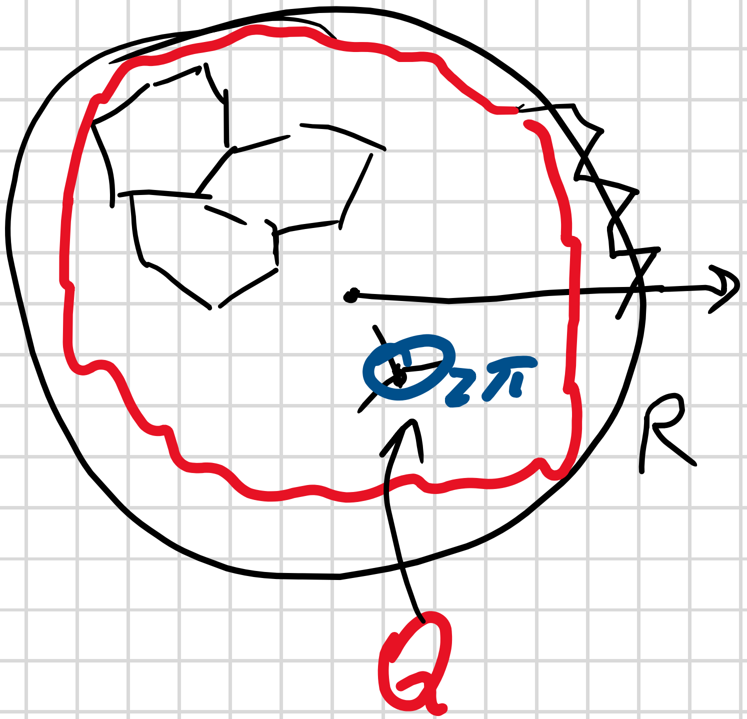
Convex Polytopes

Q: What polyhedra tile the space?

Th [Alexandrov, 1936] P – convex polygon $P \subset \mathbb{R}^2$ tiles the plane $\Rightarrow$ P has at most 6 vertices

Proof P – 7 gon



N copies of P

Σ = sum of all angles inside Q

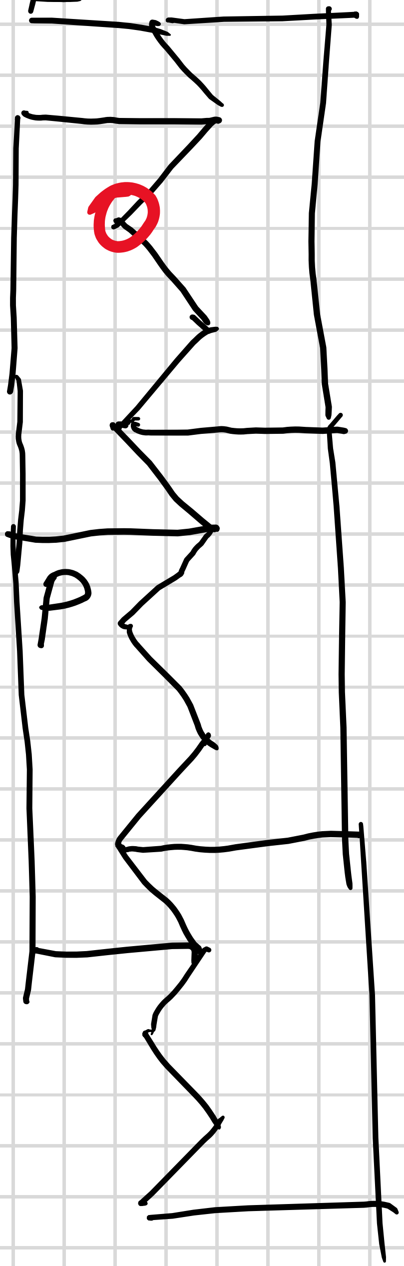
$= N\pi(7-2) = 5\pi N$

$$\sum_{\angle} - \sum_{\angle \in \partial Q} \sim 5\pi N$$

$N = \Theta(R^2)$, $|\partial Q| = \Theta(R)$ $\Rightarrow$ # int points $\sim \frac{5}{2}N$

