle $S_{n-3} \in \mathcal{M}_3$, which can be realized in $\mathbb{R}^3$. Going backwards, we obtain the desired realization $S_\circ$.

[84]Although γ_{ij} in the surface S_1 may no longer be the shortest paths, they are still geodesics and define a geodesic triangulation of S_1.

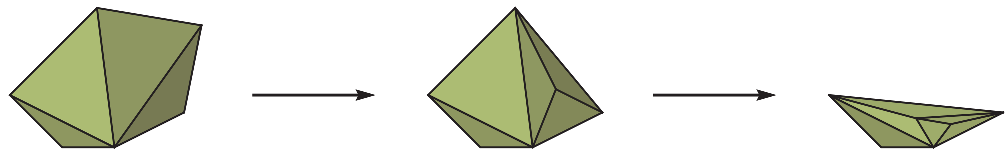

Figure 37.3. Deformation of a surface into a convex surface.

Now, let us deform the surface $S_\circ$ to a convex surface. As in the inductive construction above, let $z \in S_1$ be the center of mass of the triangle $(v_i v_i v_k)$. Move v_n along $[v_n z]$ towards z, until the resulting vertex v_n' is sufficiently close to z. Repeat the same procedure for v_{n-1}, etc. Here we change the surface triangles linearly, as the vertices v_i are moved (see Figure 37.3). At the end we obtain a convex surface sufficiently close to a doubly covered triangle $(v_1 v_2 v_3)$. By Lemma 37.4, all convex surfaces in $\mathcal{P}_n$ are connected, which implies that $\mathcal{M}_n$ is also connected. $\square$

The proof of the Alexandrov existence theorem now easily follows from the lemmas above and the mapping lemma (Theorem 41.8).

Proof of Theorem 37.1. Denote by $\mathcal{P}_n'$ the space of convex polytopes $P \in \mathcal{P}_n$ modulo the rigid motions. Let $\varphi : \mathcal{P}_n \to \mathcal{M}_n$ be the natural map giving the surface of the polytope: $\varphi(P) = \partial P$. Map φ is continuous by definition and is injective by the Alexandrov uniqueness theorem (Theorem 27.7). The group of rigid motions is 6-dimensional, so by Lemmas 37.4 and 37.5 manifolds $\mathcal{P}_n'$ and $\mathcal{M}_n$ have the same dimensions.[85] Similarly, the lemmas imply that both manifolds are connected. It remains to prove that φ is proper. By Theorem 41.8, this would imply that φ is a homeomorphism, and prove the theorem.

Consider a sequence of polytopes $P_1, P_2, \ldots \in \mathcal{P}_n$ such that $S_i = \varphi(P_i) \to S \in \mathcal{M}_n$ as $i \to \infty$. We can always assume that P_i contain the origin O. Denote by D the geodesic diameter of the limit surface S. Then, for sufficiently large N, polytopes P_i lie in the ball of radius D centered at O. Thus, a subsequence of $\{P_i\}$ converges to a convex polytope P. To prove that $P \in \mathcal{P}_n$, simply observe that the curvature ω of a vertex v_k in P is the limit of the corresponding curvatures of ω_i in P_i. Since ω_i are the curvatures in S and S_i, for all $i \geq 1$, by continuity we conclude that ω is the curvature of v_k in S. By the assumption, $S \in \mathcal{M}_n$, which implies that $\omega > 0$ and that P is strictly convex at v_k, for all $k \in [n]$. We conclude that $P \in \mathcal{P}_n$. This implies that φ is proper and finishes the proof of the theorem. $\square$

37.3. **Back to flexible polyhedral surfaces.** We can now prove a holdover result on flexible polyhedral surfaces.

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[85] Formally speaking, one needs a separate argument to show that $\mathcal{P}_n'$ is a $(3n-6)$-dimensional manifold. For that, consider planted polytopes as in Subsection 33.1 and follow the proof of Lemma 37.4.

Proof of Theorem 30.1. Let $S_0 = \partial P$ be a polyhedral surface of the convex polytope P. Denote by S_t, $t \in [0,1]$, the polyhedral surface obtained from S_0 by decreasing the length of edge e by εt. Clearly, the surfaces S_t are (intrinsically) isometric to S_0 everywhere except two triangles containing the edge e. For $\varepsilon > 0$ small enough, the surfaces S_t are *locally convex*. By the Alexandrov existence theorem (Theorem 37.1), there exist convex polytopes P_t realizing the surfaces: $S_t = \partial P_t$. Moreover, for small enough ε the combinatorial structure of P_t is the same as of P. Thus, removing two triangular faces containing e in the polyhedral surfaces S_t gives the desired flexing of the surface S as in the theorem. It remains to show that these surfaces are not globally isometric. This follows from the difference in the edge length $|e|$. □

37.4. **Exercises.**

Exercise 37.1. ♢ [1-] Suppose S is an abstract 2-dimensional polyhedral surface obtained by gluing four triangles as in a tetrahedron. Suppose also that the sum of angles at each vertex of S is $< 2\pi$. Prove that there exists a tetrahedron $\Delta \subset \mathbb{R}^3$ with the surface isometric to S.

Exercise 37.2. ♢ *a*) [1] Find the analogue of Theorem 37.1 for convex caps (cf. Subsection 25.6 and Theorem 27.5). Deduce this version from the Alexandrov existence theorem.
b) [1+] Give a topological proof of this result.

Exercise 37.3. [1-] For the unfoldings in Figures 19.4 and 19.7 draw the involutions inside the polygons.

Exercise 37.4. *a*) [1-] Find a polygon in the plane which is an unfolding of two different tetrahedra (corresponding to different gluing maps μ)
b) [1] Show that the square is an unfolding of infinitely many different convex polytopes.
c) [1+] Describe all polyhedra which have a square unfolding.

Exercise 37.5. (*Harer-Zagier formula*) Let R_n a regular $2n$-gon and denote by Π_n the set of fixed point free involutions $\mu : [2n] \to [2n]$ on edges of R_n. Clearly, $|\Pi_n| = (2n-1)!! = 1 \cdot 3 \cdot \ldots \cdot (2n-1)$. Draw μ by straight lines inside R_n, and take the orientable surface R_n/μ obtained by gluing the corresponding edges. Denote by $a_k(n)$ the number of $\mu \in \Pi_n$ such that $\operatorname{genus}(R_n/\mu) = k$.
a) [1] Prove that R_n/μ is homeomorphic to a sphere if the drawing of μ has no crossings. Conclude that $a_0(n)$ is a Catalan number.
b) [1] Give a combinatorial description of the $\operatorname{genus}(R_n/\mu)$.
c) [2] Prove that for every $N \geq 0$, we have:

$$\sum_{k=0}^{\lfloor n/2 \rfloor} a_k(n) N^{n+1-2k} \; = \; (2n-1)!! \cdot \sum_{m=1}^{N} 2^{m-1} \binom{n}{m-1} \binom{N}{m}.$$

Exercise 37.6. ♢ [1] Prove that conditions (*i*) and (*ii*) in the Example 37.3 are necessary and sufficient. Compare with part *a*) of the previous exercise.

Exercise 37.7. ♢ [1+] Let S be a 2-dimensional polyhedral surface homeomorphic to a sphere (not necessarily intrinsically convex). Use the construction in the proof of Lemma 37.5 to obtain a piecewise linear immersion of S into $\mathbb{R}^3$.

Exercise 37.8. Start with unfolding the regular icosahedron given in Figure 19.4. Perturb the edges by at most ε so that the corresponding edges under μ remain of the same length.
b) [1-] Use the Alexandrov existence theorem to prove that for sufficiently small ε this is an unfolding of a convex polytope. Find an explicit bound on ε.
b) [1] Use the algebraic approach in Section 31 to show that for sufficiently small ε this is an edge unfolding of a convex polytope combinatorially equivalent to the icosahedron.
c) [1] Use the proof of the Alexandrov existence theorem to obtain part *b*). Explain the connection to the proof of *b*).

Exercise 37.9. (*Twisting the surfaces*) Let $S = \partial P$ be the surface of a convex polytope $P \subset \mathbb{R}^3$, and let $\gamma \subset S$ be a closed geodesic. Denote by S_1 and S_2 the surfaces on both sides of γ. A surface S' obtained by gluing S_1 and S_2 along the boundary is called a *twist* of S.
a) [1-] Use the Alexandrov existence theorem to check that all surface twists are surfaces of convex polytopes.
b) [1+] Start with a regular tetrahedron. Check that all polytopes obtained by a sequence of twists must be equihedral tetrahedra (see Exercise 25.12). Prove or disprove: all equihedral tetrahedra with equal surface area can be obtained this way.
c) [$*$] Start with a square pyramid where all vertices have equal curvatures. Describe all polytopes resulted by a sequence of twists.
d) [$*$] Same question for a regular octahedron.

Exercise 37.10. (*Volkov's stability theorem*) $\Diamond$ [2+] Let S, S' two intrinsically convex 2-dimensional polyhedral surfaces homeomorphic to a sphere with geodesic diameter at most 1. Suppose $\phi : S \to S'$ is a homeomorphism such that

$$|x,y|_S - \varepsilon \, < \, |\phi(x),\phi(y)|_{S'} \, < \, |x,y|_S + \varepsilon, \quad \text{for all } \, x,y \in S.$$

Denote by $P, P' \subset \mathbb{R}^3$ the corresponding convex polytopes. Prove that there exists a rigid motion such that P and P' are "close to each other":

$$|z,\phi(z)| \, < \, C \cdot \varepsilon^{1/24}, \quad \text{for all } z \in \partial P \;\; (\text{so that } \phi(z) \in \partial P'),$$

and where C is a universal constant.

Exercise 37.11. [$*$] Prove or disprove: computing edges in the polytope corresponding to a given intrinsically convex 2-dimensional polyhedral surface is NP-hard.

37.5. **Final remarks.** The Alexandrov existence theorem and its unbounded variations (cf. Subsection 27.3) was proved by Alexandrov in a series of papers. He presented two interrelated proofs in [A1, A2], both based on elementary but delicate work with unfoldings. The proof in this section is largely new, although the use of the mapping lemma is motivated by the proof in [A2].

A variational proof for the case of polyhedral caps (see Exercise 37.2) was given by Volkov in [Vol2], and was extended to general polytopes and general unbounded polyhedra in [VP1]. This proof was motivated by Pogorelov's proof in Subsection 35.5. Most recently, a conceptual variational proof was found in [BobI], which also proved amenable to a computer implementation. This proof builds on top of Volkov's proof and uses several technical innovations, including an extension of the *Alexandrov–Fenchel inequality*, which is an advanced extension of the Brunn–Minkowski inequality and other geometric inequalities. An accessible exposition of the convex cap variation is given in [Izm].

Let us mention here an important Volkov's stability theorem (Exercise 37.10), which he proved for both bounded convex surfaces and for the convex caps [Vol3]. In a different direction, the Burago–Zalgaller theorem is a non-convex analogue of the Alexandrov existence theorem (see Exercise 39.13 and compare with Exercise 37.7).

38. Bendable surfaces

In this section we discuss *bendable surfaces* whose properties turn out to be very different from those of *flexible surfaces* in Section 30. As in the *Alexandrov existence theorem*, we place the emphasis on intrinsic geometry of surfaces rather than on geometry of faces.

38.1. **Bending surfaces is easy.** We say that a 2-dimensional polyhedral surface $S \subset \mathbb{R}^3$ is *bendable* if there exists a continuous family $\{B_t : t \in [0,1]\}$ of embeddings $B_t \subset \mathbb{R}^3$, which are intrinsically isometric to S, but every B_t and $B_{t'}$ are not globally isometric. The above family of surfaces $\{B_t\}$ is called a *bending* of S. Of course, by Pogorelov's uniqueness theorem (Theorem 27.8), at most one surface in the bending of S can be convex. On the other hand, *every* convex polyhedral surface is bendable (see Figure 38.1).

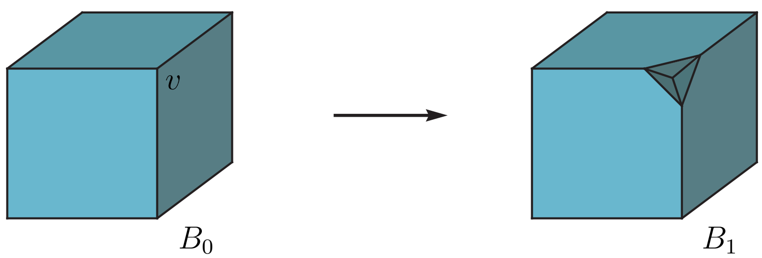

FIGURE 38.1. Surface $S = B_0$ of the cube is bendable.

Proposition 38.1. *Let $S = \partial P$ be a polyhedral surface of a convex polytope $P \subset \mathbb{R}^3$. Then S is bendable.*

Proof. Let v be a vertex of P and let H be a plane supporting v and in general position, i.e., a plane such that $H \cap P = \{v\}$. Denote by $\{H_t\}$ the family of planes parallel to H, and shifted by t in the direction of P. Let $Q_t = P \cap H_t$ be the polygon in the intersection. Finally, let B_t be a non-convex surface obtained by gluing along Q_t two parts of S divided by H_t (as in Figure 38.1). The details are straightforward. $\square$

38.2. **Convexly bendable polyhedral surfaces.** Let us restrict out attention to part of polyhedral surfaces of convex polytopes. Formally, we say that (non-closed) polyhedral surface $S \subset \mathbb{R}^3$ is *globally convex* if there exists a convex polytope P such that $S \subset \partial P$. We say that S is *convexly bendable* if there exists a bending $\{B_t : t \in [0,1]\}$ where all surfaces B_t are polyhedral[86] and globally convex. For example, the union of three sides of a tetrahedron is a convexly bendable surface (see Figure 38.2).

The following result shows that in fact this example generalizes to *every* polytope.

[86]Non-closed surfaces, of course, can have non-polyhedral bending, e.g., the surface of a cube without two opposite faces can be easily deformed into a cylinder. We restrict ourselves to polyhedral surfaces for simplicity.

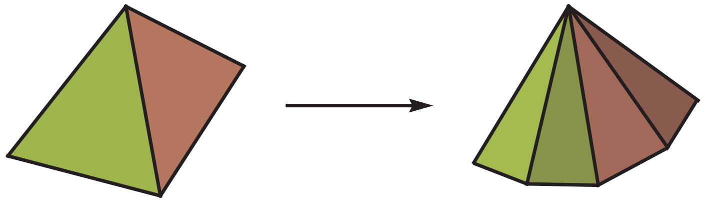

FIGURE 38.2. Three sides of a tetrahedron form a convexly bendable surface.

Theorem 38.2 (Leibin)**.** *Let F be a face in a convex polytope $P \subset \mathbb{R}^3$. Then the polyhedral surface $S = \partial P \smallsetminus F$ is convexly bendable.*

The moral is that bendability is a general property, and without additional restrictions, such as smoothness of closed surfaces, one cannot hope to have any kind of non-bendability rigidity. Once again, the proof is an easy application of the Alexandrov existence theorem.

Proof. The strategy is similar to that in the proof of Theorem 30.1. Fix an interior point $w \in F$ and consider a pyramid Z_t over F with a vertex z which projects onto w and has height εt, for every $t \in [0, 1]$. Denote by P_t a polytope obtained from $P = P_0$ by attaching a pyramid Z_t. Clearly, for sufficiently small $\varepsilon > 0$, polytopes P_t are convex. Furthermore, the convex polyhedral surface S is a polygonal region on the closed surface S_t of P_t.

Denote by A the boundary polygon of S, i.e., $A = \partial F = \partial S$. Fix a flat point $x \in A$ on an edge $e = (p, q)$ of A. For sufficiently small ε, all shortest paths on S_t between x and points $x' \in F$ lie on the surface ∂Z_t. Since the shortest path cannot go through the vertex z of Z_t (see Proposition 10.1), points $x' \neq x$ will split into two intervals depending on which side of z they have a shortest path to x. By continuity, there exists a point $y \in A$ with exactly two shortest paths γ and γ' between x and y.[87] Denote by $R_t \subset S_t$ the closed polygonal region between the paths. Consider a closed surface U_t obtained from S_t by removing R_t and gluing along the corresponding points on paths γ and γ'. It is easy to check that for small enough ε the surfaces U_t are all *locally convex*. By the Alexandrov existence theorem (Theorem 37.1), they can be realized by convex polytopes Q_t. Removing the remaining parts of the pyramid boundary ∂Z_t from ∂Q_t, we obtain a continuous family of surfaces $\{B_t, t \in [0, 1]\}$ of the original surface $S = B_0$.

We claim that the $\{B_t\}$ is the desired convex bending of S. It remains to prove that polyhedral surfaces B_t are not globally isometric. First, observe that point x is no longer flat and is a vertex of Q_t. Moreover, since the curvature of vertex x in Q_t is increasing with t, the surface angle $\alpha = \angle pxq$ is decreasing (on $U_t \smallsetminus B_t$). Since the (usual space) angle $\beta = \angle pxq$ in $\mathbb{R}^3$ between (new) edges (p, x) and (x, q)

[87]This is a basic step in the *cut locus* construction (see Section 40).

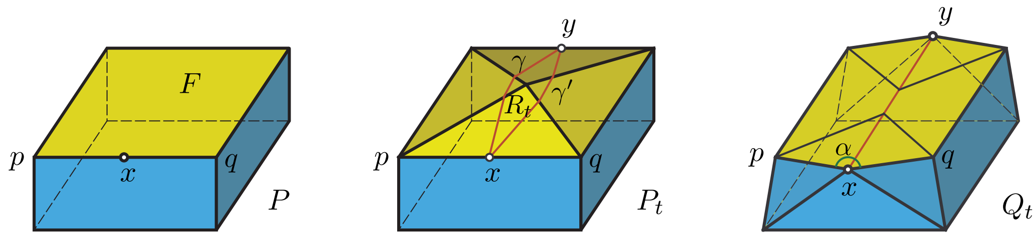

FIGURE 38.3. A convex bending $\{B_t \subset \partial Q_t\}$ of the polyhedral surface $S = \partial P - F$ with boundary $A = \partial F$.

is at most α, it is also decreasing for sufficiently small $\varepsilon > 0$.[88] Since the edges (p, x) and (x, q) lie on the boundary ∂B_t, we obtain the claim. $\square$

38.3. **Packaging liquid can be a hard science.** In the good old days milk (and other liquid) used to be sold in small containers in the shape of a regular tetrahedron (see Figure 38.4). Such containers are easy to manufacture (they have a nice unfolding which are easy to cut out of cardboard) and easy to store.[89] Unfortunately they were terrible in use, since to open them one had to use scissors, and no matter how small the hole was, the container would bend and some milk would inevitably spill. This can all be explained, of course, by Theorem 38.2 which states that all such surfaces with cuts are bendable.

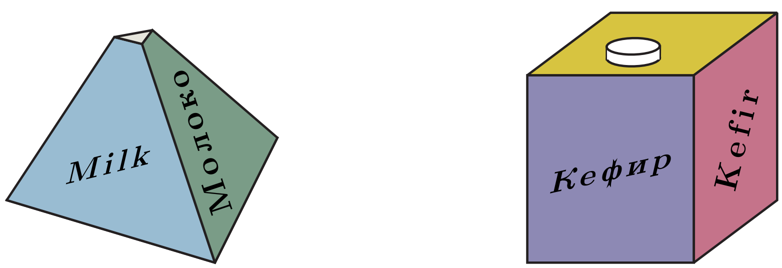

FIGURE 38.4. Old and new milk cartons.

Now, the present day milk cartons look like bricks with a big hole is in the middle of a face (see Figure 38.4). When closed, the hole is covered with a plastic cap, but even when the cap is removed, such cartons are quite rigid. Naturally, this begs a theoretical explanation. The following result does exactly that.

Theorem 38.3 (Alexandrov). *Let $A \subset \partial P$ be a polygon on a surface of a convex polytope $P \subset \mathbb{R}^3$, and suppose no vertices of P lie in the polygon A or on its boundary. Then the polyhedral surface $S = \partial P \smallsetminus A$ is* not *convexly bendable.*

[88] A priori the surface angle α may be greater than the space angle β if the triangle (pxq) is not flat on the surface U_t.

[89] There are even quite good tetrahedral packings of $\mathbb{R}^3$: `http://tinyurl.com/23y8v5e`

For example, let F is the face of a convex polytope P (milk carton), and let $A \subset F$ be a polygon in the relative interior of the F (the hole). The theorem says that $S = \partial P \smallsetminus A$ is not convexly bendable (i.e., milk is harder to spill).

Proof. Consider a convex polytope P' and a polyhedral surface $S' \subset \partial P'$ intrinsically isometric to S. By the Gauss–Bonnet theorem (Theorem 25.3, the total curvature $\omega(S) = \omega(S') = 2\pi$. Thus, all vertices of P' lie in the interior of S', and points in $A' := \partial P' \smallsetminus S'$ are all flat. Since polygons $A \subset S$ and $A' \subset S'$ may not lie in one face, denote by B and B' the unfoldings of A and A' on a plane, respectively.

Observe that by the intrinsic isometry of surface S and S', polygons B and B' in $\mathbb{R}^2$ have equal side lengths and equal outside angles. Therefore, B and B' are equal, which implies that A and A' are isometric. We conclude that surfaces $\partial P = S \cup A$ and $\partial P' = S' \cup A'$ are both closed convex polyhedral surfaces which are intrinsically isometric. By the Alexandrov uniqueness theorem (Theorem 27.7) they are globally isometric, which completes the proof. $\square$

38.4. **Exercises.**

Exercise 38.1. *a*) [1] Give a direct construction proving Leibin's theorem (Theorem 38.2) for the cube and the regular octahedron.
b) [1+] Same for the regular icosahedron and dodecahedron.

Exercise 38.2. *a*) [1+] Let $S_1 \subset \partial P_1$, $S_2 \subset \partial P_2$ be isometric polyhedral surfaces homeomorphic to a disk. Assume that all (intrinsic) angles at the boundaries $\partial S_1 \simeq \partial S_2$ are $> \pi$. Prove that there exists a convex bending from S_1 to S_2.
b) [2-] Same result under assumption that all boundary angles are $\leq \pi$.
b) [$*$] Same result without restriction on the angles.

38.5. **Final remarks.** The study of flexible and bendable surfaces was initiated by Liebmann who proved rigidity (in fact, uniqueness) of a sphere as an analytic closed surface of constant curvature (see an especially pretty proof by Hilbert in [Bla1, §91]). This study of rigidity of analytic surfaces was continued in the works of Hilbert, Blaschke, Cohn-Vossen, Weyl, Alexandrov, Pogorelov, and others. We refer to [Pog3, Sab1, Sen2] for references and historical overview.

Theorem 38.2 is proved in [Leib], and holds in much greater generality, for all convex surfaces with removed regions of strictly positive curvature (see also [Pog3, §3.6]). The terminology we use here differs from the standard, complicated by the ambiguity of Russian phraseology. The essence of the proof remains the same, however. Theorem 38.3 is due to Alexandrov, and has been later extended to unbounded and other surfaces with boundary, see [Pog3, Sho2] (also, compare with [Sho1]).

We have seen two examples when a closed convex polyhedral surface has several realizations (say, embeddings) in $\mathbb{R}^3$ (see Figure 30.1, 38.1). In each case the volume of non-convex embeddings is smaller than that of convex. One can ask if this is a general rule, i.e., whether the unique convex embedding (by Theorem 27.6) has a greater volume than all other embeddings. Surprising or not, the answer to this question is negative. In this section we exhibit volume-increasing bendings of the surface of a doubly covered triangle, regular tetrahedron, and a unit cube. We then discuss various general results in this direction.

39.1. **Burago–Zalgaller's bending.** The following construction of a bending $\{B_t\}$ has an extremal point doubly covered triangle of volume 0. Start with a rhombus divided into two equilateral triangles (ABD) and (BCD). Take two points E, F on the diagonal $[AC]$ so that $[BFDE]$ is a rhombus and $|EF| = t\,|AC|$. Now fold the rhombus $[ABCD]$ along the diagonal (BC), and glue (AB) to (BC), (AD) to (DC). Push the edge (EF) inside so that we obtain a polyhedron B_t with faces (ABE), (ADE), (CBF), (CDF), (BEF) and (DEF), as shown in see Figure 39.1. Note that is is concave along the edge (EF), and convex along other edges.

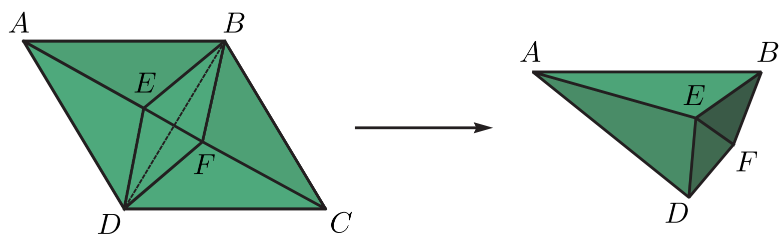

Figure 39.1. Burago–Zalgaller's bending construction.

This shows a non-convex polyhedron can have a greater volume than a convex polyhedron with an isometric surface (see also Exercise 39.5). If the reader is unhappy about the doubly covered triangle as polytope, there is a simple way to modify this construction to make it the "usual" polytope. Below we present somewhat different (and more complex) examples of the same phenomenon.

39.2. **Bleecker's bending.** Let Δ be a regular tetrahedron with vertices v_1, v_2, v_3, v_4, side lengths $|v_iv_j| = 1$, and center O. Consider a pyramid $T = (Ov_1v_2v_3)$, which can be viewed as "one quarter" of Δ. Subdivide each side of Δ symmetrically, as in Figure 39.2. Denote by $S = \partial\Delta$.

Consider a non-convex polyhedron P_t with surface $B_t = \partial P_t$ (intrinsically) isometric to S, as in the figure. Bend the edges perpendicular to the middle of the edges so that the equilateral triangles in the middle of the faces stick out. In the figure we show only one quarter Q of the resulting polytope P_t, corresponding to T. More precisely, let (w_1, w_2, w_3, O) be a rectangular pyramid homothetic to T, and fix edges (a_i, b_i)

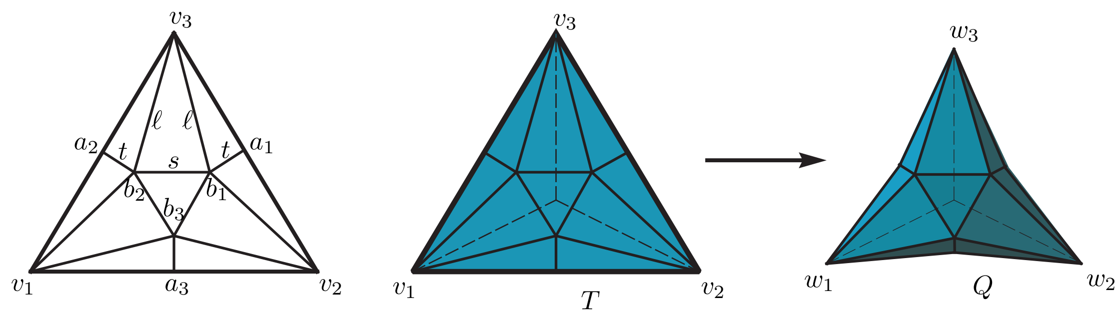

FIGURE 39.2. One quarter T of a regular tetrahedron Δ and an one quarter Q of isometric non-convex polyhedron P of bigger volume.

perpendicular to edges $(a_i w_j)$, for all $j \neq i$. Denote by $s = |b_i b_j|$ and $t = |a_i b_i|$. Note that $|a_1 a_2| = s + \sqrt{3}t$, so s is determined by t.

By construction, all surfaces B_t are intrinsically isometric to the surface $B_0 = S$ of the regular tetrahedron. Thus we obtain a bending $\{B_t\}$ of S, where parameter t can take any value between 0 and $\theta = \frac{1}{2\sqrt{3}}$.

Proposition 39.1. *We have* $\mathrm{vol}(P_t) > 1.37 \cdot \mathrm{vol}(\Delta)$*, for some* $t \in [0, \theta]$.

In fact, one can show that $\mathrm{vol}(P_t)$ is increasing for all $t > 0$ such that $t \leq \frac{s}{2}$ (see Exercise 39.2). Thus, one can think of the resulting surfaces $\{B_t\}$ as of a volume-increasing bending of a surface of a regular tetrahedron Δ. Rather than make this elaborate ad hoc calculation, let us compute directly the maximal volume of P_t.

Proof. Choose parameter t such that it is equal to half the side of the equilateral triangle $(b_1 b_2 b_3)$ in the middle: $|b_1 b_2| = s = 2t$. Since $|a_1 a_2| = \frac{1}{2}$, from above we have:

$$t = \frac{|a_1 a_2|}{(2 + \sqrt{3})} = \frac{1}{4 + 2\sqrt{3}} \approx 0.1340\,.$$

We also have

$$\ell := |v_i b_j| = \sqrt{\frac{1}{4} + t^2} \approx 0.5176\,.$$

Observe that the volume of a regular tetrahedron is equal to

$$\mathrm{vol}(\Delta) = \frac{1}{3} \cdot \mathrm{area}(v_1 v_2 v_3) \cdot \mathrm{height}(\Delta) = \frac{1}{3} \cdot \frac{\sqrt{3}}{4} \cdot \sqrt{\frac{2}{3}} = \frac{1}{6\sqrt{2}} \approx 0.1179\,.$$

On the other hand, polyhedron P is a union of four hexagonal pyramids R_i with apexes at vertices w_i, and a polytope Y shown in Figure 39.3. The height of each pyramid R_i is equal to $h := \sqrt{\ell^2 - s^2} \approx 0.4429$. Note that Y is a truncated tetrahedron with edge lengths equal to s. The volume of Y is equal to the volume of a regular tetrahedron with edge length $(3s)$ minus four times the volume of a regular tetrahedron with side length s. We obtain:

$$\operatorname{vol}(P) = \operatorname{vol}(Y) + 4\operatorname{vol}(R_i) = \frac{(3s)^3 - 4s^3}{6\sqrt{2}} + 4 \cdot \frac{1}{3}\left(6\frac{\sqrt{3}}{4}s^2\right) \cdot h \approx 0.1623\,.$$

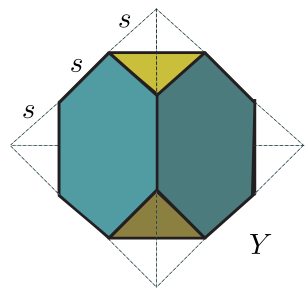

FIGURE 39.3. Truncated tetrahedron Y and a way to compute its volume.

Thus, $\operatorname{vol}(P) > 1.37 \cdot \operatorname{vol}(\Delta)$, as desired. □

For example, the volume of a tetrahedral milk carton can increase by as much as 37% under continuous (but non-convex) bending. By applying the same construction to small pyramids in P one can further increase the volume by a bending, but it is unclear how large the volume can get. Of course, such volume is bounded by the isoperimetric inequality (Theorem 7.8) which gives an upper bound of about 82%. It would be interesting to see which of these two bounds is closer to the truth.

39.3. **Milka's bending.** In this section we present a symmetric volume-increasing bending of a cube. This construction requires few calculations and motivates the general construction in the next subsection.

Consider a cube C with side length 1 and the surface $S = \partial C$. Fix a parameter $\varepsilon \in (0, \frac{1}{2})$. Think of ε as being very small. On each face of the cube, from every corner remove a square of side-length ε. Denote by $R \subset S$ the resulting surface with boundary. On every face of R there are four boundary points which form a square. Call these points corners. Translate each square directly away from the center of the cube (without expanding the squares) until all of the distances between the nearest corners on adjacent faces reach 2ε. Take the convex hull of the corners to obtain a polytope Q_ε. To each triangular face of Q_ε attach a triangular pyramid whose base is equilateral with side-length 2ε and whose other faces are right triangles. Denote by P_ε the resulting (non-convex) polyhedron.

Note that the surface ∂Q_ε without triangular faces between the corners is isometric to R. Similarly, three $\varepsilon \times \varepsilon$ squares meeting at a vertex of the cube can be divided into six right triangles which can be then bent into three faces of a pyramid (see Figure 39.4). This easily implies that the surface ∂P_ε is isometric to the surface S.

