L1 (IUM, Spring) Feb 9, 2021

Convex Polytopes

Q: What are invariants of convex polytopes?

L1 Scissor-congruence

Def $P, Q \subset \mathbb{R}^d$, $P \sim Q$ if

$\exists\ P = \bigsqcup_{i=1}^{N} P_i$ s.t. $P_i \simeq Q_i$

$Q = \bigsqcup_{i=1}^{N} Q_i$

congruence

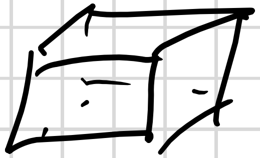
(up to rigid motions)

Th (Bolyai–Gerwien, 1830s)

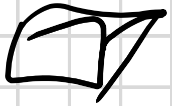
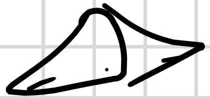
$P, Q \subset \mathbb{R}^2$ convex polytopes

$\text{area}(P) = \text{area}(Q) \Rightarrow P \sim Q$

Hilbert Third Problem

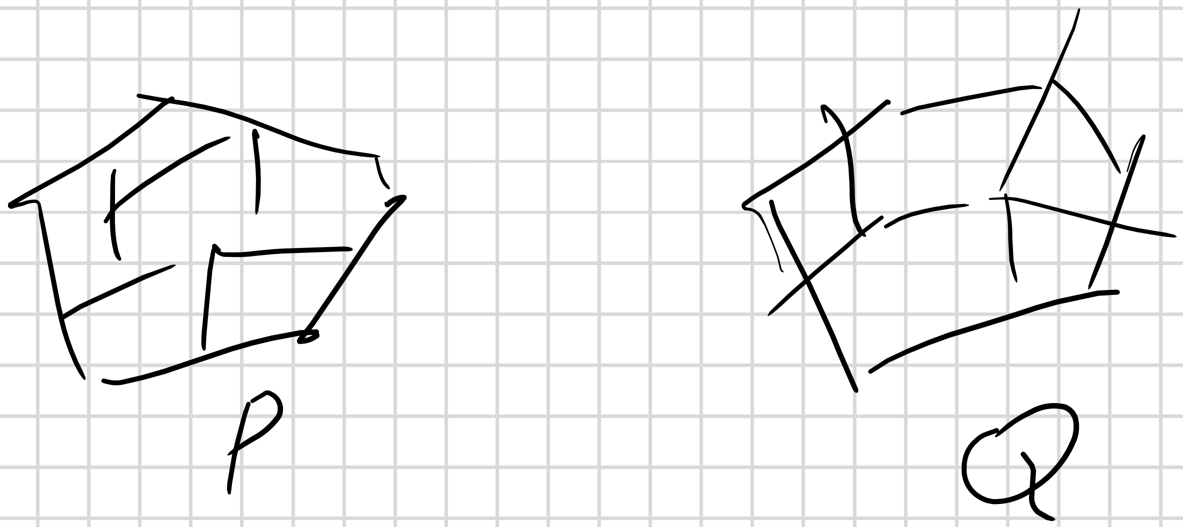
Q  $\sim ?$ 

Th (Dehn, 1903)

 $\not\sim$ 

L: $\Gamma \leftarrow$ discrete group acting on $\mathbb{R}^d$ s.t. $P, Q \leftarrow$ fundamental regions $\Rightarrow P \sim Q$

