# MIDTERM 1 (MATH 61, SPRING 2015)

**Your Name:** ------------------------------------------------

**UCLA id:** ------------------------------------------------

**Math 61 Section:** ------------

**Date:** ------------------------

**The rules:**
You MUST simplify completely and BOX all answers with an **INK PEN**.
You are allowed to use only this paper and pen/pencil. No calculators.
No books, no notebooks, no web access. You MUST write your name and UCLA id.
Except for the last problem, you MUST write out your logical reasoning and/or proof in full. You have exactly 50 minutes.

**Warning:** those caught writing after time get automatic 10% score deduction.

**Points:**

| | |
|---|---|
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| 5 | |

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**Total:** **(out of 100)**

**Problem 1.** (20 points)

Compute the number of permutations $(x_1, x_2, \dots, x_n)$ of $\{1, 2, \dots, 9\}$ such that:

*a*) $x_1 = 2$,

*b*) $x_1 \cdot x_2 \cdot x_3 = 6$,

*c*) $x_1 = x_2 = x_3 \text{ mod } 7$,

*d*) $x_1 < x_2 < 5$.

**Solutions.**

*a*) Since there are 8 other elements, the number should be 8!.

*b*) The first three numbers should be $1, 2, 3$ in some order, so the number of permutation should be 3!6!.

*c*) Since we don't have three numbers from the set such that they are all the same mod 7, the answer should be 0.

*d*) If $x_2 = 4$, $3 \cdot 7!$. If $x_2 = 3$, $2 \cdot 7!$. If $x_2 = 2$, $3 \cdot 7!$. So the answer is the sum which is $6 \cdot 7!$.

**Problem 2.** (20 points)

Let $X = \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ be the set of all integers. For each of these relations $R$, decide whether they are reflexive, symmetric or transitive (or neither).

*a*) $xRy$ if and only if $|x| = |y|$.

*b*) $xRy$ if and only if $x + 2y = 0 \bmod 3$.

*c*) $xRy$ if and only if $x^2 + 2y^2 = 0 \bmod 3$.

*d*) $xRy$ if and only if $x^3 + 122y^3 = 0 \bmod 3$.

**Solutions.**

*a*) $|x| = |x|$ for all $x$, so reflexive.
$xRy \iff |x| = |y| \iff |y| = |x| \iff yRx$, so symmetric.
If $|x| = |y|$ and $|y| = |z|$, then $|x| = |z|$, so transitive.

*b*) $x + 2x = 3x = 0 \bmod 3$. So reflextive
$xRy \iff x + 2y = 0 \bmod 3 \iff x = y \bmod 3 \iff y + 2x = 0 \bmod 3 \iff yRx$. So symmetric.
If $xRy$ and $yRz$, then $x = y \bmod 3$, and $y = z \bmod 3$. So $x = z \bmod 3$, which means the relation is transitive.

*c*) Similar to the previous problem, the relation is reflexive, symmetric and transitive.

*d*) Similar to the previous two problems, the relation is reflextive, symmetric and transitive.

**Problem 3.** (15 points)

Let $A = (0,0)$, $B = (10,10)$. Find the number of (shortest) grid walks $\gamma$ from $A$ to $B$, such that:

*a*) $\gamma$ never visits points $(0,10)$, $(10,1)$, $(5,5)$.

*b*) $\gamma$ visits all points $(1,1)$, $(2,2)$, $(3,3)$, $\ldots$, $(9,9)$.

*c*) $\gamma$ visits points $(5,0)$ and $(5,10)$, but not $(5,5)$.

**Solutions.**

*a*) $\binom{20}{10} - \binom{10}{0}\binom{10}{10} - \binom{11}{10}\binom{9}{0} - \binom{10}{5}\binom{10}{5}$.

*b*) $2^{10}$.

*c*) Since we can only go to the left or up on a grid walk, every grid walk that visits (5,0) and (5,10) must also visit (5,5). So the answer should be 0.

**Problem 4.** (15 points)

Recall the Fibonacci sequence: $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8$, etc.

Prove that $F_n \leq 2^{n-1}$.

**Solution:**

Base step:
$F_1 = 1 \leq 2^0$; $F_2 = 1 \leq 2^1$.
Induction step:
Assume that $F_{n-1} \leq 2^{n-2}$ and $F_n \leq 2^{n-1}$, then $F_{n+1} = F_n + F_{n-1} \leq 2^{n-1} + 2^{n-2} \leq 2^n$.

**Problem 5.** (30 points, 2 points each) **TRUE or FALSE?**

Circle correct answers with ink. No explanation is required or will be considered.

**T** **F** (1) The number of functions from $\{A, B, C, D\}$ to $\{1, 2, 3\}$ is equal to $4^3$.

**T** **F** (2) The sequence $10, 21, 32, 43, \ldots$ is increasing.

**T** **F** (3) The sequence 2/1, 3/2, 4/3, 5/4 is non-increasing.

**T** **F** (4) There are 20 anagrams of the word $BUBUB$.

**T** **F** (5) There are more anagrams of the words $AAAACCC$ which begin with $A$ than with $C$.

**T** **F** (6) There are infinitely many Fibonacci numbers $= 1$ mod 3.

**T** **F** (7) There are infinitely many binomial coefficients $\binom{n}{k} = 1$ mod 17.

**T** **F** (8) Each of the 14 students wrote on a paper 10 distinct numbers, from the set $\{1, 2, \ldots, 100\}$. Then there are two students who have at least 2 numbers in common on their lists.

**T** **F** (9) The probability that a random 10-subset of $\{1, 2, \ldots, 19\}$ contains 10 is equal to $1/2$.

**T** **F** (10) For every two subsets $A, B \subset U$, we must have $|A \smallsetminus B| = |B \smallsetminus A|$.

**T** **F** (11) For every two subsets $A, B \subset U$, we must have $|A \cup B| \geq |\overline{B}|$

**T** **F** (12) Every surjection that is also a bijection must be also an injection.

**T** **F** (13) Every surjection that is also an injection must be also a bijection.

**T** **F** (14) Let $\mathcal{A}$ be the set of 3-subsets of $[9] = \{1, 2, \ldots, 9\}$. Similarly, let $\mathcal{B}$ be the set of 6-subsets of [9]. Consider a map $f : A \to [9] \smallsetminus A$. Then $f$ is a bijection from $\mathcal{A}$ to $\mathcal{B}$.

**T** **F** (15) The pigeon hole principle was proved in class by induction.

**Solutions**: F T T F T T T F F F F T T T F.