Proposition 39.2. *We have* $\operatorname{vol}(P_\varepsilon) > 1$ *for* $\varepsilon > 0$ *sufficiently small. Moreover,* $\operatorname{vol}(P_\varepsilon)$ *increases for* $\varepsilon > 0$ *sufficiently small.*

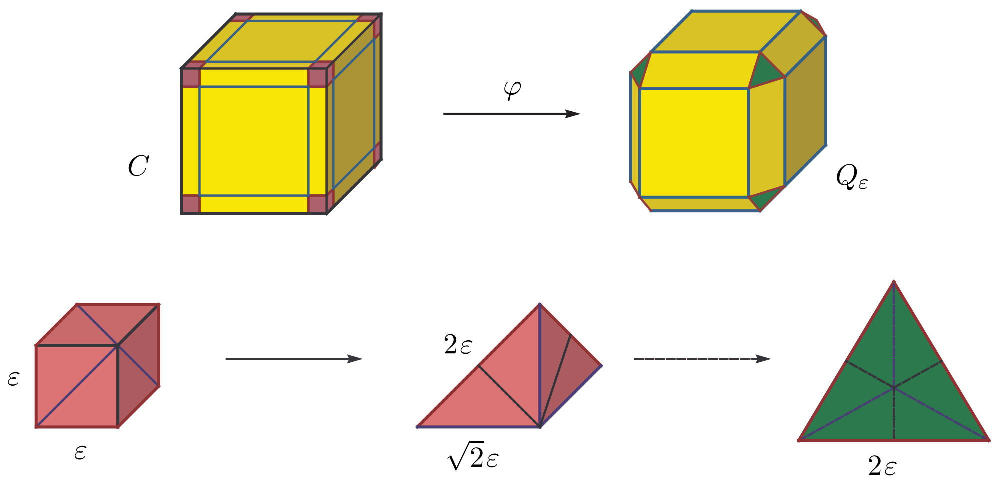

FIGURE 39.4. Construction of Milka's volume-increasing bending.

Proof. Cut Q_ε with six planes, each parallel to a square face and containing the nearest edges of its four neighboring square faces. This subdivides Q_ε into one (interior) cube, six slabs (along the faces), twelve right triangular prisms (along the edges), and eight pyramids (one per cube vertex). Observe that the cutting planes are at distance $d = \sqrt{2}\,\varepsilon$ from the square faces of Q_ε. We have:

$$\operatorname{vol}(Q_\varepsilon) \;=\; (1-2\varepsilon)^3 + 6\cdot(1-2\varepsilon)^2 d + 12\cdot(1-2\varepsilon)\,\frac{d^2}{2} + 8\cdot\frac{d^3}{6}\,.$$

Since $\operatorname{vol}(P_\varepsilon) = \operatorname{vol}(Q_\varepsilon) + 8\cdot\frac{d^3}{6}$, we conclude that

$$\operatorname{vol}(P_\varepsilon) \;=\; 1 \,+\, 6(\sqrt{2}-1)\,\varepsilon \,+\, O(\varepsilon^2),$$

This implies that for small $\varepsilon > 0$, we have $\operatorname{vol}(P_\varepsilon) > 1$, and the volume $\operatorname{vol}(P_\varepsilon)$ is increasing. $\square$

39.4. **General volume-increasing bendings.** We are ready to state the main result of this section generalizing the previous examples.

Theorem 39.3 (Pak). *Let $P \subset \mathbb{R}^3$ be a convex polytope, and let $S = \partial P$ be its surface. Then there exists an embedded polyhedral surface $S' \subset \mathbb{R}^3$ which is intrinsically isometric to S and encloses a larger volume.*

By the Alexandrov uniqueness theorem (Theorem 27.7) convex surface S is unique up to a rigid motion, which implies that the surface S' in the theorem must be non-convex. Here is a convex variation on Theorem 39.3.

We say that a surface $S' \subset \mathbb{R}^3$ is *submetric* to S, write $S' \preccurlyeq S$, if there exist a homeomorphism $\varphi : S \to S'$ which does not increase the geodesic distances: $|x,y|_S \geq |\varphi(x),\varphi(y)|_{S'}$ for all $x,y \in S$. Of course, the (intrinsically) isometric surface are submetric.

Theorem 39.4 (Pak). *Let $P \subset \mathbb{R}^3$ be a convex polytope, and let $S = \partial P$ be its surface. Then there exists a convex polytope $P' \subset \mathbb{R}^3$ such that the surface $S' = \partial P'$ is submetric to S and* $\mathrm{vol}(P) < \mathrm{vol}(P')$.

In other words, the theorem says that the surface S can be triangulated in such a way that each triangle can be made smaller to assemble into a submetric convex surface $S' \preccurlyeq S$ which encloses a larger volume. Note that if a new triangulation is consistent (i.e., the corresponding triangle edges have equal lengths) and is locally convex, then the surface S' always exists by the Alexandrov existence theorem (Theorem 37.1). Let us mention also that Theorem 39.4 implies Theorem 39.3 modulo the Burago–Zalgaller theorem in the Exercise 39.13.

To see an example of Theorem 39.4, let $P = C$ and $P' = Q_\varepsilon$ as in the previous subsection. Define a map $\varphi : \partial C \to \partial Q_\varepsilon$ as in Figure 39.4, where each of the eight attached pyramids is projected onto triangles. This shows that $\partial P' \preccurlyeq \partial P$. On the other hand, since the volume of eight pyramids is $O(\varepsilon^3)$, the same argument as above shows that $\mathrm{vol}(Q_\varepsilon) > \mathrm{vol}(P)$ for all $\varepsilon > 0$ small enough.

Remark 39.5. Here is a physical interpretation of Theorem 39.3. Imagine a polyhedron is made out of bendable, but non-stretchable material. One can then blow more air inside to make the volume larger. Of course, one can continue blowing air until a non-inflatable shape emerges (see also Exercise 39.6). The theorem says that the resulting surface cannot be a polyhedron. Two examples of such surfaces (for a doubly covered square and a cube) are given in Figure 39.5. Note that the surface shrinks in both cases and in the limit becomes submetric to the original surface (this is visible in cube).

FIGURE 39.5. The square and cubic balloons.

39.5. **Exercises.**

Exercise 39.1. $\diamondsuit$ *a*) [1-] In Milka's bending, let Q and P be the limit of polyhedra Q_ε and P_ε, as $\varepsilon \to \theta = \frac{1}{2\sqrt{3}}$. Check that Q is an octahedron, and that $\mathrm{vol}(P) < 0.95$.
b) [1-] Show that the volume of P_t maximizes at about 1.19. Compare this bound with what follows from the isoperimetric inequality (Theorem 7.8).

Exercise 39.2. $\diamondsuit$ [1] In Bleecker's bending, compute explicitly $\mathrm{vol}(P_t)$. Check that that $\mathrm{vol}(P_t)$ increases for $t \le s/2$ as in the proof, and compute the maximum value of the volume. Check that at the maximum, $t > s/2$.

Exercise 39.3. *a*) [1] Modify Bleecker's bending to prove Theorem 39.3 for a cube. Compute the maximal volume.
b) [1] Modify Bleecker's bending to prove Theorem 39.3 for a regular dodecahedron and icosahedron. Find the optimal bounds for the volume.
c) [1+] Modify Bleecker's bending to prove Theorem 39.3 for all simplicial polytopes.

Exercise 39.4. [1] Modify Milka's bending to prove Theorem 39.3 for a regular dodecahedron and a regular icosahedron. Compare the optimal bounds with those obtained in the previous exercise.

Exercise 39.5. [1] Prove Theorem 39.3 directly for the doubly covered polygons (see Figure 39.1 and Figure 39.6).

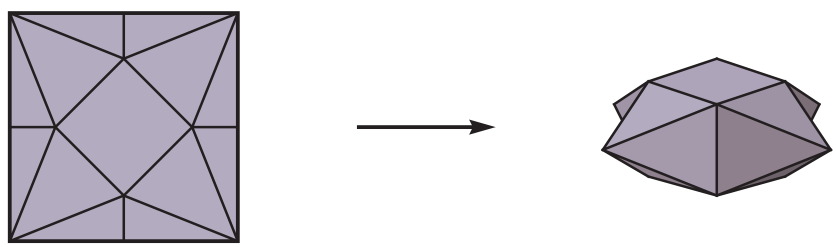

FIGURE 39.6. Volume-increasing bending of a doubly covered square.

Exercise 39.6. (*Mylar balloon*) [2-] Think of a circular party balloon as a doubly covered circle filled with gas as much as possible (see Remark 39.5). Assume the balloon retains the cyclic symmetry. Compute the volume of the balloon. Compute the surface area. Explain why the surface area decreases.

Exercise 39.7. (*Curvilinear cube*) ◊ Consider a *curvilinear surface* S defined in Figure 39.7.
a) [1-] Prove that S is isometric to a cube.
b) [1+] Show that surface S exists and is uniquely defined for every symmetric convex piecewise linear curve close enough to the edges.
c) [1] By varying the curve, give an explicit bending construction of these convex polyhedra. Prove that this bending is volume-decreasing.

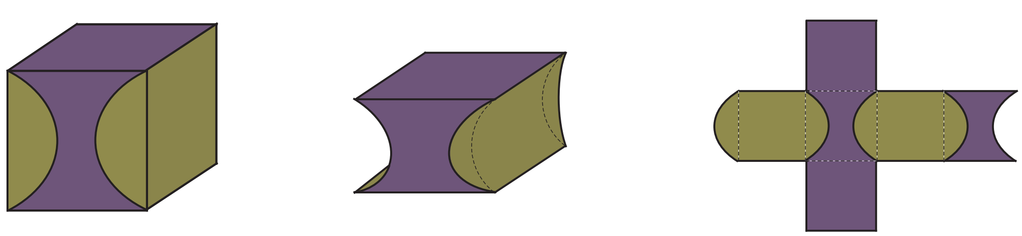

FIGURE 39.7. Curvilinear deformation (buckling) of a cube, and its unfolding.

Exercise 39.8. (*Chadwick surfaces*) Consider the curvilinear surfaces constructed in US Patent `4492723`[90] We call them *Chadwick surfaces*, after their inventor Lee Chadwick.
a) [1-] Explain why the Chadwick surfaces are isometric to doubly covered regular polygons, Platonic and Archimedean solids, etc.
b) [2-] Show that the Chadwick surfaces exist and are uniquely defined for every symmetric convex piecewise linear curve close enough to the edges.
c) [1+] By varying the curve, give an explicit bending construction in each case. Prove that all these bendings are volume-decreasing.

Exercise 39.9. We say that a surface $S \subset \mathbb{R}^3$ has a *flat region* if it contains a surface polygon lying in the plane. For example, the curvilinear surface in Figure 39.7 has two flat squares.
a) [1] Consider the surfaces in the patent from the previous exercise. Despite the appearance, check that all of them have flat regions.
b) [1+] Construct an isometric "curvilinear" embedding of the surface of the regular tetrahedron without flat regions.
c) [2-] Extend *b*) to all Platonic solids.

Exercise 39.10. $\Diamond$ An *isometry* between two closed piecewise linear curves C_1 and C_2 is a length-preserving piecewise linear map $\varphi : C_1 \to C_2$. Of course, the curves must have the same length: $|C_1| = |C_2|$. An (intrinsic) isometry $\Phi : S_1 \to S_2$ between 2-dimensional polyhedral surfaces S_1 and S_2 with boundary curves $C_1 = \partial S_1$ and $C_2 = \partial S_2$ defines an isometry between the curves.
a) [1-] Suppose now that $C_1 = \partial S_1$, where $S_1 \subset \mathbb{R}^2$ is a convex polygon, and $S_2 \subset \mathbb{R}^3$. Prove that every isometry $\Phi : S_1 \to S_2$ as above satisfies the *distance condition*: $|\varphi(x)\varphi(y)| \le |xy|$ for all $x, y \in C_1$.
b) [1] Prove that φ as above satisfies the distance condition if and only if the inequality holds for all x, y either vertices of C_1 or preimages of vertices in C_2.
c) [1] Extend part *b*) to non-convex simple polygons C_1 in the plane.

Exercise 39.11. (*Realization of polygons*) $\Diamond$ Let $S = \partial P$ be a 2-dimensional polyhedral surface. Suppose S is subdivided into triangles τ_i, $1 \le i \le k$. Suppose further that there exist a collection of triangles $\tau'_1, \ldots, \tau'_k \subset \mathbb{R}^d$ such that:

(1) the corresponding triangles are congruent: $\tau_i \simeq \tau'_i$,
(2) whenever τ_i and τ_j share an edge, τ'_i and τ'_j share the corresponding edge.

In other words, suppose the abstract polyhedral surface S' obtained as the union of triangles τ'_i is (intrinsically) isometric to S. We say that S' is a *realization* of S, and that S can be *realized* in $\mathbb{R}^d$.[91]
a) [1+] Let $C_1 = \partial S_1$, $S_1 \subset \mathbb{R}^2$, be a convex polygon in the plane, and let $C_2 \subset \mathbb{R}^3$ be a simple space polygon of the same length. Suppose $\varphi : C_1 \to C_2$ is an isometry which satisfies $|\varphi(x)\varphi(y)|_{\mathbb{R}^3} \le |xy|_{\mathbb{R}^2}$ for all $x, y \in C_1$. Prove that there exists a 2-dimensional polyhedral surface S_2 isometric to S_1, such that the isometry map $\Phi : S_1 \to S_2$ coincides with φ on C_1.
b) [2-] Extend *a*) to non-convex simple polygons C_1 in the plane.
c) [2-] Extend *b*) to general plane polygons with holes.

[90] Available at `http://www.google.com/patents?vid=USPAT4492723`.

[91] Note that in $\mathbb{R}^d$ triangles τ_i can intersect, overlap or even coincide, so S' may not be a surface in the usual sense.

d) [1] Use part a) to prove that for every $\varepsilon > 0$, the surface of a given convex polytope $P \subset \mathbb{R}^3$ can be realized inside the ball of radius ε.
e) [1+] Let $S' \preccurlyeq S$ be two convex surfaces in $\mathbb{R}^3$. Prove that for every $\varepsilon > 0$, the surface S can be realized in the ε-neighborhood of S'. Check that this is a stronger result than that in part c).

Exercise 39.12. (*Cramming surfaces*) $\Diamond$ Let $S = \partial P$ be the surface of a convex polytope $P \subset \mathbb{R}^3$.
a) [1-] Use Zalgaller's theorem (Theorem 40.10) show that for every $\varepsilon > 0$, there exists a realization of S inside the ball of radius ε.
b) [2-] Prove that for every $\varepsilon > 0$, there exists a surface S' which is isometric to S and is embedded inside the ball of radius ε.
c) [2] Prove there exists a bending (continuous piecewise linear isometric deformation) $\{S_t : t \in [0,1]\}$ such that $S_0 = S$ and $S_1 = S'$.

Exercise 39.13. (*Burago–Zalgaller's theorem*) $\Diamond$ Let S be an abstract 2-dimensional polyhedral surface homeomorphic to a sphere.
a) [2] Prove that there exists an isometric embedding of S in $\mathbb{R}^3$.
b) [2] Suppose $S' \preccurlyeq S$ for some 2-dimensional polyhedral surface $S \subset \mathbb{R}^3$. Prove that for every $\varepsilon > 0$, there exists an isometric embedding of S in the ε-neighborhood of S'.

39.6. **Final remarks.** The Burago–Zalgaller's volume-increasing bending was presented in [BZ4, §9]. Bleecker's bending was introduced in [Ble] (see also a friendly exposition in [Ale5]).

Milka's bending was discovered in [Mi5] where the other regular polyhedra were also explored and their bending analyzed. Interestingly, Milka did not notice that his bendings were volume-increasing. Our presentations follows [Pak7].

Both theorems in Subsection 39.4 are proved in [Pak8]. The proof of Theorem 39.3 uses Theorem 39.4 and a delicate result of Burago and Zalgaller [BZ4] (see Exercise 39.13). A weaker version of the Burago–Zalgaller theorem is given in Exercise 39.11, the first part of which is based on Tasmuratov's results [Tas1, Tas2]

Theorem 39.4 extends to higher dimensions and its proof uses an advanced generalization of Milka's bending construction. Similarly, Theorem 39.3 extends to non-convex (possibly self-intersecting) polyhedral surfaces [Pak8]. The curvilinear surface in Figure 39.7 is due to Shtogrin [Sht1]. However, the subject of curvilinear surfaces is quite old and in the case of the cylinder has been extensively studied by Pogorelov [Pog5].[92]

[92]According to Zalgaller, some of the work in this direction by Alexandrov, Pogorelov and others, remains classified because of rocket science applications (personal communication, not a joke).

40. Foldings and unfoldings

In this section we prove that every convex polytope can be unfolded into the plane if cuts along faces are allowed. Our tools include the shortest paths (Section 10) and Voronoi diagrams (Section 14).

40.1. **Unfoldings of convex polyhedra.** Let Γ be the graph of a convex polytope $P \subset \mathbb{R}^3$. For a spanning tree $t \subset \Gamma$, cut the surface $S = \partial P$ along the edges of t and unfold $S \smallsetminus t$ on the plane. The resulting *edge unfolding* can be either *overlapping* or *non-overlapping* (Figures 19.4, 19.7 and 40.1). We call t the *cut set* and say that P has an unfolding with the cut set t. If the unfolding is non-overlapping, the resulting polygon is called a *foldout*. Since non-overlapping unfoldings are visually very appealing, the following old conjecture says that they always exist (see also Subsection 40.6).

Conjecture 40.1 (Dürer's conjecture). *Every convex polytope $P \subset \mathbb{R}^3$ has a non-overlapping edge unfolding.*

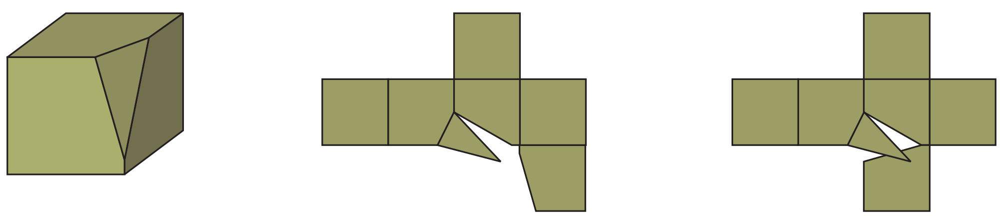

Figure 40.1. Two edge unfoldings of a snubbed cube.

Here is a way to weaken the conjecture. Recall that from the metric point of view, there is nothing significant about edges of the polytope. Suppose $\mathrm{T} \subset S$ is a piecewise linear tree, defined as a contractible union of intervals on the surface. We say that T is *spanning* if it contains all vertices of P. We can then consider an unfolding of P with the cut set T (see Figure 40.2 to appreciate the difference).

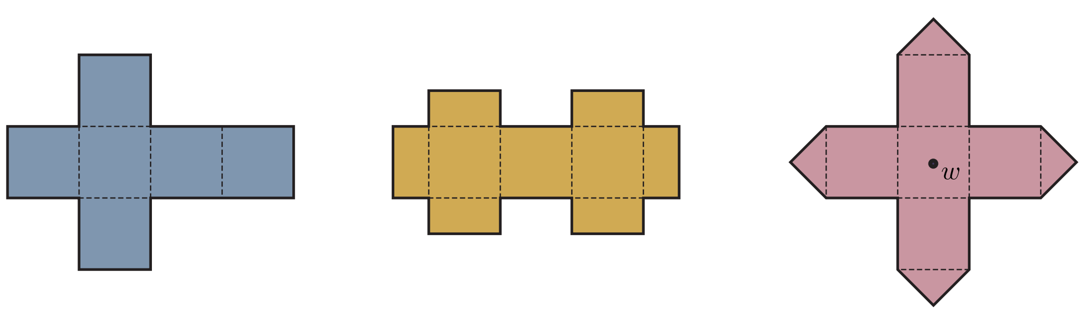

Figure 40.2. Foldouts of three unfoldings of a cube.

Let us now we present two constructions of non-overlapping unfoldings.

Theorem 40.2 (Source unfolding)**.** *Let S be the surface of a convex polytope $P \subset \mathbb{R}^3$. Fix a point $w \in S$ in the relative interior of a face and let T_w be the set of points $z \in S$ which have two or more shortest paths to w on the surface. Then the unfolding of P with the cut set T_w is non-overlapping.*

The point w is the called the *source point*, thus the *source unfolding*. The cut set in this case is called the *cut locus*. The third unfolding in Figure 40.2 is an example of a source unfolding of the cube, where the source point w is in the middle of a face. Note that the cut set can be complicated even in the most basic cases (see Figure 40.3). It is by no means obvious that T_w is a spanning tree on the surface S. We will prove this in the next section.

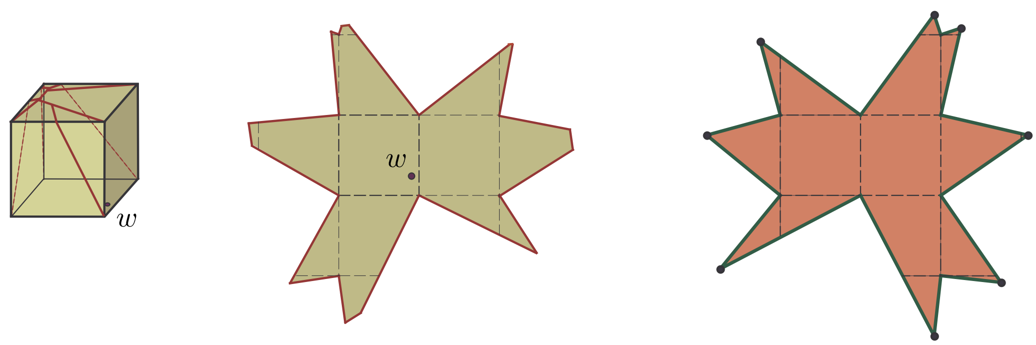

FIGURE 40.3. Source unfolding of the cube with the source on the bottom face and a foldout of the Alexandrov unfolding.

Theorem 40.3 (Alexandrov unfolding)**.** *Let S be the surface of a convex polytope $P \subset \mathbb{R}^3$. Fix a point $w \in S$ which has a unique shortest path to every vertex of P. Let T'_w be the union of these paths. Then the unfolding of P with the cut set T'_w is non-overlapping.*

We do not prove Theorem 40.3, leaving it to the reader (see Exercises 40.1 and 40.9).

Remark 40.4. Although the statements of both theorems may seem completely clear, we never really defined what is a (*general*) *unfolding*. The definition simplifies in the non-overlapping case. A *non-overlapping* (*general*) *unfolding* of the surface S with the cut set T is an isometric piecewise linear homeomorphism $\varphi : S \smallsetminus \mathrm{T} \to U$ where U is a polygon in the plane. Such a homeomorphism is called the *unfolding map*.

The great advantage of the source unfolding over the Alexandrov unfolding is the fact that the former gives a star-shaped polygon centered at $\varphi(w)$. Thus the shortest path γ_{wz} unfolds into a straight interval in U. This construction can now be used to resolve the *discrete geodesic problem* of computing the geodesic distances between points (see Exercise 40.8). Curiously, the Alexandrov unfolding is not necessarily a star-shaped polygon, even though it is usually referred in the literature as the *star unfolding*.

40.2. **Source unfolding as the source of inspiration.** As it turns out, the proof of Theorem 40.2 is straightforward once we show that the source unfolding is well defined. We prove this result in a sequence of lemmas.

Lemma 40.5. *The cut set* T_w *is a finite union of intervals.*

Proof. Suppose point w lies in the face F of the polytope P. For every face A, consider the set $X = X(w, A) \subset H$ of images under various unfoldings of the source point w onto the plane H spanned by A. Points $x \in X$ are called *source images*. Formally, every shortest path γ_{wz} joining w to a point z in the relative interior of A ends in a straight segment $\mathsf{end}(\gamma_{wz})$ in A. The *source image* corresponding to such path γ_{wz} is the point $x \in H$, such that the straight segment $[zx]$ has the same length as and contains $\mathsf{end}(\gamma_{wx})$.

The idea is that when restricted to the face A, the Voronoi diagram $VD(X)$ is a complement to T_w.[93] More precisely, denote by $C_w(e) \subset S$ the *geodesic cone* over an interval e in the edge of A through which source image "sees" the face A, i.e., a family of shortest paths from w to points in A which cross e. Denote by $A_w(e) = A \cap C_w(e)$ the region of F which can be "seen" from w through e, and thus the corresponding source image $x \in X$ (see Figure 40.4). Now, for every point $z \in A$, take all regions $A_w(e)$ containing z and decide which source image is the closest to z. Since we consider only shortest paths, no facet can appear twice in the corresponding geodesic cones, which implies that the number of different cones $C_w(e)$ is finite. Thus, face A is subdivided into a finite number of intersections of regions $A_w(e)$ with different e, and in each such intersection the cut set T_w is the complement of the Voronoi diagram of the source images. This implies that in the face A, the cut set T_w is a finite union of intervals, as desired. $\square$

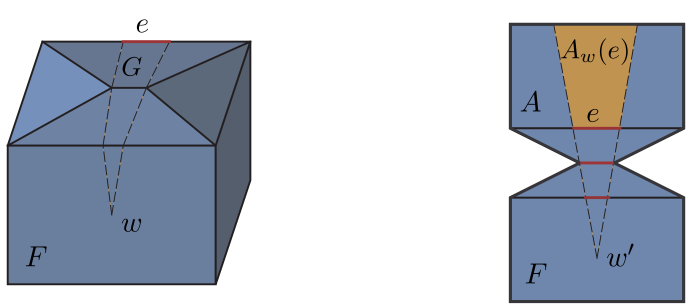

FIGURE 40.4. A geodesic cone $G = C_w(e)$ and region $A_w(e)$.

Lemma 40.6. *The complement* $S \smallsetminus \mathrm{T}_w$ *to the cut set is contractible.*

In particular, the lemma implies that the cut set T_w is connected and simply connected.

[93] This is true as stated, but requires a separate proof by a rather delicate technical argument (see Exercise 40.6). We use the geodesic cones in the proof to avoid this technicality even if that makes the cut set structure less transparent.

Proof. By Proposition 10.1, no shortest path γ_{xy} between two points $x, y \in S$ can contain a vertex of P in its relative interior. Therefore, for all $z \in S$, a shortest path γ_{wz} cannot contain *any* points in the cut set T_w in its relative interior, since for all flat points $c \in \mathrm{T}_w$ we can make the shortcut locally around c as in Figure 40.5. This shows that $S \smallsetminus \mathrm{T}_w$ is contractible to w along the shortest paths. $\square$

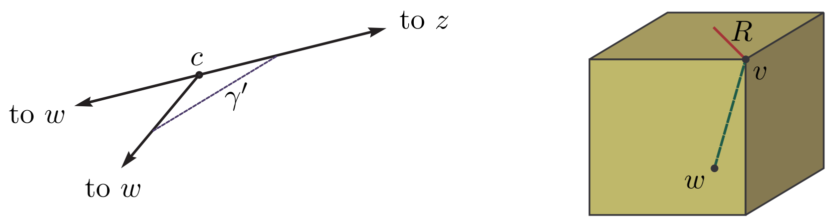

FIGURE 40.5. An intersection that is Y-shaped cannot locally minimize length in $\mathbb{R}^2$ (segment γ is a shortcut). Cut set interval R in the neighborhood of a vertex v of P

Lemma 40.7. *The cut set* T_w *is a tree in* S *which contains all vertices of* P.

Proof. By Lemma 40.5, the cut set T_w is a connected union of intervals. Since D_w is contractible by Lemma 40.6, T_w has no cycles. Therefore, T_w is a tree. To see that it is a spanning tree, consider two cases. If a vertex $v \in P$ has two or more shortest paths, then $w \in \mathrm{T}_w$. Otherwise, if there is a unique shortest path γ_{wv}, we can unfold the faces of P along γ_{wv}. Consider a bisector R of the angle at v (see Figure 40.5). Observe that for points z in the neighborhood of v, the points $z \in R$ have two shortest paths to w (cf. the cone unfolding argument in Subsection 10.1). Therefore, vertex v lies in the closure of an interval in the cut set T_w. $\square$

Proof of Theorem 40.2. Suppose point w lies in face F of P. Denote by H the plane spanned by F. Let $D_w = S \smallsetminus \mathrm{T}_w$. For a point $z \in D_w$, $z \neq w$, consider the unique shortest path γ_{wz} from w to z. When restricted to F, this path begins as a straight segment $\mathsf{start}(\gamma_{wz}) \subset F$. Following the construction of source images in the proof of Lemma 40.5, define a map $\varphi : D_w \to \mathbb{R}^2$, such that $[w\varphi(z)]$ is an interval which begins as $\mathsf{start}(\gamma_{wz})$ and has the same length: $|w\varphi(z)| = |\gamma_{wz}|$. Set $w = \varphi(w)$ and denote by $U \subset \mathbb{R}^2$ the image of φ.

Note that φ is a homeomorphism since no two distinct points have the same image: $\varphi(x) \neq \varphi(y)$ for all $x \neq y$, $x, y \in D_w$. Otherwise, we have two points with shortest paths of the same length $|\gamma_{xw}|_S = |\gamma_{yw}|_S$ which start in the same direction from w. Taking the last point of the initial segment where these paths coincide we obtain again the Y-shaped intersection as in Figure 40.5.

Now, since T_w is a tree, we conclude that the homeomorphism φ is piecewise linear. Similarly, since D_w is contractible, we conclude that U is a polygon. To prove that φ is the unfolding map, it remains to show that φ is isometric (see Remark 40.4). By definition of φ, the polygon U is star-shaped at w since for every $y \in \gamma_{wz}$ we have $\gamma_{wy} \subset \gamma_{wz}$. Consider any shortest path $\gamma_{xy} \in D_w$. Again, by definition of φ, we have

$|wx|_S = |w\varphi(x)|$, $|wy|_S = |w\varphi(y)|$, and $\angle_S xwy = \angle\,\varphi(x)w\varphi(y)$, where the $\angle_S$ is the angle between two shortest paths in S. Since the geodesic triangle $(wxy) \subset D_w$ is flat, it is isometric to $(w\varphi(x)\varphi(y)) \subset U$. This implies that $|xy|_S = |\gamma_{xy}| = |\varphi(x)\varphi(y)|_U$ for all $x, y \in D_w$, i.e., D_w and U are isometric. $\square$

Example 40.8. Consider the source unfolding of the cube shown in Figure 40.3. For the most involved 'top' face A there are 12 potential source images in the plane H spanned by A. Of which only 8 have points in A corresponding to shortest paths to the source point w (see Figure 40.6). The Voronoi diagram of these 8 points gives the part of the cut set T_w in A. To make a distinction, we mark by $\star$ the remaining four 'false' source images in the figure. Note that we are explicitly using Mount's lemma (Exercise 40.6) in this case.

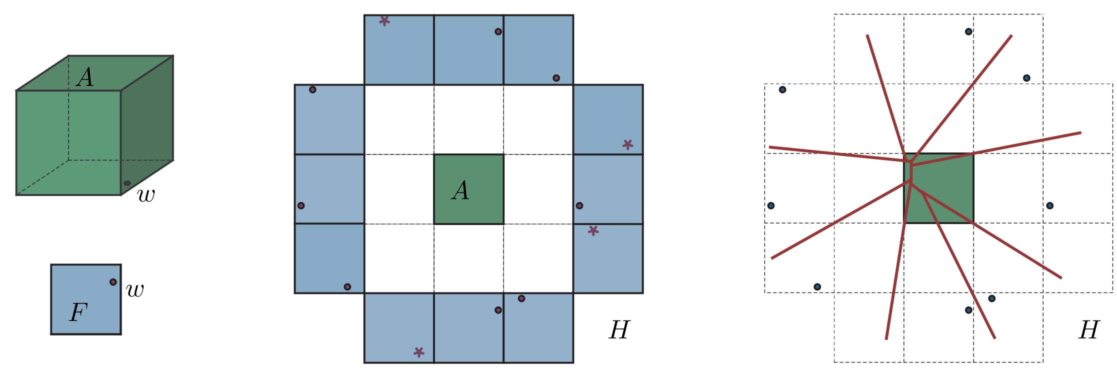

FIGURE 40.6. Source point v on the bottom face F, 12 source images for the top face A, and the Voronoi diagram of the source images.

Remark 40.9. (*Connelly's blooming conjecture*) One can think about unfoldings as a continuous process in the following sense. Take a polyhedral non-overlapping foldout made of hinged metal, is it always possible to glue its corresponding edges together? Because metal is rigid, we need not only a non-overlapping property on the foldout as it lies flat on the ground, but also a nonintersecting property as we continuously fold it up. Viewing this process in reverse, can we continuously unfold the polyhedral boundary so that all dihedral angles monotonically increase, until the whole polyhedral boundary lies flat on a plane? An example is given in Figure 40.7.