D



$Q = \bigsqcup g_i P_i$

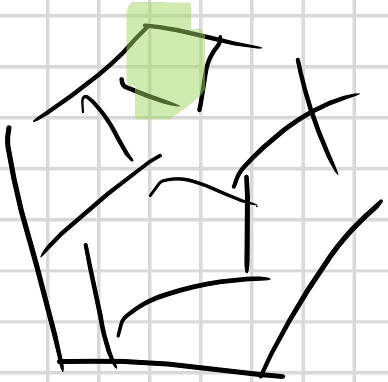
$P = \bigsqcup P_i$

$g_i \in \Gamma$

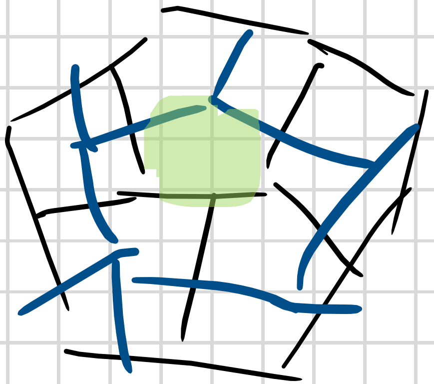
$\boxtimes$

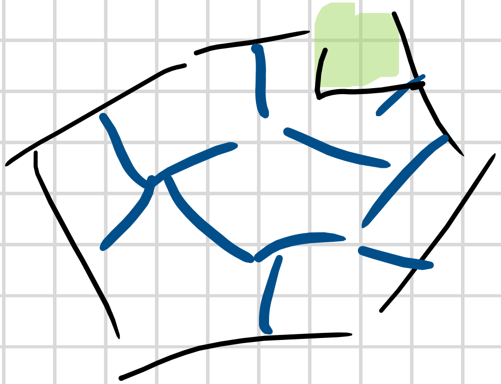
$\underline{L2}$ $\sim$ $\leftarrow$ equiv. relation

D $P \sim Q$, $Q \sim R \Rightarrow P \sim R$



P





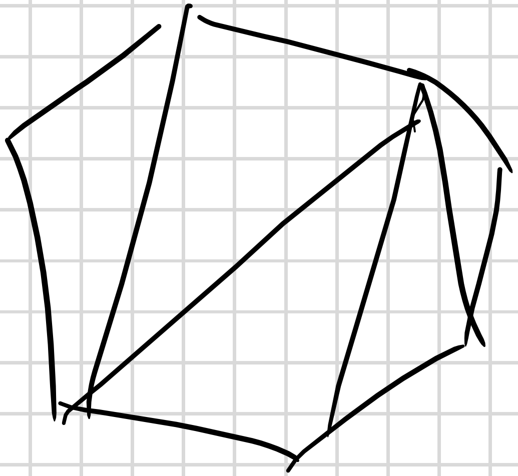
R

$$Q = \bigsqcup Q_i$$
$$= \bigsqcup Q_j'$$
$$= \bigsqcup (Q_i \cap Q_j')$$

$\boxtimes$

Proof of THI (B-G)

$P = \sqcup \Delta_i$



0) triang

1) triangles → paral.

2) paral → squares

3) squares → one big sq.

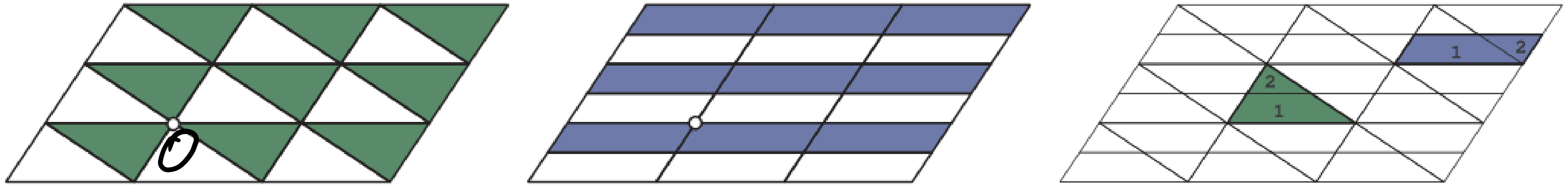
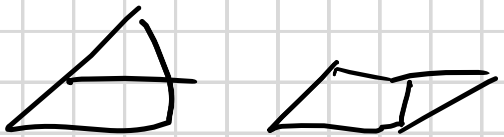


FIGURE 15.1. Converting a triangle into a parallelogram.



