MATH 61: HOMEWORK 2

3.3

25. not reflexive, not symmetric, antisymmetric, transitive, not a partial order

26. reflexive, not symmetric, antisymmetric, transitive, partial order

28. reflexive, symmetric, not antisymmetric, transitive, not a partial order (this is the same as saying $(x, y) \in R$ if $x = y \pmod 3$)

29. reflexive, symmetric, not antisymmetric, transitive, not a partial order (this relation is actually the same as the one in 28)

3.4

6. This is the same as saying $(x, y) \in R$ if $x = y \pmod 4$, and it is an equivalence relation. The equivalence classes are: $\{1, 5\}$, $\{2\}$, $\{3\}$, $\{4\}$.

8. This is not even reflexive: eg. 3 does not divide $2 - 3 = -1$, so it is not an equivalence relation.

10. This relation is not transitive: Say A and B lived in the same country, but then B moved abroad, and now lives in the same country as C. Nothing guarantees that A and C have ever lived in the same country. Hence this is not an equivalence relation.

11. This is an equivalence relation.

13. This is an equivalence relation.

14. This is an equivalence relation.

6.1

6. There are six positions, we can choose each of them to be a dot or not, this gives 2^6 options. But one of these are the case when no dot is present, which is not allowed. So there are $2^6 - 1$ possible characters.

8. If repetitions are allowed: $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10$. If repetions are not allowed: $26 \cdot 25 \cdot 24 \cdot 10 \cdot 9$

42. $5 \cdot 4 + 5 \cdot 4 \cdot 3$

43. $2 \cdot 5 \cdot 4$

88. $5 \cdot 4$ selections when Ben is chairperson, $5 \cdot 4$ selections when Alice is secretary, 4 selections when both of these happen. So by inclusion-exclusion, the number of selections when Ben is chairperson or Alice is secretary is $5 \cdot 4 + 5 \cdot 4 - 4$.

90. 6 outcomes have the blue die showing 3, $6 \cdot 3$ outcomes have an even sum, and 3 outcomes have both. The answer is $6 + 6 \cdot 3 - 3$.

91. $10000/5 = 2000$ integers from 1 to 10000 are multiples of 5, $\lfloor 10000/7 \rfloor = 1428$ of them are multiples of 7, $\lfloor 10000/35 \rfloor = 285$ of them are multiples of both 5 and 7 (or equivalently multiples of 35). By inclusion-exclusion, $2000 + 1428 - 285$ integers are multiples of at least one of the numbers 5 and 7.

6.8

3. 32 people are the pigeons, and 31 different days are the pigeonholes. So by the pigeon hole principle, at least two people receive checks on the same day.

6. According to the hint, we have 5 pigeonholes, namely the 5 parts of the partition. And the six numbers that we want to choose are the pigeons. So by pigeonhole principle, two pigeons will be in the same hole, i.e. two numbers from the same class will be chosen simultaneously. The sum of these two numbers will be 11.

8. 18 persons are 18 pigeons. The pigeonholes will be the different names, which of which there are $3 * 2 = 6$. So by the pigeonhole principle, at least three persons have the same first and last names.

1.

a) Not reflexive. Symmetric. Not transitive.

b) R is an equivalent relation, with equivalence classes $\{2x|x \in \mathbb{Z}\}$ and $\{2x+1|x \in \mathbb{Z}\}$.

c) Not reflexive. Not symmetric. Not transitive.

d) Not reflexive. Symmetric. Not transitive.

e) R is an equivalence relation. $xRy \iff |x+1| = |y+1|$, so the equivalence class of x is $\{x, -2-x\}$.

f) R is an equivalence relation. The equivalence class of x is $\{x, -(x+6)\}$.

g) Not reflexive. Symmetric. Not transitive.

h) Not reflexive. Symmetric. Transitive.

2.

a) R is an equivalence relation, and the equivalence classes are the circles centered at the origin.

b) Not reflexive. Not symmetric. Not transitive.

c) Not reflexive. Symmetric. Not transitive.

d) R is an equivalence relation, and the equivalence classes are the lines of slope 1.

e) R is an equivalence relation, and the equivalence classes are the parabolas of the form $\{(a,b)|a^2+b=r\}$ for $r \in \mathbb{R}$.

f) R is an equivalence relation, and the equivalence classes are the hyperbolas of the form $\{(a,b)|ab=r\}$ for $r \in \mathbb{R}$.

g) Not reflexive. Symmetric. Transitive.

h) R is an equivalence relation, and the equivalence classes are of the form $\{(a,b)|a^3+b^3=r\}$ for $r \in \mathbb{R}$.

3.

a) Let $a_1, ..., a_{100}$ be the elements of A. Let $s_i = a_1 + ... + a_i$ for $0 \leq i \leq 100$. There are 101 integers from 0 through 100, and there are 100 possible values of an integer mod 100, so by the pigeonhole principle, there is some n such that $s_i = n$ and $s_j = n$ for 2 distinct i and j (let j be the larger one). Let $X = \{a_{i+1}, a_{i+2}, ..., a_{j-1}, a_j\}$. $s_j = s_i + a_{i+1} + a_{i+2} + ... + a_{j-1} + a_j$ and $s_j = s_i \mod 100$, so $a_{i+1} + a_{i+2} + ... + a_{j-1} + a_j = 0 \mod 100$.

b) The circle of radius 1 fits inside a square of side length 2. Divide this square into a 30 by 30 grid, so there are 900 grid cells, each of which is a square with side length $\frac{1}{15}$. Let each of the grid cells that overlaps with the circle be a pigeonhole. The grid cells in the corners of the square do not overlap with the circle, so there are fewer than 900 pigeonholes. Each of the 900 points is in one of the pigeonholes, so there is some pigeonhole with at least two points in it. The distance between any two points in a square with side length $\frac{1}{15}$ is at most $\frac{\sqrt{2}}{15}$, which is < 0.1, so the two points are at distance < 0.1.

c) The average age of the members of the music club is greater than 13, so the average of the ages of the 20 oldest members must also be greater than 13, and hence the sum of their ages is greater than 260.