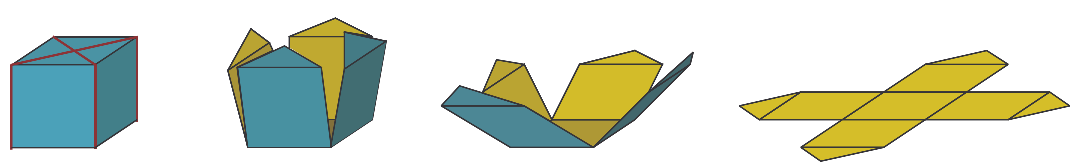

FIGURE 40.7. Blooming of the source unfolding of a cube.

This problem is called the *blooming conjecture*, and it can be stated in a number of different (inequivalent) ways. It is believed that such blooming always exists for the source unfolding, but as of now it is not known if *any* non-overlapping unfolding of a convex polyhedron can be always bloomed (see [MilP] for a precise statement).

40.3. **Folding polyhedra is easier than unfolding.** Let $S = \partial P$ be the surface of a convex polytope $P \subset \mathbb{R}^3$. Suppose S can be subdivided into a finite number of triangles which can be placed on a plane $\mathbb{R}^2$ in such a way that adjacent triangles in S are also adjacent along the corresponding edges.[94] The resulting piecewise linear continuous map $\varphi : S \to \mathbb{R}^2$ is called a *flat folding*. By construction, the map φ is locally isometric everywhere except on some edges of P.

For example, the surface of a cube has a flat folding shown in Figure 40.8. Here we subdivide the surface of a cube into 24 congruent right triangles which are mapped onto the same triangle according to the coloring as in the figure. Since triangle edges with equal colors are still adjacent, this is indeed a flat folding.

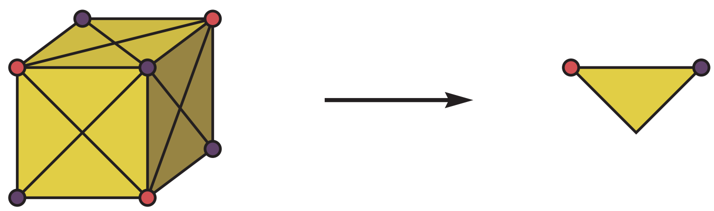

FIGURE 40.8. The surface of a cube can be folded onto a plane.

Theorem 40.10 (Zalgaller)**.** *Let $S = \partial P$ be the surface of a convex polytope $P \subset \mathbb{R}^3$. Then S has a flat folding.*

In other words, every convex surface can be folded onto a plane. The proof idea is in fact more general and works for all 2-dimensional polyhedral surfaces (see Exercise 40.2).

Proof. Let $U \subset \mathbb{R}^2$ be the foldout of the source unfolding constructed above. Recall that U is a star-shaped polygon with two corresponding edges per every interval in the cut set, i.e., polygon U has edges (a_1, b_1) and (a_2, b_2) for every $(a, b) \in \mathrm{T}_w$. The triangles (wa_1b_1) and (wa_2b_2) are congruent since straight intervals to point $z_1 \in (a_1, b_1)$ and the corresponding point $z_2 \in (a_2, b_2)$ are the images under φ of the shortest paths to the same point in the cut set.

Subdivide U into cones over the edges and consider the corresponding subdivision of S. The surface S is now a union of quadrilaterals $[wawb]$, for all $(a, b) \in \mathrm{T}_w$. In every quadrilateral, cut each triangle (wab) with a bisector (wc) as in Figure 40.9. Place all resulting triangles in the plane so that their vertices w are at the origin and the edges (w, a) and (w, b) are in the positive part of the x axis. By construction, the adjacent triangles in each quadrilateral $[wawb]$ remain adjacent. Similarly, the adjacent triangles in different quadrilaterals can only be adjacent along the edges (wz), where z is a vertex of U, and all such edges now lie on the x axis. This finishes the proof. $\square$

[94] This is called a realization of the surface S in $\mathbb{R}^2$ (see Subsection 31.1 and Exercise 39.11).

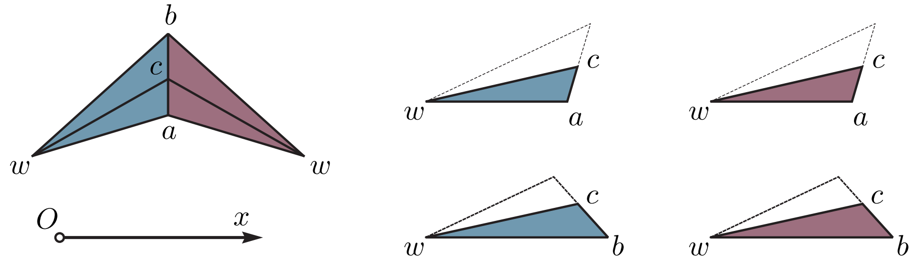

FIGURE 40.9. Zalgaller's flat folding construction.

Example 40.11. For the construction in Figure 40.8, it is not at all obvious that this flat folding can be obtained continuously. Here is an alternative, *continuous folding* of the cube. Start with the back side of the cube and push it forward until the back half is 'glued' to the front half (see Figure 40.10). Do this two more times as in the figure. The resulting surface is 8-folded and looks like corner of a cube, consisting of three squares. This surface can now be folded onto the plane in a number of ways, e.g., as in the figure. It is an open problem whether this procedure is possible for every convex polyhedron (see Exercise 40.14).

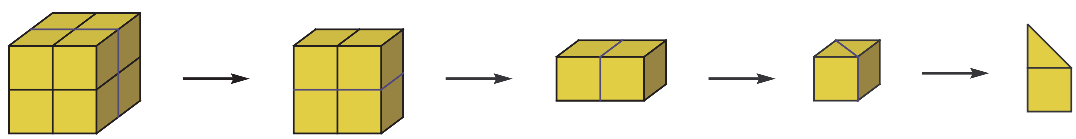

FIGURE 40.10. A step-by-step construction of a continuous folding of the cube.

40.4. **Folding a napkin.** Let $Q \subset \mathbb{R}^2$ be a unit square which we view as a 2-dimensional polyhedral surface with boundary. The *perimeter* of a flat folding $\varphi : Q \to \mathbb{R}^2$ is the perimeter of the set of points in the plane with one or more preimage. It was an open problem for decades whether this perimeter is always at most 4. As it turns out, one can make this perimeter as large as desired.

Theorem 40.12 (Napkin folding problem)**.** *The perimeter of a flat folding of a unit square is unbounded.*

In other words, the square napkin can be folded into a figure whose perimeter is as large as desired. The theorem is only one way to formalize this problem, and there are several ways to strengthen the folding condition (see Exercise 40.16).

Proof. Subdivide the square into k^2 congruent squares. Subdivide each square into $N = 8m$ triangles of with equal angle at the center, as in Figure 40.11. Fold triangles in each square on top of each other, into N layered triangular shaped *booklet*. Place all k^2 booklets on top of each other. This is a flat folding since by the symmetry the adjacent triangles in different booklets are congruent and thus adjacent in the plane.

Observe that the booklets are adjacent along edges outside the circles inscribed into the squares. Thus the booklets can be further folded to spread them out as in the

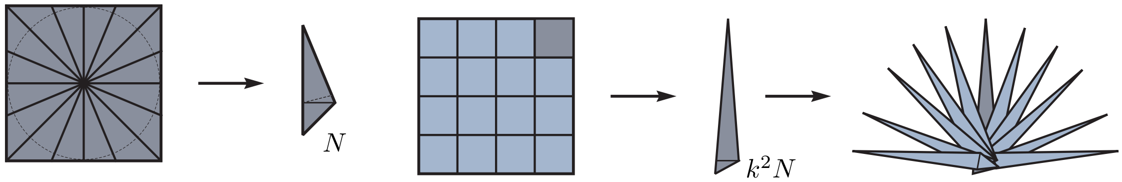

FIGURE 40.11. Flat folding of a square with large perimeter.

figure. To compute the perimeter of the resulting flat folding, observe that one can make the triangles as slim as desired. Thus, the overlap of the booklet perimeters can be made $O(\frac{1}{k^2})$. Since the height of each triangle (or booklet) is at least $\frac{1}{2k}$, and there are k^2 of them, this gives the total perimeter $k - O(1)$. This implies the result. $\square$

40.5. **Exercises.**

Exercise 40.1. $\Diamond$ [1-] Let $w \in S$ be a generic point on the surface of a convex polytope $P \subset \mathbb{R}^3$. Take the union of cut sets of the source and the Alexandrov unfolding with the same point w. Show that this subdivides S into n convex polygons, called *peels*, where n is the number of vertices in P (see Figure 40.12).

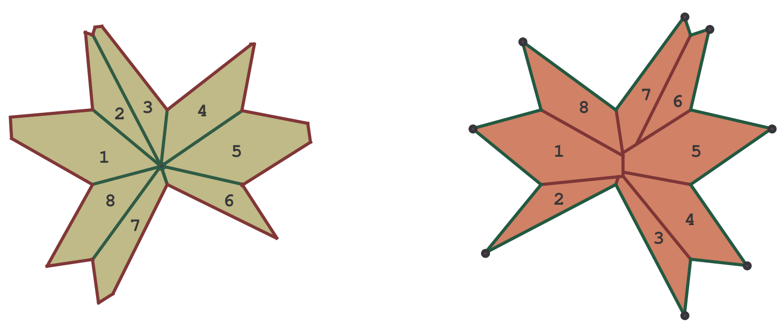

FIGURE 40.12. Source and Alexandrov unfoldings of the cube, subdivided into peels (cf. Figure 40.12).

Exercise 40.2. (*Extended Zalgaller's theorem*) $\Diamond$ Let $S \subset \mathbb{R}^3$ be a 2-dimensional polyhedral surface. For a finite set of points $X = \{x_1, \ldots, x_k\} \subset S$ define the *geodesic Voronoi diagram* $VD_S(X)$ to be a union of sets D_i of points $z \in S$, which are closest to x_i and have a unique shortest path to x_i, $1 \le i \le k$. Define the *cut locus* $\mathcal{C}(X)$ to be the complement to $VD_S(X)$. Note that for convex surfaces S, the cut locus of a single point is exactly the cut set in the source unfolding.
a) [1-] Check that the cut locus $\mathcal{C}(X)$ is not necessarily 1-dimensional (for non-convex surfaces).
b) [1] Let V be the set of vertices of S. Prove that the cut locus $\mathcal{C}(V)$ is 1-dimensional.
c) [1] Prove or disprove: sets D_i are connected, i.e., the geodesic Voronoi diagram $VD_S(V)$ has exactly k cells.

d) [1-] Extend Theorem 40.10 to all abstract 2-dimensional polyhedral surface (defined by a collection of triangles with given edge lengths and combinatorics).
e) [1+] Extend Theorem 40.10 to spherical polyhedra.
f) [2-] Extend Theorem 40.10 to convex polytopes in $\mathbb{R}^4$.

Exercise 40.3. $\Diamond$ Consider the geodesic Voronoi diagram *VD* of vertices on the surface of a convex polytope $P \subset \mathbb{R}^3$.
a) [1-] Give an example of a convex polytope P where two polygons in *VD* have two or more common edges.
b) [1-] Give an example of a convex polytope P where a polygon in *VD* has self-adjacent edges.

Exercise 40.4. *a*) [1] Prove that for all $n \geq 7$, the regular tetrahedron cannot be unfolded into a convex n-gon. Show that this possible for $3 \leq n \leq 6$.
b) [1+] For every convex polytope $P \subset \mathbb{R}^3$ find an explicit bound on $N = N(P)$ such that for all $n \geq N$, polytope P cannot be unfolded into a convex n-gon.

Exercise 40.5. *a*) [1+] Let $P \subset \mathbb{R}^3$ be a pyramid with the base a convex polygon Q. Prove that when other faces are *collapsed* (rotated around the edges onto the plane spanned by Q), they cover the whole of Q.
b) [1] Similarly, if the faces are rotated around the edges onto the *outside* of Q, they do not intersect, and thus give the edge unfolding of P.
c) [2-] Generalize parts *a*) and *b*) to higher dimension and to general polytopes P whose facets intersect a given facet Q of P by a facet of Q.

Exercise 40.6. (*Mount's lemma*) $\Diamond$ *a*) [1+] In notation of the proof of Lemma 40.7, prove that if a point $z \in F'$ lies in the Voronoi cell of the source image $x \in X(w, F')$, then $|xz| = |\gamma_{xz}|_S$.
b) [2-] Generalize part *a*) to higher dimensions.

Exercise 40.7. (*Number of shortest paths*) $\Diamond$ *a*) [2-] Let S be the surface of a convex polytope $P \subset \mathbb{R}^3$ with n vertices. Prove that the number of shortest paths between every pair of points on S is at most polynomial in n.
b) [1-] Show that part *a*) fails for non-convex polyhedral surfaces.
c) [2-] A *combinatorial type* of a shortest path is a sequence of faces it enters. Prove that the number of combinatorial types shortest paths on S is at most polynomial in n.
d) [$*$] Generalize part *a*) to higher dimensions.

Exercise 40.8. (*Discrete geodesic problem*) $\Diamond$ *a*) [2-] Convert the source unfolding construction into a polynomial time algorithm to solve the *discrete geodesic problem*: compute the geodesic distance between two points on the surface of a convex polytope in $\mathbb{R}^3$.
b) [2] Find a polynomial time algorithm to compute the geodesic diameter of a polytope in $\mathbb{R}^3$.

Exercise 40.9. (*Alexandrov unfolding*) $\Diamond$ [2] Prove Theorem 40.3.

Exercise 40.10. [2] Generalize Theorem 40.2 to higher dimensions.

Exercise 40.11. $\Diamond$ *a*) [2-] The *shortest path problem* asks to find a shortest path between points $x, y \in \mathbb{R}^3$ which avoids a given set of convex polytopes $Q_1, \ldots, Q_k$. Prove that this problem is NP-hard, if the polytopes are given by their vertices.
b) [1] Conclude from part *a*) that the discrete geodesic problem on 3-dimensional surfaces in $\mathbb{R}^4$ is also NP-hard.

Exercise 40.12. Let $S = \partial P$ be the surface of a convex polytope, and let $\gamma = \gamma_{xy}$ be a shortest path between some two points $x, y \in S$.
a) [1] For every $\varepsilon > 0$, give a construction of P, such that for some shortest path γ we have for the total curvature: $\varkappa(\gamma) > 2\pi - \varepsilon$.
b) [2] Give a construction of P, such that for some shortest path γ we have for the total curvature: $\varkappa(\gamma) > 2\pi$.
c) [2-] Define the *folding angle* $\upsilon(\gamma)$ to be the sum of exterior angles of edges of P intersected by γ. Check that $\upsilon(\gamma) \geq \varkappa(\gamma)$. Show that $\upsilon(\gamma)$ is unbounded, i.e., for every $N > 0$ give a construction of P and γ, such that $\upsilon(\gamma) > N$.
d) [$*$] Prove or disprove: $\varkappa(\gamma)$ is bounded for all P and γ.

Exercise 40.13. (*Volkov*) Let $S = \partial P$ be the surface of a convex cap $P \subset \mathbb{R}^3$ (see Subsection 25.6). Suppose $x, y \in S$ are two points in the bounded faces of S.
a) [1+] Prove that the length of all geodesics between x and y is bounded.
b) [2-] Conclude that there is at most a finite number of geodesics between x and y.

Exercise 40.14. $\diamondsuit$ *a*) [1-] Formalize the *continuous folding* introduced in Example 40.11. Show that there exists a continuous folding which produces the folding in Figure 40.8.
b) [1-] Find a continuous folding of the truncated cube (see Figure 16.4).
c) [1] Find a continuous folding of the regular icosahedron and the regular dodecahedron.
d) [1] Prove that every tetrahedron has a continuous folding.
e) [1+] Show that every convex surface has a continuous folding.

Exercise 40.15. Let $S = \partial P$ be the surface of a convex polytope in $\mathbb{R}^3$. Define a *layered folding* to be a flat folding map $\varphi : S \to \mathbb{R}^2$ with the ordering map $\pi : S \to \mathbb{N}$ defined so that adjacent triangles in the layers can be physically glued together.
a) [1-] Formalize the definition. Check that every continuous folding of S gives a layered folding.
b) [1+] Prove that every S as above has a layered folding.
c) [1+] Generalize *b*) to abstract 2-dimensional polyhedral surfaces homeomorphic to a sphere.

Exercise 40.16. (*Napkin problem*) $\diamondsuit$ *a*) [1] Define the *reflection folding* of a polygon to be a flat folding obtained sequentially, by reflecting the whole polygon along a line (ab) as in Figure 40.13. Show that the perimeter does not increase under these reflections. Conclude that the perimeter of every reflection folding of a napkin is at most 4.
b) [$*$] Define a *natural folding* of a polygon to be a flat folding obtain sequentially, by reflecting *layers* along a line (see Figure 40.13). Show that every natural folding of a unit square has perimeter at most 4.
c) [1+] Find a layered folding (see Example 40.15) of a unit square with perimeter as large as desired.
d) [2-] Same for continuous foldings (see Example 40.11).
e) [2] Same for continuous piecewise linear foldings.

Exercise 40.17. *a*) [1-] Find a non-convex polytope P in $\mathbb{R}^3$ which does not have a non-overlapping unfolding.
b) [1+] Same for P with convex faces.

Exercise 40.18. [$*$] Prove Dürer's conjecture for zonotopes (see Exercise 7.16).

Exercise 40.19. [$*$] Prove that every star-shaped surface in $\mathbb{R}^3$ has a (general) non-overlapping unfolding.

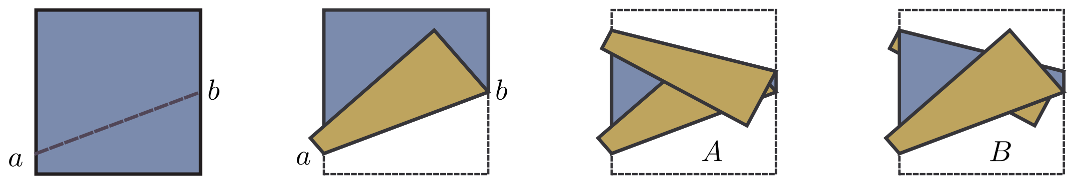

FIGURE 40.13. A reflection folding A and a natural folding B of a square.

Exercise 40.20. [1-] Find a painting by Salvador Dali which exhibits an unfolding of a 4-dimensional cube (among other things).

40.6. **Final remarks.** Dürer's conjecture (Conjecture 40.1) is named in honor of Albrecht Dürer, who described a number of unfoldings in his classical monograph (c. 1500). In modern times it is usually attributed to Shephard (1975). The conjecture inspired a lot of passion and relatively little evidence in either direction (a rare negative result was recently obtained in [Tar3]). Personally, I would bet against it, while most experts probably would bet on it. Either way, as it stands, the conjecture remains unassailable.

The non-overlap of the source unfolding (Theorem 40.2) was first proved in [ShaS], although most supporting results about the cut loci on convex polyhedra were proved much earlier in [VP2]. Mount's lemma (Exercise 40.6) is due to D. M. Mount (1985, unpublished), and was generalized to higher dimension in [MilP]. It can also be deduced from the *Toponogov's theorem* on geodesics in all metric spaces with curvature bounded from below (see [BBI, §10.3]).[95]

For non-convex 2-dimensional polyhedral surfaces $S \subset \mathbb{R}^3$ the *discrete geodesic problem* was resolved in [MMP] by a *continuous Dijkstra algorithm*, generalizing the classical graph connectivity algorithm. The problem is NP-hard in higher dimensions (Exercise 40.11). For convex surfaces an optimal time algorithm was found in [SchrS].

The Alexandrov unfolding was introduced by Alexandrov in [A1, §6.1]. It was proved to be non-overlapping in [AroO]. The algorithmic consequences were further explored in [Aga+]. Zalgaller's theorem is a special case of a general result outlined in [Zal2] (see also Exercise 40.2).

The napkin problem goes back to V. I. Arnold (1956) and is often called *Arnold's rouble problem* or *Margulis's napkin problem*. The problem is stated in [Arn3, pp. 2, 158] in ambiguous language, perhaps intentionally. Variations on the problem were resolved in [Yas], [Lang, §9.11] and [Tar2], the latter paper giving the strongest version. As Lang explains in his book, the construction we present in Subsection 40.4 is standard in the *origami* literature, going back several centuries.

[95]Ezra Miller, personal communication.

Part III

Details, details...

41.1. **The area of spherical polygons.** In the next two subsections we present several basic definitions and classical results in spherical geometry. A few preliminary words. We consider a unit sphere $\mathbb{S}^2$ with center at the origin O. The role of lines play *great circles*, defined as circles of radius 1 with centers at the origin O. Triangles, polygons, areas, etc. are defined by analogy with the plane geometry. The angle between two 'lines' is defined as the dihedral angle between planes containing the corresponding great circles. Finally, recall that $\operatorname{area}(\mathbb{S}^2) = 4\pi$.

Theorem 41.1 (Girard's formula)**.** *Let T be a spherical triangle with angles α, β and γ. Then* $\operatorname{area}(T) = \alpha + \beta + \gamma - \pi$.

Proof. Let A, B and C be the triangular regions attached to the triangle as in Figure 41.1. Observe that $\operatorname{area}(A \cup T) = \frac{\alpha}{2\pi} \times \operatorname{area}(\mathbb{S}^2) = 2\alpha$, $\operatorname{area}(B \cup T) = 2\beta$, and $\operatorname{area}(C \cup T) = 2\gamma$. Now for the upper hemisphere H we have:

$$\begin{aligned}\operatorname{area}(H) &= \operatorname{area}(A) + \operatorname{area}(B) + \operatorname{area}(C') + \operatorname{area}(T) = \big(\operatorname{area}(A) + \operatorname{area}(T)\big) \\ &\quad + \big(\operatorname{area}(A) + \operatorname{area}(T)\big) + \big(\operatorname{area}(C) + \operatorname{area}(T)\big) - 2\operatorname{area}(T) \\ &= 2\big(\alpha + \beta + \gamma - \operatorname{area}(T)\big)\end{aligned}$$

On the other hand, $\operatorname{area}(H) = \operatorname{area}(S)/2 = 2\pi$, which implies the result. $\square$

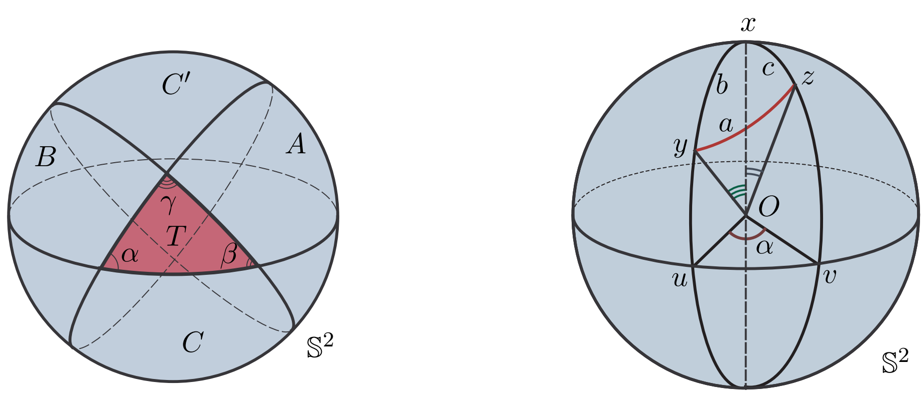

Figure 41.1. The area and the law of cosines for spherical triangles.

Theorem 41.2 (The area of a spherical polygon)**.** *Let Q be a spherical n-gon with angles $\alpha_1, \ldots, \alpha_n$. Then* $\operatorname{area}(Q) = \alpha_1 + \ldots + \alpha_n - (n-2)\pi$.

We refer to the formula in the theorem as *Girard's formula for polygons*. For the proof, subdivide the polygon into triangles and sum the areas of all triangles according to Girard's formula. The details are straightforward.

41.2. **The law of cosines for spherical triangles.** In Section 21 we repeatedly use the following claim: if the spherical triangles (xyz) and $(x'y'z')$ satisfy $|xy| = |x'y'|$, $|xz| = |x'z'|$, and $\sphericalangle yxz > \sphericalangle y'x'z'$, then $|yz| > |y'z'|$. In the plane this follows immediately from the (first) *law of cosines*:

$$|yz|^2 \,=\, |xy|^2 \,+\, |xz|^2 \,-\, 2\,|xy|\,|xz|\,\cos\widehat{yxz}.$$

For spherical triangles there is a similar formula which also implies the result. We present a simple proof below.

Proposition 41.3. *Let $(xyz) \subset \mathbb{S}^2$ be a spherical triangle with edge lengths $a = |yz|$, $b = |xy|$, and $c = |xz|$. Let $\alpha = \sphericalangle\, yxz$. Then:*

$$\cos a \,=\, \cos b\, \cos c \,+\, \sin b\, \sin c\, \cos \alpha\,.$$

Proof. Consider the unit sphere $\mathbb{S}^2$ centered at the origin O. Place point x at the North Pole of the sphere, and let u, v be the intersections of meridians (xy) and (xz) with the equator (see Figure 41.1). Denote by $\boldsymbol{s} = \overrightarrow{Os}$, for any point $s \in \mathbb{S}^2$. We have:

$$\boldsymbol{y} \,=\, \cos b \cdot \boldsymbol{x} \,+\, \sin b \cdot \boldsymbol{u}\,, \quad \mathbf{z} \,=\, \cos c \cdot \boldsymbol{x} \,+\, \sin c \cdot \boldsymbol{v}\,.$$

Now observe that $(\boldsymbol{y}, \mathbf{z}) = \cos a$. We conclude:

$$\cos a = (\boldsymbol{y}, \mathbf{z}) = (\cos b \cdot \boldsymbol{x} + \sin b \cdot \boldsymbol{u},\, \cos c \cdot \boldsymbol{x} + \sin c \cdot \boldsymbol{v}) \,=\, \cos b\, \cos c \,+\, \sin b\, \sin c\, \cos \alpha,$$

since $(\boldsymbol{u}, \boldsymbol{v}) = \cos\alpha$ and $(\boldsymbol{x}, \boldsymbol{u}) = (\boldsymbol{x}, \boldsymbol{v}) = 0$. □

41.3. **The irrationality of** $(\arccos \frac{1}{3})/\pi$**.** Let $\alpha = \arccos \frac{1}{3}$ be the dihedral angle in a regular tetrahedron (see Sections 20 and 15.1). In this subsection we prove that $(\alpha/\pi) \notin \mathbb{Q}$. More precisely, by induction on n we show that $\cos(n\alpha) \notin \mathbb{Z}$, for all $n \in \mathbb{N}$. Now, if $\alpha = \frac{m}{n}\pi$ for some $m, n \in \mathbb{N}$, then $\cos(n\alpha) = 0$, a contradiction.

Let us make an even stronger inductive claim: for every $n \in \mathbb{N}$ we have $\cos n\alpha = r/3^n$, where $3 \nmid r$. The base of induction is clear: $\cos\alpha = \frac{1}{3}$. Now recall that

$$\cos(\beta + \gamma) \,+\, \cos(\beta - \gamma) \,=\, 2 \cos\beta\, \cos\gamma\,.$$

Substituting $\beta = n\alpha$ and $\gamma = \alpha$, we obtain:

$$\cos{(n+1)\alpha} \,=\, \frac{2}{3}\, \cos n\alpha \,-\, \cos{(n-1)\alpha}\,.$$

The inductive claim follows immediately from here. □

41.4. **The Minkowski inequality.** The main result of this subsection in the following inequality used in Section 7 to prove the Brunn–Minkowski inequality (Theorem 7.4). As the reader shall see this is really a disguised form of the arithmetic mean vs. geometric mean inequality.

Theorem 41.4 (The Minkowski inequality)**.** *For every $x_1, \ldots, x_n, y_1, \ldots, y_n > 0$ we have:*

$$\left[\prod_{i=1}^n (x_i + y_i)\right]^{1/n} \,\geq\, \left[\prod_{i=1}^n x_i\right]^{1/n} \,+\, \left[\prod_{i=1}^n y_i\right]^{1/n}.$$

Moreover, the inequality becomes an equality if and only if $x_i = c y_i$, for all $i \in [n]$, and some $c > 0$.

Proof. Recall the arithmetic mean vs. geometric mean inequality:

$$\frac{a_1 + \ldots + a_n}{n} \,\geq\, (a_1 \cdots a_n)^{1/n} \;\text{ for all }\; a_1, \ldots, a_n > 0,$$

and the inequality becomes an equality if and only if $a_1 = \ldots = a_n$. We have:

$$\frac{\left(\prod_{i=1}^n x_i\right)^{1/n} + \left(\prod_{i=1}^n y_i\right)^{1/n}}{\prod_{i=1}^n (x_i+y_i)^{1/n}} = \prod_{i=1}^n \left(\frac{x_i}{x_i+y_i}\right)^{1/n} + \prod_{i=1}^n \left(\frac{y_i}{x_i+y_i}\right)^{1/n}$$
$$\le \frac{1}{n}\sum_{i=1}^n \frac{x_i}{x_i+y_i} + \frac{1}{n}\sum_{i=1}^n \frac{y_i}{x_i+y_i} = 1,$$

and the inequality becomes an equality if and only if there exists $c > 0$, such that $x_i/y_i = c$, for all $i \in [n]$. $\square$

41.5. **The equality part in the Brunn–Minkowski inequality.** Let us start by noting that one can view the second (equality) part in Theorem 7.4 as a uniqueness result: for every A there exists a unique B of given volume such that the Brunn–Minkowski inequality becomes an equality. Of course, the claim does not follow from the convergence argument in Subsection 7.7: just because the equality holds for a pair of convex sets (A, B) does not imply that it must hold for pairs of brick regions (A_n, B_n). Nevertheless, one can still use the brick-by-brick approach to prove the claim.

Proof of the equality part. Denote the unit vectors in the direction of axis coordinates of $\mathbb{R}^d$ by $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_d$. Let $C \subset \mathbb{R}^d$ be a convex set. Divide A into two parts C_0, C_1 of equal volume by a hyperplane orthogonal to $\boldsymbol{e}_1$. Then divide each part into two parts by (separate) hyperplanes orthogonal to $\boldsymbol{e}_2$, to obtain four parts C_{00}, C_{01}, C_{10}, and C_{11} of equal volume. Continue cutting each part into two, cyclically changing the normals $\boldsymbol{e}_i$ of the hyperplanes. After $n = kd$ iterations (k rounds of all d directions) we obtain 2^n convex regions C_i, where $0 \le i < 2^n$ and corresponds to the binary expression of i.

We say that C_i is a *boundary region* if it contains points of the surface: $C_i \cap \partial C \ne \varnothing$. Clearly, non-boundary regions are bricks. Denote by $R_n \subset C$ the union of all non-boundary regions (bricks) C_i after n iterations. The process is illustrated in Figure 41.2.

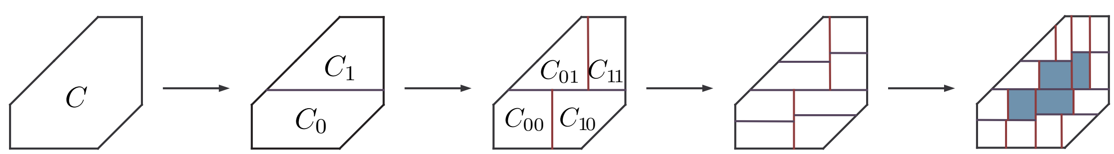

FIGURE 41.2. Division of the region C into 16 equal parts, 12 boundary and 4 non-boundary (shaded).

For a convex region $X \subset \mathbb{R}^d$ denote by $W(X) = [w_1(X) \times \ldots \times w_d(X)]$ the smallest brick containing X. Here $w_i(X)$ denotes the width of X in direction $\boldsymbol{e}_i$ (see Exercise 3.6).