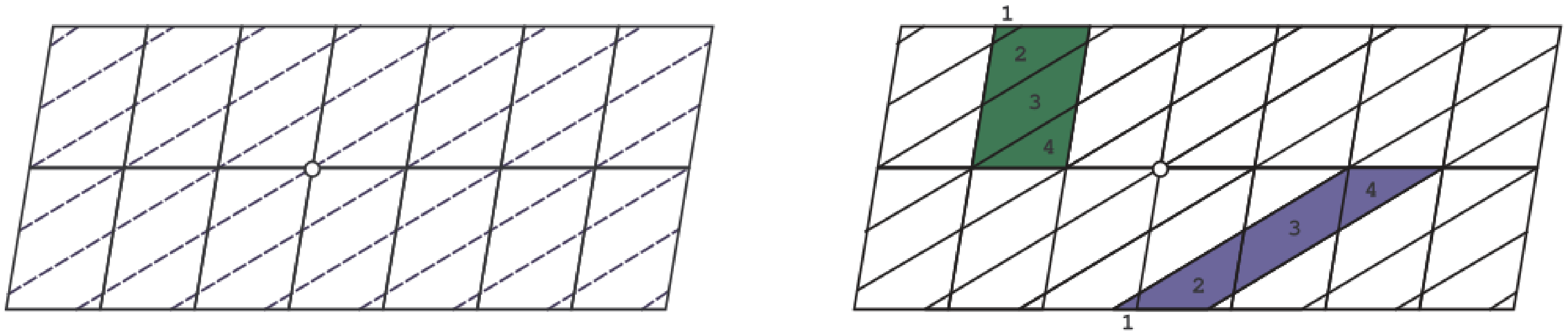
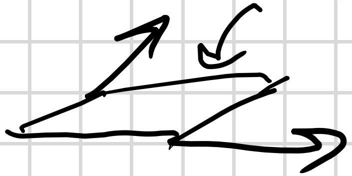
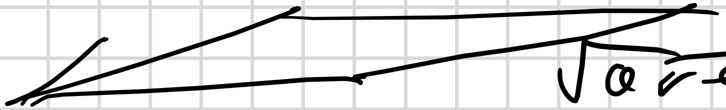


FIGURE 15.2. Converting a parallelogram into another parallelogram.

$\mathbb{Z}^2$



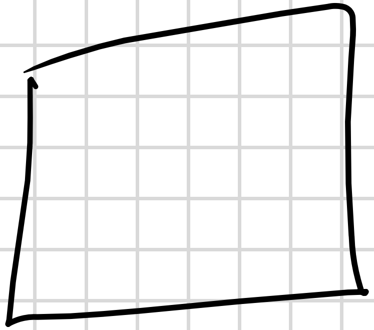
$> \sqrt{area}$



$\sqrt{area}$

$\sqrt{\text{area}}$

$\Rightarrow P \sim \bigsqcup \square_i$

$\bigcup \square_i \longrightarrow$ 

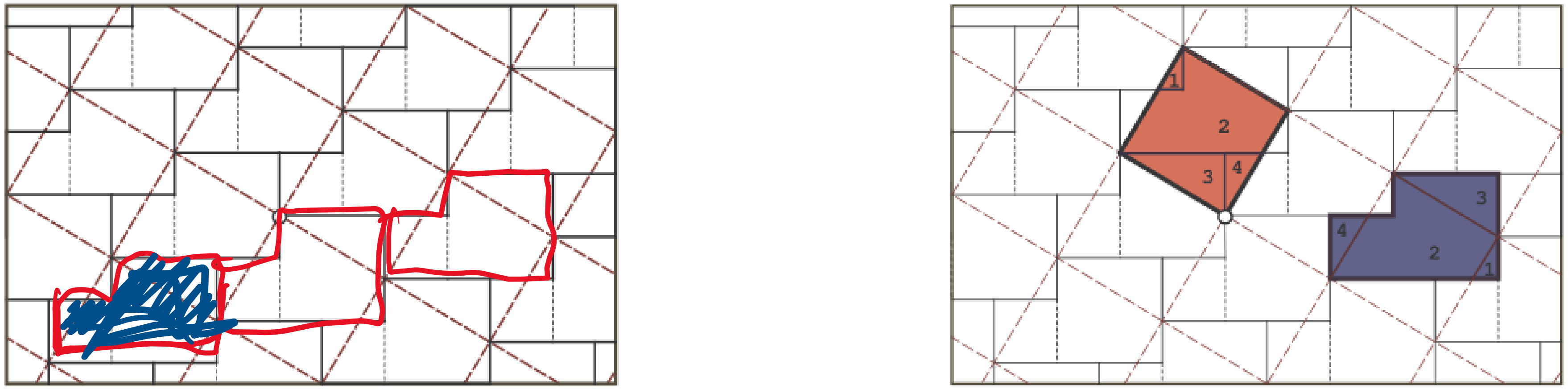
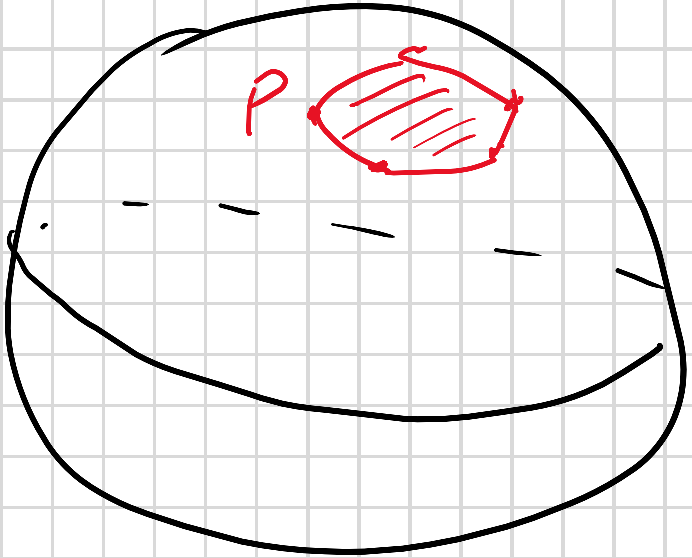