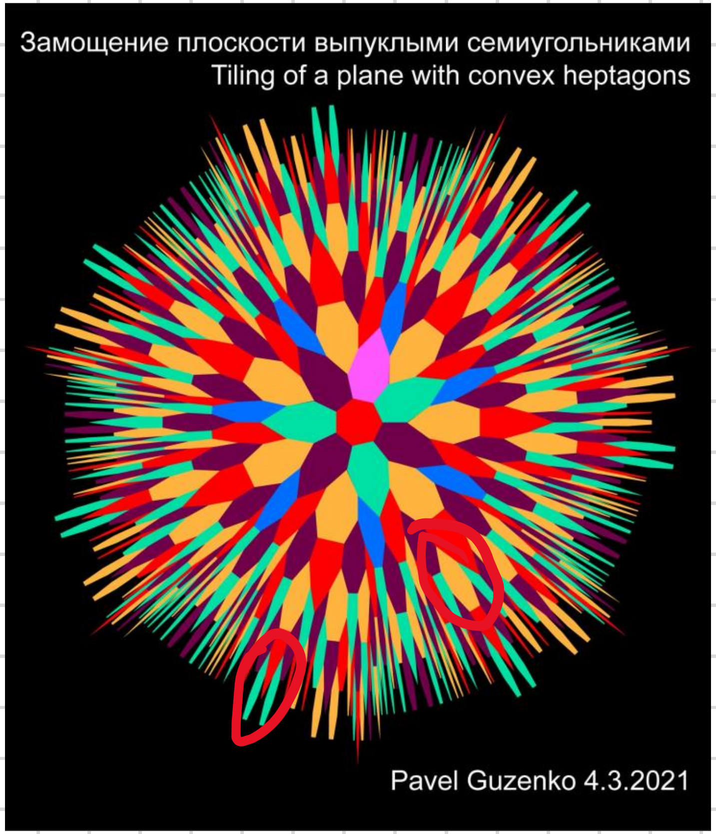
OTOH # int points $\leq \frac{7 \cdot N - |\partial Q|}{3} \sim \frac{7}{3}N$ ☐

Ex



P - 7 gon which tiles $\mathbb{R}^2$

Q $\exists$ tiling of $\mathbb{R}^2$ w/ 7-gons (non necessarily congruent)?



Th (Grünbaum, Mani-Levitska, Shephard, 1984)

$\forall P \subset \mathbb{R}^3$ convex polytope

$\exists$ tiling of $\mathbb{R}^3$ w/ convex polytopes combin. equivalent to P

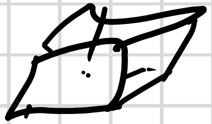
Q: /Open problem/ $P \subset \mathbb{R}^3$ convex polytope tiles $\mathbb{R}^3 \Rightarrow$ # vertices of $P < 10^6$

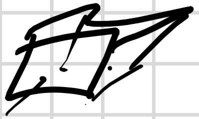
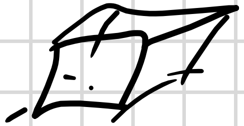
Th [Lagarias-Moews]

$P \subset \mathbb{R}^3$ – convex polytope, P tiles $\mathbb{R}^3$

Then $P \in \mathcal{Q} \,/\, \Longleftrightarrow P \sim$  /

Th –||– P tiles $\mathbb{R}^3$ periodically

$\Longrightarrow P \sim$ 

$\forall P \sim$  $\underset{\text{Sydler}}{\Longrightarrow} P \sim$ 

L [M-L] $P \subset \mathbb{R}^3$ tiles $\mathbb{R}^3$

$\Rightarrow$ Dehn invariant of $P = 0$

$f : \mathbb{R} \to \mathbb{R}$ - additive function

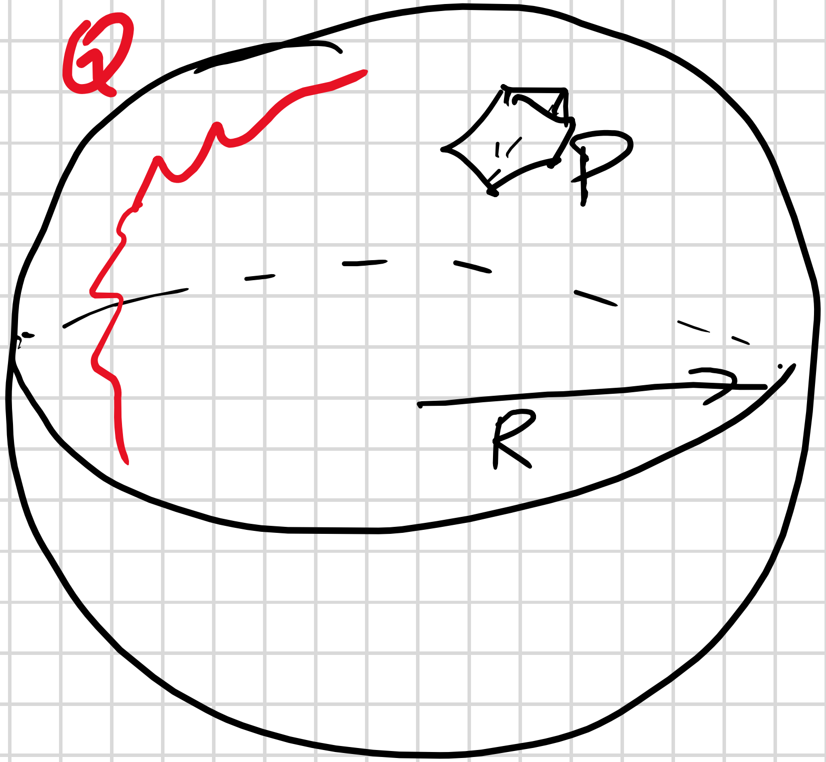
$f(\pi) = 0$ $\Leftarrow$ Kagan function

$\underline{\varphi_f(P)} := \sum_{e \in E(P)} \ell_e \cdot f(\underline{\alpha_e})$ — α_e: dihedral angle.