Now let A and B be any convex sets in $\mathbb{R}^d$ for which the Brunn–Minkowski inequality becomes an equality. We can always assume that $\mathrm{vol}(A) = \mathrm{vol}(B)$, or take an appropriate expansion of B otherwise. Rather than apply the above process to A and B, we alter the process and make it dependent of both sets. First, divide A into two parts A_0, A_1 a hyperplane orthogonal to $\boldsymbol{e}_1$, such that $w_1(A_0) = w_1(A_1) = w_1(A)/2$. Similarly, divide B into two parts B_0, B_1 by another hyperplane orthogonal to $\boldsymbol{e}_1$, such that $\mathrm{vol}(B_0) = \mathrm{vol}(A_0)$ and $\mathrm{vol}(B_1) = \mathrm{vol}(A_1)$. Then, divide parts A_0, A_1 by hyperplanes orthogonal to $\boldsymbol{e}_2$ to four parts A_{00}, A_{01}, A_{10}, and A_{11} such that $w_1(A_{00}) = w_1(A_{01}) = w_1(A_0)/2$ and $w_1(A_{10}) =$

$w_1(A_{11}) = w_1(A_1)/2$. Then divide sets B_0, B_1 by hyperplanes into four parts B_{00}, B_{01}, B_{10}, and B_{11}, such that the corresponding sets in A and B have equal volume. Then repeat this iteration of cuts for $\boldsymbol{e}_3,\ldots,\boldsymbol{e}_d$, making a total of d iterations of cuts. Then switch the roles of A and B and in the next d cuts halve the widths of regions in B. This make the total of $2d$ cut, which we call a *round*. Now make k rounds of such cuts. In the notation above, we obtain regions A_i and B_i, $0 \le i < 2^n$, where $n = 2kd$.

Denote by X_n and Y_n the unions of all non-boundary regions A_i and B_i, respectively. Clearly, there may be regions in X_n that do not have corresponding regions in Y_n and vice versa. Let $X_n' \subset X_n$ be a union of non-boundary regions A_i such that the region B_i is also non-boundary. Define $Y_n' \subset Y_n$ analogously. A priori X_n' and Y_n' can be disconnected, so let X_n^*, Y_n^* be their connected components containing centers of mass $\mathrm{cm}(A)$ and $\mathrm{cm}(B)$, respectively.

Observe that after k rounds of cuts as above we have $w_r(A_i) \le w_r(A)/2^k \le \mathrm{diam}(A)/2^k$. Letting $D = \max\{\mathrm{diam}(A), \mathrm{diam}(B)\}$ we conclude that $w_r(A_i), w_r(B_i) \le \varepsilon$, where $\varepsilon = (D/2^k) > 0$. For large enough $k = n/2d$ we can ensure that neither center of mass lies in a boundary region. Further,

$$\mathrm{vol}(X_n^*) \,\ge\, \mathrm{vol}(A) \,-\, [\varepsilon \cdot \mathrm{area}(A)],$$

and thus $\mathrm{vol}(X_n^*), \mathrm{vol}(Y_n^*) \to \mathrm{vol}(A) = \mathrm{vol}(B)$ as $n \to \infty$.

Let us further restrict our regions: denote by X_n° and Y_n° the union of bricks which have no points at distance $\le \varepsilon$ from the boundary, where $\varepsilon > 0$ is as above. From the reasoning as above, for large enough n we obtained two connected brick regions X_n°, Y_n° with corresponding bricks $A_i \subset A$, $B_i \subset B$, and such that $X_n^\circ \to A$, $Y_n^\circ \to B$, as $n \to \infty$.

Now comes the key observation. In the inductive step in the proof of the Brunn–Minkowski inequality in Section 7, we divided the regions A, B into two disjoint parts by a hyperplane in the same ratio, and concluded that the *inequality* holds for sets A, B only if it holds for the corresponding parts. Clearly, the *equality* also holds only if the equality holds for the corresponding parts. Repeating the procedure we get again the equality for the four corresponding pairs of smaller parts, etc. By induction, we conclude that the equality in the Brunn–Minkowski inequality holds for all pairs (A_i, B_i), $0 \le i < 2^n$.

Now, if both A_i and B_i are bricks the equality can hold only if one is an expansion of the other. Since $\mathrm{vol}(A_i) = \mathrm{vol}(B_i)$ by construction, bricks A_i, B_i must be equal up to a translation. By itself, this does not imply that X_n° and Y_n° are equal. It does, however, imply that they are converging to the same convex set, by the following argument.

We say that two bricks A_i and $A_{i'}$ *touch* each other if their boundaries intersect in points interior to the faces. Observe that the corresponding bricks B_i and $B_{i'}$ do not necessarily have to touch, but they must both touch a hyperplane separating them, a hyperplane corresponding to a hyperplane separating A_i and A_i'.

Denote by A_j and B_j, $0 \le j < 2^d$, the regions resulted after the first round of cuts. Consider $A_j^\circ = A_j \cap X_n^\circ$ and $B_j^\circ = B_j \cap Y_n^\circ$. Both brick regions contain a large number of smaller regions, but none of them boundary regions. Furthermore, since point of the boundary region can lie in an interval parallel to $\boldsymbol{e}_r$ between two points in A_j° or in B_j°.

Now, we claim that $A_j^\circ \simeq B_j^\circ$ for all j. In other words, these regions can be obtained from each other by a translation. Indeed, start at a corner which must contain equal corresponding bricks. Start adding bricks along the straight interior edges of A_j°, and all of them will sequentially touch each other. Thus, the interior edges of B_j° will be lined up with equal corresponding bricks. Now start filling all 2-dimensional interior faces in A_j° and B_j°, again by the equal corresponding bricks, etc. Eventually we obtain two equal fillings of the regions, implying that the regions themselves are equal. Note that we are implicitly using the fact that by convexity of A, B and the construction, there are no 'holes' in A_j°, B_j°.

While the corresponding regions A_j°, B_j° are equal, we can no longer use the same argument to prove that $X_n^\circ = Y_n^\circ$. Instead, let us put them together in the reverse order they were obtained by separation with hyperplanes. When put together, regions A_j° and B_j° can be shifted along the axes. Observe that by convexity of A and B, they cannot be shifted by more than (2ε) in each direction. Thus in fact the distance between the corresponding points in X_n°, Y_n° is at most $\sqrt{d}\,(2\varepsilon) \to 0$ as $n \to \infty$.

Make a translation of Y_n° so that it has the came center of mass as X_n°. From above, we have $\operatorname{vol}(Y_n^\circ \smallsetminus X_n^\circ) \to 0$. Since $X_n^\circ \to A$ and $Y_n^\circ \to B$, this implies that $A \simeq B$, and completes the proof. $\square$

41.6. **The Cayley–Menger determinant.** Here we give an explicit computation of the Cayley–Menger determinant defined and studied in Section 34. This is an important ingredient in the proof of the bellows conjecture (Theorem 31.2).

Theorem 34.5 (Cayley–Menger). *For every simplex $\Delta = (v_0 v_1 \dots v_d) \subset \mathbb{R}^d$, we have:*

$$\operatorname{vol}^2(\Delta) \; = \; \frac{(-1)^{d-1}}{2^d \, d!^2} \cdot \det \begin{pmatrix} 0 & 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & \ell_{01}^2 & \ell_{02}^2 & \dots & \ell_{0d}^2 \\ 1 & \ell_{01}^2 & 0 & \ell_{12}^2 & \dots & \ell_{1d}^2 \\ 1 & \ell_{02}^2 & \ell_{12}^2 & 0 & \dots & \ell_{2d}^2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \ell_{0d}^2 & \ell_{1d}^2 & \ell_{2d}^2 & \dots & 0 \end{pmatrix},$$

where $\ell_{ij} = |v_i v_j|$, for all $0 \le i < j \le d$.

Proof of Theorem 34.5. Suppose $v_i = (x_{i1}, \dots, x_{id}) \in \mathbb{R}^d$, for all $0 \le i \le d$. Define matrices A and B as follows:

$$A \; = \begin{pmatrix} x_{01} & x_{02} & \dots & x_{0d} & 1 \\ x_{11} & x_{12} & \dots & x_{1d} & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{d1} & x_{d2} & \dots & x_{dd} & 1 \end{pmatrix}, \qquad B \; = \begin{pmatrix} x_{01} & x_{02} & \dots & x_{0d} & 0 \\ x_{11} & x_{12} & \dots & x_{1d} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{d1} & x_{d2} & \dots & x_{dd} & 0 \end{pmatrix}.$$

Clearly, $\det(B) = 0$, and

$$\det(A) \; = \; \det \begin{pmatrix} x_{01} & x_{02} & \dots & x_{0d} & 1 \\ x_{11} - x_{01} & x_{12} - x_{02} & \dots & x_{1d} - x_{0d} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{d1} - x_{01} & x_{d2} - x_{02} & \dots & x_{dd} - x_{0d} & 0 \end{pmatrix} \; = \; (-1)^d \, d! \operatorname{vol}(\Delta).$$

Note that

$$A \cdot A^T = \big(\langle v_i, v_j \rangle + 1\big)_{0 \le i,j \le d}, \quad \text{and} \quad B \cdot B^T = \big(\langle v_i, v_j \rangle\big)_{0 \le i,j \le d}.$$

From here we have:

$$\det(A\cdot A^T) \,=\, \det\begin{pmatrix} \langle v_0,v_0\rangle+1 & \langle v_0,v_1\rangle+1 & \dots & \langle v_0,v_d\rangle+1 & 0\\ \langle v_1,v_0\rangle+1 & \langle v_1,v_1\rangle+1 & \dots & \langle v_1,v_d\rangle+1 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ \langle v_d,v_0\rangle+1 & \langle v_d,v_1\rangle+1 & \dots & \langle v_d,v_d\rangle+1 & 0\\ 1 & 1 & \dots & 1 & 1 \end{pmatrix}$$

$$= \,-\,\det\begin{pmatrix} \langle v_0,v_0\rangle & \langle v_0,v_1\rangle & \dots & \langle v_0,v_d\rangle & 1\\ \langle v_1,v_0\rangle & \langle v_1,v_1\rangle & \dots & \langle v_1,v_d\rangle & 1\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ \langle v_d,v_0\rangle & \langle v_d,v_1\rangle & \dots & \langle v_d,v_d\rangle & 1\\ 1 & 1 & \dots & 1 & -1 \end{pmatrix}.$$

Since $\det\bigl(\langle v_i,v_j\rangle\bigr)=\det(B\cdot B^T)=\det^2(B)=0$, in the determinant above we can replace the bottom right entry -1 with 0. Denote the resulting matrix by M.

Now observe that $\ell_{ij}^2=\langle v_i,v_i\rangle-2\langle v_i,v_j\rangle+\langle v_j,v_j\rangle$. Denote by C the matrix as in the theorem. Using row and column operations we obtain:

$$\det(C) \,=\, \det\begin{pmatrix} -2\langle v_0,v_0\rangle & -2\langle v_0,v_1\rangle & \dots & -2\langle v_0,v_d\rangle & 1\\ -2\langle v_1,v_0\rangle & -2\langle v_1,v_1\rangle & \dots & -2\langle v_1,v_d\rangle & 1\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ -2\langle v_d,v_0\rangle & -2\langle v_d,v_1\rangle & \dots & -2\langle v_d,v_d\rangle & 1\\ 1 & 1 & \dots & 1 & 0 \end{pmatrix} \,=\, (-2)^d\det(M).$$

Since $\det(M)=-\det(A\cdot A^T)=-\bigl(d!\,\mathrm{vol}(\Delta)\bigr)^2$, we conclude that

$$\det(C) \,=\, (-1)^{d-1}\,2^d\,d!^2\,\mathrm{vol}^2(\Delta),$$

as desired. $\square$

41.7. **The theory of places.** The proof of the bellows conjecture (Theorem 31.2) is based on the integrality criteria (Theorem 34.2), a classical result in the theory of places. In this section we present a reworking of the presentation in [Lan], making it self-contained and, hopefully, more accessible to the reader unfamiliar with the field.

Let L be a field containing ring R. We will always assume that R contains 1. To prove the integrality criteria we need to show:

A. if $x\in L$ is integral over R, then every place that is finite over R, is also finite over x;

B. if $x\in L$ is *not* integral over R, then there exists a place that is finite over R, and infinite over x.

While the proof of the bellows conjecture uses only part **B**, we include a simple proof of part **A** for the sake of clarity and completeness.

Proof of part **A**. Since $x\in L$ is integral over R, by definition we have

$$x^n\,+\,a_{n-1}x^{n-1}\,+\,\ldots+\,a_1x\,+\,a_0\,=\,0\,,$$

where $a_i\in R$. Dividing both sides by x^n we obtain:

$$1\,+\,\frac{a_{n-1}}{x}\,+\,\ldots+\,\frac{a_1}{x^{n-1}}\,+\,\frac{a_0}{x^n}\,=\,0\,.$$

Recall that $\widehat{F} = F \cup \{\infty\}$ and let $\varphi : L \to \widehat{F}$ be a place which is finite on R. Suppose that $\varphi(x) = \infty$. Then $\varphi(\frac{1}{x}) = 0$. Applying φ to both sides of the above equation, we obtain $1 = 0$, a contradiction. $\square$

Proof of part **B**. Since x is not integral over R, we have $x \neq 0$. Let us construct a place $\varphi : L \to \widehat{F}$, for some (possibly very large) field F, such that $\varphi(\frac{1}{x}) = 0$. This would imply that $\varphi(x) = \infty$ and prove the claim.

Denote by $R\langle\frac{1}{x}\rangle$ the ring generated by R and $\frac{1}{x}$. Consider the ideal $I = \frac{1}{x}R\langle\frac{1}{x}\rangle$ generated by $\frac{1}{x}$. Note that $I \neq R\langle\frac{1}{x}\rangle$. Indeed, otherwise $1 \in \frac{1}{x}R\langle\frac{1}{x}\rangle$ and we can write

$$1 = \frac{a_1}{x} + \frac{a_2}{x^2} + \ldots + \frac{a_n}{x^n},$$

for some $a_i \in R$. Multiplying both sides by x^n gives a monic polynomial equation for x, implying that x is integral over R, a contradiction.

From above, there exists a maximal ideal $\mathsf{m} \subset R\langle\frac{1}{x}\rangle$ containing I. Since m is a maximal ideal, we can consider a field $F_1 = R\langle\frac{1}{x}\rangle/\mathsf{m}$. This defines a natural map $\varphi : R\langle\frac{1}{x}\rangle \to F_1$, such that $\varphi(\frac{1}{x}) = 0$. Denote by F the algebraic closure of F_1. Let us extend the map φ from F_1 to F, still calling it φ. We have $\varphi : R\langle\frac{1}{x}\rangle \to F$. [96]

Now, consider a set of ring homomorphisms $\psi_S : S \to F$ such that $R\langle\frac{1}{x}\rangle \subset S \subsetneq L$, and $\psi_S(y) = \varphi(y)$ for all $y \in R\langle\frac{1}{x}\rangle$. There is a natural partial order $\psi_S < \psi_{S'}$ on these homomorphisms, where $S \subset S'$ and $\psi_{S'}$ is an extension of ψ_S.

Consider an increasing sequence of ring homomorphisms $\psi_1 < \psi_2 < \ldots < \psi_i < \ldots$, corresponding to an increasing sequence of rings $R\langle\frac{1}{x}\rangle \subset S_1 \subset S_2 \subset \ldots \subset S_i \subset \ldots$, where $S_i \subsetneq L$. We can construct a ring $S = \cup_i S_i$ and a homomorphism $\psi : S \to F$, where $\psi_i < \psi$ for all $i = 1, 2, \ldots$ Clearly, $S \subsetneq L$, since if $S = L$ then $x \in S_i$ for some i, and $1 = \varphi(1) = \varphi(x \cdot \frac{1}{x}) \neq \varphi(x) \cdot \varphi(\frac{1}{x}) = 0$. Therefore, homomorphism ψ is an upper bound on ψ_i. Clearly, the same argument works for all infinite chains of homomorphisms (not necessarily countable), and by Zorn's lemma the partial order defined above has a maximal element, i.e., a homomorphism which cannot be extended.

The following lemma characterizing the ring of the maximal homomorphism is the key result which allows us to finish the construction. We postpone the proof of the lemma until the end of the proof of the theorem.

Lemma 41.5. *Let $\psi : S \to F$ be a* maximal homomorphism*, where $\varphi < \psi$ and $R\langle\frac{1}{x}\rangle \subset S \subsetneq L$. Then for every $y \in L$ either $y \in S$ or $\frac{1}{y} \in S$.*

We can now define $\widehat{\psi} : L \to \widehat{F}$ by letting $\widehat{\psi}(y) = \psi(s)$ for all $y \in S$, and $\widehat{\psi}(y) = \infty$ for all $y \notin S$. Since $S \subsetneq L$ is a subring, it is easy to see (by checking all relations in the definition of a place) that

$$(\ominus) \qquad \widehat{\psi}\left(\frac{1}{y}\right) = 0 \quad \text{for all } \ y \notin S$$

implies that $\widehat{\psi}$ is a place. Therefore, to prove part **B** it suffice to establish $(\ominus)$.

By the lemma, for every $y \notin S$ we have $\frac{1}{y} \in S$. Now suppose that $\psi(\frac{1}{y}) \neq 0$. Take a bigger ring $S' = S\langle y\rangle$ generated by S and y, with a homomorphism $\psi' : S' \to F$ defined by $\psi'(y) = 1/\psi(\frac{1}{y})$ and $\psi'(s) = \varphi(s)$, for all $s \in S$. To see that the map ψ' is indeed a ring homomorphism, simply note that $\sum_{i=0}^n s_i y^i = 0$ is equivalent to $\sum_{i=0}^n s_i (\frac{1}{y})^{n-i} = 0$,

[96] The importance of taking the algebraic closure $F \supset F_1$ will come only at the end of the proof of Lemma 41.5. Until then, the arguments work for any extension of F_1.

for all $s_i \in S$. Since $y \in S'$, we conclude that $\psi < \psi'$, a contradiction with ψ being maximal. Therefore, $\psi(\frac{1}{y}) = 0$ and we proved ($\ominus$). This completes the proof of part **B** modulo Lemma 41.5. $\square$

Proof of Lemma 41.5. Suppose $y, \frac{1}{y} \notin S$ for some $y \in L$. Define $S_1 = S\langle y\rangle$ and $S_2 = S\langle \frac{1}{y}\rangle$. We obtain a contradiction with maximality of $\psi : S \to F$ by showing that ψ can be extended to at least one of the rings S_1 or S_2.

Let $\mathsf{m} = \mathrm{Ker}(\psi : S \to F) \subset S$ be the ideal of elements mapped into $0 \in F$. Let us start with the following two technical lemmas whose proofs we postpone.

Sublemma 41.6. *The ideal* $\mathsf{m} \subset S$ *is maximal. In addition, for every* $y \notin \mathsf{m}$*, we have* $\frac{1}{y} \in S$.[97]

Sublemma 41.7. *Either* $\mathsf{m}S_1 \neq S_1$ *or* $\mathsf{m}S_2 \neq S_2$.

By Sublemma 41.6, the ideal $\mathsf{m} \subset S$ is maximal and S/m is a field. Thus, we obtain a natural field homomorphism $\overline{\psi} : S/\mathsf{m} \to F$. By Sublemma 41.7, we can assume that $\mathsf{m}S_1 \subsetneq S_1$.

Since $\mathsf{m}S_1$ is an ideal in S_1, we can consider a maximal ideal $\mathsf{m}_1 \supset \mathsf{m}$ in S_1. Then S_1/m_1 is a field extension of S/m, obtained by adding $\alpha = \overline{y}$, defined as the class of y. Note here that this implies that α is algebraic over S/m, since otherwise $S_1/\mathsf{m}_1 = S/\mathsf{m}[\alpha]$ is not a field.

Now, to prove the lemma it suffices to construct an extension $\overline{\psi}_1 : S_1/\mathsf{m}_1 \to F$ of $\overline{\psi}$, since this would give a ring homomorphism $\psi_1 : S_1 \to F$ by letting $\psi_1 : \mathsf{m}_1 \to 0$. This would imply that $\psi < \psi_1$, i.e., $\psi : S \to F$ is not maximal, a contradiction. Clearly, we need only to define $\overline{\psi}_1(\alpha)$ as both homomorphisms must coincide on S/m. Let $\overline{\psi}_1(\alpha)$ be a root of $\psi(f)$ over F. This is possible since $F \supset F_1$ is algebraically closed. Thus, we obtain the desired extension $\overline{\psi}_1$. This completes the proof of the lemma. $\square$

Proof of Sublemma 41.6. Let $\psi : S \to F$ be the maximal homomorphism as in the lemma, let $\overline{\psi} : S/\mathsf{m} \to F$ be the corresponding homomorphism modulo ideal m, and let $\gamma : S \to S/\mathsf{m}$ be a natural projection.

Let $y \in S$, $y \notin \mathsf{m}$, be an element of the ring, and let $\alpha = \gamma(y) \in S/\mathsf{m}$ be its projection. From above, $\alpha \neq 0$. By the definition of $\mathsf{m} = \mathrm{Ker}(\psi)$ we conclude that $\overline{\psi}$ is an injection. Since $\alpha \neq 0$, we conclude that $\overline{\psi}(\alpha) \neq 0$ and $\psi(y) \neq 0$.

Now take $S' = S\langle \frac{1}{y}\rangle$. Let $\psi'(s) = \psi(s)$ for all $s \in S$, and let $\psi'(\frac{1}{y}) = \frac{1}{\psi(y)}$. It is easy to see that ψ' uniquely extends to a homomorphism $\psi' : S' \to F$. Since $\psi < \psi'$, from the maximality of ψ we conclude that $S = S'$, and $\frac{1}{y} \in S$. This proves second part of the sublemma.

For the first part, note that there is an element $\frac{1}{\alpha} = \gamma(\frac{1}{y})$. Therefore, every nonzero element in S/m is invertible, and S/m is a field. This implies that m is maximal. $\square$

Proof of Sublemma 41.7. From the contrary, suppose $\mathsf{m}S_1 = S_1$ and $\mathsf{m}S_2 = S_2$. Then there exist polynomials

$$(\boxtimes) \qquad 1 = \sum_{i=0}^{n} a_i\, y^i \quad \text{and} \quad 1 = \sum_{j=0}^{k} b_j \left(\frac{1}{y}\right)^j,$$

[97]This says that S is a *local ring*. To make the presentation self-contained we will refrain from using this notion.

for some $a_i, b_j \in \mathsf{m}$, where the degrees k, n are chosen to be the smallest possible. By symmetry, we can assume that $k \le n$. Rewriting the second polynomial, we have:

$$y^k = \sum_{j=0}^{k} b_j\, y^{k-j}. \quad \text{Equivalently,} \quad (1-b_0)\, y^k = \sum_{j=1}^{k} b_j\, y^{k-j}.$$

Now observe that $(1-b_0)$ is invertible. Indeed, since $1 \notin \mathsf{m}$ and $b_0 \in \mathsf{m}$ we have $(1-b_0) \notin \mathsf{m}$, and the second part of Sublemma 41.6 implies that $(1-b_0)^{-1} \in S$. This gives:

$$y^k = \frac{1}{1-b_0} \sum_{j=1}^{k} b_j\, y^{k-j}. \quad \text{Equivalently,} \quad y^n = \frac{1}{1-b_0} \sum_{j=1}^{k} b_j\, y^{n-j}.$$

This allows us to decrease the degree of the first polynomial in (⊠):

$$1 = \sum_{i=0}^{n-1} a_i\, y^i + a_n\, y^n = \sum_{i=0}^{n-1} a_i\, y^i + \frac{a_n}{1-b_0} \sum_{i=n-k}^{n-1} b_{n-i}\, y^i,$$

a contradiction with the minimality of n. □

41.8. **The mapping lemma.** In Sections 35 and 37 we use the following standard result.

Theorem 41.8 (The mapping lemma)**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be two manifolds of the same dimension. Suppose a map $\varphi : \mathcal{A} \to \mathcal{B}$ satisfies the following conditions:*

1) *every connected component of $\mathcal{B}$ intersects the image $\varphi(\mathcal{A})$,*
2) *map φ is injective, i.e., $\varphi(a_1) = \varphi(a_2)$ implies that $a_1 = a_2$,*
3) *map φ is continuous,*
4) *map φ is proper.*

Then φ is a homeomorphism; in particular, φ is bijective.

Here by a *proper map* φ we mean that for every sequence of points $\{a_i \in \mathcal{A}\}$ and images $\{b_i = \varphi(a_i)\}$, if $b_i \to b \in \mathcal{B}$ as $i \to \infty$, then there exists $a \in \mathcal{A}$, such that $b = \varphi(a) \in \mathcal{B}$, and a is a limit point of $\{a_i \in \mathcal{A}\}$.

To get better acquainted with the lemma, consider the following examples which show that the conditions in the lemma are necessary.

Example 41.9. Suppose $\mathcal{A} = S^1$ and $\mathcal{B} = S^2$, i.e., we have two manifolds of different dimensions. Then any embedding $\varphi : S^1 \to S^2$ satisfies conditions 1) to 4), but φ is obviously not bijective.

Similarly, if $\mathcal{A} = S^1$ and $\mathcal{B} = S^1 \cup S^1$, i.e. $\mathcal{B}$ is a union of two disjoint circles. Then the map sending $\mathcal{A}$ into the first circle in $\mathcal{B}$ satisfies conditions 2) to 4), but is not a bijection. If in this example one takes $\mathcal{B}'$ to be two circles attached at a point, then all conditions 1) to 4) are satisfied. Clearly, φ is not bijective, which may be puzzling at first, until one realizes that $\mathcal{B}'$ is no longer a manifold.

Further, if now $\mathcal{A} = S^1 \cup S^1$, $\mathcal{B} = S^1$ and φ sends both circles into one, then all conditions are satisfied, except for condition 2). Finally, suppose both $\mathcal{A}$ and $\mathcal{B}$ are open unit disks and let φ be a map contracting $\mathcal{A}$ into a disk of smaller radius. Then all conditions are satisfied except for the condition 4), and φ is obviously not bijective.

Proof of Theorem 41.8. By continuity of φ, different connected components of $\mathcal{A}$ are mapped into different connected components of $\mathcal{B}$. Thus, it suffices to prove the result when $\mathcal{B}$ is connected.

Let $\mathcal{B}' = \{\varphi(a), a \in \mathcal{A}\} \subset \mathcal{B}$ be the image of the map φ. By injectivity of φ, the map $\varphi^{-1} : \mathcal{B}' \to \mathcal{A}$ is well defined. Furthermore, condition 4) implies that φ^{-1} is also continuous. Therefore, $\varphi : \mathcal{A} \to \mathcal{B}'$ is a homeomorphism, and $\mathcal{B}'$ is open in $\mathcal{B}$. On the other hand, condition 4) implies that $\mathcal{B}'$ is also closed in $\mathcal{B}$. Since $\mathcal{B}$ is connected, this implies that $\mathcal{B}' = \mathcal{B}$ and completes the proof. $\square$

41.9. **Final remarks.** The results in spherical geometry can be found in virtually all classical geometry textbook (see [Hada]). The presentation of Girard's formula and the law of cosines follows [Ber1, §18.3, 18.6]. Our proof of the Minkowski inequality (Theorem 41.4) is standard. See [BecB] for a different proof, and numerous proofs of the arithmetic mean vs. geometric mean inequality. For a similar computation of the Cayley–Menger determinant, and several related results we refer to [Ber1, §9.7], while further references can be found in [GriK, §3.6]. For the original expanded presentation of the mapping lemma (Theorem 41.8) see [A2, §2.2]. See also [Ale6] for a background and a survey of applications to convex polyhedra.

42.1. **Problems on polygons and polyhedra.**

Exercise 42.1. [1-] A quadrilateral in $\mathbb{R}^3$ is tangent to a sphere. Prove that the tangency points are coplanar.

Exercise 42.2. Let $Q = [x_0, x_1, \ldots, x_n]$, $x_i \in \mathbb{R}^3$ be a 3-dimensional n-gon, where $x_0 = x_n$. We say that Q is *regular* if $|x_{i-1}, x_i| = 1$ and $\angle x_{i-1}x_ix_{i+1} = \alpha$, for all $1 \le i \le n$ and some fixed $0 < \alpha < \pi$. Space polygon Q is called *non-degenerate* if no four points of Q are coplanar.
a) [1] Prove that every regular pentagon lies on a plane.
b) [1] Prove that for all $n \ge 6$ there exist non-planar regular n-gons.
c) [1+] Prove that for n large enough there exist non-degenerate regular n-gon.

Exercise 42.3. (*Sylvester's problem*) [1+] Let A be a convex set and let x_1, x_2, x_3, x_4 be chosen at random from A. Denote by $p(A)$ the probability that four points form a convex quadrilateral. Compute $p(A)$ for a circle, a square, and an equilateral triangle.

Exercise 42.4. [2] Let Q be a piecewise linear curve in the plane. Prove that one can add straight shortcuts of total length at most $c_1|Q|$ such that the distance between every two points x, y along Q is at most c_2 times the Euclidean distance $|xy|$, for some universal constants c_1, c_2.

Exercise 42.5. [1-] Prove that in every simplicial polytope $P \subset \mathbb{R}^3$ there exists an edge such that all face angles adjacent to it are acute.

Exercise 42.6. [1-] Let $P \subset \mathbb{R}^3$ be a convex polytope with faces F_i, $1 \le i \le n$, and let $X_i \subset F_i$ be convex polygons inside the faces. Prove that for every $\varepsilon > 0$ there exists a convex polytope $Q \subset P$ such that all X_i are faces of Q and $\text{vol}(Q) > (1-\varepsilon)\text{vol}(P)$.

Exercise 42.7. [1] Let $\Delta \subset \mathbb{R}^3$ be a tetrahedron and let x, y be the midpoints of opposite edges. Prove that every plane passing through x, y divides Δ into two polytopes of equal volume.

Exercise 42.8. [1] Let $P \subset \mathbb{R}^d_+$ be a convex polytope lying in the positive orthant. Prove that for every interior point $a = (a_1, \ldots, a_d) \in P$ there exists a vertex $v = (v_1, \ldots, v_d)$, such that $v_i \le d \cdot a_i$ for all $1 \le i \le d$.

Exercise 42.9. [1] Find the maximal number of sides in a polygon obtained as the intersection of a unit hypercube and a 2-dimensional plane.

Exercise 42.10. [1+] Prove that for every tetrahedron $\Delta \subset \mathbb{R}^3$ there exist two planes L_1, L_2, such that the areas A_1, A_2 of projections of Δ satisfy $A_1/A_2 \ge \sqrt{2}$. Check that this bound is optimal.

Exercise 42.11. [1+] Let $P \subset \mathbb{R}^3$ be a convex polytope with n facets, and let Q be its projection on a plane. What is the maximal possible number of edges the polygon Q can have?

Exercise 42.12. [1-] Let $D \subset \mathbb{R}^3$ be a unit cube centered at the origin O. Suppose C is a cone with O as a vertex and all face angles equal to $\pi/2$. Prove or disprove: at least one vertex of D lies in C.

Exercise 42.13. *a*) [2] Suppose $P \subset \mathbb{R}^3$ is a convex body, such that all projections of P are convex polygons. Prove that P is a convex polytope.
b) [2-] Suppose further, that all projections are polygons with at most n sides. Then P has $n^{O(n)}$ facets.
c) [1] Give an explicit construction of polytopes P with $e^{\Omega(n)}$ facets.

Exercise 42.14. *a*) [2] Suppose $P \subset \mathbb{R}^3$ is a convex body, such that all projections of P are convex polygons. Prove that P is a convex polytope.
b) [2-] Suppose further, that all projections are polygons with at most n sides. Then P has $n^{O(n)}$ facets.

Exercise 42.15. *a*) [1] Let $P \subset \mathbb{R}^3$ be a convex body with countably many extremal points. Prove or disprove: the set of extremal points is closed, i.e., the limit of every converging sequence of extremal points is also extremal.
b) [2] Let $P \subset \mathbb{R}^3$ be a convex body such that every projection of P on a plane is a convex polygon. Prove that P is a convex polytope.

Exercise 42.16. (*Pohlke's theorem*) [1+] Prove that every convex quadrilateral $Q \subset \mathbb{R}^2$ is an oblique projection of a regular tetrahedron.

Exercise 42.17. Let Q be a 3-dimensional unit cube.
a) [1] Denote by X a projection of Q onto a plane. Prove that $1 \le \operatorname{area}(X) \le \sqrt{3}$.
b) [1+] Denote by X the projection of Q onto a plane H, and by ℓ the length of a projection of Q onto a line orthogonal to H. Prove that $\ell = \operatorname{area}(X)$.
c) [1+] Denote by Y a *cross section* of Q, i.e., the intersection of Q with a plane. Prove that $1 \le \operatorname{area}(Y) \le \sqrt{2}$.

Exercise 42.18. *a*) [1] Let $P \subset \mathbb{R}^3$ be a convex polytope, and let Δ be a tetrahedron with vertices at vertices of P of largest volume. Prove that for every plane H, an orthogonal projection of Δ onto H has area at least $1/9$ of the area of the projection of P onto H.
b) [1+] Improve the above bound to $1/7$.

