HOME ASSIGNMENT 4 (MATH 206A, FALL 2020)

I. *a*) A *labeled tree* T is tree with vertices labeled $1, \ldots, n$. Place a root at vertex 1. We say that T is *increasing* if every shortest path from i to 1 contains only vertices j with $1 \le j \le i$. Prove that there are $n!$ increasing trees with $n+1$ vertices.

b) Orient all edges away from the root. An increasing tree is called *narrow* if every out-degree is at most 2. Denote by $\mathcal{L}_n$ the set of narrow trees on n vertices. Prove that $|\mathcal{L}_n| = a_n$, where a_n is the number of alternating permutations $\sigma(1) < \sigma(2) > \sigma(3) < \ldots$ For example, for $n = 3$, there are two narrow trees on 3 vertices, and three alternating permutations (132) and (231). See also Figure 1 below.

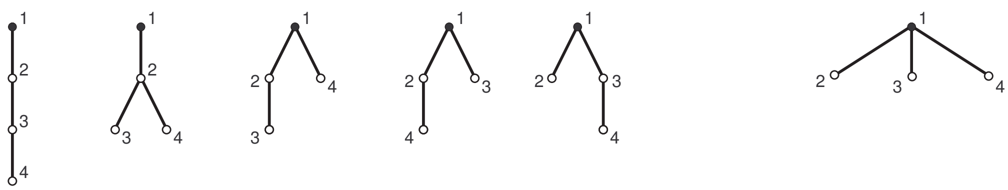


FIGURE 1. Five narrow and one non-narrow trees on $n = 4$ vertices. On narrow trees in $\mathcal{L}_4$, left to right: α-values: $(3, 2, 1, 2, 3)$ and β-values: $(4, 3, 3, 4, 2)$.

c) For an increasing tree T on n vertices, denote by $\alpha(T)$ the label of a vertex adjacent to n. Prove that the number of $T \in \mathcal{L}_n$ such that $\alpha(T) = k$ is equal to $a_{n,n-k}$, where $a_{n,k}$ is defined as the number of alternating permutations $\sigma \in S_n$, s.t. $\sigma(1) = k$. For example, $a_{4,1} = 2$, $a_{4,2} = 2$, and $a_{4,3} = 1$.

d) For an increasing tree T on n vertices, consider the DFS path defined as follows: from 1 to the smallest adjacent label (always 2, of course, for $n \ge 2$), to the smallest adjacent label, etc. Denote by $\beta(T)$ the end of this path. Prove that the number of $T \in \mathcal{L}_n$ such that $\beta(T) = k + 1$ is equal to $a_{n,n-k}$.

II. Let T_k be a *full binary tree* with k levels, e.g. B_4 is given in the Figure below. Denote by $\mathcal{L}_k$ the set of linear extensions of the tree poset associated with T_k, where the root is the minimal element. Note that $T_k \subset T_n$, for $k \le n$. Fix two vertices $x, y \in T_k$ and consider the probability

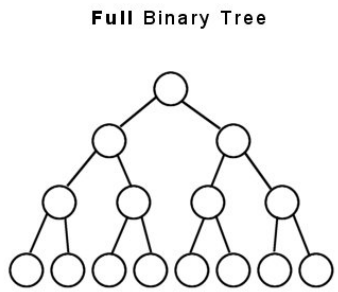
$$p_n(x, y) := \mathbb{P}\big[A(x) < A(y)\big] \quad \text{over uniform random } A \in \mathcal{L}_n$$

where $n \ge k$. Compute the limit

$$\pi(x, y) = \lim_{n \to \infty} p_n(x, y)$$

in terms of the heights $a := h(x)$, $b := h(y)$ and $c := h(x \wedge y)$.

III. *a*) Denote by a_m the number of labeled domino tilings of a $4m \times 4m$ square (increasing in rows and columns). Similarly, denote by b_m the number of labeled domino tilings of a $8m \times 2m$ rectangle. Compute a_m and b_m exactly. Compare their asymptotics. Are there domino tilings of a rectangle which cannot be labeled to become a labeled domino tilings?



b) Denote by c_n the number of B_n-domino tableaux, where B_n is the Boolean lattice. Compute the asymptotics of c_n. Are there domino tilings of a n-dimensional cube with edge length 2, which cannot be labeled to become a B_n-domino tableaux?

c) Prove that every two domino tilings of a $2m \times 2m$ rectangle are connected by a sequences of 2-*flips*, defined as flips which use exactly two dominoes. Prove of disprove a similar result for the usual $2 \times 1 \times 1$ domino tilings of a $2m \times 2m \times 2m$ cube.

IV. Prove that the Bruhat order is a lattice.

V. Let $P = (X, \prec)$ be a poset, $|X| = n$. Prove that there is a chain C in P, such that

$$\sum_{x \in C} \frac{1}{e(P-x)} \;\geq\; \frac{n}{e(P)}$$

VI. Let $\mathcal{A}, \mathcal{B} \subset B_n$ two subsets of elements of the Boolean lattice. Suppose for all $X \in \mathcal{A}$ and $Y \in \mathcal{B}$ we have: X and Y are incomparable in B_n. Prove that $|\mathcal{A}| \cdot |\mathcal{B}| \leq 4^{n-2}$.

VII. Let $L = (X, \prec)$ be a lattice. For $A, B \subset X$, define $A \wedge B = \{a \wedge b \,:\, a \in A, b \in B\}$, and $A \vee B = \{a \vee b \,:\, a \in A, b \in B\}$. Suppose

$$|A| \cdot |B| \;\leq\; |A \wedge B| \cdot |A \vee B|$$

for all $A, B \subset X$. Prove that L is distributive.

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This Homework is due Friday Dec 4, at 1:59 pm (right before class). The solutions must be uploaded to the Gradescope. Please read the collaboration policy on the course web page. Feel free to look for collaborators on the CCLE chat. Do not discuss there any solution ideas, and only use the chat to clarify the statements of the problems. Make sure you write your **name**, your **UCLA id number**, and your collaborators' names on the first page.

P.S. Each problem above has the same weight.