HOME ASSIGNMENT 2 (MATH 206A, FALL 2020)

I. Let $I_i = [a_i, b_i] \subset \mathbb{R}$, $1 \le i \le n$ be a family of intervals. Denote by $G = ([n], E)$ a graph on these intervals with edges $(i,j) \in E \Leftrightarrow I_i \cap I_j \ne \varnothing$. Prove that G is perfect.

II. In the proof of WPGC, recall how each vertex $x \in G$ is replaced with a clique G_x and the resulting graph $\widehat{G}$ remains perfect by the replication lemma. Suppose now, more generally, that all G_x are perfect graphs. Prove that the resulting graph $\widehat{G}$ remains perfect.

III. Let $G = (V, E)$, $\alpha = \alpha(G)$, $\omega = \omega(G)$. Observe that every independent set A and every clique K intersect at zero or one vertex. We say that G is *nice* if every α-independent set A and every ω-clique K intersect at exactly one vertex. We say that G is *terrific* if G is nice and every induced subgraph of G is nice. Prove that all terrific graphs are perfect.

IV. (a) Let $P = (X, \prec)$ be a finite poset. Denote by h the height of P, and by h' the maximal size $|C_1 \cup C_2|$ over all nonintersecting chains C_1, C_2 in P, $C_1 \cap C_2 = \varnothing$. Prove or disprove: one can always choose C_1, C_2 as above such that $|C_1| = h$, $|C_2| = h' - h$.

(b) Let $P = (X, \prec)$ be a finite poset. Denote by w the width of P, and by w' the maximal size $|A_1 \cup A_2|$ over all nonintersecting antichains A_1, A_2 in P, $A_1 \cap A_2 = \varnothing$. Prove or disprove: one can always choose A_1, A_2 as above such that $|A_1| = q$, $|A_2| = w' - w$.

(c) Independently of (a) and (b), prove or disprove both claims for $P = P_\sigma$, a poset associated with a permutation $\sigma \in S_n$.

(d) Independently of (a) and (b), prove or disprove both claims for $P = B_n$, the Boolean lattice.

V. Let $A \subset \mathbb{Z}/p\mathbb{Z}$, $|A| = n$, where $p > 2^{2^n}$ is a prime. Denote by $s_x(A)$ the number of subsets $S \subseteq A$ such that $\sum_{a \in A} a = x$. Prove that $s_x(A) = o(2^n)$ for all $x \in \mathbb{Z}/p\mathbb{Z}$.

VI. Let $n = 2k$. A cyclic sequence $(a_1, \ldots, a_N)$, $a_i \in [n]$, is called (n,k)-*universal* if it contains every k-permutation as a subword. Clearly, we must always have $N \ge n!/k!$, and such sequence exists for $N = k(n!/k!)$. Can you find a shorter (n,k)-universal sequence?

VII. State and prove the q-*Bollobás Theorem* for subspaces $A_i, B_i \subset \mathbb{F}_q^n$, $1 \le i \le m$ as in Lecture 6. Deduce the q-LYM Theorem.

VIII. (a) Suppose every maximal antichain in poset P contains at least 2 elements. Prove that P contains 2 disjoint maximal chains.

(b) Prove or disprove: If every maximal antichain in poset P contains at least 3 elements, then P contains 3 disjoint maximal chains.

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This Homework is due Monday Nov 2, at 1:59 pm (right before class). The solutions must be uploaded to the Gradescope. Please read the collaboration policy on the course web page. Feel free to look for collaborators on the CCLE chat. Do not discuss there any solution ideas, and only use the chat to clarify the statements of the problems. Make sure you write your **name**, your **UCLA id number**, and your collaborators' names on the first page.

P.S. Each problem above has the same weight.