Exercise 42.19. Let $P, P' \subset \mathbb{R}^3$ be two convex polytopes. We say that P *passes through* P' if there exist projections Q and Q' (of P and P') such that Q fits inside Q'. When $P = P'$ we say that P *passes through itself*.[98]
a) [1] Prove that every tetrahedron passes through itself.
b) [1-] Prove that the cube passes through itself.
c) [1] Prove that the regular octahedron passes through itself.
d) [$*$] Prove or disprove: every convex polytope passes through itself.

Exercise 42.20. (*Shadows of polytopes*) *a*) [2] Let $X \subset L$ be a polygon in the plane $L \subset \mathbb{R}^3$. Suppose $Y \subset H$ is an orthogonal projection of X onto another plane H. Prove that there exists a rigid motion ρ such that $\rho(Y) \subset X$. In other words, the shadow of any polygon is smaller than the polygon itself.
b) [1-] Show that part *a*) does not generalize to 3-dimensional convex polytopes in a hyperplane in $\mathbb{R}^4$.
c) [1+] Prove that when $d \ge 3$ no d-dimensional polytope P can cover all its shadows. For example, when $d = 4$, *every* convex polytope satisfies *b*).

[98]The idea is that in a solid Q one can make a hole so that P can pass through that hole.

Exercise 42.21. *a*) [1-] Let $C_d \subset \mathbb{R}^d$ be a unit cube, $d \geq 3$. Prove that there exists a hyperplane which intersects every facet of C_d.
b) [1+] Prove that for every convex polytope $P \subset \mathbb{R}^2$ and two vertices v, w of P there exists a plane L containing v, w, and such that at least three faces of P are not intersected by L.
c) [1+] Prove that for every simplicial convex polytope $P \subset \mathbb{R}^3$ and two vertices v, w of P there exists a hyperplane H containing v, w, and such that at least three facets of P are not intersected by H.
d) [2-] Find a polytope $P \subset \mathbb{R}^4$ and two vertices v, w, such that every hyperplane H containing v and w intersects all but at most two facets.

Exercise 42.22. [1+] Let $P \subset \mathbb{R}^4$ be a product of two triangles:

$$P = \{(x_1, x_2, y_1, y_2) \mid (x_1, x_2) \in \Delta_1, (y_1, y_2) \in \Delta_2\}.$$

Prove that there are no 9-gon projections of P on a 2-dimensional plane.

Exercise 42.23. [2-] Let $P \subset \mathbb{R}^3$ be a convex polytope with the property that for every face F of P there is a parallel face F' and a parallel plane L, such that L contains all vertices of P that are not in F or F'. Prove that K has at most 14 faces. Show that this bound is tight.

Exercise 42.24. [1+] Find a polyhedral embedding of a torus which has 7 (non-convex) faces, such that every two faces have a common edge.

Exercise 42.25. [1] Find a polyhedral embedding of a torus which has 7 vertices, 14 triangular faces, and such that every two vertices are connected by an edge.

42.2. **Volume, area and length problems.**

Exercise 42.26. [1-] Let $\Delta \subset \mathbb{R}^3$ be a tetrahedron. Prove that Δ has a vertex v, such that the lengths of three edges of Δ adjacent to v satisfy the triangle inequality.

Exercise 42.27. [1] For a tetrahedron $\Delta \subset \mathbb{R}^3$, denote by $h(\Delta)$ the minimal height in Δ, and by $w(\Delta)$ the minimal distance between the opposite edges in Δ. Prove that $\frac{2}{3} h(\Delta) \leq w(\Delta) \leq h(\Delta)$.

Exercise 42.28. (*Sarron's formula*) [1] Let $X = [x_1 x_2 \ldots x_n] \subset \mathbb{R}^2$ be a simple polygon and let $\alpha_1, \ldots, \alpha_n$ be the angles defined as in Figure 42.1. Denote by $\ell_i = |x_i x_{i+1}|$ the edge lengths of X. Prove:

$$\operatorname{area}(X) \,=\, \frac{1}{2} \sum_{i<j} \ell_i \ell_j \sin(\alpha_j - \alpha_i)\,.$$

Exercise 42.29. [1] Suppose points $x_1, \ldots, x_n$ lie on a unit sphere $\mathbb{S}^2$. Prove that

$$\sum_{1 \leq i < j \leq n} |x_i x_j|^2 \,\leq\, n^2.$$

Exercise 42.30. (*Law of sines in* $\mathbb{S}^2$) [1] Let $(ABC) \subset \mathbb{S}^2$ be a triangle on a unit sphere with edge lengths $|BC|_{\mathbb{S}^2} = a$, $|AC|_{\mathbb{S}^2} = b$, and $|AB|_{\mathbb{S}^2} = c$. Suppose also $\alpha = \sphericalangle A$, $\beta = \sphericalangle B$ and $\gamma = \sphericalangle C$ be the spherical angles in the triangle. Then

$$\frac{\sin(a)}{\sin(\alpha)} \,=\, \frac{\sin(b)}{\sin(\beta)} \,=\, \frac{\sin(c)}{\sin(\gamma)}\,.$$

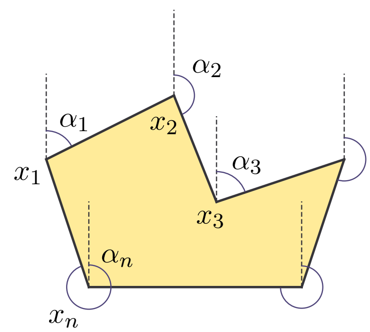

FIGURE 42.1. Polygon $X = [x_1 x_2 \ldots x_n]$ and angles α_i in Sarron's formula.

Exercise 42.31. (*Law of sines in* $\mathbb{R}^3$) [1-] Let $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$ and $\mathbf{c} = \overrightarrow{OC}$ be three non-collinear vectors in $\mathbb{R}^3$. Denote by K the cone spanned by these vectors at the origin O. Define the 3-*dimensional sine function* as

$$ {}^3\!\sin(O, ABC) \, = \, \frac{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}{[\mathbf{a}, \mathbf{b}] \cdot [\mathbf{a}, \mathbf{c}] \cdot [\mathbf{b}, \mathbf{c}]} \, , $$

where $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ is the volume of a parallelepiped spanned by the vectors; similarly, $[\mathbf{a}, \mathbf{b}]$ is the area of a parallelogram spanned by $\mathbf{a}, \mathbf{b}$. Consider a tetrahedron $(OABC) \subset \mathbb{R}^3$. Prove the following *law of sines*:

$$ \frac{\operatorname{area}(ABC)}{{}^3\!\sin(O, ABC)} = \frac{\operatorname{area}(ABO)}{{}^3\!\sin(C, ABO)} = \frac{\operatorname{area}(ACO)}{{}^3\!\sin(B, OAC)} = \frac{\operatorname{area}(BCO)}{{}^3\!\sin(A, OBC)} \, . $$

Exercise 42.32. (*Polar sine in* $\mathbb{R}^3$) [1] Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be three non-collinear vectors in $\mathbb{R}^3$. Define the *polar sine function* as

$$ \operatorname{psin}(\mathbf{a}, \mathbf{b}, \mathbf{c}) \, = \, \frac{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}{\|\mathbf{a}\| \cdot \|\mathbf{b}\| \cdot \|\mathbf{c}\|} \, . $$

For every nonzero vector $\boldsymbol{u}$ in $\mathbb{R}^3$, prove the following inequality:

$$ \operatorname{psin}(\mathbf{a}, \mathbf{b}, \mathbf{c}) \, \le \, \operatorname{psin}(\boldsymbol{u}, \mathbf{b}, \mathbf{c}) \, + \, \operatorname{psin}(\mathbf{a}, \boldsymbol{u}, \mathbf{c}) \, + \, \operatorname{psin}(\mathbf{a}, \mathbf{b}, \boldsymbol{u}) \, . $$

Exercise 42.33. (*The law of cosines in* $\mathbb{R}^3$) [1+] Let $P \subset \mathbb{R}^3$ be a convex polyhedron with $n+1$ faces $F, G_1, \ldots, G_n$. Denote $B = \operatorname{area}(F)$, and $A_i = \operatorname{area}(G_i)$, $1 \le i \le n$. Finally, let α_{ij} be an angle between between planes spanned by G_i and G_j. Prove that

$$ B^2 \, = \, \sum_{i=1}^{n} A_i^2 \, - \, \sum_{i \ne j} A_i A_j \cos \alpha_{ij} \, . $$

Generalize the result to non-convex polyhedra and to higher dimensions.

Exercise 42.34. [1-] Let $x_1, x_2, x_3 \in \mathbb{S}^2$ be points on a unit sphere in $\mathbb{R}^3$ centered at the origin O, and let $\Delta = (O, x_1, x_2, x_3)$ be the simplex spanned by these points. Denote by α_{ij} the angles between (O, x_i) and (O, x_j). Prove that

$$ \operatorname{vol}^2(\Delta) \, = \, \frac{1}{36} \det \begin{pmatrix} 1 & \cos \alpha_{12} & \cos \alpha_{13} \\ \cos \alpha_{12} & 1 & \cos \alpha_{23} \\ \cos \alpha_{13} & \cos \alpha_{23} & 1 \end{pmatrix} . $$

Exercise 42.35. [1] Fix a vertex v of a given tetrahedron Δ. Let a, b, c be the lengths of edges adjacent to v, and let α, β, γ be the face angles between these edges. Prove that

$$\operatorname{vol}(\Delta) \, = \, \frac{abc}{3}\sqrt{\rho(\rho-\alpha)(\rho-\beta)(\rho-\gamma)}\,, \quad \text{where} \quad \rho = \frac{\alpha+\beta+\gamma}{2}.$$

Exercise 42.36. (*Kahan's formula*) [1+] Let $\Delta = (x_1x_2x_3x_4)$ be a tetrahedron with edge lengths

$$|x_1x_2| = a, \quad |x_1x_3| = b, \quad |x_2x_3| = c,$$
$$|x_3x_4| = p, \quad |x_2x_4| = q, \quad |x_1x_4| = r.$$

Define

$$u = (p+q-c)(p+q+c), \;\; v = (p+r-b)(p+r+b), \;\; w = (q+r-a)(q+r+a),$$
$$s = (p-q+c)(q-p+c), \;\; t = (p-r+b)(r-p+b), \;\; z = (q-r+a)(r-q+a),$$
$$\text{and} \quad \alpha = \sqrt{uvz}, \;\; \beta = \sqrt{uwt}, \;\; \gamma = \sqrt{vws}, \;\; \lambda = \sqrt{stz}.$$

Prove that

$$\operatorname{vol}(\Delta) \, = \, \frac{1}{192pqr}\,\sqrt{(\alpha+\beta+\gamma-\lambda)(\alpha+\beta+\lambda-\gamma)(\alpha+\gamma+\lambda-\beta)(\beta+\gamma+\lambda-\alpha)}.$$

Exercise 42.37. [1] For every six points $x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{R}^3$ define the *Cayley–Menger bideterminant* $\mathrm{CM}(\cdot)$ as follows:

$$\mathrm{CM}(x_1, x_2, x_3 \,|\, y_1, y_2, y_3) \, = \, \det\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & |x_1y_1|^2 & |x_1y_2|^2 & |x_1y_3|^2 \\ 1 & |x_2y_1|^2 & |x_2y_2|^2 & |x_2y_3|^2 \\ 1 & |x_3y_1|^2 & |x_3y_2|^2 & |x_3y_3|^2 \end{pmatrix}.$$

Prove that the dihedral angle φ between the planes $X = (x_1x_2x_3)$ and $Y = (y_1y_2y_3)$ satisfies:

$$\cos^2\varphi \, = \, \frac{CM^2(x_1, x_2, x_3 \,|\, y_1, y_2, y_3)}{CM(x_1, x_2, x_3)\cdot CM(x_1, x_2, x_3)}\,.$$

Exercise 42.38. Let $P \subset \mathbb{R}^2$ be a convex polygon containing the origin O in its relative interior. Denote by $f_P(r) = \operatorname{area}(P \cap B_r)$, where B_r is the circle centered at O of radius $r > 0$.
a) [1] Suppose $f_T = f_{T'}$ for two triangles $T, T' \subset \mathbb{R}^2$. Prove that $T \simeq T'$.
b) [1-] Prove or disprove: if $f_P = f_{P'}$, then $P \simeq P'$.
c) [1+] What happens in higher dimensions?

Exercise 42.39. (*Euler's inequality*) *a*) [1-] Let R and r be the circumradius and the inradius of a triangle. Prove that $R \geq 2r$.
b) [1] Let R and r be the radii of circumscribed and inscribed sphere for a simplex $\Delta \subset \mathbb{R}^d$. Prove that $R \geq dr$.
c) [1+] Show that the volume of a simplex inscribed into a unit sphere in $\mathbb{R}^d$, is maximal for a regular simplex.
d) [1+] Show that the volume of a simplex circumscribed into a unit sphere in $\mathbb{R}^d$, is minimal for a regular simplex.
e) [1-] Deduce part *b*) from *c*) and *d*).

42.3. **Miscellaneous problems.**

Exercise 42.40. *a*) [1] Let $Q \subset \mathbb{R}^3$ be an unbounded convex polyhedron such that every projection of Q onto a plane is a cone. Prove that Q is a 3-dimensional cone.
b) [1] Same result for $Q \subset \mathbb{R}^d$ and projections on all 2-dimensional planes.

Exercise 42.41. *a*) [1-] Let Q and Q' be two convex polygons in the XY and XZ planes, respectively such that projections of Q and Q' on the X line are the same. Prove that there exists a convex polytope P with projections Q and Q'.
b) [1] Generalize this to simple (not necessarily convex) polygons.

Exercise 42.42. *a*) [1-] Prove that every tiling of a d-dimensional cube by smaller cubes contains at least 2^d cubes.
b) [1] Prove that a 3-dimensional cube cannot be tiled by a finite number of cubes of distinct size.
c) [2-] Prove that a square can be tiled by a finite number of squares of distinct size.

Exercise 42.43. [1+] A trapezoid is called *isosceles* if its non-parallel sides have equal lengths. Prove that every convex polygon has a dissection into isosceles trapezoids.

Exercise 42.44. *a*) [1] Prove that an equilateral triangle cannot be tiled by a finite number of equilateral triangles of different sizes.
b) [1+] Prove that no convex polygon can be tiled by a finite number of equilateral triangles of different sizes.
c) [2-] Prove that the plane cannot be tiled with distinct equilateral triangles whose side length are at least 1.
d) [1] Prove that an isosceles right triangle can be tiled by a finite number of isosceles right triangles of different sizes.
e) [1+] Prove that every non-equilateral triangle T can be tiled by a finite number of triangles similar to T and of different sizes.
f) [2-] Prove that a square can be tiled by a finite number of isosceles right triangles of different sizes.

Exercise 42.45. Two convex polytopes in $\mathbb{R}^3$ are called *adjacent* if they touch by a face.
a) [1-] Find eight pairwise adjacent tetrahedra.
b) [1+] Prove that there are no 100 pairwise adjacent tetrahedra.
c) [1+] For every n, find a family of n pairwise adjacent convex polytopes.
d) [2] For every n, find a family of n pairwise adjacent congruent convex polytopes.

Exercise 42.46. Let Q be a convex polygon in the plane. Points $x_1, \ldots, x_n \in \partial Q$ are said to *fix* Q, if for every direction on the plane an infinitesimal translation of Q in that direction contains one of the x_i in its relative interior.
a) [1] Prove that one can always find six points which fix Q.
b) [1] Prove that six points is necessary only when Q is a hexagon with parallel edges.

Exercise 42.47. Let $P_1, P_2, \ldots, \subset \mathbb{R}^d$ be a family of non-intersecting convex bodies. We say that P_i can be *extracted* if there exists a continuous vector valued function $f : [0, \infty) \to \mathbb{R}^d$ such that $\|f(t)\| = t$ and a polytopes $P_1, \ldots, P_i + f(t), \ldots, P_n$ are non-intersecting for all $t \geq 0$.
a) [1] Prove that for $d = 2$ and a finite family of non-intersecting convex polygons as above, there is always a polygon P_i which can be extracted.

b) [1] Prove that for $d = 2$ and a finite family of three or more non-intersecting convex bodies, at least three of them can be extracted.
c) [1+] Prove that for all d and any finite family of $d+1$ or more balls in $\mathbb{R}^d$ (not necessarily of the same radius), then at least one can be extracted.
d) [1+] For $d = 3$, construct an infinite family of non-intersecting convex polytopes P_i which lie between two parallel planes, and such that none can be extracted.
e) [2-] For $d = 3$, construct a finite family of non-intersecting convex polytopes P_i such that none can be extracted.
f) [2-] For $d = 3$, construct a finite family of non-intersecting unit cubes, such that none can be extracted.
g) [$*$] Is it always possible, for a given convex polytope $P \subset \mathbb{R}^3$, to construct a finite family of polytopes congruent to P such that none can be extracted?
h) [2+] We say that a family of polytopes $Q_1, \ldots, Q_k \subset \mathbb{R}^3$ can be *extracted* from a family of polytopes $Q'_1, \ldots, Q'_\ell \subset \mathbb{R}^3$ if for some $f : [0, \infty) \to \mathbb{R}^3$, $\|f(t)\| = t$, and some fixed k, $1 < k \leq n$, polytopes $Q_1, \ldots, Q_k$, $Q'_1 + f(t), \ldots, Q'_\ell + f(t)$ are non-intersecting. A family of polytopes $\mathcal{P} = \{P_1, \ldots, P_n\}$ in $\mathbb{R}^3$ can be *taken apart with two hands* if some subset $\mathcal{A}$ of $\mathcal{P}$ can be extracted from $\mathcal{P} \smallsetminus \mathcal{A}$. Find a finite family of non-intersecting convex polytopes which cannot be taken apart with two hands.

Hints, solutions and references to selected exercises

1.2. For b), consider all halfspaces defining P, and suppose all normals are in general position.

1.3. By Corollary 1.7, it suffices to check that every three vertices of Q lie inside a circle of radius $L/4$. For obtuse triangles the claim is trivial, and for acute triangles the claim for the circumradius can be checked directly. For a generalization and another proof see section 24.1.

1.4. This problem is given by V. Proizvolov in [Kvant], M1665 (1993, no. 3).

1.5. Start with the largest area triangle Δ with vertices at z_i.

1.8 and 1.9. See [BajB].

1.10. See [BolS, §19].

1.11. a), b) and c) These are special cases of results by Molnár and Baker. See [Bak] for the references.

1.12. a) From part a) of the previous exercise and the logic as in the proof of Corollary 1.7, it suffices to check that no four spherical circles of radius r can cover $\mathbb{S}^2$. This follows from the area argument when $r < \pi/3$.
For b), when $r = \pi/2 - \varepsilon$, the claim is not true; vertices of a regular tetrahedron give a counterexample. The bound $r < 2 \arcsin \frac{1}{\sqrt{3}}$ given by this example is optimal.

1.13. See [Leic, §1.7] and [DGK, §5].

1.14. Part b) is a discrete version of [Hal1]. Similarly, parts c) and d) are discrete versions of special cases of [Hal2].

1.15. See [YagB], Problem 20.

1.16. Part a) is due to Berge (1959) and part b) is due to Breen (1990). We refer to [BárM] for references, details and extensions.

1.17. This is proved in [BolS, §21].

1.19. An ingenious solution of this problem is given by N. B. Vasiliev in [Kvant], M30 (1971, no. 4).

1.20. See [ErdP, §6.4] for references and related results.

1.22. Both results are due to Breen [Bre].

1.23. A short proof is given in [Gug4].

1.24. This is due to Dvoretzky [Dvo] (see also [BZ2, §6]).

1.25. Part a) is due to Schreier (1933) and the generalization to higher dimension is due to Aumann (1936). See [BZ2, §5] for further results and references.

1.26. See [Rub].

2.1. Note that in $\mathbb{R}^4$ the facets can intersect by a 2-dimensional face which can have different triangulations on each side.

2.3. a) Denote by n, ℓ, and m the number of points, line intervals and regions in the plane. Prove by induction that $n - \ell + m = 1$ (or use Euler's formula). We have:

$$n = \sum_{i \geq 2} p_i\,, \quad \ell = \sum_{i \geq 2} i p_i\,, \quad m = \sum_{j \geq 3} q_j\,, \quad 2\ell = \sum_{j \geq 3} j\, q_i\,,$$

$$3 = (3n - \ell) + (3m - 2\ell) = \sum_{i \geq 2} (3 - i) p_i + \sum_{j \geq 3} (3 - j) q_i\,,$$

$$p_2 = 3 + \sum_{i\geq 4}(3-i)p_i + \sum_{i\geq 4}(3-j)q_i \geq 3.$$

This proof follows [BorM].

2.4. A simple proof, references and generalizations are given in [ErdP, §3.3].

2.5. Denote by m_i the number of line passing through point x_i. Summing the numbers over all these lines, we obtain $(m_i - 1)a_i + s = 0$, where $s = a_1 + \ldots + a_n$. We conclude that $(m_i - 1)a_i = (m_j - 1)a_j$ for every $1 \leq i, j \leq n$. Since not all points line on the same line, we conclude that all a_i have the same sign, a contradiction. This proof is given by F. V. Vainshtein in [Kvant], M451 (1978, no. 5).

2.7. Use the previous exercise. See [Bár] for details.

2.8. For b), note that center of a cross-polytope $Q \subset \mathbb{R}^d$ does not lie in the interior of any subset of vertices of Q.

2.9. See [BKP].

2.11. For each x_i, consider $n-2$ disjoint triangles with a vertex at x_i. Now place y_1 close to x_1 in the rightmost of these triangles around x_1, place y_2 in the second rightmost triangle around x_2, etc., until the leftmost triangle around x_{n-2}. Check that $y_1, \ldots, y_{n-2}$ are as desired. This simple solution is given by N. B. Vasiliev in [Kvant], M551 (1980, no. 2).

2.12. For a), take a Voronoi diagram of points X in P (see Subsection 14.2). Choose the cell of largest volume. For b), prove the result by induction on k, for $n = 3(2^k - 1)$. For $k = 1$, we have $n = 3$ points. Cut P with a plane which contains X and choose the larger part. For $k > 1$, take two points in X and cut P with a plane through these points which divides P into two parts with equal volume. Now apply the inductive assumption to the half with fewer points in X. This part of the problem is based on a solution by L. Lipov in [Kvant], M375 (1976, no. 11).

2.14. Carathéodory theorem says that every such v belongs to a tetrahedron Δ with vertices in Q. Consider all triangles with two vertices the vertices of Δ and the third vertex on Q. Check that one of them work.

2.15. The proof is given in [Ani, §1]. See also [Scho] for classical references and applications of convex polygons in $\mathbb{R}^d$.

2.16. This result was proved in [Naz]. See also [Kara] for extensions and further references.

2.17. This is a classical result of Lovász (1974).

2.18. This is a recent result of Holmsen and Pach (2008).

2.19. See [BáFu].

2.20. This result is due to Dolnikov and has appeared in the 2004 Russian Math. Olympiad.

3.1. Take the furthest pair of points of the polygon to be the median of a square.

3.2. For both parts, note that in every centrally symmetric set X, the points $x, y \in X$ with $|xy| = \operatorname{diam}(X)$ must be opposite. For a), intersect the polytope with a generic hyperplane through the center of symmetry and note that opposite vertices lie on different sides. For b), note that $X \subset B$, where B is the ball of radius $\operatorname{diam}(X)/2$. The result now follows from the proof of the Borsuk conjecture for B.

3.4. For a) and b), see [BolG, HDK]. Part c) is proved in [BojF, §3.38].

3.6. Part *b*) is called *Barbier's theorem*; see [BolG, YagB] and Exercise 24.6. Part *c*) is due to Blaschke and Lebesgue, and a simple proof can be found in [Egg1, §7.2] and [YagB, §7].

3.7. Take four copies of the same curve joined at the vertices of a square.

3.8. See [Hep2, Mel] and the references therein. Part *b*) was proved by Neville (1915) who found the optimal disk configuration. Part *c*) is proved in K. Bezdek (1983).

3.9. This problem is discussed on the *Math. Overflow*, see `http://tinyurl.com/ykzgfxh`

4.1. Take a function $g(x) = f(x+a) - f(x)$. Either $g(x)$ changes sign or it does not. In the first case, g has a zero. In the second case, if $g(x) > 0$ for all x, then $g(x) \to \infty$, a contradiction. Alternatively, this follows from the mountain climbing lemma (Theorem 5.5), when two climbers start climbing at two minima.

4.2. For $a \in (0,1] \smallsetminus \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$, Levy used the following function:

$$f(x) \,=\, \sin^2\left(\frac{\pi x}{a}\right) \,-\, x\sin^2\left(\frac{\pi}{a}\right).$$

See [Lyu, §34] for an elementary proof and further references.

4.3. This is proved in [Ros].

4.5. For *a*), the argument in the proof of Proposition 4.1 does not work since the lines ℓ_1, ℓ_2 are not necessarily unique. Note that for generic ℓ the number of lines ℓ_1 (or ℓ_2) must be odd. Now consider the set of pairs of lines which work and use the parity argument as in Section 5.

4.7. See [Buck].

4.8. Take a $1 \times t$ rectangle, where $t > 0$ is large enough.

4.10. It is easy to make a mistake in [BorF] as pointed out in [BMN]. In the correct construction, take n points on the curve $y = 9x^3$ very far apart from each other. See the proof in [BMN].

4.11. See [Schu1].

4.13. For *c*), see [Alon].

4.14. Choose the closest two pearls of the same size and cut between them, right after the first one. This problem is given by V. Proizvolov in [Kvant], M1684 (1999, no. 6).

4.15. This is equivalent to the inscribed chord theorem (Theorem 4.5). See [Tot] for proofs and several related results.

5.1. For *a*), consider a polygon X' symmetric to X with respect to O. By the area argument, $X \cap X' \neq \varnothing$, which gives the desired triple. For *b*), use the same construction and Lemma 9.6. For *c*), take any concave vertex x_i. Such a vertex always exists by Exercise 24.2.

5.2. An easy uniqueness argument for part *a*) is given in [Emch]. For *b*), the continuity is immediate from the second proof, in all directions except for those orthogonal to the edges. On the other hand, the limit of rhombi is also a rhombi. Now the uniqueness from part *a*) implies the result.

5.3. For *a*), denote the inscribed quadrilateral by $Q = [v_1v_2v_3v_4]$. Clearly, vertex v_1 determines v_2, v_3 and v_4. Suppose Q is not cyclic, i.e., $f(v_1) = \angle v_1 - \angle v_2 + \angle v_3 - \angle v_4 \neq 0$. As v_1 moves along X, function f changes sign: $f(v_1) = -f(v_2)$, and by the intermediate value theorem we have $f(z) = 0$ for some $z \in X$.

For *b*), consider $f(v) = \mathrm{vol}(v_1v_2v_3v_4)$, the signed volume of a tetrahedron and proceed analogously. While both parts are probably well known, [Stru] is the only reference we were able to find.

5.4. See [Egg2].

5.5. Start with an inscribed equilateral triangle near the vertex x and continuously expand it until one of the endpoints is reached. See [Mey1] for details.

5.6. The following proof by Erdős is presented in [Jac]. Denote by $A \subset \mathbb{R}^2$ the region enclosed by Q and by Q_i the arcs of Q separated by x_i. Let $d_i(z)$ be the distance between $z \in A$ and Q_i. Finally, define A_i to be a subset of A consisting of points $z \in A$ such that $d_i(z) \le d_j(x)$ for all j. Observe that each A_i is connected since for every $z \in A_i$ we have z is connected to Q_i within A_i by a straight line to the closest point in Q_i. Now, if $A_1 \cap A_2 \cap A_3 = \varnothing$, then $A_1 \cap (A_2 \cup A_3)$ consists of two nonempty sets $A_1 \cap A_2$ and $A_1 \cap A_3$, and thus disconnected. But this is impossible since A is simply connected and thus unicoherent (see Exercise 6.14). Finally, a circle at $z \in A_1 \cap A_2 \cap A_3$ with radius $d_1(z) = d_2(z) = d_3(z)$ is the desired circle.

5.7. This was proved in [Mey1] (see also [Mey3]).

5.9. For a), consider the location of the third vertex for equilateral triangles with two vertices on given parallel lines, and use the continuity argument. For b), use induction on d to construct various regular simplices with $d+1$ points on given hyperplanes, and use the continuity argument again.

5.11. a) This is false. A counterexample is given by a cone with face angles $\pi/2$, $\pi/2$ and $\pi/3$. Let O be the vertex of C and suppose (A, B, C) are the vertices of an equilateral triangle, such that $\angle AOB = \pi/3$. Since $|CA| = |CB|$ we have $|OA| = |OB|$, which in turn implies that $|AB| = |OA|$. But $|CA| > |OA|$, a contradiction. Part b) is also false for all cone face angles $< \pi/3$.
d) Denote the faces of the cone by F_1, F_2, F_3, F_4 (in cyclic order), and let $\ell = F_1 \cap F_3$, $\ell' = F_2 \cap F_4$. Check that every plane parallel to ℓ, ℓ' intersects the cone by a parallelogram.

5.12. For a), take L parallel to opposite edges. Continuously move it between the edges and compare the edge lengths of the resulting parallelograms.

5.13. This is false. Consider a polygon $Q = [x_1x_2x_3x_4]$, where $x_1 = (0,0,0)$, $x_2 = (0,0,1)$, $x_3 = (1,0,0)$, and $x_4 = (1,1,0)$. This construction is given in [MeyA] (see also [Mey3]).

5.14. See [Koe] and further references in [CFG, §B2].

5.15. For general k and smooth curves part a) was proved in [Wu]. The piecewise linear case follows by a limit argument.

5.16. Part a) is proved in [Kake], while part b) is proved in [Gri]. For d), take a polygon X approximating a circle. This implies that Q must be cyclic (inscribe into a circle). Similarly, take a triangle X with sides ℓ, ℓ, and $2\ell - \varepsilon$. Observe that a quadrilateral similar to Q is inscribed into X, for all ℓ large enough, only if Q has parallel edges. Together these two conditions imply the result.

5.17. For a), see Subsection 23.6. The same idea works in other parts as well. For d), there are 24 ways to arrange vertices on different lines ℓ_i, there are $4 \cdot 24$ ways to arrange vertices on lines ℓ_i, such that two adjacent vertices lie on the same line, and there are $2 \cdot 24$ ways where two diagonal vertices lie on the same line. This gives a total of 168 distinct quadrilaterals.

5.18. This proof is outlined in [Mak2].

5.19. For a), use the proof idea of Proposition 5.9. See [Kra] for the details. Parts b) and c) are given in [HLM].

5.20. Part a) was claimed by Pucci (1956), but the proof was shown to be incorrect in [HLM]. Even though the question was listed as an open problem in [KleW], the proof of a much more general result is outlined in [Gug2].

5.21. Take the intersection of two tall pencil-like cones at a fixed acute angle to each other. See [Biel] for details.

5.22. Part a) was proved in [Fenn], part b) in [Zak1] (see also [KroK]). See [Mey2] for a construction in c).

5.23. For a), take $f_1(x) = \frac{1}{2}$ for all $x \in [\frac{1}{3}, \frac{2}{3}]$, and let f_2 have $x \sin(\frac{1}{x})$ singularity at $\frac{1}{2}$. See [Kel] For b), show that a variation on the first proof works.

5.24. Use the second proof of the mountain climbing lemma (Theorem 5.5). Alternatively, prove the claim by induction, since $f_i(g_i(t)) = f_j(g_j(t))$ immediately implies $f_i(g_i(h(t))) = f_j(g_j(h(t)))$, for every $h : [0, 1] \to [0, 1]$.

5.25. Consider the graph of points (x, y) such that $|xy| = 1$, where $x, y \in C$. Then use the idea of the second proof of the mountain climbing lemma (Theorem 5.5). See [GPY] for details and further references.

5.26. Parts a) and b) were proved in [GPY]. Part c) was proved in [Ger].

6.3. This was proved in [NieW].

6.6. A smooth case is outlined in [Mak1].

6.9. This was proved in [Liv, Zar].

6.10. This follows immediately from the proof of the Kakutani theorem (Theorem 6.3).

6.11. This follows immediately from the tripod theorem (Theorem 6.4), with $\alpha = \pi/3$ and $\beta = 2\pi/3$.

6.12. This was proved in [YamY].

6.13. This was proved in [HMS].

7.3. Prove the claim for polygons first and then use the limit argument. See [Grub, §9.1] for a concise proof of parts a) and b).

