HINTS TO HOMEWORK 1 (18.319, FALL 2006)

1) Let $X_d \subset \mathbb{R}^d$ be a union of $(2d+1)$ unit cubes, where one cube in the center is attached to all others by a $(d-1)$-dimensional face. Then for all d one can tile $\mathbb{R}^d$ with translations of X_d. The idea is to place the centers of the cube in points $(a_1, \ldots, a_d) \in \mathbb{Z}^d$ such that $a_1 + 2 \cdot a_2 + \ldots + d \cdot a_d = 0 \mod 2d+1$.

History/References: For $d = 2, 3$ this can be found in numerous recreational math books. For general d this can be found e.g. in the book "Algebra and tiling" by Stein and Szabó.

2) a) The idea is to to show that every line can be drawn without adding new vertices except on the line. This shows that every existing interval can be extended to the boundary, dividing triangle into two pieces. Then use induction on the number of interior points. At the end, use an easy ad hoc argument for triangulations with no interior vertices.

b) Use the same argument as in a) to superimpose a fixed star triangulation of P (connecting a given vertex by diagonals to all others) onto a given triangulation. Clean up each triangle separately.

History/References: For general d and general (not necessarily convex) polyhedra this is shown in "Elementary moves on triangulations" by Ludwig and Reitzner.

3) In the literature these are called *curves of constant width*. For a), examples can be obtained by taking regular $(2n+1)$-gon (say, with longest diagonal equal to 1) and adding segments to each side to make it have constant width. For b) there are several analytic ways depending on the regularity of the boundary. In general, subdivision of the perimeter by n points, taking convex hull and proving a convergence result is the most straightforward. A complete solution is given in "Convex figures" by Jaglom and Boltjanskiĭ.

History/References: The circle and the *Reuleaux triangle* are the most standard examples which optimize the area of the curves. Look up `Wikipedia` page for pictures and links, and check out the references in `MathWorld` for the entry "curve of constant width".

4) The proof of a) was given in class. For b) see e.g. the "great stellated dodecahedron" (google it to see the picture or simply check it out in the little MIT hallway a cross 2-135).

History/References: The problem is a folklore. I heard it from János Pach, I think.

5) We solved this problem in class. I made it up to prove the point.

6) Let $P \subset \mathbb{R}^d$ be a centrally symmetric convex polytope. First check that every two points $x, y \in P$ if $|x, y| = \mathrm{diam}(P)$, they they are the opposite (hint: otherwise

the opposite x' of x would have $|x, x'| > |x, y|$). Thus $P \subset B$, where B is the ball of radius $\text{diam}(P)/2$. It remains to prove the Borsuk conjecture for B which we sketched in class.

History/References: I took this from the Boltyansky-Gohberg book on problems in combinatorial geometry (in Russian), but it's probably mentioned in all standard surveys on the Borsuk conjecture.

7) a) You need to take a circle or radius R and compute the average angle of polygons in two different ways: one via the average in each octagon, and another from the fact that at each vertex there are at least three angles meeting. Letting $R \to \infty$ obtain a contradiction.

History/References: There must be a standard reference, but I don't know it. I learned this theorem years ago and forgot where. Let me know if you do.