L′ $\forall f$ - Kagan function

$P \subset \mathbb{R}^3$ tiles $\mathbb{R}^3$ $\Rightarrow$ $\varphi_f(P) = 0$

Proof of L'



$Q :=$ union of copies of P inside Ball of radius R

Assume $a = \varphi_f(P) \neq 0$

$$\varphi_f(Q) = \Theta(R^3 a)$$

/# copies of P in $Q = \Theta(R^3)$/

OTOH $\varphi(Q) = \sum_{e \in \partial Q} \ell_e \cdot f(\alpha_e) = O(R^2)$

$< L$ $\quad |f| < F$

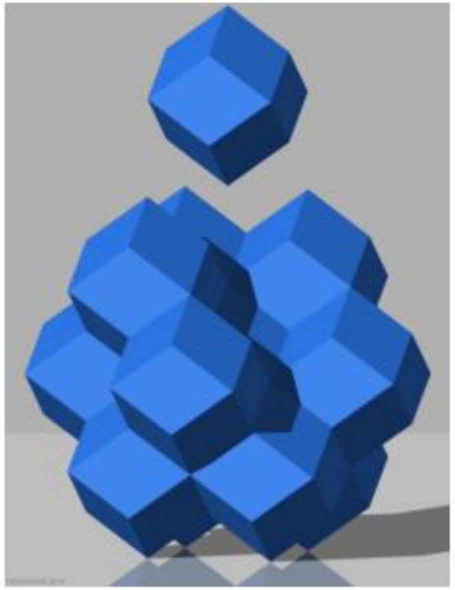
$\Rightarrow a = 0$ ✗

Tiling spaces with congruent polyhedra

"Perhaps our biggest surprise when we started collecting material for the present work was that so little about tilings and patterns is known. We thought, naively as it turned out, that the two millenia of development of plane geometry would leave little room for new ideas."

B. Grünbaum & G. C. Shephard, *Tilings and Patterns*, 1986.

Motivation: Hilbert's 18th Problem (1900)



Question 1: What polyhedra tile $\mathbb{S}^d$, $\mathbb{E}^d$ and $\mathbb{H}^d$ with congruent copies?

Question 2: Are all such polyhedra fundamental regions of group actions?

People: Fricke, Klein, Fëdorov, Voronoy, Schoenflies, etc.

Answer to 1: What is known is very small compared to what is **not** known.

Answer to 2: Not at all. Which explains the previous answer.

The Good:

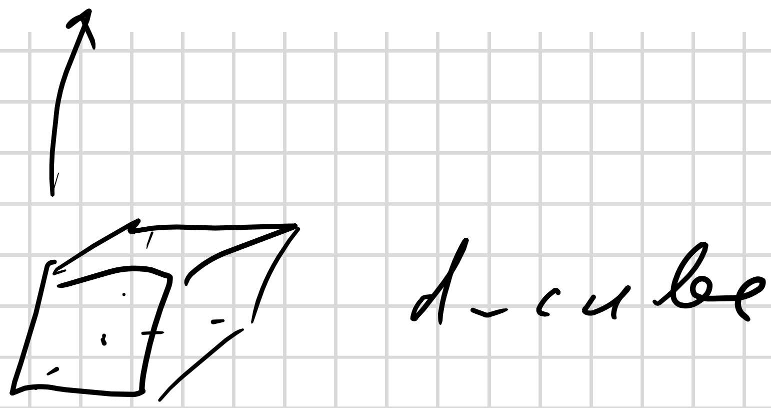
Theorem [Bieberbach, 1911]

Crystallographic groups Γ (discrete cocompact subgroups of $\mathrm{SO}(d,\mathbb{R}) \ltimes \mathbb{R}^d$) are finite extensions of $\mathbb{Z}^d$ by a finite $G \subset \mathrm{GL}(d,\mathbb{Z})$.

Theorem [Minkowski, 1910]: $|G| \leq (2d)!$ Thus, # of such Γ is $C(d) < \infty$.

Sequence $C(d)$ grows rapidly: 2, 17, 230, 4894, 222097, 28934974, ...

Theorem [Feit, 1996]: $|G| \leq 2^d d! = |B_d|$ (this uses CFSG).



From groups to tilings:

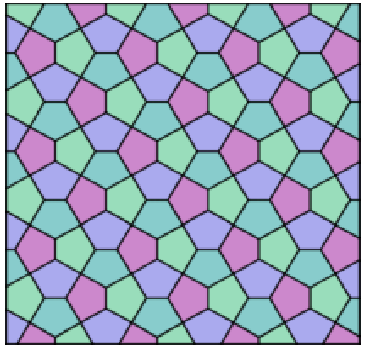
Let Γ be crystallographic, and let $R = \Gamma(p)$ be an orbit of a generic point p.