7.4. This is a special case of [GarM].

7.5. This is almost always false.

7.7. This is a classical result of Bonnesen (see [Sant, §7.5]).

7.9. Without loss of generality, we can assume that the angle of vectors v_i increases clockwise. Consider a convex polygon $Q = [O\, a_1 \ldots a_n]$ where $a_i = v_1 + \ldots + v_n$ (we may have $O = a_n$). The perimeter of Q is at least 1. Suppose the diameter is achieved on a diagonal (a_i, a_j). Then choose $I = \{i + 1, \ldots, j\}$. Finally, use the isoperimetric inequality to show that the diameter of a convex polygon is at least $1/\pi$ times perimeter. In the opposite direction, the constant $1/\pi$ is obtained in the limit of polygons Q_n which approach a unit circle. This proof follows [SCY], Problem 6.

7.10. See [Grub, §8.3].

7.11. For a), project Q_2 onto Q_1. Check that this map shrinks the distances. The same approach works in any dimension.

7.12. For a), use monotonicity of the mean curvature (Exercise 28.2). For b), consider parallel projections of the edges of P_1 onto the faces of P_2. Sum the lengths of projections, compare these to L_2 and note that the sum of three projections of the same edge is at most the length of the edge. This part is based on a solution by A. Kh. Shen and V. O. Bugaenko in [Kvant], M1687 (1999, no. 6).

Part c) is false. The idea of a counterexample is given in Figure 42.2. When the edge lengths of "long edges" increases, four such edges of inner tetrahedron Δ_1 will overcome the three edges of the outer tetrahedron Δ_2. This problem and solution appeared at the Moscow Math. Olympiad in 2002; available at `http://tinyurl.com/2vftjn`
Part d) is also false. Consider any two convex polytopes P_1, P_2 with $L_1 > L_2$, as e.g., in c). Now the polytopes $Q_1 = P_1 + \varepsilon P_2$, $Q_2 = \varepsilon P_1 + P_2$ for sufficiently small $\varepsilon > 0$ is the desired pair of polytopes (cf. Section 36).

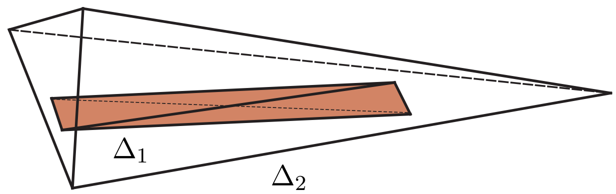

FIGURE 42.2. Tetrahedra $\Delta_1 \subset \Delta_2$ with $L_1 > L_2$.

7.13. For a), take a plane through a face F and observe that it fits inside a circle of radius ≤ 1. Therefore, perimeter$(F) \leq 2\pi$. By the isoperimetric inequality for the plane, we obtain: area$(F) \leq L^2/4\pi \leq L/2$. Summing this inequality over all faces and taking into account that all edges will be used twice, we obtain the result. To see that the inequality is sharp, consider doubly covered polygons inscribed into the equator of $\mathbb{S}^2$.
For b), consider projections onto every F of all triangles (Ovw) as in the statement of the exercise. The area of each projection is at most $\xi\,|vw|/2$, since the distance from O to edge (v, w) is at most 1. Now sum this over all edges e of P.

7.14. a) This was proved in [Fej1]. b) See [Lin].

7.15. a) See [BesE]. b) This was observed in [Bes2] (see also [She1]).

7.16. For a), b), c) and e) see the hint to Exercise 14.25, part a). In all cases, remove the layers of parallelograms (parallelepipeds) one at a time and prove the claims by induction. See [SCY], problems 112–119, for the easy proofs and [Zie1, §7.3] for the references.

8.2. An elegant presentation of this is given in [Zie1, §9.2].

8.3. By the argument as in the section, all vertices on one side of the hyperplane H are connected to a vertex lying on the final face with respect to H.

8.4. See [BrøM].

8.5. Part a) is given in [VasE], Problem 461. For part b), P must be a tetrahedron. Indeed, if P has n faces, it has exactly $(2n-4)$ vertices and $(3n-6)$ edges. Since the number of edges of each color must be at least $2n-5$, we have $2\cdot(2n-5) \leq 3n-6$, which implies $n = 4$. For c), let $v_1, \ldots, v_n$ be the set of vertices ordered according to some Morse function φ, so that the simplex $\Delta = (v_1, \ldots, v_d)$ is the smallest facet and each v_i, $i \geq d+1$, is connected by at least $d-1$ edges to vertices v_j with $j < i$. Color the vertices of Δ with $d-1$ colors to satisfy the requirements. Color the remaining vertices v_i by coloring with different colors the $d-1$ edges pointing to v_j, $j < i$. This gives a coloring where every v_i is connected to v_1 by any of the $d-1$ colors, as desired.

8.6. The result is obvious for $d = 2$. For $d \geq 3$, cut P with a hyperplane H around a vertex v. By induction, the $d-1$-dimensional polytope $Q = P \cap H$ has an embedding of K_d. Project away from v the vertices and edges of this embedding onto the vertices and edges in Γ. Connecting them by the edges with v gives the desired embedding of K_{d+1}. This result is attributed to Grünbaum, while our proof follows the outline in [Kuh, §2].

8.7. Observe that the total number of edges of each color must be the same, since it is equal to one quarter the total number of sides of all faces. Now take P to be any polytope as in the theorem with an odd number of edges. This elegant solution is given by S. Tokarev in [Kvant], M1365 (1993, no. 2).

8.8. For *a*), use Euler's formula to conclude that the average degree of a vertex is strictly smaller than 6. Parts *b*) and *c*) are due to Kotzig (1955, 1963) and use more delicate applications of Euler's formula (see [Bor] for extensions and references). To obtain sharp bounds in part *d*), take polytopes dual to the *truncated dodecahedron* (an Archimedean solid with two decagons and one triangle adjacent to every vertex) and *icosidodecahedron* (an Archimedean solid with two pentagons and two triangles adjacent to every vertex).

8.9. *a*) Denote by ℓ the line through two vertices of P at distance $\operatorname{diam}(P)$. Every plane $H \perp \ell$ intersects at least three edges. Therefore, projections of edges of P onto ℓ cover every point at least three times. Since the sum of these projections is at most L, this implies the result.

b) By Menger's theorem, for every two vertices $x, y \in \Gamma$ there exists three non-intersecting paths from x to y. Thus, for the diameter d of Γ we have $3\text{th}(d-1)+2 \le n$. To see that this is sharp, stack $d-1$ triangular prisms and two regular triangular pyramids attached on the opposite ends. Now perturb the surface to make it strictly convex. See [GM1, JucM] for complete proofs.

8.10. Part *a*) follows from Euler's formula. For *b*), consider paths on Γ which are defined by moving right and left, alternatively. Prove that every such path is closed and has even length. Conclude from here the claim. We refer to [GM2] for details. Part *c*) is proved in [Mot] (see also [Grü1]).

8.11. This ia variation on Exercise 24.2.

8.12. In $\mathbb{R}^4$, take two equilateral triangles lying in two orthogonal 2-planes, with centers at the origin. Their convex hull is the desired neighborly polytope. In $\mathbb{R}^d$, $d \ge 5$, take 6 points in a 4-dimensional subspace as before, and the remaining points in general position. See [NagC] for the details.

8.13. The result is obvious in $d = 2$. In general, take any facet $F \subset P$. If F is not simplex, use inductive assumption. If all faces are simplices, choose any adjacent pair of facets. See [Tve].

8.14. For *b*), in the same way as in *a*), consider two cases: when P is not simple and not simplicial, and use induction on d. See [Dev] for the easy details.

8.15. For *a*) and *b*), assume every cross section of P is a k-gon. Intersect P by a plane near a vertex. Then every vertex has degree k. Now intersect P by plane parallel to an edge e and near e. The intersection if a $(2k-2)$-gon, a contradiction. This implies part *c*) as well. For *d*), fix a plane in general position and move it across the polytope. When crossing a vertex, observe that the parity is unchanged.

8.16. For *a*), take P obtained by attaching triangular pyramids to the faces of an octahedron. Check that P does not have a Hamiltonian cycle. For *b*), use an iterative construction.

8.17. For *c*), let $\varphi : Q_{n,k} \to P_{n,k}$ be defined by $\varphi(x_1, \ldots, x_n) = (y_1, \ldots, y_n)$, where $y_1 = 1 - x_1$ and

$$y_i \, = \begin{cases} \quad x_{i-1} - x_i & \text{if } \ x_i < x_{i-1} \\ 1 + x_{i-1} - x_i & \text{if } \ x_i \ge x_{i-1} \end{cases} \qquad \text{for } \ i \ge 2\,.$$

This elegant construction was discovered in [Sta1].

8.18. For a), prove combinatorially the recurrence relation $B_{n+1,k} = (n-k+1)B_{n,k-1} + (k+1)B_{n,k}$ and use induction on n.

8.19. Part b) is a classical result of Birkhoff, (see [Barv, EKK]). For c), see [EKK]. For f) and g), see [LovP].

8.20. Parts $a) - d)$ are due to Klee and Witzgall (1968). We refer to [EKK] for a nice exposition. For e), see [Pak2].

8.21. Assume that no two particles enter vertices at the same time. Take a dual graph Γ^* to the graph Γ of the polytope. For every vertex in Γ^* corresponding to a face F, orient an edge in Γ^* along the edge where the particle is moving. Since the number of oriented edges is equal to the number of vertices in Γ^*, there is an oriented cycle. Consider how the cycle changes as one particle passes through a vertex. Check that if there are no collisions, the cycle gets smaller every time, a contradiction.
This result and its group theoretic applications are described in [FenR] (see also a friendly exposition in [Ols]).

8.22. For part a), assume P is simplicial. Then it has $n \geq 6$ vertices. By Euler's formula, it has $3n-6$ edges and $2n-4$ faces. We obtain:

$$f_0 + f_1 + f_2 + f_3 \, = \, n + (3n-6) + (2n-4) + 1 \, = \, 6n - 9 \, \geq \, 27.$$

If P is not simplicial, it has a non-triangular face F. Denote by F' the centrally symmetric face and by G the intersection of P with a hyperplane through the origin and parallel to F. Since the vertices and edges of G correspond to the edges and faces of P, we obtain:

$$\begin{aligned} f_0(P) \, &\geq \, f_0(F) + f_0(F') \, \geq \, 4+4 \, = \, 8, \\ f_1(P) \, &\geq \, f_1(F) + f_1(F') + f_0(G) \, \geq \, 4+4+4 \, = \, 12, \\ f_2(P) \, &\geq \, f_2(F) + f_2(F') + f_1(G) \, \geq \, 1+1+4 \, = \, 6, \end{aligned}$$

which implies the result. The above argument, the proof of part b), various extensions and references are given in [SWZ]. The last part is an open problem due to Kalai.

9.1. a) Take $e = (x_1, x_2)$ to be the longest edge. Let ℓ_1 and ℓ_2 be the lines perpendicular to e going through x_1 and x_2, respectively. If no vertex projects onto e, at least one edge intersects both ℓ_1 and ℓ_2, contradicting the choice of e.
For b), note that the $\min\max$ in ($\sharp$) can never be achieved on a cut through two vertices. For c), take, e.g., the great stellated dodecahedron. This problem appeared in [CroW].

9.2. For a), take a triangle of the largest area.

9.4. a) This is false as stated. Think of a generic curve C made of metal and place on the floor. In the equilibrium C either has three tangent points, or two point which have contact with C of order 2 and 1, or one point which has contact with C of order 3. For b), consider a plane through any face of the convex hull of Q.

9.5. For b), consider a triangle $\Delta' = -2\Delta$, and such that vertices of Δ are midpoints of Δ'. Use the fact that Δ has maximal area to conclude that Q is inscribed into Δ'. Note also that $\operatorname{cm}(\Delta') = \operatorname{cm}(\Delta)$. Now bound the desired ratio by that of inside Δ' and inside Δ. For details, see e.g., [Pro1, §9].

9.6. This is a discrete analogue of the result in [Tab3], which uses a version of the four vertex theorem (Theorem 21.1). While the discrete result easily follows from the smooth result by the limit argument, we do not know a simple direct proof.

9.7. These are discrete versions of several known results gathered in [Gug3].

9.8. This was proved in [DeoK].

9.10. For parts a) and b) see [CGG, Daw2]. See [Hep1] for part c). Part d) is a delicate result due to Dawson [Daw2]. The main idea is to apply the Minkowski theorem 36.2 to the simplices. Part e) is in two followup papers by Dawson and coauthors (1998, 2001).

9.11. Consider a degenerate (flat) parallelepiped as in Figure 42.3. Perturb the dihedral angles to make it non-degenerate (still nearly flat). When placed on one of the non-square faces, the center of mass lies outside the face, so the polytope rolls.

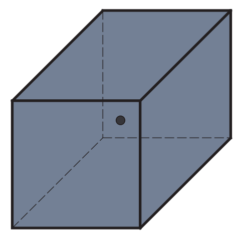

FIGURE 42.3. Parallelepiped which can stand only on two faces.

9.12. For b), start with a construction of a 17-gon using the idea of a heptagon in Figure 9.3, optimizing it in such a way that point O can be taken as close to the center of mass as possible. Now use this polygon as a cross-section of the slanted cylinder described in Example 9.9. Make the sides slanted far enough so that the center of mass of the resulting polyhedron projects onto O. We refer to [CGG] for details.
Parts c) and d) were recently obtained in [VD1, VD2]. In the plane, think of the projection points as minima of the distance function from $\mathrm{cm}(Q)$. Now apply Lemma 9.6.

9.13. For a), this is called the *orthic triangle* (see [CoxG, §1.6]). For b), see [Phi].

9.15. This nice observation was communicated by Joe O'Rourke.

9.17. Denote by $L = \ell(\Delta)$ the length of the longest triangle $\Delta = [x_1x_2x_3]$ inscribed into C. For every point $z \in C$, denote by R_z the longest triangle inscribed into C with z as a vertex. Clearly, $\ell(R_z) = L$ when $z = x_1$ and $z = x_2$. Take R_z with the smallest length $\ell(R_z)$, where z lies between x_1 and x_2. Check that this is the desired billiard trajectory. See [Bir, §6.6], [Tab6, §6] and [CroS].

9.18. This is proved in [Weg1].

9.19. A direct proof is given in [Kui]. By analogy with the 2-dimensional case, one can reduce this problem to the number of simple closed geodesics on convex bodies in $\mathbb{R}^4$, and conclude that there exist at lest four double normals (see [Alb]). We do not know an elementary proof of this result.

9.21. See [Tab2, §4] and [Tab6, §9], for these and other related results, background and references (see also [Tab7]). Parts i) and j) are due to R. E. Schwartz (2007).

10.1. For a), consider a regular pyramid Δ with a face angle τ on the bottom of side triangles. The face angle at the top of side triangle is $\delta = 1 - 2\tau$. Now, by the proof above, we need only to check that $\alpha_1 + \alpha_2 \neq 2\pi$, i.e., $\tau \neq \pi/3$. For example, a standard tetrahedron Δ corresponding to $\tau = \pi/4$ gives the desired example (see Figure 10.5).
b) Consider a truncated regular pyramid as in Figure 42.5. Let the sums of face angles be $\frac{2\pi}{3} + \epsilon$ for the top vertices, and $\frac{2\pi}{3} - \epsilon$ for the bottom three. When $\epsilon/\pi \notin \mathbb{Q}$ there are no simple closed geodesics.

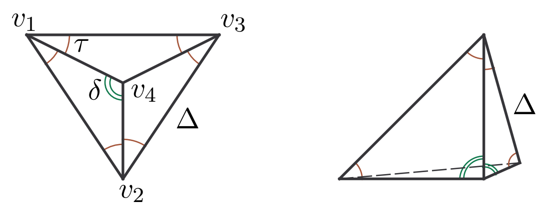

FIGURE 42.4. An example of a regular pyramid Δ without simple closed geodesics: $\tau = \pi/4$, $\delta = \pi/2$ (different views).

FIGURE 42.5. An example of a polytope P with only non-simple geodesics.

10.2. Consider a unit cube in each case and a geodesic starting with an interval $[ab]$, where $a = (0, \frac{1}{2}, 0)$, $b = (\frac{1}{8k}, 0, 0)$, and $k \in \mathbb{N}$.

10.3. These calculations were made in [FucF].

10.4. a) Recall that every geodesic divides the curvature into equal halves, and that pairs of separated vertices are different for the three geodesics. We obtain three equation which imply that the curvature of every vertex is π. The result now follows from Exercise 25.12. Parts b) and c) are given in [Pro2] (see also [StrL]).
d) For an equihedral tetrahedron Δ, tile a plane with copies of its faces. Consider lines which do not contain vertices and have irrational slope with the respect to the resulting grid. Such lines when folded back onto Δ give infinite simple geodesics. In the other direction, unfold the tetrahedron and extend the idea of the proof of Theorem 25.6 to show that the curvature in every vertex is π. For a complete proof of part d) and e), see [Post].

11.1. For c), see [Shar], Problem 221.

11.2. Start with two adjacent faces F_1 and F_2 which can be inscribed into a sphere $\mathbb{S}^2$. Check that all faces adjacent to both F_1, F_2 are also inscribed into $\mathbb{S}^2$ (use the fact that P is simple here). Continue with faces adjacent to two previously inscribed faces. Show that this process halts when all faces are checked.

11.3. See [ShaS], Problems 5.21 and 5.35.

11.4. For a), assume there exists an inscribed sphere. For every face F, take a tangent point a_F of the sphere and triangulate F by taking cones from a_F over the edges of F. Now calculate the surface are of faces colored each color and compare the results. For b), take a 1!×!3!×!3 box and color large squares black. For c), take a symmetric bipyramid Q over 1!×!ε rectangle, with $\varepsilon > 0$ sufficiently small, so that Q has no inscribed sphere. Then there is a unique proper coloring with equal white and black areas by the symmetry. Now perturb the vertices of Q to break the equality.

11.5. See [Grü4, §13.5] for the proof and some background.

11.6. Part a) is proved in [Bar1]. For b), start with a tetrahedron and construct a pyramid on each face. The resulting polyhedron is dual to the truncated tetrahedron, has 8 vertices and 12 faces. Show that it cannot project onto a regular octahedron. This solution is given in [Bar2]. Note that this problem is a generalization of Exercise 14.16, corresponding to the case where P is simplicial and C is a Hamiltonian cycle. Thus part a) of this exercise is a generalization of c) of Exercise 14.16. Similarly, a counterexample described in the solution of part b) of Exercise 14.16 also works for part b) in this exercise.

11.7. See [GMS] and [Schu1].

11.8. This is a special case of the main result in [Tho1].

12.2. This is due to Zamfirescu and Gleason (see [Gle] and references therein).

12.3. Both parts are proved in [Schr2] (see also [Var2]).

12.5. Draw two parallel lines and take midpoints of intersections with the parabola. Show that the line through the midpoints is parallel to the y axis. See [Vard], Problem 23.

12.6. Consider the right triangle with vertices: the center O, the vertex v and the edge midpoint z of a pentagon. If the pentagon is rational, then $\cos\frac{3\pi}{10} = |vz|/|vO| = \sqrt{a}$ for some $a \in \mathbb{Q}$, a contradiction.

12.8. See a construction in [Zie3].

13.4. This result is due to A. Galitzer (see [KM3, §3]).

14.4. See [Epp, ShaH].

14.5. For triangulations of the middle regions, encode them with the 0–1 sequences, which corresponds to $k-1$ up triangles and $k-1$ down triangles. The number of these sequences is $\binom{2k-2}{k-1}$, and two remaining Catalan numbers C_{k-2} correspond to triangulations of the top and bottom k-gons. See [HNU] for the details (see also [DRS, §3.4]).

14.6. This problem was on a 2007 Putnam Mathematical Competition. See a solution here: `http://tinyurl.com/2crvn9w`

14.7. See [DRS, §3.3].

14.8. For a) see [HNU]. For b) and c), see [Gal+].

14.10. For part a), use at most $n-2$ moves to transform a given triangulation to a triangulations with all diagonals meeting at the same point. Part b) is proved in [STT] using hyperbolic geometry; a combinatorial proof is given in [Deho]. See also [DRS, San2] for further references.

14.11. Part b) is false (see [BerE]).

14.12. For each of these functionals check that they behave accordingly along increasing flips (essentially, prove the claim for all quadrilaterals). See [BerE] and [Mus2] for more on this and further references, and [Lam] for g).

14.13. See [Tut, §10] (see also [PouS] for further references, applications and a bijective proof).

14.14. For a) and b) use the idea in the example. For c), consider a grid fine enough and add grid vertices. For parts c)$-e$) and the references see [BZ4, Mae1, Sar, Zam]. For the progress towards f), see [ESU]. Part g) was proved in [Kri].

14.15. For a), triangulate the projection and lift it to X. For this and other parts, see [BDE].

14.16. For a), use a binary tree as in Section 8 and lift the triangles up depending how far they are from the root. Alternatively, take the function ξ as in Example 8.5. For b), take a regular hexagon $Q = [x_1 \dots x_6]$, triangulations T_1 with diagonals $(x_1, x_5), (x_2, x_4), (x_2, x_5)$, and T_2 with diagonal $(x_1, x_3), (x_1, x_4), (x_4, x_6)$. See [Dek] for details and further references. See also Exercise 11.6 for a special case of part *b*) and a generalization of part *c*).

14.17. For *a*), take six 1×1×11 bricks and attach each of them in the middle to a unit cube. Now slightly decrease the size of the bricks to make sure they do not intersect.
For *b*), one can use the construction in *a*). Here is an alternative approach. Solve a 2-dimensional version first and then add two large parallel slabs in $\mathbb{R}^3$.

14.18. *a*) This is one of the oldest results in discrete geometry [Len]. The idea is to find a diagonal inside Q and then use induction. First, project Q onto the x axis and let v be the leftmost vertex. Either the diagonal (u, w) between vertices neighboring v is inside Q, or triangle (uvw) contains at least one vertex of Q. Then the diagonal (v, z) connecting the leftmost such vertex z with v cannot intersect any edge of Q, and thus lies in the interior of Q.
For parts *b*) and *c*), take the Schönhardt's polyhedron [Schö] (see also an extension in [Ramb] and [Bag]). Simply take a triangular prism and twist the top triangle (see Figure 42.6). We refer to [BerE] and [DRS, §3.5] for the context and further references. Part *d*) is given in [Bin].[99]
For part *e*), take e.g., six bricks as in part *a*) of the previous problem and connect them by short tubes far away from the center.

14.19. See [RupS].

FIGURE 42.6. Schönhardt's polyhedron and a non-regular triangulation.

14.20. For *a*), the standard example is given in Figure 42.6. For *b*), this is also false, e.g., for a cuboctahedron (see Figure 16.4). There is also an example with 6 vertices. For the rest of the exercise see [San1].

14.21. See [Dil].

14.22. For *a*), denote by R_t the intersection of a simplex Δ in the d-cube with a hyperplane $x_1 = t$. Show that $\mathrm{vol}(R_t) = ct^i(1-t)^{d-i}$, for some $c > 0$, where i depends on the number of vertices in Δ with $x_1 = 1$. Now use the fact that there is a unique linear combination of polynomials $t^i(1-t)^{d-i}$ equal to a constant. This argument is given in [Glaz]. For *b*) and further references, see [DRS].

14.23. For *a*), here is a simple ad hoc argument: Draw the edges between centers of squares inside each domino. Overlap two tilings. Find the innermost circle C. Since both tiling coincide inside C, use 2-moves to remove this circle. Repeat this procedure until the tilings coincide.
For *b*), the counterexamples to the first two questions are given in Figure 42.7. For the final part, take an $n \times n$ square region G in $z = 0$ level and in the checkerboard fashion add

[99] An easier example can be found in `http://tinyurl.com/radztn`.

squares in $z = 1$ and $z = -1$ level, above or below every square except for the boundary of G. This fixes the $(n-2)^2$ vertical dominoes and allows only two possibilities the horizontal dominoes in $z = 0$ level.

FIGURE 42.7. The 2-move connectivity fails for non-simply connected and 3-dimensional regions.

Parts $c)$–$g)$ are based on Chaboud's exposition (1996) of Thurston's construction (1990). For $g)$, simply observe that height functions are exactly the integer functions on Γ which around every square are different by 1 on three edges and by 3 on one edge. Now check that $\vee$ and $\wedge$ preserve this.
For $j)$, define $|h| = \sum_{x\in\Gamma} h(x)$. Observe that under a 2-move $|h|$ changes by 4 and that $|h_{\max}| - |h_{\min}| = \theta(n^3)$ in this case. Part $k)$ is due to C. and R. Kenyon (1993) and uses height functions with values in the free product $\mathbb{Z}_k * \mathbb{Z}_\ell$. See [Pak3] for details, generalizations and references to all parts.

14.24. Part $a)$ for rectangles is due to Pak (2000). Part $b)$ is due to Sheffield (2002), who used multidimensional height functions. Parts $c)$–$e)$ are due to Conway and Lagarias (1990). See [Pak3] for the proof outline and the references.

14.25. Draw the dotted lines through the parallelograms with one side parallel to the same side (see Figure 42.8). Number the lines from 1 to n. Define a map φ from mosaics of a $2n$-gon to certain reduced decompositions of a permutation $\omega = (n, n-1, \ldots, 2, 1) \in S_n$ into a product of adjacent transpositions $s_i = (i, i+1)$ by stretching them as in the figure and projecting the intersection points on the horizontal line (transposition s_i corresponds to the intersection of i-th and $(i+1)$-th line from the top). These intersection points are in bijection with pairs of elements in $\{1, \ldots, n\}$ corresponding to lines, so the number of possible orders is at most $\binom{n}{2}!$

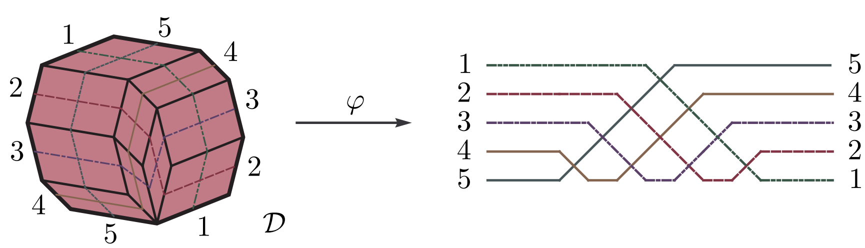

FIGURE 42.8. Example of the map $\varphi : \mathcal{D} \to s_4 s_3 s_2 s_4 s_1 s_3 s_2 s_4 s_3 s_4$.

Observe that there is an ambiguity created by the order of transposition s_i and s_j, $|i-j| \geq 2$. Check that if reduced decompositions are taken modulo commutations as above, the map φ is a bijection (Elnitsky, 1997). Also, the flips correspond to 3-relations $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$. These and commutativity relations give all Coxeter relations for S_n, which implies that flips connect all mosaics. Observe that every reduced decomposition of ω must contain at least

one s_1, s_{n-1}, and at least two s_i for all $2 \le i \le n-2$, because all elements $\le i$ must move across i-th position. Use this to conclude that one can always apply at least $n-2$ different 3-relations.
For d) and e), one can find a small non-regular mosaic by hand and then check that a random mosaic contains lines with the same pattern. Here is a more general argument. Consider n vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ which project on the edges of P. Removing all lines but three lines i, j, k gives one of the two mosaics of a hexagon. This "submosaic" gives an inequality on triples of vectors. Thus, for a fixed P, the space of these vectors is n-dimensional, so the $\binom{n}{3}$ inequalities correspond to $\binom{n}{3}^n = e^{O(n \log n)}$ regions in $\mathbb{R}^n$, since k hyperplanes in $\mathbb{R}^n$ divide the space into $k^{O(n)}$ regions. In a different direction, use induction to show that the number of mosaics is at least $e^{\Omega(n^2)}$.[100] This implies that the number of regular mosaics is asymptotically smaller than the number of all mosaics of $2n$-gon.

15.1. Subdivide P into a large number of small cubes and approximate the boundary.

15.2. Extend the proof of Theorem 15.2.

15.3. It suffices to prove the result for symmetric tetrahedra. For a tetrahedron $\Delta = (a_1, a_2, a_3, a_4) \subset \mathbb{R}^3$, let O be the center of the circumscribed sphere. Assume for now that $O \in \Delta$. Denote by b_i the orthogonal projections of O onto the face F_i opposite to a_i, for all $1 \le i \le 4$. Subdivide Δ into 12 tetrahedra with vertices at O, b_i and two vertices of the face F_i. Observe that each of the resulting tetrahedra is mirror symmetric, which implies that Δ is scissor congruent to its mirror image Δ' in this case. When $O \notin \Delta$, use the same argument combined with Theorem 16.3. This proof is based on [Bri1].
15.4. Use Lemma 15.3 repeatedly or brute force to obtain scissor congruence of any parallelepiped with a brick $1 \times 1 \times \ell$.

15.5. Since every polygon can be triangulated, it suffices to prove the claim for the triangular prisms. In the latter case, observe that two triangular prisms tile a parallelepiped and use Theorem 16.4.

15.7. See [HadG] and [Bolt, §9–11, 19].

15.8. For both a) and b), start with the scissor congruence and "fix" it by switching simplices on the boundary with those in the interior. For c), take unit height prisms over a unit 3-cube and a regular tetrahedron of the same volume. See [Schn1] for a related problem.

15.10. Place the centers of the cross tiles in the lattice points $(a_1, \ldots, a_d) \in \mathbb{Z}^d$, where $a_1 + 2a_2 + \ldots + da_d = 0 \bmod 2d+1$. See [SteS, §3] for the complete proof and references.

15.11. For a) and b), see [Ste1] and [Schme], respectively. Part c) is given in [HocR].

15.12. For d), see [HocR]. Part g) follows from [Schme] or by a direct argument.

15.13. See [Ada1, Schmi] and references therein.

15.14. The idea of this problem is due to Yuri Rabinovich (unpublished). For a), note that every translation of the unit square is a fundamental region of the natural action of $\mathbb{Z}^2$. Thus the orbit $\mathcal{O}(x)$ of every $x \in Q$ has exactly one point in each Q_i. Not all these points are necessarily distinct, of course, but since the number of squares n is odd, at least one point in $\mathcal{O}(x)$ is covered by an odd number of squares. In other words, for every point $x \in Q$ there exists a point $y \in A$ in the orbit of x. Thus, area$(A) \ge$ area(Q).

[100] One can obtain a much better bound using the exact formula due to Stanley (1984) for the total number of reduced decompositions of ω (see [Sta3, Ex. 7.22]).

For c), take three subsets of a 1×7 rectangle, obtained by replacing 1 with unit squares in the following sequences: 1101100, 0110110, 0011011. In this case area$(A) = 2$ while area$(Q) = 4$. This construction is due to R. Connelly (personal communication).
For d), consider three symmetric translations of a triangle with unit sides, which have a common intersection: similar triangle with sides 1/2. This construction is due to A. Akopyan (personal communication).

15.15. This follows by elementary counting [Pak6, §8.3]. See also [KonP] for generalizations and bijective proofs.

16.1. Observe that $Q(\lambda)$ is the complement to eight copies of standard tetrahedra $\lambda\Delta_1$. Use complementarity and tiling lemmas to conclude that $Q(\lambda)$ is rectifiable only when $\lambda = 0$. Now assume that $Q(\lambda) \sim cQ(\mu)$, for some $\lambda < \mu$. The equality of the volumes gives

$$c = \frac{1 - 8(\lambda^3/6)}{1 - 8(\mu^3/6)} > 1.$$

Denote by D the regular octahedron comprised of eight tetrahedra Δ_1. From above, $Q(\lambda) \oplus \lambda D \sim H$, where $H = Q(0)$ is the unit cube. We have:

$$\begin{aligned} H \oplus c\mu D \,\sim\, Q(\lambda) \oplus c\mu D \oplus \lambda D \,\sim\, cQ(\mu) \oplus c\mu D \oplus \lambda D \\ \sim\, c\big(Q(\mu) \oplus \mu D\big) \oplus \lambda D \,\sim\, cH \oplus \lambda D. \end{aligned}$$

Since $c > 1$ as above, the equality of the volumes on both sides of the equation implies that $\lambda < c\mu$. Using $s = \lambda/c\mu$, we obtain $D \oplus R_1 \sim sD \oplus R_2$, where $s \neq 1$ and $R_1, R_2 \in \mathcal{R}$. Now Theorem 16.2 gives a contradiction.

