HOMEWORK 3 (18.319, FALL 2006)

1) Suppose the faces of a tetrahedron have equal area. Prove that the opposite edges have equal length, i.e. the faces are congruent triangles. Is this also true in higher dimensions?

2) Recall permutohedron P_n obtained as a convex hull of $(\sigma_1, \ldots, \sigma_n) \in \mathbb{R}^n$, where $\sigma \in S_n$. Prove that permutohedron is scissor congruent to a cube of equal volume.

3) Let $Q \subset \mathbb{R}^2$ be a non-convex polygon and let C be the convex hull of Q. Choose edge e of C that is not in Q and reflect the portion of Q across e, as in Figure 1. Repeat this transformation for a resulting polygon until a convex polygon is obtained. Prove that the process stops after finitely many steps.

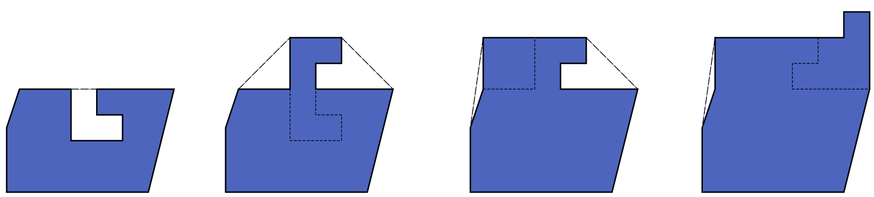

FIGURE 1. First few transformations of a polygon.

4) All faces of a convex polytope $P \subset \mathbb{R}^3$ are centrally symmetric. Prove that P has at least eight simple vertices (vertices adjacent to exactly three edges). Note that this is tight since the cube has exactly eight simple vertices.

5) Define *pseudo billiard trajectories* in a convex polygon Q defined as follows. The reflections at the edges are defined as for the usual billiard trajectories, but now we allow the trajectories to enter the vertices with one "pseudo" condition: if α, β are the reflection angles as in Figure 2, then $|\alpha - \beta| \leq \pi - \gamma$, where γ is the vertex angle. Prove the following analogue of the Birkhoff theorem: for every $x, y \in Q$ there exist infinitely many pseudo billiard trajectories from x to y.

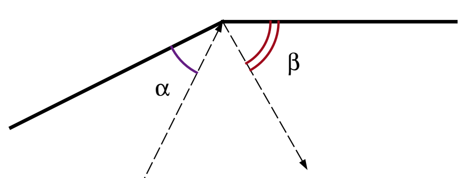

FIGURE 2. Angles of reflection in a pseudo billiard trajectory.

6) Let $P \subset \mathbb{R}^3$ be a convex polytope. We say that an orientation of edges of P is *balanced* if every vertex has at least one ingoing and one outgoing edge (i.e. the oriented graph of P has no sinks and no sources). Prove the edges of at least two faces of P form oriented cycles. Any idea how to generalize this to higher dimension?

7) Let $P \subset \mathbb{R}^3$ be a convex polytope whose faces are colored in black and white such that no two black faces are adjacent. Suppose total area of black faces is larger than the total area of white faces. Prove that P does not have an inscribed sphere. Find an example of a polytope with an inscribed sphere and a coloring, where the areas are equal.

8) a) As above, let $P \subset \mathbb{R}^3$ be a convex polytope whose faces are colored in black and white such that no two black faces are adjacent. Suppose the number of white faces is smaller than the number of black faces. Prove that P does not have an inscribed sphere.

b) Let Q be a parallelepiped and let P be obtained from Q by cutting off the vertices (an example is shown in Figure 3). Use part a) to show that P does not have an inscribed sphere.

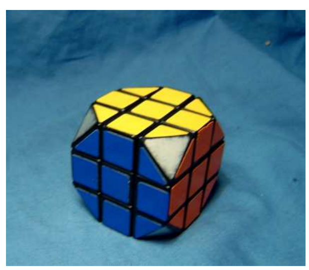

FIGURE 3. Truncated cube.

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This Homework is due Friday November 3 at 11:05 am.

P.S. Do not forget: some of these problems are quite difficult.
By no means you are expected to solve all or even most of them.