FIGURE 15.3. Converting two squares into a bigger one.

$\mathbb{Z}^2$

$$P \sim \underbrace{\square}_{\sqrt{\text{area}}} \sim Q \qquad \blacksquare$$

Q: Is B-G true for spherical
Exc and hyperbolic polygons?



Th B-G holds for S^2, $\mathbb{H}^2$

area $(P) = ??$ – easy

exc

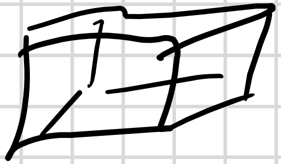
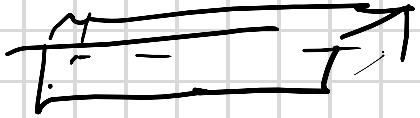
Def $P \subset \mathbb{R}^3$ ← convex polytope.

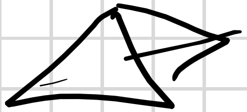
P ← *fortunate* if $\exists\, c_i \in \mathbb{Q}_+$

$$\boxed{c_1 \alpha_1 + \ldots + c_n \alpha_n = \pi}\,,$$

α_i = dihedral angle at edge e_i

$E = \{e_i\}$ ← edges of P

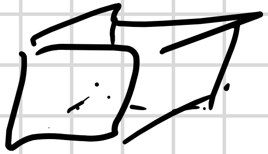
Ex $P =$   ← fortunate.

$P =$  ← not fortunate.

$$\boxed{\begin{aligned} &\alpha = 2\arcsin\frac{1}{\sqrt{3}} \\ &\angle \quad \alpha/\pi \notin \mathbb{Q} \end{aligned}}$$

