HINTS TO HOMEWORK 2 (18.319, FALL 2006)

1) This is a famous Mohr-Mascheroni theorem. See an attractive proof in "A Short Elementary Proof of the Mohr-Mascheroni Theorem" by Hungerbühler. Search the web for the history and numerous references.

2) This formula is called the "Cayley-Menger Determinant". It's known, but a bit obscure. See `MathWorld` for the references. A complete solution is posted in a separate file. It is a mild simplification of the proof in "Géométrie" by M. Berger.

4) c) Consider sequences of parallelograms with the same parallel edge. Show that every two such "rings" will always intersect in two parallelograms. The result follows by counting.

a), b) Consider one such ring and remove it from the surface. Show that the remaining two parts can be "glued" together. Deduce the results by induction.

History/References: These results go back to Minkowski, and were certainly proved later by Alexandrov. Read up on zonotopes in Ziegler's book or elsewhere. For other applications see references in the "Zonotopal tilings" section in "Oriented Matroids" by Bjorner et al.

5) This is called "Krasnoselsky Theorem" in the literature. Of course the idea is to use the Helly theorem.

History/References: The standard reference is "Convex figures" by Jaglom and Boltjanskiĭ, problem 20 (if you can't find this one, see also the Russian original and the German translation). Also "Convex sets" by Valentine.

7) a) Assume $Q = [x_0, x_1, \ldots, x_5] \subset \mathbb{R}^3$ is regular, $x_0 = x_5$. If it is not flat, its convex hull is either a bipyramid or a pyramid over a 4-gon. Either way, we can always assume that x_1 and x_4 lie on the same side of the plane spanned by x_2, x_3 and x_5. By assumptions, all diagonals in Q have equal length. Thus tetrahedra (x_4, x_2, x_3, x_5) and (x_1, x_2, x_3, x_5) are congruent. Since x_1, x_4 lie on the same side of the isosceles triangle (x_2, x_3, x_5), we conclude that x_1 and x_4 are symmetric with respect to the plane bisecting (x_2, x_3). Thus points x_1, x_2, x_3, x_4 lie in the same plane. Same argument shows that x_5, x_2, x_3, x_4 lie on the same plane, and we obtain a contradiction.

b) For even n, consider the regular *antiprism* and a polygon obtained by the side edges. One can construct examples for all odd $n \geq 7$. To see why this is true for odd n large enough, consider a long metal chain of small equilateral triangles attached as in the regular antiprism, but make the chain open and flexible along the edges. Clearly we can twist is and close it up if the chain is long enough (try it!) Again the side edges in the chain give us the desired polygon.

History/References: I got solution to the part a) from a popular Russian "problem book" in 3-dimensional geometry for high schoolers (Prasolov-Sharygin).

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More is coming up!!!