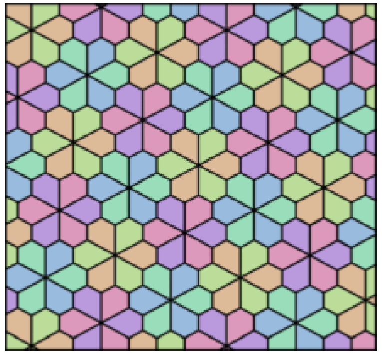
Take the **Voronoi diagram** $\mathcal{V}(R)$ of R (= **Dirichlet domain**).

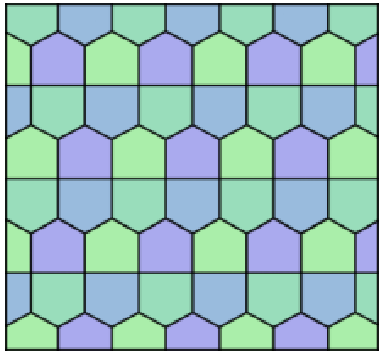
Then Γ acts transitively on $\mathcal{V}$ and the cell $V(p)$ tile the space.

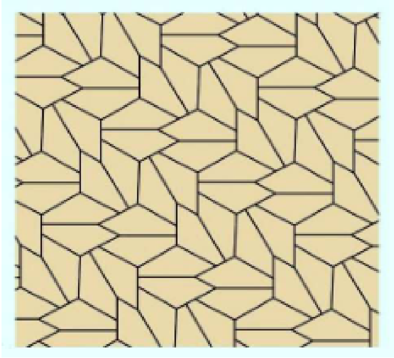
More generally, let P be a **fundamental region** of the action of Γ.

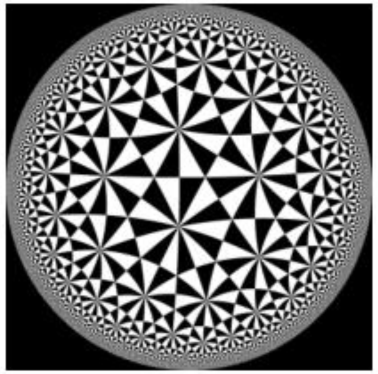
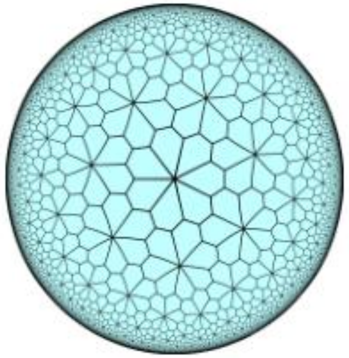
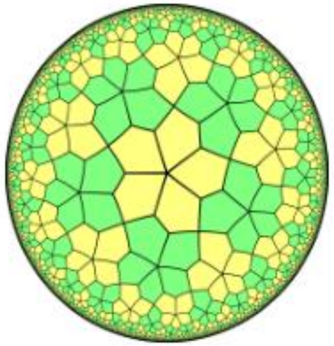
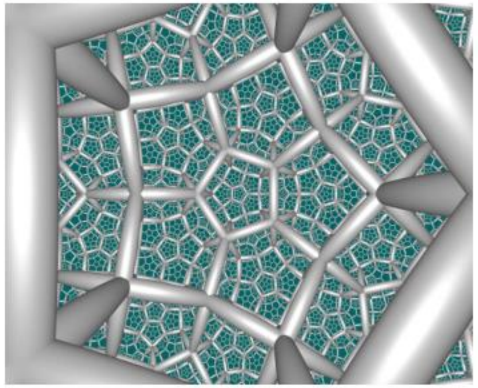
Take $Q \subset P$ s.t. congruent copies of Q tiles P (not necessarily face-to-face).









"[In $\mathbb{H}^3$] there is absolutely no hope of giving any reasonable kind of answer to this question; there is a plethora of possible groups, and each group has a continuum of orbits, which can lead to a variety of Voronoi polyhedra."

John H. Conway (Wed, 13 Dec 1995, 11:26:55)

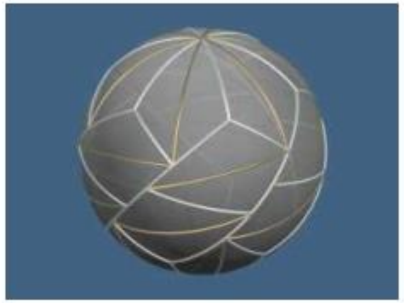
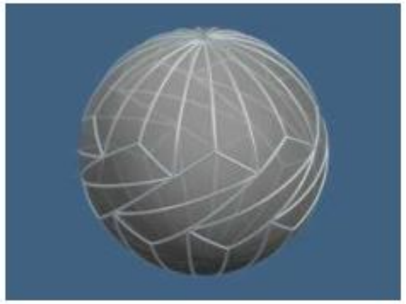
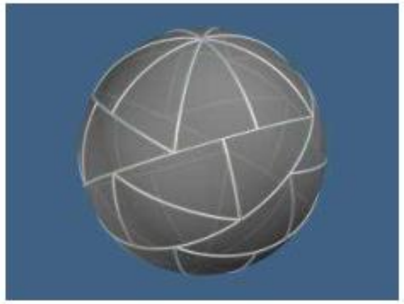
usenet newsgroups