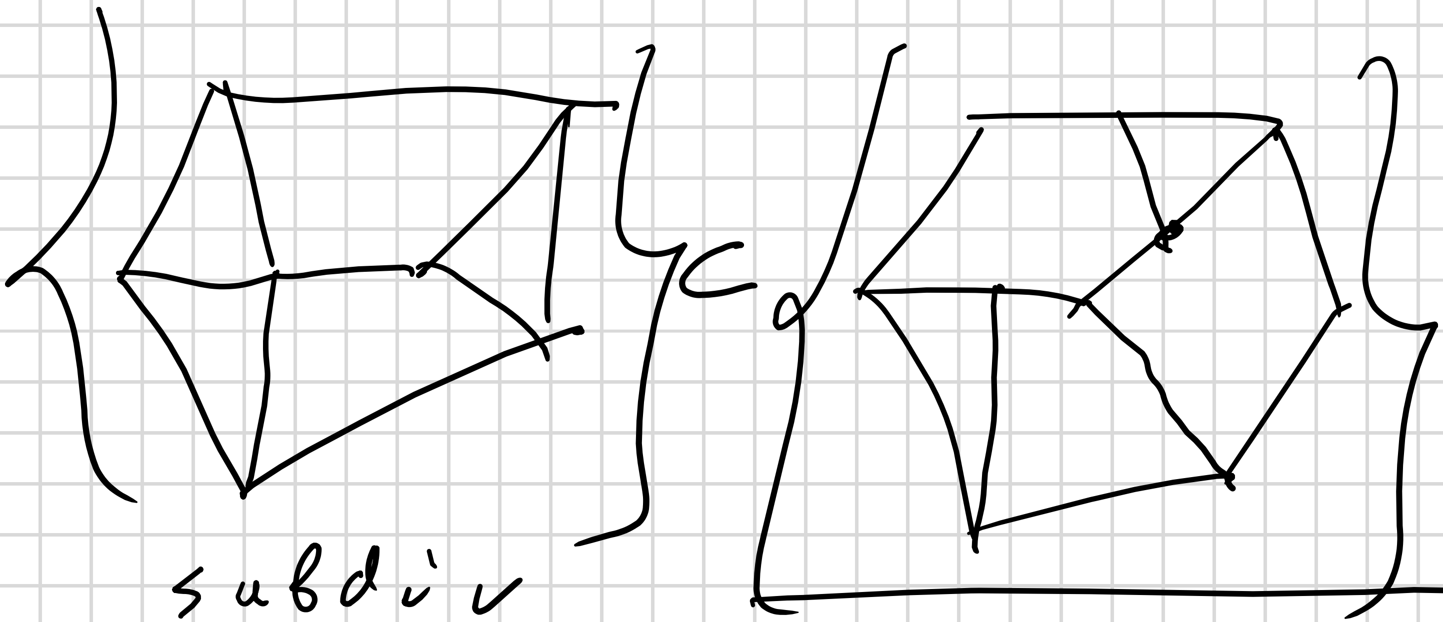
Exc

Th (Bricard, 1890s)

P – fortunate $\Longleftarrow$ $P \sim$ 

Def (Bricard)

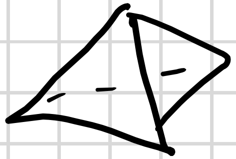
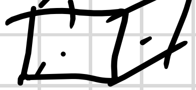
$P = \bigsqcup P_i$ ← polytopal subdivision



subdiv

face-to-face

partition

Cor  $= \bigsqcup P_i$,  $= \bigsqcup P_i'$

$P_i \sim P_i'$

Def $P \subset \mathbb{R}^3$ convex polytope.

$$\sigma(P) = \sum_{i=1}^{n} \alpha_i \qquad \alpha_i = \angle \text{ at } e_i$$

$P \sim Q$ / es subdiv /

$$\begin{array}{ccc} P & \sim & Q \\ \| & & \| \\ \sqcup P_i & & \sqcup Q_i \end{array}$$

$P_i \simeq Q_i$

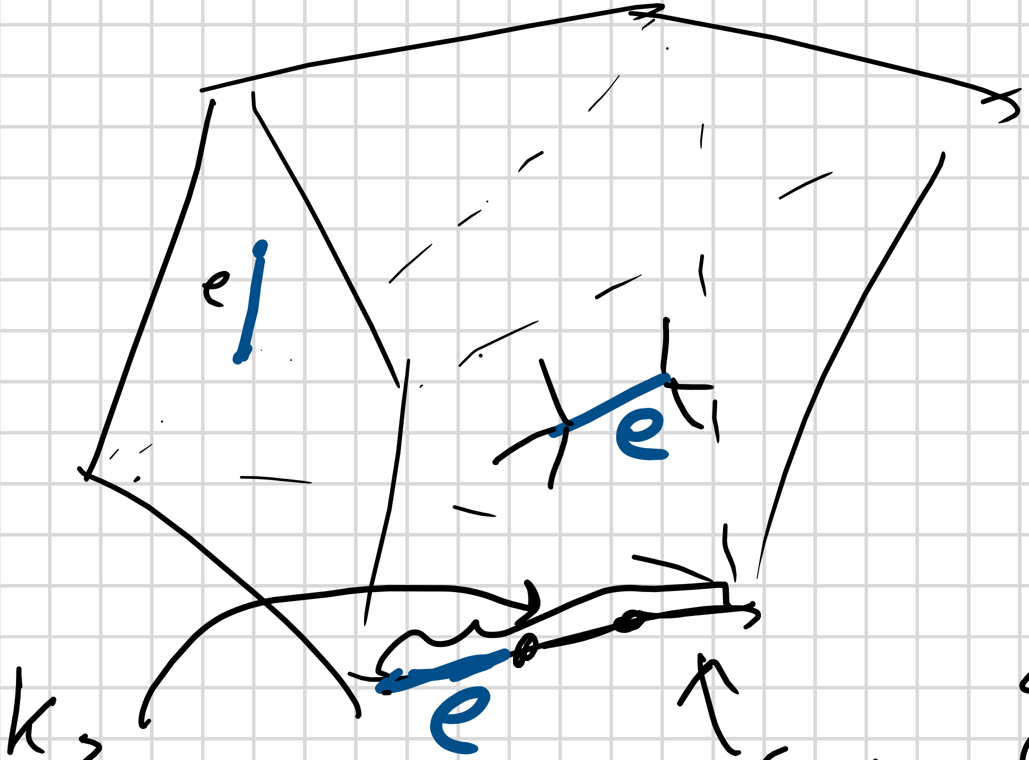
$$\sum := \sum \sigma(P_i) = \sum \sigma(Q_i)$$

