HOME ASSIGNMENT 3 (MATH 206A, FALL 2020)

I. *a*) Prove by induction the product formula

$$e(P_T) \,=\, n! \prod_{v\in T} \frac{1}{b(v)}$$

for the number of linear extensions of the tree poset with shape tree T.

b) For a vertex v of a rooted tree T, denote by $h(v)$ the height of v in T = number of vertices on the shortest path from v to the root. Prove the following equation

$$\sum_{v\in T} h(v) \,=\, \sum_{v\in T} b(v)$$

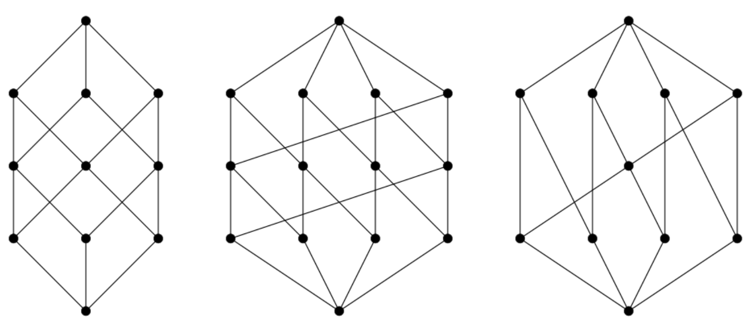
and the inequality:

$$\prod_{v\in T} h(v) \,\geq\, \prod_{v\in T} b(v).$$

Describe for which trees the inequality above is an equality.

For example, take tree T on 4 vertices $x \to y \to z$ and $x \to w$, so the root is at x. We then have: $b(x) = 4$, $b(y) = 2$, $b(w) = b(z) = 1$, $h(w) = 1$, $h(y) = h(w) = 2$, and $h(z) = 3$. Then we have the equality $4+2+1+1 = 8 = 3+2+2+1$, and the inequality $4\cdot2\cdot1\cdot1 = 8 \leq 3\cdot2\cdot2\cdot1 = 12$.

II. Which of the following are lattices? Argue your answer.



III. Let L be a lattice on finite set X. Prove that L is distributive if and only if

$$\big[z \vee x = z \vee y, \quad z \wedge x = z \wedge y\big] \implies x = y \quad \text{for all } x, y, z \in X.$$

IV. *a*) Let $P \subset \mathbb{R}^d$ be a convex polytope. Denote by $\mathcal{F}(P)$ the set of faces of all dimensions ordered by inclusion. Prove that $\mathcal{F}(P)$ is a lattice. For example, $\mathcal{F}(\Delta) = B_{d+1}$, where Δ is a d-simplex.

b) For a hypercube $H_d = [01]^d$ compute the number of elements in $\mathcal{F}(H_d)$. Is this lattice distributive? Compute the number of maximal chains in $\mathcal{F}(H_d)$. Is $\mathcal{F}(H_d)$ Sperner?

c) Consider the set $\mathcal{C}(P)$ of maximal chains in $\mathcal{F}(P)$. Define a *flip graph* $\Gamma(P) = (V, E)$, where $V = \mathcal{C}(P)$ and E consists of pairs (C, C'), s.t. $C, C' \in \mathcal{C}(P)$, and C and C' coincide in all but one dimension. For example, for $Q_n \subset \mathbb{R}^2$ convex n-gon, flip graph $\Gamma(Q_n)$ is a $2n$-cycle. Prove that $\Gamma(P)$ is connected for all P.

V. Let $L = (X, \prec)$ be a lattice. Denote by $\Gamma(L)$ the graph on all maximal chains in L with edges corresponding to pair of chains which differ in only one place.

a) Suppose L is finite and distributive. Prove that $\Gamma(L)$ is connected. Check that this claim fails for general finite lattices.

b) Suppose L is infinite, distributive and has finite height. Prove or disprove: graph $\Gamma(L)$ is connected.

VI. Let $L = (X, \prec)$ be a poset on the set of integers $X = \{1, 2, 3, \ldots\}$ such that $a \prec b$ if and only if $a|b$. Prove that L is a distributive lattice. Prove or disprove: there is a poset P such that $L = J(P)$.

VII. Denote by X the set of all finite $0-1$ sequences, including $\varnothing$. Denote by Λ the graph on X with edges of the form

$$(\alpha_1, \ldots, \alpha_n, 0, 1, \ldots, 1) \to (\alpha_1, \ldots, \alpha_n, 0, 0, 1, \ldots, 1)$$

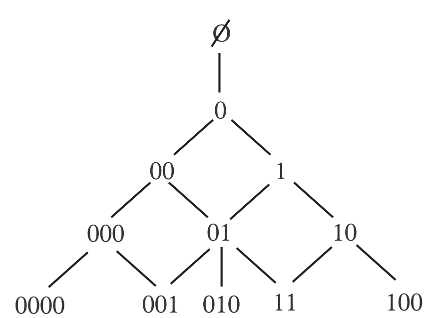
and

$$(\alpha_1, \ldots, \alpha_n, 0, 1, \ldots, 1) \to (\alpha_1, \ldots, \alpha_n, 1, 1, \ldots, 1)$$

where $(\alpha_1, \ldots, \alpha_n) \in X$, the first sequence has an extra 0, the second has $0 \to 1$. Note that the sequence of 1's can be empty at the end, in which case we simply add 0 on the right. For example we allow appending 0 at the end of an all-0 sequence: $(0, \ldots, 0) \to (0, \ldots, 0, 0)$. An example is given in the figure below.

a) Prove that Λ is a Hasse diagram of a ranked poset L. Prove that L is a lattice. Check that L is not distributive. Compute the rank generating function of L.

b) For a sequence $\overline{\alpha} = (\alpha_1, \ldots, \alpha_n) \in X$, denote by $f(\overline{\alpha})$ the number of shortest paths from $\varnothing$ to $\overline{\alpha}$. Prove that $f(\overline{\alpha})$ can be computed in polynomial time.



VIII. What is the maximum of $e(P)$ over all posets with n elements of width at most w? What is the minimum of $e(P)$ over all posets with n elements of height at most h?

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This Homework is due Monday Nov 16, at 1:59 pm (right before class). The solutions must be uploaded to the Gradescope. Please read the collaboration policy on the course web page. Feel free to look for collaborators on the CCLE chat. Do not discuss there any solution ideas, and only use the chat to clarify the statements of the problems. Make sure you write your **name**, your **UCLA id number**, and your collaborators' names on the first page.

P.S. Each problem above has the same weight.