HOMEWORK 2 (18.319, FALL 2006)

1) a) Prove that the center of a circle can be constructed by using only a compass (i.e. with no ruler).

b) Prove that anything that can be constructed with ruler and compass can be also done without ruler. (First, you need to figure out how to make this statement precise.)

2) For every simplex $\Delta = (v_0 v_1 \ldots v_d) \subset \mathbb{R}^d$, prove that:

$$\mathrm{vol}^2(\Delta) \; = \; \frac{(-1)^{d-1}}{2^d \, d!^2} \cdot \det \begin{pmatrix} 0 & 1 & 1 & 1 & \ldots & 1 \\ 1 & 0 & \ell_{01}^2 & \ell_{02}^2 & \ldots & \ell_{0d}^2 \\ 1 & \ell_{01}^2 & 0 & \ell_{12}^2 & \ldots & \ell_{1d}^2 \\ 1 & \ell_{02}^2 & \ell_{12}^2 & 0 & \ldots & \ell_{2d}^2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \ell_{0d}^2 & \ell_{1d}^2 & \ell_{2d}^2 & \ldots & 0 \end{pmatrix},$$

where $\ell_{ij} = |v_i v_j|$, for all $0 \le i < j \le d$.

3) Let $P \subset \mathbb{R}^3$ be a convex polytope with n facets, and let Q be its projection on a plane. What is the maximal possible number of edges polygon Q can have?

4) Let $P \subset \mathbb{R}^3$ be a convex polytope whose faces are parallelograms.

a) Prove that P can be subdivided into *parallelepipeds*.

b) Prove that P is centrally symmetric.

c) Prove that the number of faces of P is twice a triangular number.

5) Let Q be a (possibly non-convex) polygon on a plane. We say that Q is a *star polygon* if there exists a point u such that for every $x \in Q$ the interval $[x, u] \subset Q$. Suppose for every three points $x, y, z \in Q$ there exists a point w such that $[x, w]$, $[y, w]$ and $[z, w] \subset Q$. Conclude that Q is a star polygon.

6) A polytope $P \subset \mathbb{R}^3$ is said to have *limited visibility* if for every vertex v of P there exists a point $O \notin P$ such that all vertices in P except possibly v can be seen from O. Suppose there is no point from which *all* vertices of P can be seen. What is the largest number of vertices polytope P can have?

(two more problems on the next page!!!)

7) Let $Q = [x_0, x_1, \ldots, x_n]$, $x_i \in \mathbb{R}^3$ be a 3-dimensional n-gon, where $x_0 = x_n$. We say that Q is *regular* if $|x_{i-1}, x_i| = 1$ and $\angle x_{i-1}x_ix_{i+1} = \alpha$, for all $1 \leq i \leq n$ and some fixed $0 < \alpha < \pi$.

a) Prove that every regular pentagon lies on a plane.

b) Are there any other $n > 5$ for which one can make the same conclusion?

8) Let P be a (not necessarily convex) polyhedron in $\mathbb{R}^3$ (think of P as a 2-dim surface), such that the neighborhood of every vertex is a convex cone. Prove or disprove: P is a convex polyhedron. What happens in higher dimensions?

⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆

This Homework is due Friday October 13 at 11:05 am. Please make sure you have read the collaboration policy on the course web page.

P.S. Do not forget: some of these problems are quite difficult. By no means you are expected to solve all or even most of them.