16.2. For a), see [Ker]. For part b), see [Gol5]. See also similarly titled papers by M. Goldberg on convex tiles with other number of faces.

16.3. For a), take a circle of radius R and compute the average angle of polygons in two different ways: one via the average in each octagon, and another from the fact that at each vertex there are at least three edges meeting. Letting $R \to \infty$, obtain a contradiction. For b), tile a plane with T-tetrominoes. For more on tiling a plane with polygons of ≥ 7 sides see [Niv].

16.4. Consider the asymptotic behavior of the summations $\sum$ defined in the proof of Lemma 15.4. Calculate the asymptotics in two different ways (cf. Exercise 16.3). This result is due to Debrunner (1980), and an elementary proof was given in [LM].

16.5. Take a rhombus R with irrational angles and consider two triangular prisms with isosceles triangular faces orthogonal to R. Attach these prisms along the rhombus to form a polytope P as in Figure 42.9. If the heights of the prisms are chosen generically, the only way to tile the space is by attaching triangular faces to each other and forming layers as in the figure. Since the layers are at irrational angle to each other, in the resulting tiling has copies of P oriented in infinitely different way. This implies that all such tilings are aperiodic. This example is due to J. Conway and outlined at `http://tinyurl.com/3y4o4j`

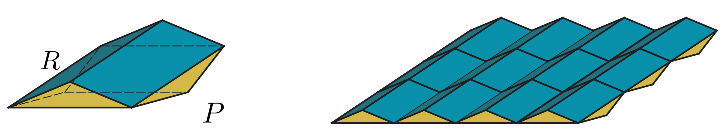

FIGURE 42.9. An example of a polytope which tiles the space aperiodically.

16.6. For c), consider the affine Weyl group $\widehat{A}_n$. Compare the second part with Exercise 7.16 a). For f), the edges correspond to the vectors $\boldsymbol{v} = e_i - e_j$, for all $i < j$. These vectors correspond to the edges in a complete graph K_n. Check that $n-1$ of these vectors are independent if and only if they correspond to a spanning tree in K_n. Compute a determinant to show that the volume of every parallelepiped spanned by such vectors is 1. The last two steps are exactly the same as in the matrix-tree theorem (see [Tut, §4]).

16.10. The first such decomposition was given in [Syd2]. The decomposition in Figure 42.10 is given in [TVS]. Yet another decomposition (with only three pieces) is due to Schöbi (1985). See [SloV] for a description, references and generalizations.

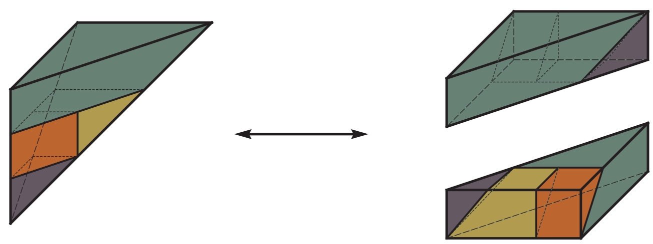

FIGURE 42.10. A scissor congruence between tetrahedron Δ_1 and the right triangular prism (view from the top and from the bottom).

16.11. In fact, all these tetrahedra can tile the space, which is why they were discovered by Sommerville (1923) and Goldberg [Gol3]. See also [Sene] for the history of these constructions and their underlying symmetries. For a), this tetrahedron tiles a unit cube. The tiling lemma (Theorem 16.4) implies the result. For b), this tetrahedron can be tiled by two tetrahedra in a). Similarly, four copies of a tetrahedron in c) or two copies of a tetrahedron in d) tile Δ_1. Part e) is a special case of f). For f), observe that three such tetrahedra tile a triangular prism and use the tiling lemma.

16.12. This result in part a) is due to Hadwiger (1951). We refer to [SloV] for a direct proof, extensions and references. For b), see [Gol3].

16.13. See [Syd2] for details and references.

16.14 and 16.15. See [Fie] and [Cox2].

16.16. See [Deb2] (rediscovered by Brandts, Korotov and Křížek in 2007).

16.17. See [Cox1, §11] and [Cox2].

16.18. Sommerville's 1923 classification was recently finished in [Edm2].

16.19. See [Jes2].

16.20. See [Leb1] and [Jes1] for the necessary and sufficient conditions on scissor congruence of aggregates of Archimedean solids. For part c), see also [CRS].

16.22. For b) and d) see [Deb1]. For c), use the E_8 lattice (see [Cox1]).

16.23. See [Bolt, §16].

16.25. An elementary proof is given in [Bolt, §20].

17.1. In higher dimension, extend Lemma 17.15 by taking height functions of star triangulations as follows. Set $\xi(a) = 1$ for a center of $\mathcal{D}$, $\xi(b) = \varepsilon$ for every center of a facet triangulation, $\xi(c) = \varepsilon^2$ for every center of a triangulation in a face of codimension 2, etc. It is easy to see that when $\varepsilon > 0$ is small enough, the height function ξ is as desired (use induction on d). Check that when ξ changes, the triangulation remains stable except for 2–d moves inside a bipyramid in $\mathbb{R}^d$.

17.4. For d), use induction on the dimension and number of facets. Suppose first that P is not simple. Cut P with a hyperplane near a vertex v with $\deg(v) > d$. The polytope Q in the intersection has a smaller dimension and by induction can be cut into simplices. Extend these cuts to hyperplanes through v. Check that the resulting polytopes have fewer facets and use inductive assumption. Suppose now that P is simple. By Exercise 8.13, we can choose two adjacent facets F_1, F_2 and a vertex $v \notin F_1, F_2$. Cut P with a hyperplane spanned by v and $F_1 \cap F_2$. As in the proof of Proposition 17.12, observe that both P_1 and P_2 have fewer facets than P. Use inductive assumption to finish the proof.

17.5. For a), observe that the graph dual to the triangulation is a tree, and the desired triangles corresponds to its endpoints.
For b), use the following construction. Start with a region $A_1 \subset Q$ adjacent to the boundary ∂Q. If $Q \smallsetminus Q_1$ is not simply connected, take a connected component $B_1 \subset A_1$ with the smallest number of regions. Let $A_2 \subset B_1$ be a region adjacent to the boundary ∂Q. Again, if $Q \smallsetminus A_2$ is not simply connected, take a connected component $B_1 \subset Q \smallsetminus A_2$ with the smallest number of regions. Repeat this procedure until one of the regions $Q \smallsetminus A_i$ is simply connected. Use induction on the number of regions in B_i. See [MucP] for a complete proof.
For c), take three 3-dimensional tiles as in Figure 42.11. For d), a delicate construction of one such triangulation is given in [Rud].

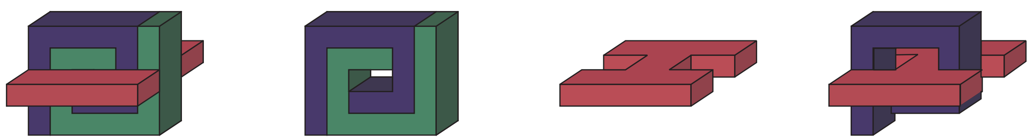

FIGURE 42.11. An impossible puzzle with three pieces.

17.6. See [IzmS] for an accessible proof of c). For d), suppose $\mathcal{D}_1, \mathcal{D}_2 \vdash P$ are two dissections of $P \subset \mathbb{R}^d$. For every simplex $\Delta \in \mathcal{D}_1$ with points on the boundary, use elementary moves to obtain a barycentric subdivision of Δ. This gives a subdivision $\mathcal{D}_1' \vdash P$ refining $\mathcal{D}_2$. Construct $\mathcal{D}_2'$ similarly. By assumption, $\mathcal{D}_1'$ and $\mathcal{D}_2'$ are connected by stellar moves. Check that each stellar move can be obtain as a composition of elementary moves.

18.1. It suffices to show that the global change can be made with the same sequence $(*, c, \ldots, c)$, where $c = \min\{a_i, a_j'\} > 0$. For that, connect 1 to all other vertices in G and make local exchanges to concentrate all values at 1 while keeping c elsewhere.

18.3 and 18.4. See [Sta2].

18.5. Part c) is proved in [Pak1] (see also [Pak6, §9.1]).

19.2. See [Kis, §86].

19.3. Parts a) and b) were communicated to me by Ezra Miller and Günter Rote.

19.4. The first unfolding is that of a polytope with two non-square rhombic faces. The second and third are the Johnson solids called *sphenocorona* and *disphenocingulum*.

19.5. Part a) follows from the fact that the center of mass $\mathrm{cm}(P)$ is a fixed point of every symmetry line. For part b), fix a symmetry line ℓ. The remaining symmetry lines are split into pairs symmetric with respect to ℓ, unless they are orthogonal to ℓ. In the latter case, observe that if $\ell \perp \ell'$ are symmetry lines, then the line ℓ'' is orthogonal to both ℓ and ℓ'. Therefore, the lines orthogonal to ℓ are split into pairs $\{\ell', \ell''\}$, which implies the result. This friendly solution is given by N. B. Vasiliev, V. A. Senderov, and A. B. Sossinsky in [Kvant], M623 (1981, no. 3).

19.6. Consider six points on the edges of C at distance x from either $(0,0,0)$ or $(1,1,1)$. Then use monotonicity (in fact, $x = 3/4$ works).[101]

19.7. See [Mill].

19.8. For a), take a regular bipyramid. For b), take the *trapezohedron*, a dual polytope to the biprism. Finally, for c), consider a classification of finite subgroups of $\mathrm{SO}(3,\mathbb{R})$, the automorphism group of P. We do not know a direct proof.

19.9. Consider a regular n-prism $P_n(h)$ with varying height h. There are two types of cones over the faces, and by continuity there exists h so that $P_n(h)$ is fair (see [DiaK]). For $n = 5$, see also the US Patent `6926275`.[102]

19.12. a) Take a unit cube centered at the origin and attach to each face of the cube a regular pyramid with dihedral angles $\pi/4$ in the square faces. The resulting polytope P is called *rhombic dodecahedron*, has 6 cube vertices of degree 3, and 8 new vertices of degree 4. Note that P has 12 rhombic faces, is face-transitive and edge-transitive around the origin. Thus, P is midscribed. On the other hand, P is clearly not inscribed.
b) Take the *cuboctahedron* (see Figure 16.4). This is a variation on Problem 15.10 in [PraS].

19.13. See a solution by V. A. Senderov in [Kvant], M1192 (1990, no. 4).

19.14. a) Such polytopes are called *deltahedra*. There are only eight of them (see [Cun]). For b), note that there is only a finite number of combinations of faces around each vertex, giving a lower bound $\omega_i \geq \varepsilon$ on the curvature of every vertex v_i, for some constant $\varepsilon = \varepsilon(k) > 0$. By the Gauss–Bonnet theorem (Theorem 25.3), every such polytope has at most $4\pi/\varepsilon$ vertices. Thus, there is only a finite number of combinatorial types. By the Cauchy theorem (Theorem 26.1), we conclude that there is only a finite number of such polytopes. For c), these are *prisms* and *antiprisms*.

19.16. For a) and other polyhedra with symmetries see [Cox1, Crom, McS]. For b) and c), see solution to Exercise 7.16 and the US Patent `3611620`.[103]

19.17. See the Jessen's original paper [Jes2].

19.18. For a), change the edge lengths one by one. For b), take $\ell_{01} = \ell_{12} = \ldots = \ell_{d0} = 1 + \delta$ and $\ell_{ij} = 1$ otherwise [MarW]. For c), consider combinatorics of different edge lengths [Edm1].

20.1. See [Mus5] for references to optimal bounds.

20.2. The maximum is equal to six (take the longest diagonals in the icosahedron). To prove this, fix two lines, which we represent as two points $x, y \in \mathbb{S}^2$ on a sphere and their opposite points x', y'. Now consider all $z \in \mathbb{S}^2$ at equal distance to the closest of the pair, i.e., such that $\min\{|zx|, |zx'|\} = \min\{|zy|, |zy'|\}$. Each of the four possible choices for the minima corresponds to at most one additional lines, bringing the total to six. This observation is due to I. F. Sharygin.

20.3. Denote by B_1 the ball of radius 10 as in the theorem and suppose $\mathrm{diam}(P) \leq 21$. Then P lies inside the ball B_2 of radius 11 with the same center as B_1. By Exercise 7.11,

[101] This problem was given at the 28th *Tournament of the Towns*: `http://www.turgor.ru/28/`

[102] Available online at `http://www.google.com/patents?vid=USPAT6926275`.

[103] Available at `http://www.google.com/patents?vid=USPAT3611620`.

the surface area of P must be greater than area(B_1). On the other hand, each face F of P lies in a circle $H \cap B_2$, where H is a plane tangent to B_1. This gives an upper bound on the area of F. Now use the condition that P has 19 faces to obtain a contradiction. Two solutions of this problem (including this one) are given by A. G. Kushnirenko in [Kvant], M35 (1971, no. 6).

20.4. Take spherical circles around each x_i of radius $\pi/8$ and use the area argument to show that these circles cannot be disjoint. This solution is given by A. K. Tolpygo in [Kvant], M656 (1981, no. 8).

20.6. For a), consider a line not parallel to the symmetry axis of parabolas. Check that only a finite portion of the line is covered. For b), translate the cones $C_1, \ldots, C_n$ to have the symmetry point at the origin. Observe that these cones are still disjoint except at the boundary. Indeed, otherwise there exist two translations of a (small) cone which lie in two cones C_i and C_j, a contradiction with $C_i \cap C_j = \varnothing$. This problem is given by A. Kuzminykh in [Kvant], M748 (1982, no. 11).

20.7. Part b) was given at the Moscow Math. Olympiad in 1966. An easy solution is given in `http://tinyurl.com/n47h9l` .

20.8. See [Daw3].

20.10. For a), take all cylinders to be parallel. For b), start with six parallel cylinders and rotate three of them as a solid block. For c), consider the area of intersection or a cylinder and a sphere of radius r. Optimize for r (see [BraW] for the full proof). This problem is due to W. Kuperberg (1990).

20.11. For a), take two copies of $\mathcal{B}$ and put them together so some two balls touch. For b), take the arrangements of balls in $\mathbb{S}^3$ at the vertices of the 600-cell. Make a stereographic projection. For c), take the graph $\Gamma = \Gamma(\mathcal{B})$ with vertices corresponding to balls and edges if they touch. Orient the edges from balls of smaller to larger radius. Since the outdegree of all vertices in Γ is $\le K_d$, conclude the result. For more on this problem, see [KupS].

20.12. Part a) follows from part a) of Exercise 20.1 and the proof of part c) of the previous exercise. For b), observe that n spheres in $\mathbb{R}^d$ are determined by their centers and radii, and thus have $nd + n$ degrees of freedom. For pairwise touching, they need to satisfy $\binom{n}{2}$ equations. Now proceed as in Section 31. For c), make an inversion at a point where two spheres are touching. They become two parallel hyperplanes, which implies that all other spheres have the same radius. Parts a) and c) were communicated to me by Itai Benjamini (2010).

21.1. b) An example is shown in Figure 42.12 (see [Mus1]). Here $r_1 < r_2 < r_3 < r_4$.

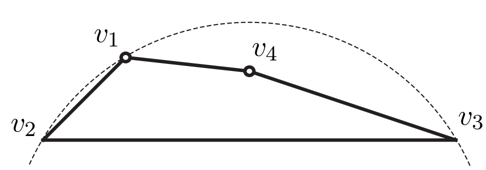

FIGURE 42.12. A non-coherent quadrilateral.

21.2. This follows from Corollary 21.12. For a direct simple proof see [SCY], Problem 9.

21.4. The proof follows roughly along the same lines as the proof of Theorem 21.7. In this case one needs to consider the set of points at equal distance to 3 or more lines and prove that this is a binary tree.

21.6. See [Weg3].

21.10. For b), take an immersed closed curve C in Figure 9 in [Arn2]. Inscribe an equilateral polygon Q and the same polygon Q' with a shifted order of vertices.

21.11. See [OT1].

21.12. This is a discrete analogue of a result by Ghys (see [OT1] and references therein). To see a connection to the four vertex theorem, consider a polygon $Q \subset \mathbb{R}^2$ with vertices $v_1 = (x_1, y_1)$, ..., $v_n = (x_n, y_n)$. For every i, consider a unique projective linear transformation $\varphi_i : \mathbb{RP}^1 \to \mathbb{RP}^1$, which maps x_{i-1}, x_i and x_{i+1} into y_{i-1}, y_i and y_{i+1}, respectively. Denote by Γ the graph $(x, \varphi(x))$ of function φ. Graph Γ now plays the role of circumscribed circles since by definition it contains v_{i-1}, v_i and v_{i+1}. Now check that $\alpha_i - \beta_i$ is a local minimum or maximum is the cyclic sequence if and only if the vertices v_{i-2} and v_{i+2} of Q lie on the same side of Γ.

21.13. Part a) is a classical result of Szegö (1920) and Süss (1924). We refer to [Ball] for the related results and the references.

22.1. This proof is given in [A2, §9.1]. For a), see Figure 36.1. For b), take the signed area of triangles spanned by the origin and the edges of P. For c), translate P so that the origin in inside and differentiate the area:

$$2 \cdot \text{area}(P_t)' = (h_1 \ell_1' + \ldots + h_n \ell_n') + (h_1' \ell_1 + \ldots + h_n' \ell_n).$$

On the other hand, $\text{area}(P_t)' = h_1' \ell_1 + \ldots + h_n' \ell_n$ by definition of the area, which implies that $\text{area}(P_t)' = h_1 \ell_1' + \ldots + h_n \ell_n'$. Since $\text{area}(P_t)' = 0$ by assumption, and $h_i > 0$ by construction, we conclude that some ℓ_i' must be negative.
For d), suppose vertices v and w separate the increasing and decreasing edges in P. Place the origin in the intersection of the lines supporting P at v and w. Now all $h_i \ell_i'$ have the same sign in the expression for $\text{area}(P_t)'$, a contradiction.

22.3. Two proofs are given in [OR].

22.4. In part a) use the arm lemma (Lemma 22.3) and in b) use the extended arm lemma (Exercise 22.3). The proofs are given in [OR, ORS].

23.1. Consider separately all pairs of convex polygons in Y and then take a sum.

23.3. Part a) is a classical result on crossing numbers which can be found in [PacA, §14] and [Mat1, §4.3]. Part b) was proved in [AloG].

23.5. For b), see [Hal4]. For parts c) and d), see [Oza1].

23.6. This is studied in [Oza].

23.7. An involved version of this result is proved in [BolG], where it was proved by using the cut locus construction and deformations of polygons.

23.8. For a), an elegant smooth example is given in [Mor]. For b), see [Heil].

23.9. This is proved in [Hal3] via reduction to the smooth case.

23.10. This exercise follows closely [Carv].

23.11. The first explicit construction of a "stuck unknot" (space hexagon which cannot be rigidly flatten) was constructed in [CanJ]. Another example is given in Figure 42.13 (see [Tou]).

23.12. This is proved in [CocO].

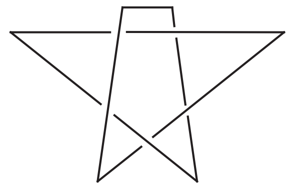

FIGURE 42.13. A stuck unknot.

23.13. This is a classical Erdős problem. We refer to [DGOT] for an overview of the history, several proofs and references.

24.1. *a*) Of all $(n-1)!$ polygons on X, let Q have the shortest length. If two edges of Q intersect, they can be replaced by two others to make a polygon of shorter length. This implies that Q is simple. For *b*), project X onto a generic plane and take a polygon as in part *a*). For *c*), project X and take the first triple to form the first triangle, the second triple to form the second triangle, etc. For *d*), this is true in fact for all embeddings of K_6 into $\mathbb{R}^3$, not necessarily by straight lines (see [Ramí]). The easiest way to prove this is by deforming the configuration and checking that the number s of pairs of linked triangles is always odd. Indeed, for every two edges there are two way these edges can connected to the remaining two points. When two edges cross, both pairs of triangles change from linked to unlinked or vice versa, so the parity of s is unchanged.

24.2. For two points $x, y \in Q$, denote by γ the shortest path between x and y which lies inside Q. If $(xy) \notin Q$, then γ is not an interval, and thus must contain at least one vertex v of Q. Now observe that locally around v we must have $\angle v > \pi$, a contradiction. For another elementary proof of this classical result see [Cher, §1].

24.3. This is a classical corollary from a number of advanced results in knot theory (see, e.g., [Ada2]). The following proof is given in [Fox, §7]. Denote by $K\#L$ the (connected) sum of knots K and L (see Figure 42.14). Observe that $K\#L = L\#K$. Suppose K is non-trivial and $K\#L$ is an unknot. Consider a wild knot $Q = K\#L\#K\#L\#\ldots$ On one hand, Q is isotopic to $(K\#L)\#(K\#L)\#\ldots$, and thus an unknot. On the other hand, Q is isotopic to $K\#(L\#K)\#(L\#K)\#\ldots = K$. Therefore, K is also an unknot, a contradiction.[104]

24.4. Follow the proof of Fenchel's theorem, and use induction on the dimension for the equality part.

24.5. For this result, generalizations and references see [Sant].

24.7. See [Sant, §7.5].

24.8. For *a*), this follows from the fact that the average length of a random projection has length $|Q|/\pi$ (see Exercise 24.6). See [Rama]. Parts *b*) and *c*) are presented in [Sul, §8].

24.9. For *a*), if A is a connected component of $\mathrm{sh}(Q)$ and $x \in \mathrm{conv}(A)$, then every plane H through x intersects A at some point a. Since H goes through a, it contains at least four points in Q, which implies that A is convex. The closure of A is a convex polytope by piecewise linearity.

[104]The reader may consider this method a "cheating" for we do not give a delicate convergence argument implicitly used in the proof. There are several known discrete proofs of this exercise, but none are elementary and sufficiently succinct to be included here.

For b), there are two cases. Either L has a knotted polygon, in which case the second hull theorem implies the result. Otherwise, L has two polygons Q_1, Q_2 whose convex hulls intersect. The result now follows from $\operatorname{conv}(Q_1) \cap \operatorname{conv}(Q_2) \subset \operatorname{sh}(L)$.

24.10. For a), take the sum $K\#L$ of two non-trivial knots K and L as shown in Figure 42.14. While $\operatorname{sh}(K), \operatorname{sh}(L) \neq \varnothing$, the "middle points" $x \notin \operatorname{sh}(K\#L)$.
For b), take an unknot Q and x as in the figure and observe that $x \in \operatorname{sh}(Q)$. For c), deform Q appropriately.

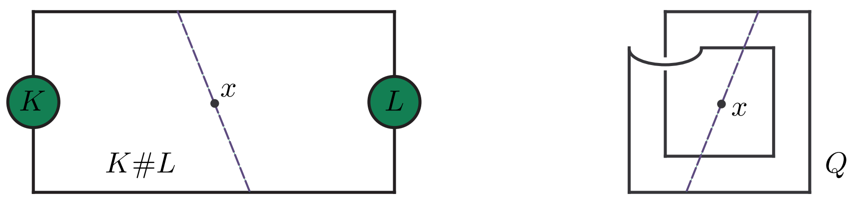

FIGURE 42.14. Sum of knots $K\#L$ with a disconnected second hull and an unknot Q with a nonempty second hull.

24.11. This is proved in [CKKS, §6].

24.12. A stronger form of this result was proved in [Izm].

24.13. This inequality is due to A.D. Alexandrov (1947). See [Res2] for the proofs and references.

24.14. Part a) goes back to Radon (1919) and is completely straightforward: the curvature minimizes when C is convex, in which case, when the projection is simple, it becomes an equality. Part b) is a corollary of Exercise 25.2 applied to the cone. For advanced generalizations and references see [Res1].
For c), consider a point $z \in \operatorname{sh}(Q)$ and take Q' to be the result of a translation of Q which moves z to O. By definition of the second hull, all generic hyperplanes through O intersect Q' four or more times. Now use Crofton's formula (Lemma 24.6) and the proof of Fenchel's theorem (Theorem 24.4) to conclude that $\varkappa(Q) \geq \varphi(Q') \geq 4\pi$. We refer to [CKKS] for the sketch of this proof and related results.

24.15. Let S be the set of midpoints of (x, y), where $x, y \in Q$. Following Subsection 5.3, the surface S consists of triangles and parallelograms with the boundary at Q. Since Q is knotted, the interior of S intersects with Q, giving the desired triple.

24.16. For a), given a point $x \in Q$ consider all triangles spanned by x and edges of Q. Denote by S the resulting surface. If S is embedded, then Q is unknotted, a contradiction. If two triangles $T_1, T_2 \subset S$ intersect, the intersection is the desired line. For b) and c) see [Den, Kup1].

24.17. Part a) is due to Fáry and has been generalized in various directions [BZ3, §28]. Three elementary proofs are given in [Tab5]. Part b) is proved in [Cha], and part c) in [LR, NazP].

24.18. For b) and c), see [CSE]. For d), use a small circle C_2.

24.19. The net total curvature and the corresponding Crofton's formula are proved in [Gul]. Part b) is proved in [AbrK] (see also [FSW] for a simple proof).

24.20. The idea is to adapt the proofs of the Fenchel and Fáry–Milnor theorems in this case [Gul].

24.21. Part a) was proved in [Bes1]. For b), take three symmetric geodesic arcs from the North Pole of length $\pi - \varepsilon$ and add a small geodesic equilateral triangle around the South Pole. The resulting net has total length $3\pi + O(\varepsilon)$, where $\varepsilon > 0$. Part c) is a generalization of a) given in [Crof].

25.3. Region A contains all vertices of P, has $n = 2\,|E|$ vertices, while ∂A has $r = |\mathcal{F}|$ connected components. Observe that every angle of A is a complement of the corresponding face angle to 2π, which gives

$$\alpha(A) \,=\, n \cdot (2\pi) \,-\, \sum_{v \in V} \alpha(v) \,=\, (n - |V|) \cdot 2\pi \,+\, \sum_{v \in V} (2\pi - \alpha(v))$$
$$=\, 2\pi \cdot (2|E| - |V|) \,+\, \sum_{v \in V} \omega(v) \,=\, 4\pi \cdot |E| \,-\, 2\pi \cdot |V| \,+\, 4\pi\,.$$

On the other hand, by the previous exercise, we have $\alpha(A) = (2|E| + 2|\mathcal{F}| - 4) \cdot \pi$. Together these two equations imply Euler's formula.

25.5. For a), consider the dual cones C^* and D^*. Use the supporting hyperplanes to show that $C^* \supset D^*$. Conclude from here that $\omega(C) = \sigma(C^*) \geq \sigma(D^*) = \omega(D^*)$.

25.6. For a), note that the centrally symmetric faces have at least four vertices. Now use Euler's formula. For b), consider the Minkowski sum of n vectors in convex position (take e.g., the side edges of a symmetric n-pyramid). The resulting zonotope has at least n faces, all parallelograms (see Exercise 7.16).

25.8. For b), here is a solution from [Grub, §15.1]. Denote by a_i the number of vertices of degree i, and by f_i the number of i-sided faces. For the numbers n, m and k of vertices, edges and faces, respectively, we have $n = a_3 + a_4 + a_5 + \ldots,\ \ k = f_3 + f_4 + f_5 + \ldots$, and $2m = 3a_3 + 4a_4 + 5a_5 + \ldots = 3f_3 + 4f_4 + 5f_5 + \ldots$ By Euler's formula, we conclude:

$$8 \,=\, (4n - 2m) + (4k - 2m) \,=\, (4a_3 + 4a_4 + 4a_5 + \ldots - 3a_3 - 4a_4 - 5a_5 - \ldots)$$
$$+\, (4f_3 + 4f_4 + 4f_5 + \ldots - 3f_3 - 4f_4 - 5f_5 - \ldots) \leq a_3 + f_3\,.$$

25.9. For a), observe that the curvature of all vertices are bounded: $\omega_i \leq \pi/2$. Now the Gauss–Bonnet theorem implies the result. Part b) is analogous. Note that both bounds are tight: take the cube and the icosahedron, respectively.

25.10. Use Girard's formula to write each solid angle in terms of dihedral angles. Summing over all vertices, each dihedral angle will be counted twice. Then use Euler's formula.

25.11. This is a dual result to the Gauss–Bonnet theorem (Theorem 25.3).

25.12. $(ii) \Rightarrow (i)$ Write out equations for the edge lengths.
$(iii) \Rightarrow (i)$ From the Gauss–Bonnet theorem (Theorem 25.3), conclude that the curvature of each vertex is equal to π. Now write out the equations for the angle sums around each vertex and inside each face.
$(iv) \Rightarrow (i)$ Use an argument based on Subsection 25.3.
$(v) \Rightarrow (iv)$ Write out equations for the solid angles in terms of dihedral angles.

25.13. This follows from Euler's formula. See Subsection 26.3 for the proof.

25.14. This is a restatement of Euler's formula. It is proved in [Gla] (see also [GGT, §2]).

25.15. Both parts easily follow from Exercise 25.14. For a), if there are no sinks, sources and faces with oriented cycles, then the index of every vertex and face is ≤ 0, while by the formula in the exercise, their sum is zero, a contradiction. Alternatively, one can prove part a) in the same manner as Lemma 32.7. Since there are no sinks in the graph, every oriented path can be extended until it self-intersects. If the resulting cycle is not along a face, start moving inside the cycle until a smaller cycle (in the area) is obtained. Repeat until the desired face cycle is found. This simple result seems to have been rediscovered a number of times, but [Gla] is a definitive reference.

25.16. Take a triangular bipyramid where the top and bottom vertices have curvatures ε and $(2\pi - \varepsilon)$, respectively, for $\varepsilon > 0$ small enough.

25.18. *a*) Let C be a convex cone with n faces and acute dihedral angles γ_i, $1 \le i \le n$. The sum of dihedral angles is equal to the sum of face angles of the dual cone C^*, i.e.,

$$\sum_{i=1}^{n} \gamma_i = \pi n - \omega(C^*) \ge \pi(n-2).$$

If $\gamma_i < \pi/2$, then $n = 3$ and all face angles of C are acute. The only polygons with acute angles are triangles. We conclude that P is simple and all faces are acute triangles, i.e., P is a tetrahedron.
b) By the same argument, P is simple and all faces are either non-obtuse triangles or rectangles. Now apply Euler's formula.
c) This is false. Consider a prism of height h and two sides regular n-gons with side 1. Connect adjacent midpoints as in Figure 42.15 and remove $2n$ resulting tetrahedra. Note that the dihedral angle γ between triangles and n-gons decreases continuously with $h \in (0, \infty)$, from π to $\pi/2$. Take h to be such that $\gamma = 3\pi/2$. It remains to check that the dihedral angle γ' between triangles and rhombi $\to 3\pi/2$ as $n \to \infty$. A more involved example is given in [Pach] (see Exercise 40.12).

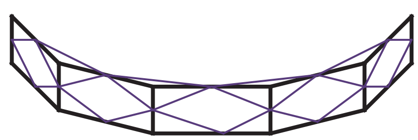

FIGURE 42.15. Truncated prism with dihedral angles $\approx 3\pi/2$.

25.20. See [Grü4, §13.3] for the elementary proof of *b*) and some background.

26.1. Parts *b*) and *c*) are given in [Karc].

26.2. This was proved in [Schl2].

27.1. See [DSS2].

27.2. Part *a*) is false. Take $\Delta = (v_1v_2v_3v_4)$ with $|v_1v_2| = |v_1v_3| = |v_4v_2| = |v_4v_3|$, an obtuse dihedral angle at (v_2, v_3) and equal other dihedral angles. This proof is given by N. B. Vasiliev and V. A. Senderov in [Kvant], M1301 (1992, no.3). For *b*), observe that equal dihedral angles imply equal face angles.

27.3. By definition, the local convexity condition divides all vertices into concave and convex (via orientation of the cone at a vertices). Note that the edges in convex cones have dihedral angles $< \pi$, while the edges in concave cones have dihedral angles $> \pi$. Thus, no edge can connect concave and convex vertices. The connectivity of the edge graph implies that all dihedral angles are $< \pi$. Now use the local convexity criterion (Lemma 14.7).

28.1. To remove the second assumption, attach to two polygons the same 'big' spherical triangle so that the obtained polygons satisfy conditions of the theorem.

