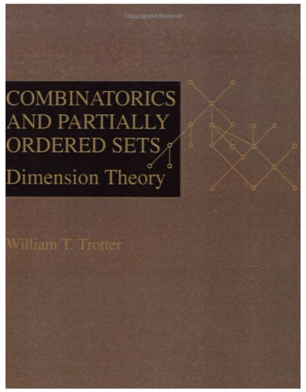
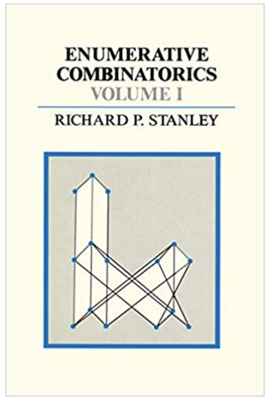
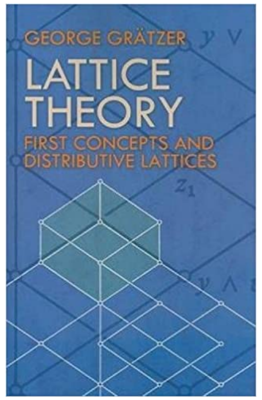
Combinatorial Theory (Math 206A)

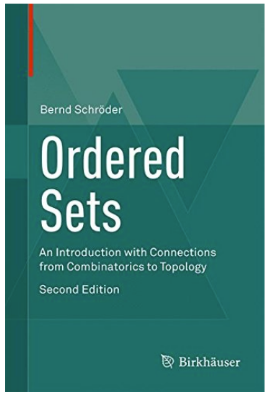
Igor Pak, UCLA

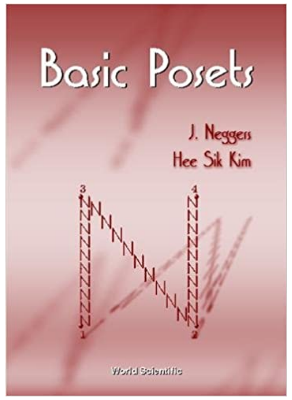
Zoom Lecture 1 (Oct 2, 2020)


COMBINATORICS
AND PARTIALLY
ORDERED SETS
Dimension Theory
William T. Trotter


ENUMERATIVE
COMBINATORICS
VOLUME I
RICHARD P. STANLEY


GEORGE GRÄTZER
LATTICE
THEORY
FIRST CONCEPTS AND
DISTRIBUTIVE LATTICES


Bernd Schröder
Ordered
Sets
An Introduction with Connections
from Combinatorics to Topology
Second Edition
Birkhäuser


Basic Posets
J. Neggers
Hee Sik Kim
World Scientific

What this class is about?

Assorted recent and classical results on *combinatorics of posets*.

Special emphasis on counting *linear extensions*.

Examples of subjects:

- Chain and antichain decomposition, Dilworth's theorem, applications to graph theory
- LYM and Sperner properties, Greene—Kleitman theorem, Bollobás's theorem
- Linear extensions, lower and upper bounds, Young tableaux, Stanley's P-partition theory
- Two poset polytopes, geometric inequalities, log-concavity properties
- 1/3–2/3 conjecture and variations, Linial and Kahn–Sacks theorems
- 1/3–2/3 conjecture for skew Young diagrams via [Olson–Sagan] and [Chan–P.–Panova]
- Random linear order, different models and analysis, after [Brightwell]
- Complexity of counting linear extensions, positive and negative results, after [Felsner–Manneville], [Brightwell–Winkler], [Dittmer–P.]
- FKG inequality, probabilistic applications to percolation
- Coupling from the past on posets, after [Propp–Wilson]

Chains & Anti-chains

206A

Def: $\mathcal{P} = (X, \prec)$, $X \leftarrow$ finite set
poset $\quad \prec \leftarrow$ partial order

poset = partially ordered set

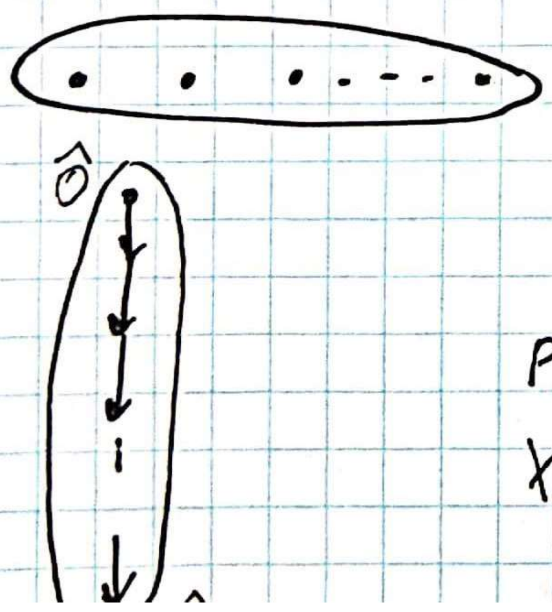
$x \prec y, \quad y \prec z \Rightarrow x \prec z \quad \forall x, y, z \in X$