(double counting)

$$\sum = \sum_{i=1}^{N} \sum_{e \in \mathcal{E}} \alpha_i(e) =$$

$\mathcal{E} \leftarrow \{$ all edges in $P_i \}$

$$\boxed{\Sigma_P} = \sum_{e \in \mathcal{E}} \sigma(e), \qquad \sigma(e) = \sum_{i=1}^{n} \alpha_i(e)$$



$s \leftarrow$ edge in P

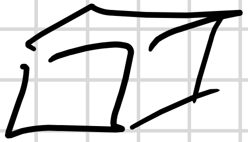
$\sigma(e) = 2\pi$

$e \leftarrow$ internal edge

$\sigma(e) = \pi$

$$\Sigma = \underset{\mathbb{N}}{M} \cdot \pi + \sum k_s \alpha_s$$

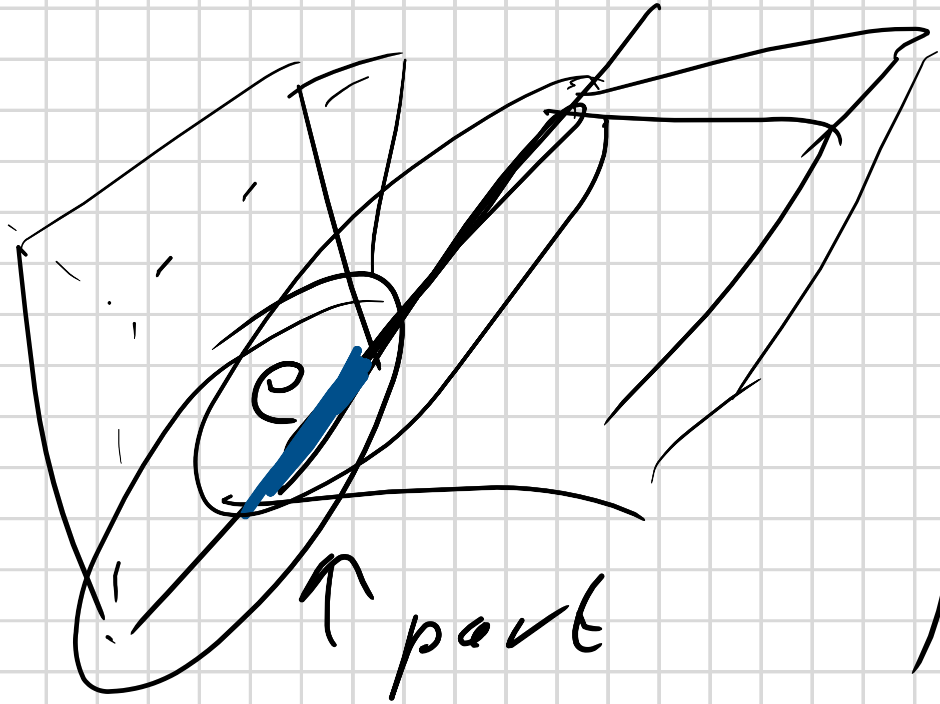
$k_s \in \mathbb{N}$

$Q =$  $\Rightarrow \quad \alpha_i = \frac{\pi}{2}$

$$\Sigma \in \mathbb{Q} \cdot \pi$$

$$\Sigma = \sum_{i=1}^{n} \underbrace{k_s}_{\in \mathbb{N}} \underset{\text{dih. angle in } P}{\alpha_s} + M\pi$$