The Bad:

Theorem [Sommerville, 1923]: Classification of **tetrahedra** which tile $\mathbb{E}^3$ when rotations are not allowed. **Open** in full generality.

Theorem [Davies, 1965]: Classification of **triangles** which tile $\mathbb{S}^2$.

Warning: In both cases there are nontrivial NON-face-to-face tilings.

$(60^\circ\ 80^\circ\ 100^\circ)$ tiles $\mathbb{S}^2$

The Ugly:

Open Problem 1. [aka the *einstein conjecture*]

Does there exist a (non-convex) polygon which tiles the plane $\mathbb{E}^2$, but only aperiodically?

Open Problem 2.

Is the tileability problem by a convex polyhedron in $\mathbb{H}^d$ decidable?

Open Problem 3.

Does there exist a tile such that the tileability problem is independent of ZFC?

Remark: NO on OP2 implies YES on OP3 (easy).

Main question: how bad can it get?

Open Problem [Fëdorov, Voronoy, etc.]

Does every P which tiles $\mathbb{E}^3$ has a bounded number of facets?

More generally, let $\mathbb{X}$ be either $\mathbb{S}^d$, $\mathbb{E}^d$ or $\mathbb{H}^d$. Denote by $\phi(\mathbb{X})$ the maximum $f_{d-1}(P)$ over P which tiles $\mathbb{X}$, or ∞ if max does not exist.

Question: What can be said about all $\phi(\mathbb{X})$?

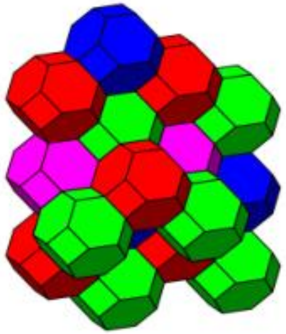
Easy: $\phi(\mathbb{E}^2) = 6$, $\phi(\mathbb{S}^2) = 5$, $\phi(\mathbb{H}^2) = \infty$, $\phi(\mathbb{H}^3) \geq 12$ (just wait!).

Current champion: $\phi(\mathbb{E}^3) \geq 38$ [Engel, 1980].

Euclidean tilings: parallelohedra

Theorem [Minkowski, 1911]
In $\mathbb{E}^d$ is tiled by parallel translations of P, then $f_{d-1}(P) \leq 2^{d+1} - 2$.
We have $f_{d-1}(P) = 2^{d+1} - 2$ when $P = \Pi_{d+1}$ is a permutohedron.

Π_4



Note: This proof is an application of the Minkowski Uniqueness Theorem (that the polytope is uniquely determined by the facet volumes).

Note: Fëdorov proved there are exactly five *parallelohedra* in $\mathbb{E}^3$ (1885).

Euclidean tilings: stereohedra

Theorem [Delone & Sandakova, 1969]

If P is a fundamental region of crystallographic Γ acting on $\mathbb{E}^3$, then

$f_{d-1}(P) \leq 2^d(h+1) - 2, \quad$ where $\; h = |G|,\, G = G(\Gamma)$.

Moral: aperiodic constructions are needed to show $\phi(\mathbb{E}^3) = \infty$.

Note: Using Feit's estimate $H = |G| \leq 2^d d!$, in $\mathbb{E}^3$ this gives $f_2(P) \leq 390$.

This bound was improved by Tarasov (1997) to 378.

Spherical tilings: the unbounded number of facets

Theorem [Dolbilin & Tanemura, 2006]

$\phi(\mathbb{S}^d) = \infty$ for $d \geq 3$.

Construction: Let $d = 3$, $S^3 \hookrightarrow \mathbb{R}^4$. Fix $n \geq 2$.

Let R_1 be the set of points $(\sin \frac{2\pi j}{n}, \cos \frac{2\pi j}{n}, 0, 0)$, $0 \leq j < n$

Let R_2 be the set of points $(0, 0, \sin \frac{2\pi j}{n}, \cos \frac{2\pi j}{n})$, $0 \leq j < n$.

The set $R = R_1 \cup R_2$ has a transitive group of symmetries.

Now take the Voronoi diagram of R with Voronoi cell P.

Check that P is combinatorially an n-prism, so $f_2(P) = n + 2$.

n polytopes tiles S^3

Question: Can we get larger $f_2(P)$ for spherical tiles of $\mathbb{S}^3$?