28.2. *a*) Let $P_t = P + B_t$, where B_t is a ball of radius t, so that $P_0 = P$. Observe that

$$\operatorname{area}(P_t) \,=\, \operatorname{area}(P) \,+\, t \cdot \sum_e \ell_e(\pi - \gamma_e) + 4\pi t^2.$$

On the other hand, since $P_t \subset B_{R+t}$, we have $\operatorname{area}(P_t) \le \operatorname{area}(B_{R+t}) = 4\pi(R+t)^2$. Therefore,

$$\operatorname{area}(P) \,+\, t \cdot \sum_e \ell_e\,(\pi - \gamma_e) + 4\pi t^2 \,\le\, 4\pi(R+t)^2.$$

Letting $t \to \infty$, we obtain the result.
b) Define $P_t = P + B_t$, $Q_t = Q + B_t$, and use $\operatorname{area}(P_t) \le \operatorname{area}(Q_t)$ as $t \to \infty$.

28.4. See [CSE].

28.6. For *a*), move a vertex of a tetrahedron along one of the edges. Check the formula by a direct calculation. Obtain the general formula from an (infinitesimal) composition of such transformations. For *b*), triangulate the polyhedron enclosed by the surface, apply *a*) and use the additivity of the Schläfli formula.

28.8 and 28.9. See [Kor2, Kor1].

30.1. For *a*), let us first prove that tightness implies both conditions. Suppose the edge condition fails for the edge $e = (v, w) \subset \operatorname{conv}(P)$, where v and w are vertices in P. Then the plane along e and separating from other vertices of P divides P into at least three parts. Similarly, if the vertex condition fails at $v \notin \partial\operatorname{conv}(P)$, we can separate v from other vertices by a plane near v. Now observe that every such plane divides P into three or more parts. In the opposite direction, suppose plane L intersects P and creates three or more connected components. Choose a side with two or more such components. Recall that the graph of edges of $\operatorname{conv}(P)$ lying on one side of L is connected (see Exercise 8.3). If all components contain vertices in $\operatorname{conv}(P)$, then by the edge condition all components must be connected to each other via the edges of $\operatorname{conv}(P)$, a contradiction. Finally, if $Q \subset P$ is a component without vertices in $\operatorname{conv}(P)$, then the furthest from L vertex $v \in Q$ violates the vertex condition. This result is due to Banchoff, while our proof follows [Kuh, §2].
b) The edge condition is clearly insufficient as evident from a non-convex bipyramid in Figure 30.1. Similarly, the vertex condition holds for a non-tight Jessen's icosahedron (see Exercise 19.17).
d) If an intersection $Q = P \cap L$ of P with a plane L is disconnected, then P has at least three connected components, since n curves separate the surface homeomorphic to a sphere into $n+1$ components. Now, if every intersection is connected, the interior of A is a contractible polygon, and the convexity criterion (Exercise 1.25) implies the result.

30.2. See [Spi] for one such construction.

30.3. Remember, it is very hard to make a flexible polyhedron!

30.4. The first part remains an open problem despite numerous attempts (and incorrect constructions). The second part is a conjecture of Stachel [St1].

31.2. See [FedP, §7.5].

31.3. Parts $a), b), c)$ are due to Robbins, who also conjectured parts *d*) and *e*, which were proved in [FedP]. See [Pak4] for a survey and [Var1] for an independent approach.

31.4. See [Con1].

31.5. For *a*), let D be the square length of the diagonal and let R be the squared radius. Clearly, $D/4 + R = L$ is a constant given by the squared side edge length. Now make a substitution of $(L - D/4)$ into the Sabitov polynomial for R. For b), use the fact that Z_n has exponentially many realizations. For c), further details and references see [FedP].

31.6. a) Use $\ell_{12} = \ell_{23} = \ell_{13} = 2$ and $\ell_{01} = \ell_{02} = \ell_{03} = 1 + \varepsilon$, for $\varepsilon > 0$ small enough. b) Consider $\Delta, \Delta' \subset \mathbb{R}^3$ with edge lengths

$$\ell_{01} = \ell_{02} = \ell_{03} = 1\,, \qquad \ell_{13} = \ell_{23} = \sqrt{2}\,, \qquad \ell_{12} = \varepsilon\,,$$
$$\ell'_{01} = \ell'_{02} = \ell'_{03} = 1\,, \qquad \ell'_{12} = \ell'_{23} = \sqrt{2}\,, \qquad \ell'_{13} = \varepsilon\,,$$

where $\varepsilon > 0$ is small enough. Check that Δ, Δ' with such edge lengths exist, and that there is no tetrahedron with $(\ell_{ij} + \ell'_{ij})/2$ edge lengths. For c) and the references see [Riv].

34.8. Parts a) and b) are proved in [Mon1]. This technique was later extended in [Mon2] to prove c) and d). Parts e) and f) are proved in [Pok]. See [Ste2] for a survey.

32.2. This result is due to Pogorelov [Pog6].

32.3. From the contrary, suppose there exists a continuous deformation of C'. Denote by G the graph of P. As in the proof of the Cauchy theorem, consider a subgraph $H \subset G$ of edges where the dihedral angles change. Check that every vertex of H has degree at least 4. Use Exercise 25.7 to conclude that at least eight triangular faces of H (as a planar graph) are triangular. In the other direction, show that even-sided faces of G cannot subdivide these triangles. The second part is similar. See [DSS1] for details and references.

32.4. This can be proved using the combinatorial approach in Subsection 32.3. See [A2, §10.3] for the original proof.

33.2. Consider the following deformation $\{P_t\}$ of the coordinates of Jessen's orthogonal icosahedron:

$$\big(\pm 2, \pm(1+t), 0\big), \quad \big(\pm(1+t), 0, \pm 2\big), \quad \big(0, \pm 2, \pm(1+t)\big).$$

Observe that the lengths of the long edges are unchanged under the deformation, while the lengths of the short edges are

$$\sqrt{(1-t)^2 + (1+t)^2} \;=\; \sqrt{2 + 2t^2} \;=\; \sqrt{2} + O(t^2) \quad \text{as} \;\; t \to 0\,.$$

This implies that the polyhedron P_0 is not infinitesimally rigid. Continuous rigidity follows from the second order rigidity. First, check that the rigidity matrix has corank 1, i.e., the above first order deformation is the only possible (up to scaling). Then check that the second order terms implied by $O(t^2)$ are strictly positive, for all $t \neq 0$.

33.5. Part d) is a proved in [Con3]. See [Con5, ConS, IKS] for more second order rigidity.

33.7 and 33.8. See [RotW, §6, 7].

33.10. See [Luo] for the proof, extensions and references.

31.7. For a), consider normals $\boldsymbol{u}_i$ to the faces. Observe that $\angle\, \boldsymbol{u}_i \boldsymbol{u}_j = \pi - \gamma_{ij}$. Now take $\boldsymbol{w}_i = \operatorname{area}(F_i) \cdot \boldsymbol{u}_i$, where F_i is the face of Δ which does not contain vertex v_i. Recall that $\boldsymbol{w}_1 + \boldsymbol{w}_2 + \boldsymbol{w}_3 + \boldsymbol{w}_4 = \boldsymbol{0}$ (see Proposition 36.1). Thus there exists a closed space quadrilateral Q with edges $\boldsymbol{w}_i$, and the sum of angles in Q is $\le 2\pi$. Since the sum of angles in Q is equal to $\gamma_{12} + \gamma_{23} + \gamma_{34} + \gamma_{14}$, we obtain the result.

For b), observe that $\gamma_{12} + \gamma_{13} + \gamma_{14} > \pi$, since the curvature of the dual cone $\omega(C_1^*) = (\pi - \gamma_{12}) + (\pi - \gamma_{13}) + (\pi - \gamma_{14}) < 2\pi$. Adding such inequalities for all four vertices we obtain the lower bound. The upper bound follows by averaging of the inequalities in a).

For c), we have:

$$\sum_{ij} \cos\gamma_{ij} \;=\; \sum_{ij} -\langle \boldsymbol{u}_i, \boldsymbol{u}_j \rangle \;=\; 2 - \frac{1}{2}\langle \boldsymbol{u}_1 + \boldsymbol{u}_2 + \boldsymbol{u}_3 + \boldsymbol{u}_4, \boldsymbol{u}_1 + \boldsymbol{u}_2 + \boldsymbol{u}_3 + \boldsymbol{u}_4 \rangle \;=\; 2 - \frac{|\boldsymbol{r}|^2}{2},$$

where $\boldsymbol{r} = \boldsymbol{u}_1 + \boldsymbol{u}_2 + \boldsymbol{u}_3 + \boldsymbol{u}_4$. Since $|\boldsymbol{r}| \leq 2$, we immediately obtain both lower and upper bounds. For d), observe that the equality holds only if $\boldsymbol{r} = \mathbf{0}$. From the proof of the previous exercise, we conclude that Δ is equihedral.
Parts a)-d) are given by I. F. Sharygin in [Kvant], M353 (1976, no. 7). For part e), show that one can monotonically increase the dihedral angles to obtain dihedral angles of a Euclidean polytope in the limit.

33.11. This is a small special case of Schramm's theorem in [Schra]. The proof uses an advanced generalization of inversions and a similar inversion counting argument.

34.1. For part a), see [Hada, § 255]. For part b), see [FedP, §8].

34.2. For b), write both sides of the Cayley-Menger determinants $\mod 8$. Part c) was on a Putnam Mathematical Competition in 1993.

34.3. For a), take two 4-tuples of points in the plane:

$$|a_1a_2| = |a_1a_3| = \sqrt{2},\ |a_2a_3| = 2,\ |a_2a_4| = |a_3a_4| = \sqrt{10},\ |a_1a_4| = 4.$$

$$|b_1b_2| = |b_3b_4| = \sqrt{2},\ |b_1b_3| = 2,\ |b_1b_4| = |b_2b_3| = \sqrt{10},\ |b_2b_4| = 4.$$

For b), fix the first coordinates of all points $a_i = (x_i, y_i)$, $1 \leq i \leq n$. For every six distances which can potentially form a flat 4-tuple, write out their Cayley-Menger determinant. Check that the resulting polynomial equations on y_i are nontrivial unless these distances form the usual 4-point pattern. Conclude that for almost all values of y_i the set $\{a_i\}$ has no equivalent and non-congruent sets. See [BouK] for this and further results.

34.6. This was first proved in [Var1]. A simple proof using the theory of places is given in [Con6] (see also [Pak4]).

35.1. For c), use the fact that $\sigma(C)$ is monotone on cones C. See [Pog3, §7.7] for the details and further generalizations.

35.5. Take a set of points $b_1, \ldots, b_k$ such that polygon $B = [b_1 \ldots b_k]$ contains all a_i in its relative interior. Let $Z = \partial B$. For every $\epsilon > 0$, use use Theorem 35.7 to obtain P_ϵ on $n + k$ vertices a_i and b_j, with boundary Z and curvatures $\omega_1 - \epsilon, \ldots, \omega_n - \epsilon$. Let P'_ϵ be a shifted P_ϵ with the sum of heights of interior vertices equal to 0. Check that P'_ϵ converges to the desired convex cap as $\epsilon \to 0$.

35.8. For a), we implicitly use combinatorics of P when lifting v_i, since for $\varepsilon > 0$ small enough the combinatorics is unchanged. For a greedy variational algorithm, start with the upper surface of the convex hull conv(Z). Take the smallest i such that $\nu_i < \omega_i$. Lift it up until the equality is reached. Repeat. For b), presumably the exponential upper bound on the number of full triangulations must play a role (see Exercise 14.7).

35.9. Find an appropriate direction of rays and use Theorem 35.4. This result is obtained in [Zil] (cf. [Thu, §7]).

36.2. Compare the corresponding faces in polytopes P, $Q = \frac{1}{2}(P + P')$ and P'. Since the normals to the corresponding faces are equal, these polygons are parallel (have parallel corresponding edges). Here we allow zero edge lengths in case the parallel edge is not present. Observe that the edge lengths in Q are the averages of the corresponding edge lengths in P and P'. Write signs $(+)$, (0) and $(-)$ on the edges of Q (including zero length edges), if their edge lengths in P are greater, equal or smaller than that of edges in P', respectively.

Note that there are two types of faces in Q: those coming from faces in one polytope and a vertex from another, and those coming from one edge in each. Apply Alexandrov's lemma to show that around each face of Q, either all signs are zero or there are four sign changes.

Take the dual graph to the graph of Q and keep the same signs, so that there are at least four sign changes around every vertex. Modify counting argument in Subsection 26.3, to conclude that all signs must be zero.

36.3. In notation of Subsection 36.2, define a map $\varphi : \mathcal{P} \to \mathcal{Z}$. Prove that $\mathcal{Z} \subset W_+$ is a convex polyhedron by a direct argument (cf. the proof of Lemma 35.3). Similarly, prove directly that $\mathcal{P}$ is connected. Consider only polytopes $P \in \mathcal{P}$ of volume 1. In the topological lemma, the injectivity follows from the Alexandrov theorem and the continuity is trivial. To show that φ is proper, check that the limit polyhedron cannot become unbounded or degenerate. A complete proof is given in [A2, §6.3].

36.5. Note that in the proof we use the equal facet areas only to avoid $F_k^* \subset F_k$, and other conditions can be used in its place. Use monotonicity of the perimeter, surface area, and the mean curvature (see Exercises 7.11 and 28.2).

36.9. For a), consider normals $\boldsymbol{u}_i$ to the faces and use the equation in Proposition 36.1 to conclude that their sum is zero. Check that the resulting 2-dimensional family of $\{(\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4)\}$ (up to rotations around O), is the same as the family of normals of equihedral tetrahedra.
For b), recall that equihedral tetrahedra have equal opposite edges (see Exercise 25.12). From a), this implies that all non-adjacent edges have equal length. This easily implies the result.

36.10. Much of this exercise is due to Shephard (1963). See proofs and references in [McM1, Mey]. Part e) is given in [Kall].

36.12. For a), translate P by a vector $\boldsymbol{w}$. We have:

$$\upsilon(P, P' + \boldsymbol{w}) \,=\, \sum_{i=1}^{n} \bigl(z_1 + \langle \boldsymbol{w}, \boldsymbol{u}_i \rangle\bigr)\, a_i' \,=\, \sum_{i=1}^{n} z_1 a_i' \,+\, \left\langle \boldsymbol{w}, \sum_{i=1}^{n} a_i' \boldsymbol{u}_i \right\rangle \,=\, \upsilon(P, P'),$$

since the second term is zero. See [A2, §7.2] for the rest of this exercise.

36.13. Here $\upsilon(P, P')$ is the usually called the *mixed volume* $\mathrm{vol}(P, P', \ldots, P')$, and the exercise gives an equivalent way to state the Brunn–Minkowski inequalities. This approach leads to the further inequalities, as discussed in [BonF, BZ3, Schn2].

36.14. By the uniqueness part of Theorem 36.2, polytope P is a translate of $(-P)$. This implies the result.

36.15. A concise proof of part a is given in [Grub, §18.2].

36.16. For d) and e) see [A2, §7.1]. The truncated octahedron tiles the space and has 14 faces (see Exercise 16.6). The upper bound in f is due to Minkowski. We refer to [Grub, §32.2] for the proofs, references and the context.

36.18. See [BezB].

36.19. This example was recently constructed in [Pan].

37.1. For a), cut the surface of a convex cap by a hyperplane and combine two copies to obtain a convex polytope with reflection symmetry. Show that this can be done intrinsically. For b), see [A2, §4.4].

37.4. See [ADO].

37.5. Part c) was proved by Harer and Zagier in (1986). See combinatorial proofs in [GouN, Lass].

37.8. For a), it suffices to check that the sum of angles around each vertex is $< 2\pi$. Use the law of cosines to obtain stability result for the angles (e.g., $\varepsilon < 0.01$ easily works). For b), check what happens to the convexity of realizations when the inverse function theorem is applied.

37.10. See [Vol3].

37.11. This is a restatement of an open ended problem communicated to me by Victor Zalgaller.

38.2. For a), repeatedly use convex bendings in the proof of Leibin's theorem (Theorem 38.2). Both parts a) and b) are outlined in the Appendix to [Pog3].

39.3. See [Ble].

39.4. The bendings were constructed in [Mi5].

39.6. See [Pau] and [MlaO].

39.8. Part a) is implicitly stated in the "description" part of the patent.

39.9. Parts b) and c) are due to Shtogrin in a series of papers [Sht1, Sht2, Sht3]

39.10. For a), we have $|xy| = |xy|_{S_1} = |\Phi(x)\Phi(y)|_{S_2} \geq |\Phi(x)\Phi(y)| = |\varphi(x)\varphi(y)|$, for all $x, y \in C_1$. Parts b) and c) are announced in [Tas1].

39.11. For a), use induction on the total number of vertices in C_1 and C_2. Start at a vertex v of C_1 and consider a ray R starting at $\varphi(v)$ such that the sum of angles R forms with edges of C_2 is equal to the angle at v. Extend the partial diagonal inside S_1 as far as possible, until one distance condition inequality becomes an equality. Use this construction repeatedly until bigger polygons split into smaller polygons. Part a) and b) are outlined in [Tas1], part c) in [Tas2].[105]

39.12. For a), once the surface S is folded onto a plane, continue folding it in halves until it fits a disk of radius ε. This gives the desired realization.

39.13. See [BZ1] and [BZ4].

40.2. For a), see Figure 42.16. The outline of the rest in given in [Zal2].

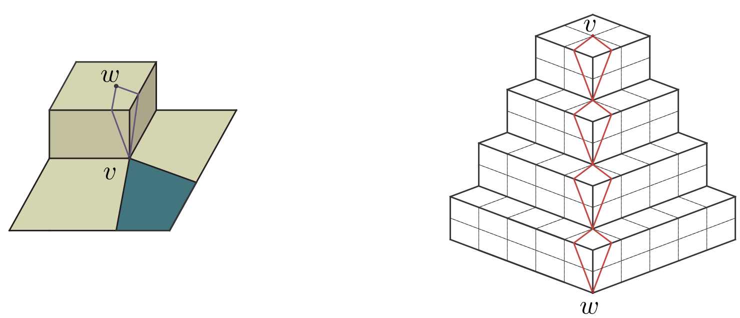

FIGURE 42.16. Points in the shaded region have exactly two shortest paths to w (they all go through vertex v). Non-convex polyhedral surface in $\mathbb{R}^3$ and shortest paths between points v and w.

40.3. For a), take a tetrahedron $\Delta = [v_1v_2v_3v_4]$ with edge lengths $|v_1v_3| = |v_2v_3| = |v_1v_4| = |v_2v_4| = 1$, $|v_1v_2| = \varepsilon$, and $|v_3v_4| = 2 - \varepsilon$, for $\varepsilon > 0$ sufficiently small.
For b), take a tetrahedron $\Delta = [v_1v_2v_3v_4]$ with edge lengths $|v_1v_2| = |v_1v_3| = |v_1v_4| = 1$, $|v_2v_3| = \varepsilon$, and $|v_2v_4| = |v_3v_4| = 2 - \varepsilon$, for $\varepsilon > 0$ sufficiently small.

40.4. For a), see [AHKN].

[105]The inductive proof of b) in [Tas1] has a serious gap.

40.5. For a), see [PakP]; for b) see [Pinc].

40.6. The original proof of a) by D. M. Mount is unpublished and has a crucial gap. The full proof of both parts is given in [MilP].

40.7. For b), see Figure 42.16 above. For a) and c) see [Aga+]. For a discussion on d) see [MilP].

40.8. See [Aga+, ShaS] for the proofs and [SchrS] for further references.

40.9. The proof in [AroO] uses induction on the number of vertices by deforming the peels (see Exercise 40.1) and using the Alexandrov existence theorem.

40.10. Use the higher dimensional generalization of the Mount lemma (Exercise 40.6) to generalize Lemma 40.5. See the complete proof in [MilP].

40.11. Part a) is proved in [CanR] who proved it for equilateral triangles Q_i. Reduction b) is proved in [MilP, §9].

40.12. For a), consider an obtuse triangle and points x, y on side edges close to acute vertices (see Figure 42.17), and take a tall prism with this triangle as an equator. For b) and a partial result in the direction of d), see [BKZ]. Part c) was a problem of Pogorelov resolved in [Pach] (cf. Exercise 25.18). Part d) is a conjecture of Har-Peled and Sharir (1996), still open.

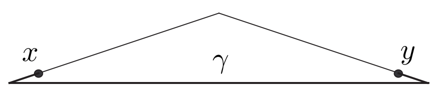

FIGURE 42.17. Shortest path γ between x and y.

40.13. The proof is outlined in [Vol2].

40.14. For a), use the idea in Example 40.11, i.e., glue the halves of the cube separated by the plane through diagonals in the top and bottom faces. Same for the other diagonal. The resulting 4-layered surface is already flat. For the general surfaces, see the next exercise.

40.15. Here is a sketch of my solution of c). Use the realization of S given in the proof of Lemma 37.5. Give an ad hoc construction of a layered (continuous) folding of three faces of a tetrahedron onto the plane containing the fourth one, such that the boundary is unchanged. Bend the folding so it fits inside triangle. Repeat until the whole surface is folded. See [BerH] for another approach.

40.16. For a), observe that the interval $[ab]$ is smaller than the covered part of the boundary, which is a polygon between a and b. For c) see constructions in [Yas] and [Lang, §9.11]. Although stated in a continuous form, it is unclear if these constructions can be used to obtain d), as the authors do not prove the isometry of intermediate stages. The most general part e) was recently proved in [Tar2].

40.17. For a), place a small cube in the middle of the top face of a bigger cube. For b), take a truncated cube (see Figure 16.4) and attach to each face a very tall triangular pyramid [Tar1] (see also [Grü3] for further references).

40.20. See `http://tinyurl.com/6jm5az` .

42.1. Three simple proofs can be found in [Vard, pp. 12–17].

42.2. a) Assume $Q = [x_1 \ldots x_5] \subset \mathbb{R}^3$ is regular and not flat. Then the convex hull $\operatorname{conv}(Q)$ is either a bipyramid or a pyramid over a 4-gon. Either way, we can always assume that

x_1 and x_4 lie on the same side of the plane spanned by x_2, x_3 and x_5. By the assumptions, all diagonals in Q have equal length. Thus tetrahedra $(x_4x_2x_3x_5)$ and $(x_1x_2x_3x_5)$ are congruent. Since x_1 and x_4 lie on the same side of the isosceles triangle $(x_2x_3x_5)$, we conclude that x_1 and x_4 are symmetric with respect to the plane bisecting (x_2, x_3). Thus points x_1, x_2, x_3 and x_4 lie in the same plane. The same argument shows that x_5, x_2, x_3 and x_4 lie on the same plane, and we obtain a contradiction. See [PraS], Problem 8.29.
b) For even n, consider the regular n-*antiprism* and a polygon obtained by the side edges. For n odd, find a non-planar regular 7-gon and use induction. To see why this is true for odd n large enough, consider a long metal chain of small equilateral triangles attached as in the regular antiprism, but make the chain open and flexible along the edges. Clearly, we can twist it and close it up if the chain is long enough.

42.4. This result is due to P. W. Jones (1990). We refer to [KenK] for an elegant presentation and references.

42.5. Take the longest edge. See [Shar], Problem 170.

While there is a large literature on the subject, there is no standard name for these tetrahedra. We refer to [Arn3, pp. 2, 188–191] for the references, and to [PraS, §6.4] (see also [Shar], Problem 304) for simple proofs of these and other basic results on equihedral tetrahedra. Let us mention that these tetrahedra are also called *isosceles* (see [Lee, Thé]) and *equifacial* (see [HajW]). See also Exercise 10.4 for more on this.

42.8. Consider a hyperplane $H \subset \mathbb{R}^d$ defined by the equation

$$\frac{x_1}{a_1} + \ldots + \frac{x_d}{a_d} = d.$$

Clearly, $a \in H$. Since a is an interior point, there exists a vertex v of P on the same side of H as the origin O. We conclude:

$$\frac{v_i}{a_i} \le \frac{v_1}{a_1} + \ldots + \frac{v_d}{a_d} \le d.$$

This solution follows [SGK, §3.2].

42.10. See Problem 266 in [VasE].

42.11. The upper bound $(2n-4)$ is sharp. See the solution to Problem 8.6 in [PraS].

42.12. This is false (see [PraS], Problem 13.12). Check that two vertices on the same edge can lie in C. Extend the planes of C to divide the space into 8 cones. Conclude that at least one of them will have two vertices of the cube.

42.13. Part a) is proved in [Klee]. Part b), was originally proposed in [She2] (see also [CFG], section B10) and completely resolved in [CEG]. For c), start with a regular tetrahedron and add a vertex at distance ε from the barycenter of every face. Then add a vertex at distance ε^2 from the barycenter of every new face, etc. Repeat this $O(n)$ times.

42.15. For a), the claim is false. Start with a unit circle $C \subset \mathbb{R}^3$ defined by $x^2 + y^2 = 1$ and $z = 0$. Take a cone over C from $a = (-1, 0, 1)$ and $b = (-1, 0, -1)$. The resulting convex body B has extremal points at $C - (-1, 0, 0)$ and a, b. To make only countably many extremal points, modify this example by taking a convex hull of a, b and a converging sequence to $(-1, 0, 0)$ of points in C. We are unaware of an elementary proof of b).

42.17. See [Zon2] for proofs, generalizations and references. For b), see also a hint on p. 74 in [Tao].

42.18. This problem was given at the Moscow Math. Olympiad in 2008. See `http://tinyurl.com/ma34td` for a complete solution.

42.19. *a*) Take a projection of a tetrahedron Δ onto a plane L containing face (abc). Assume the remaining vertex d is projected into the interior of (abc). Choose two lines $\ell_1 = (ab)$ and $\ell_2 \perp \ell_1$. Rotate Δ by an angle $\epsilon > 0$ around ℓ_1 and then around ℓ_2. The new projection is a triangle $(a'b'c')$ that fits inside (abc), for ϵ small enough.
b) For the unit cube: the unit square projection fits inside the regular hexagonal projection (with side $\sqrt{2/3}$).
c) For the regular octahedron with edge length 1: start with a unit square projection. Rotate the octahedron around a diagonal of the square by an angle $\epsilon > 0$. Rotate the octahedron now around the second diagonal by an angle ϵ^2. The resulting projection fits inside the unit square, for ϵ small enough.
d) Although we have not checked, the regular icosahedron looks like a potential counterexample.

42.20. Part *a*) was proved by Kovalëv (1984), while part *c*) was proved in [DML]. See [KósT] for a simple proof of *a*) and the references.

42.21. *a*) For $C_d = \{(x_1, \ldots, x_d), 0 \le x_i \le 1\}$, take a hyperplane $x_1 + \ldots + x_d = 2$. For *b*), *c*) and *d*), see [JocP].

42.22. Use the fact that $K_{3,3}$ is not planar. See [Sany] for details and generalization.

42.23. See [Pol]. The example of cuboctahedron shows that the bound is tight.

42.24. This is called the *Szilassi polyhedron*. It is easy to see that this is possible only if it has 14 vertices and 28 edges. See `http://tinyurl.com/2qqzgc` for the references.

42.25. This construction is called the *Császár polyhedron* [Csá].

42.26. Let $\Delta = (v_1v_2v_3v_4)$ and suppose (v_1v_2) is the longest edge. We have:

$$2|v_1v_2| \le \bigl(|v_1v_3| + |v_2v_3|\bigr) + \bigl(|v_1v_4| + |v_2v_4|\bigr) = \bigl(|v_1v_3| + |v_1v_4|\bigr) + \bigl(|v_2v_3| + |v_2v_4|\bigr).$$

Therefore, either $|v_1v_2| \le |v_1v_3| + |v_1v_4|$, or $|v_2v_1| \le |v_2v_3| + |v_2v_4|$, as desired. The problem comes from the Tournament of Towns in 1994 (see `http://tinyurl.com/lk6he3`)

42.27. An elegant proof of this is given in [GriL, §2].

42.28. Let $\boldsymbol{v}_i = \overrightarrow{x_1x_i}$ and $\boldsymbol{w}_i = \overrightarrow{x_ix_{i+1}}$, for all $1 \le i \le n$. Clearly, $\boldsymbol{v}_i = \boldsymbol{w}_1 + \ldots + \boldsymbol{w}_{i-1}$. By analogy with the volume formula in Subsection 34.1, we have $2\text{area}(X) = |\boldsymbol{u}|$, where

$$\begin{aligned} \boldsymbol{u} &= \sum_{i=1}^{n} \boldsymbol{v}_i \times \boldsymbol{v}_{i+1} = \sum_{i=1}^{n} \boldsymbol{v}_i \times (\boldsymbol{v}_i + \boldsymbol{w}_i) = \sum_{i=1}^{n} \boldsymbol{v}_i \times \boldsymbol{w}_i \\ &= \sum_{i=1}^{n} (\boldsymbol{w}_1 + \ldots + \boldsymbol{w}_{i-1}) \times \boldsymbol{w}_i = \sum_{i<j} \boldsymbol{w}_i \times \boldsymbol{w}_j \,. \end{aligned}$$

Taking the norm of both sides immediately implies Sarron's formula.

42.31. See [Eri].

42.32. For this and other properties of the polar sine function see [LerW].

42.33. This is a result of Lagrange (1783). See [GooT] for a short proof.

42.36. See [Kah].

42.38. For *b*), this is false. Take a quadrilateral inscribed into a circle with O as a center. Now permute triangles spanned by O and edges.

42.39. Part *a*) is usually attributed to Euler (1765), while parts *b*)-*e*) are due to L. Fejes-Tóth (1964). We refer to [KlaT] for references and a simple proof of *b*).

42.40. See [BolS, §22].

42.41. See the *Math. Overflow* question `http://tinyurl.com/yd998rw`

42.42. For a), note that no cube can contain more than two vertices. For b), take the smallest cube H_1 on the bottom plane. Take then the smallest cube H_2 on top of H_1, etc. For c), see [Tut, §1] and [Yag1].

42.43. A complete solution is given by Vsevolod Lev in [Kvant], M607 (1980, no. 11).

42.44. For elementary proofs of b), see [Yag1, §4.5] and [Tuza]. For c), see [Sche]. For d) and e), see [Kais]. Finally, for d), see [SST, §11].

42.45. For a), take two copies of four pairwise adjacent tetrahedra as shown in Figure 42.18. Join them along the subdivided triangular faces after slightly twisting them. The bound in part b) is not sharp and can be improved to 9 (see [Zak2]). Part c) is due to Tietze (1905), while part d) (long conjectured to be false) was recently established in [EriK].

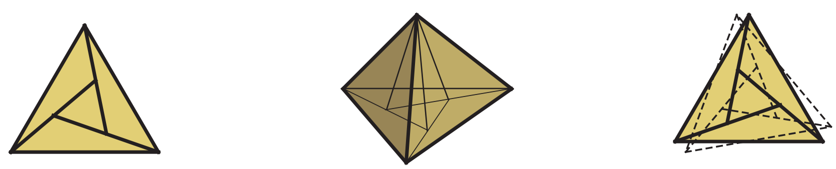

FIGURE 42.18. Four pairwise adjacent tetrahedra and the twisting direction.

42.46. See [Boll], Problem 8.

42.47. For a)–c), one can always extract along a line: $f(t) = t \cdot \boldsymbol{u}$ for some unit vector $\boldsymbol{u}$. We prove part a). Define $P_i \prec P_j$ if there is a point in P_i which lies directly below a point in P_j. Check that $\prec$ defines a partial order on P_i. Now the largest element can be moved upward.

d) Tile a plane L with unit squares. Place the regular tetrahedra with side 2 around L, one per square, so that vertices of squares are midpoints of edges. There are two ways to fit a tetrahedra around a square then; use both ways alternatively, in a checkerboard fashion, so that the tetrahedra fit together. Shrink each tetrahedron around its center by ε, for some small $\varepsilon > 0$. Check that for sufficiently small ε, none of these tetrahedra can be extracted (by symmetry, it suffices to check this for one tetrahedron).

e) To construct a finite family, take a $2m \times 2m$ square S region on L. Attach the opposite sides of S and embed it into $\mathbb{R}^3$. By taking m large enough, the resulting torus can be made as flat as desired, so that the images of the squares are at α-distance from the unit squares. Now arrange the tetrahedra as in part b). If α is much smaller than ε, the tetrahedra will be non-intersecting and neither one can be extracted.

f) A similar construction, but in this case start with a tiling of L with regular hexagons. Take a quotient S of L by two long vectors which preserve the grid. We omit the details.

Parts a)–c) are given in [Daw1]. Construction in d) is given in [Kan] (see also [DEKP]). The first construction for e) is given in [Daw1] (see also [SnoS]). The constructions in e), f) are new and based of infinite constructions in [DEKP, Kan]. Part g) is open. For part h), references and pictures see [SnoS].

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