$\Rightarrow$ Bricard Th. $\square$



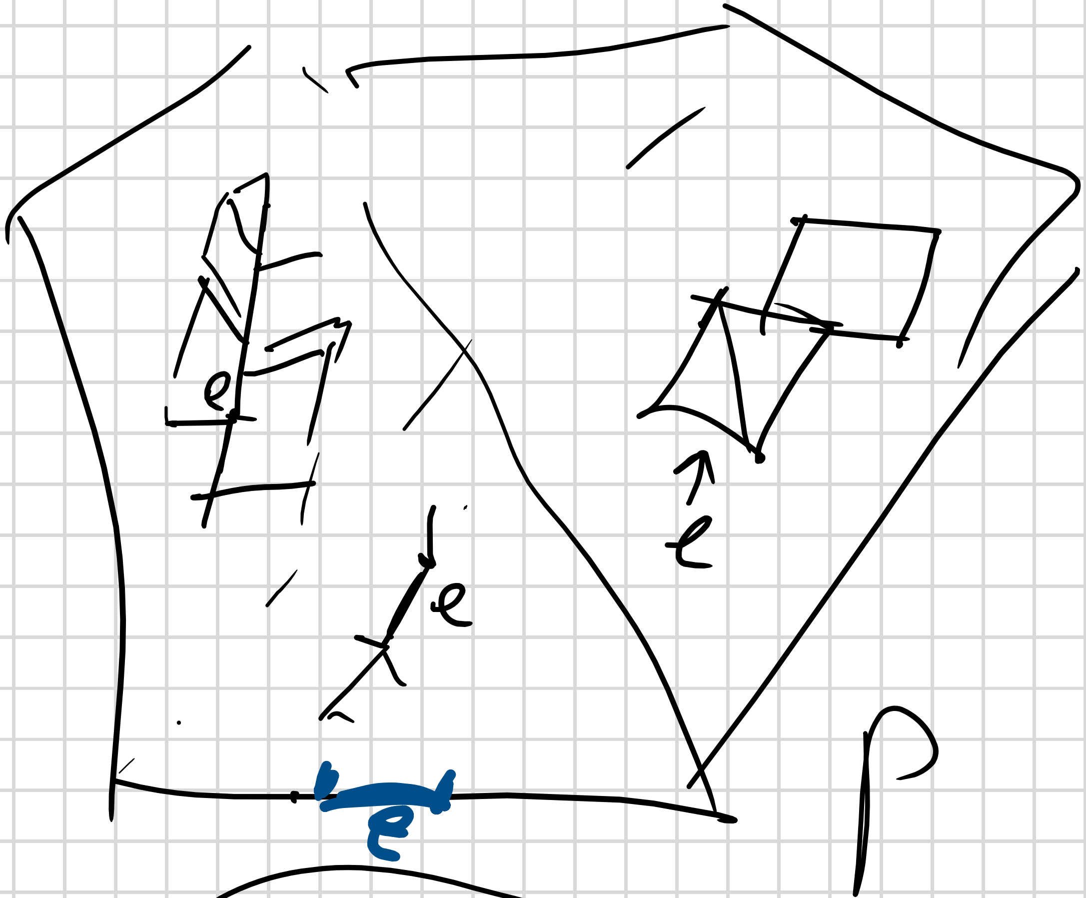
Proof of Th2 (Dehn)

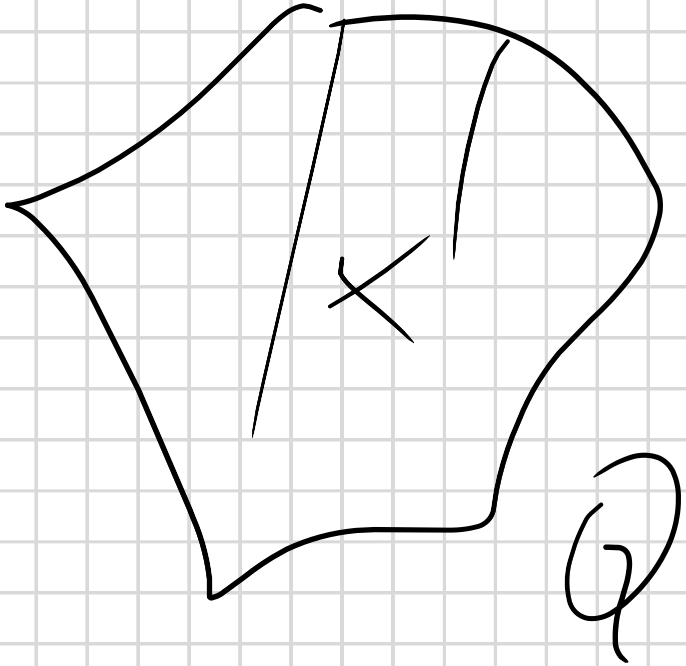
Assume all edge length are rational

$|e| = \ell_e \in \mathbb{Q}$

$$\Sigma = \sum_{i=1}^{n} \sum_{e \in \mathcal{E}} \ell_e \cdot \underbrace{\alpha_i(e)}_{\text{dihedral angle at } e \text{ in } P_i}$$