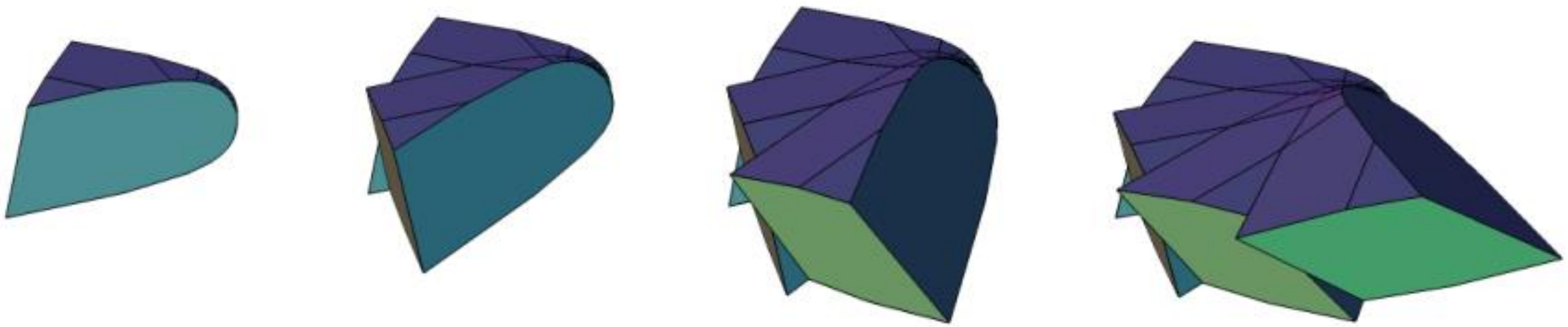
Theorem [Erickson, 2001; Erickson & Kim, 2003]: For every $n \geq 1$, there is a tiling of $\mathbb{E}^3$ with infinitely many congruent (unbounded) polyhedra with n facets.

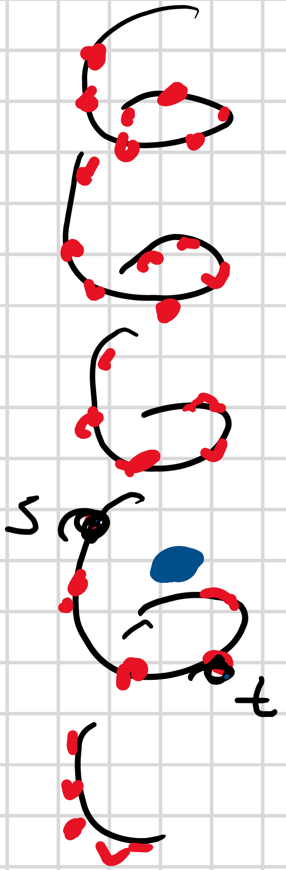
Erickson's construction: points on a helix

Let $R_n = \left\{ \left(t, \sin\frac{2\pi t}{n}, \cos\frac{2\pi t}{n}\right), t \in \mathbb{Z} \right\}$.

Take the Voronoi diagram of R_n with Voronoi cells P_t.

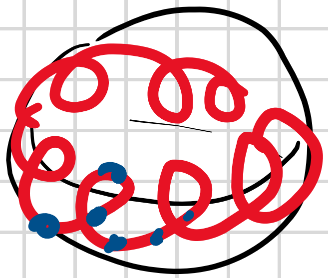
Check that P_s and P_t have common facet if $|s - t| \leq n/2$.





Spherical tilings: the neighborly construction

Definition: A (finite) tiling is *neighborly* if every two tiles have a common facet.



Theorem [Nguyen & P., 2015]: For $n \geq 2$ and $d \geq 3$, there is a neighborly tiling of $\mathbb{S}^d$ with n congruent polyhedra.

Corollary [Nguyen & P., 2015]: For $n \geq 2$ and $d \geq 4$, there is a neighborly tiling of $\mathbb{E}^d$ with n congruent (unbounded) polyhedra.

Our construction: points on a spherical helix

Fix $0 < \theta < \pi/2$, $m \geq 2$. Let $A_{\theta,n}(\alpha) = \big(\cos\theta\cos\alpha, \cos\theta\sin\alpha, \sin\theta\sin m\alpha, \sin\theta\cos m\alpha\big)$.

Take $R_n = \big\{ A_{\theta,m}\big(\frac{2\pi j}{n}\big), 0 \leq j < n \big\}$ and the Voronoi diagram of R_n.

Our construction: points on a spherical helix

Fix $0 < \theta < \pi/2$, $m \geq 2$. Let $A_{\theta,n}(\alpha) = \big(\cos\theta\cos\alpha, \cos\theta\sin\alpha, \sin\theta\sin m\alpha, \sin\theta\cos m\alpha\big)$.