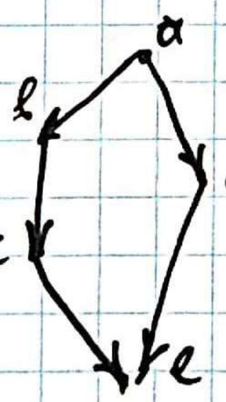
Notation: $x \preccurlyeq x \quad \forall x \in X$

$\hat{0} \leftarrow$ global min $\qquad \hat{0} \preccurlyeq x \; \forall x$

$\hat{1} \leftarrow$ global max $\qquad \hat{1} \succcurlyeq x \; \forall x$

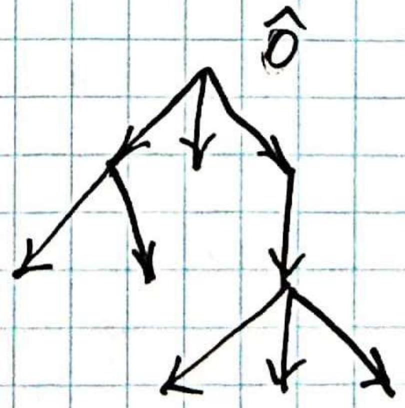
Examples:





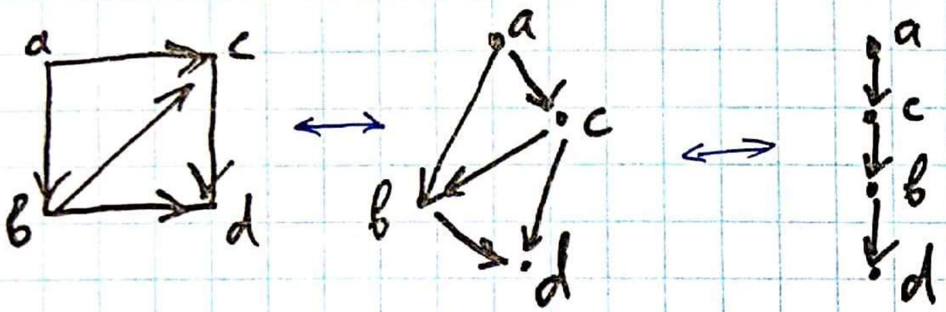
$a \prec b \prec c \prec e$
$a \prec d \prec e$

$P = (X, \prec)$
$X = \{a\, b\, c\, d\, e\}$



Orogins of posets

① acycluc graphs, tournaments



$P = (X, \prec)$

$X = \{a\,b\,c\,d\}$

$\prec$ ← linear order

---

② unfinished sorting

$X = \{a_1\ a_2 \dots a_n\}$, $a_i \in \mathbb{N}$

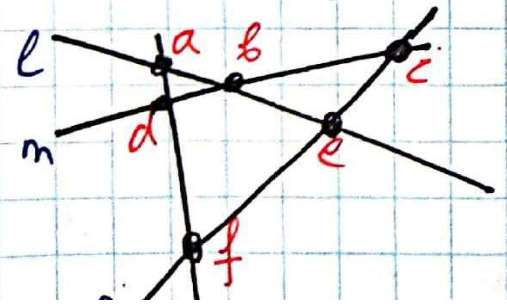
$a_i \precsim a_j \implies a_i \le a_j$ / but not $\Leftarrow$ /

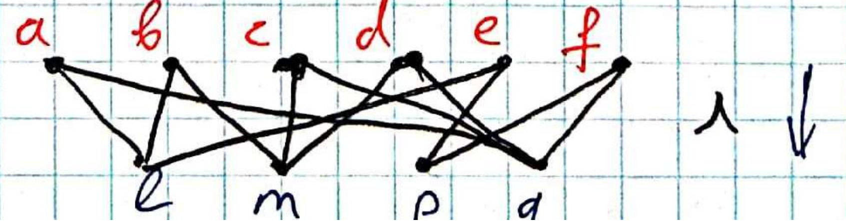
---

③ inclusion relation

$X = \{V_1, V_2, \dots V_n\}$, $V_i \subseteq W$ subspaces, $W = \mathbb{k}^d$

$V_i \precsim V_j \iff V_i \subseteq V_j \quad \forall\, i,j$





④ d-dim. data set

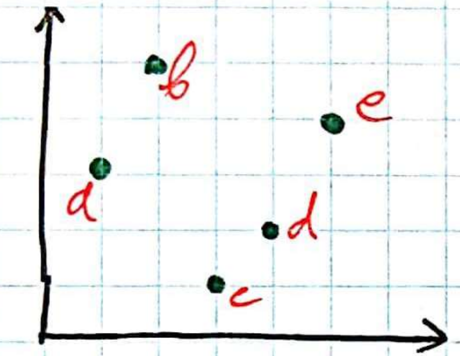
$$X = \{ \bar{x}, \bar{y}, \dots \} \subset \mathbb{R}^d$$