$\mathcal{E}$: all edges





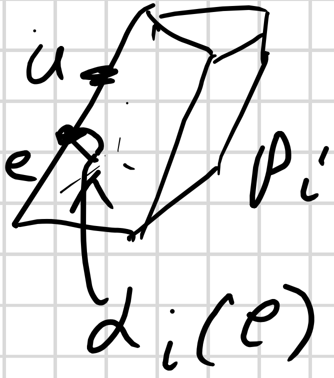
$l_e \cdot 2\pi$

$\mathbb{Q}$ $l_e \cdot \pi$

French proof

$$\Sigma := \sum l_e \otimes_{\mathbb{Z}} \alpha_e$$

$$\Sigma_p = \sum_{i=1}^{n} \sum_{u \in E(P_i)} \sum_{e \in u} l_e \, \alpha_i(e) \qquad \boxed{f(e)}$$



$$= \sum_{i=1}^{N} \sum_{u \in E(P_i)} l_u \, \alpha_i(u) \qquad Q = \Phi' \qquad = \Sigma_Q$$

$$= \sum_{i=1}^{n} \boxed{\sigma(P_i)} = \sum_{i=1}^{n'} \sigma(Q_i)$$

$$\sigma(P_i) := \sum_{u \in P_i} \underbrace{l_u}_{Q^\lambda} \, \alpha_i(u) \qquad Q^\lambda_\pi$$

Proof of Th 2 (Dehn version)

[V. F. Kagan]

$\Sigma_f' = \sum_{i=1}^{n} \sum_{e \in \mathcal{E}} f(e)\, \alpha_i(e)$ P

$\|$

$\Sigma_g' = \sum \sum g(e)\, \beta_i(e)$ Q

$f, g : \mathcal{E} \to \mathbb{Q}_+$

$D = \dim R \leq 2M$

$f : \mathcal{E} \to \mathbb{R}_+$

all edges

$g : \mathcal{E}_M \to \mathbb{R}_+$

set of such $\{f, g\}$

Obs (1) set $\in$ convex polytope.

(2) $f = g = \ell_e$

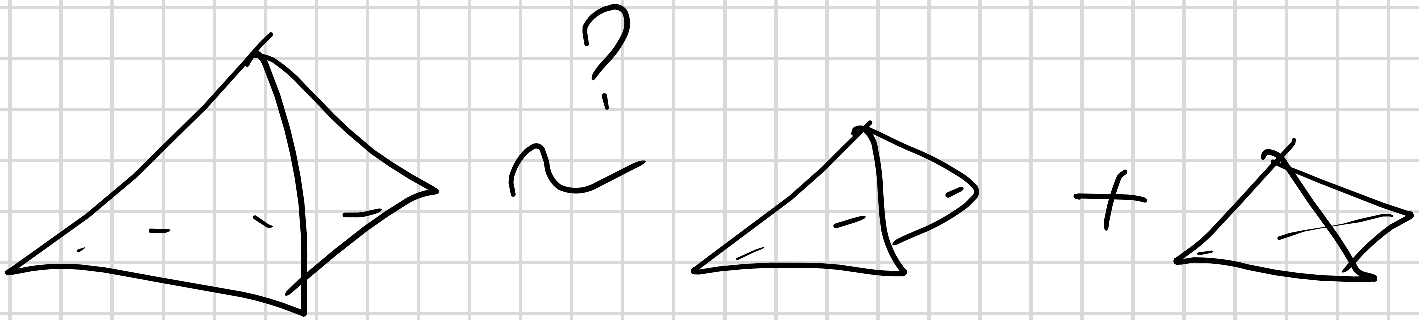
$\{f, g\} \in$ set

Obs $R \subset \mathbb{R}^D$ ← defined over $\mathbb{Q}$

and $R \neq \emptyset$ $\Longrightarrow$ R has a rational point.

$\Longrightarrow$ back to rational case. ∎

Q:



Sydler