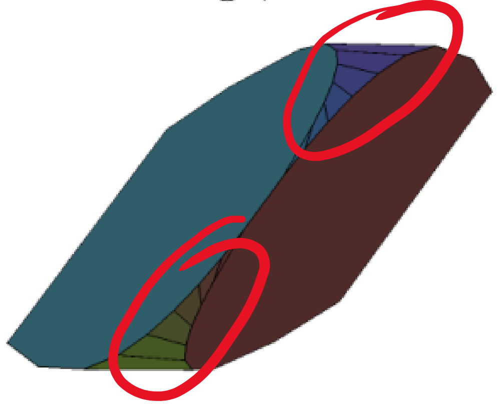
Take $R_n = \big\{ A_{\theta,m}\big(\frac{2\pi j}{n}\big), 0 \leq j < n \big\}$ and the Voronoi diagram of R_n.

Explanation: Spherical helix $A_{\theta,n}(\alpha)$ winds m times around the torus

$T_\theta = \big\{(x_1, x_2, x_3, x_4),\, x_1^2 + x_2^2 = \cos^2\theta,\, x_3^2 + x_4^2 = \sin^2\theta\big\} \subset S^3$.

Now observe that $\mathbb{Z}_n$ acts transitively on R_n.

Note: Drawing spherical tiles is a challenge, but for m large, the front end looks like this:

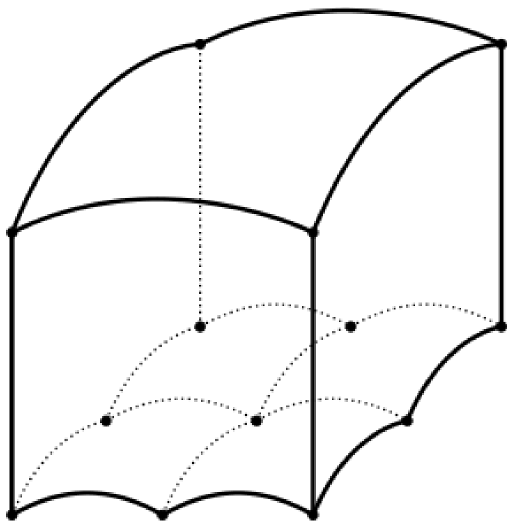


— locally similar to Ericson

Hyperbolic tilings: the generalized Böröczky construction

Theorem [Böröczky, 1974; Zare, 1995; etc.]: $\phi(\mathbb{H}^d) = \infty$ for $d \geq 3$. Specifically, for $n \geq 2$, there exist a polyhedron P_n with (n^2+5) facets, which tiles $\mathbb{H}^3$.

Construction: In the upper half-space $\mathbb{H}^3$, let $A_n = \{(1,i,j) : 0 \leq i,j \leq n\}$, $B_n = \{(n,0,0),(n,n,0),(n,0,n),(n,n,n)\}$, and $P_n = \operatorname{conv}(A \cup B)$.



P_2

Hyperbolic tilings: combinatorial constructions

Theorem [Pogorelov, 1967; see also Andreev, 1970]:

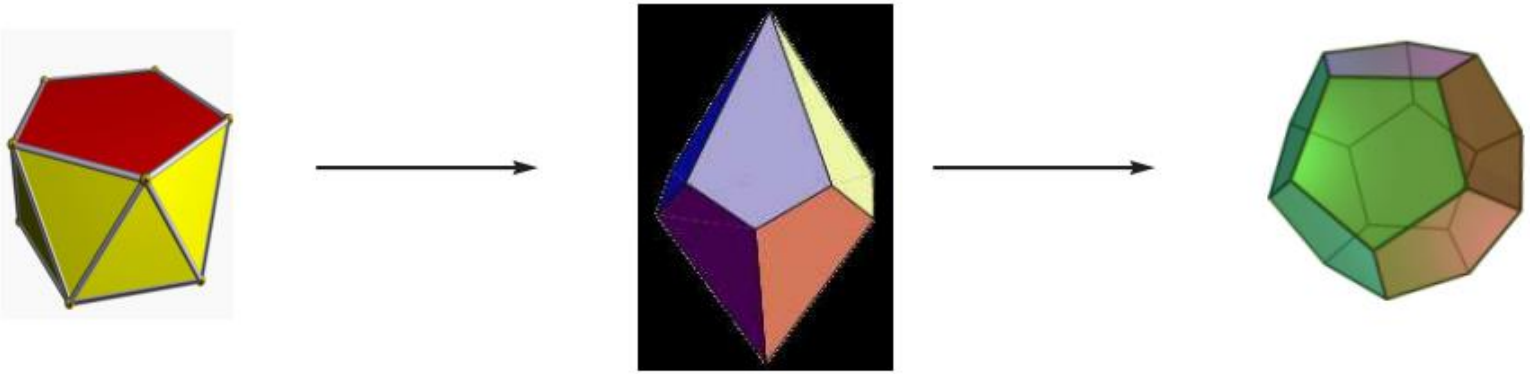
A 3-connected planar graph G can be realized in $\mathbb{H}^3$ as a **bounded right-angled polyhedron** if and only if it is cubic, every face is a k-gon with $k \geq 5$, and every simple closed circuit in G^* which separates some two faces intersects at least 5 edges.

Poincaré's Polyhedron Theorem (1883):

Sufficient combinatorial conditions on T, which can be checked locally to prove that T tiles $\mathbb{H}^d$.

Theorem [Löbell, 1931]:

Let P_n be right-angled hyperbolic polyhedron with two n-gonal and $2n$ pentagonal faces (see the Figure). Then they tile $\mathbb{H}^3$.



Hyperbolic tilings: basic arithmetic constructions

Recall: $\mathrm{PSL}(2,\mathbb{C})$ acts on $\mathbb{H}^3$ by isometries.

Matrix $A \in \mathrm{PSL}(2,\mathbb{C})$ is **loxodromic** if $\mathrm{tr}^2 A \notin [0,4]$
(as opposed to **elliptic** or **parabolic**).

Theorem [Jørgensen, 1973; Drumm & Poritz, 1999]:

Let $A \in \mathrm{PSL}(2,\mathbb{C})$ be loxodromic, $\Gamma = \langle A \rangle$. Take Voronoi diagram $\mathcal{V}\big(\Gamma(p)\big)$.

Then number of facets of the (unbounded) polyhedron $V(p)$ can be arbitrary large.

Note: This is a hyperbolic analogue of Erickson's construction.

Hyperbolic tilings: nested property

Let $\Gamma_1 \supset \Gamma_2 \supset \Gamma_3 \supset \ldots$ be a chain of subgroups acting on $\mathbb{E}^d$ or $\mathbb{H}^d$.
Let $P_1 \subset P_2 \subset P_3 \subset \ldots$ be the corresponding Voronoi cells on the same point.

Question: Can we have $f_{d-1}(P_n) \to \infty$ as $n \to \infty$?

Note: Erickson's construction is suited for *ascending*, not *descending* chains.

Theorem [Nguyen & P., 2015+]:
For every $\mathbb{H}^d$, $d \geq 3$, there exists such a chain.

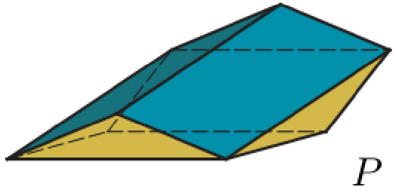
Proof is based on two difficult results: [Millson, 1976], [Lubotzky, 1996] and an observation that $f_{d-1}(P_n) \geq \text{rank}(\Gamma_n)$.

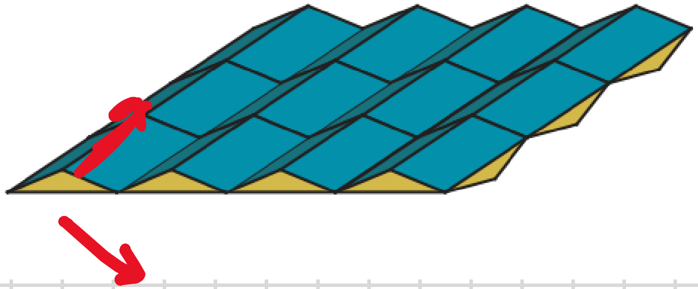
Aperiodicity of Euclidean tilings

Tile T is *weakly aperiodic* if no tiling of $\mathbb{E}^d$ with T is invariant under $\mathbb{Z}^d$.

Tile T is *strongly aperiodic* if no tiling of $\mathbb{E}^d$ with T is invariant under $\mathbb{Z}$.

Theorem [Conway, 1995]
In $\mathbb{E}^3$, there exists a weakly aperiodic tile P. There is an action of $\mathbb{Z}$, however.





Aperiodicity of Euclidean tilings: some questions

Question: Does there exist a weakly aperiodic tile in any $\mathbb{E}^3$ with a dense set of rotations in $SO(3, \mathbb{R})$?

Question: Does there exist a strongly aperiodic tile in any $\mathbb{E}^d$?

Question: Is **self-similarity** decidable in $\mathbb{E}^2$?

Aperiodicity of hyperbolic tilings

Tile T is *weakly aperiodic* if there is no tiling with T of a compact $\mathbb{H}^d/\Gamma$, for any Γ.

Tile T is *strongly aperiodic* if no tiling of $\mathbb{H}^d$ with T is invariant under $\mathbb{Z}$.

Theorem [Margulis & Moses 1998]
In $\mathbb{H}^2$, there are weakly aperiodic n-gons, for all $n \geq 3$. There is an action of $\mathbb{Z}$, however.

Proposition: In $\mathbb{H}^d$, $d \geq 3$, Böröczky polyhedra P_n are weakly aperiodic. There is an action of $\mathbb{Z}$, however.

Question: Does there exist a weakly aperiodic right-angled polyhedral tile $\mathbb{H}^3$, with an unbounded number of facets?