$$\bar{x} = (x_1 \dots x_d) \qquad \bar{y} = (y_1 \dots y_d)$$

$$\bar{x} \precsim \bar{y} \iff x_i \le y_i \quad \forall i = 1 \dots d$$

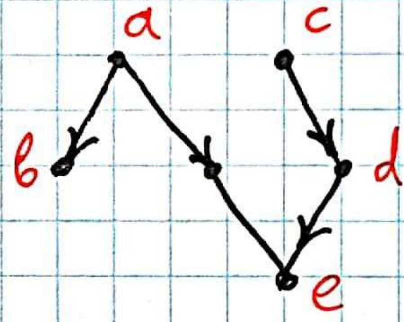
Ex

d=2



$a = (1,3)$
$b = (2,5)$
$c = (3,1)$
$d = (4,2)$
$e = (5,4)$

$(35124) \in S_5$

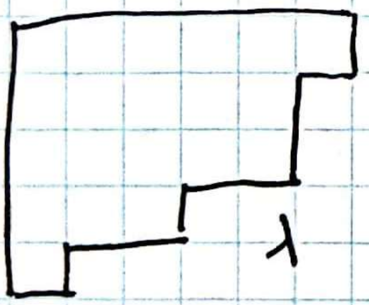


$\textcircled{\lambda}$

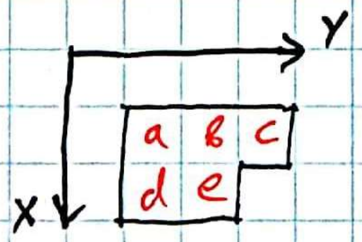
⑤ partitions, Young diagrams

$$\lambda = (\lambda_1 \dots \lambda_\ell) \vdash n, \quad \lambda_1 + \dots + \lambda_n = n, \quad \lambda_1 \geq \dots \geq \lambda_\ell$$

$\mathcal{P}_\lambda \leftarrow$ 2-dim. poset

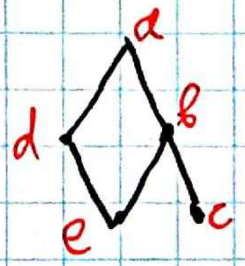


$\lambda = (6 5 5 3 1) \vdash 20$



$\lambda = (3,2)$

$a = (1,1)$
$b = (1,2)$
$c = (1,3)$
$d = (2,1)$
$e = (2,2)$



perm. posets

$$\sigma \in S_n \longrightarrow P_\sigma = (X, \precsim)$$

$$P_\sigma \leftarrow \text{2-dim}$$

Def chain $\leftarrow$ $x_1 < x_2 < \dots < x_\ell$, $x_i \in X$

$P=(X,<)$ antichain $\leftarrow$ $A \subseteq X$, $A = \{a_1 \dots a_\ell\}$

s.t. $\underbrace{a_i \nless a_j \, , \; a_j \nless a_i}_{\text{independent elt's}}$ $\forall i j$

Def $P=(X,<)$ $\leftarrow$ fin. poset

height of P $\leftarrow$ $\max\limits_{C\text{-chain}} |C|$

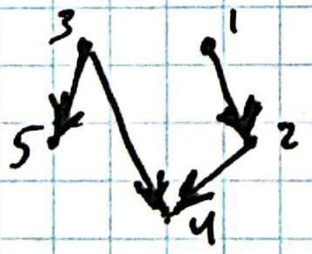
width of P $\leftarrow$ $\max\limits_{A\text{-antichain}} |A|$

Ex $\sigma \in S_n$, P_σ $\leftarrow$ perm. poset

height of P_σ = $LIS(\sigma)$ $\leftarrow$ size of longest increasing subs

width of P_σ = $LDS(\sigma)$ $\leftarrow$ —//— decreas. —//—

$\sigma = (35124)$



$LIS(\sigma) = 3$ $\leftarrow$ $\underline{124}$

$LDS(\sigma) = 2$ $\leftarrow$ $\underline{54}$

Dilworth Theorem

Prop $P = (X, \prec)$, heigth of $P = k$ [Jukna, Th 9.1]

Then $\left[\begin{array}{l} P \text{ can be partititioned into} \\ k \text{ antichains} \end{array}\right.$

$X = A_1 \sqcup \dots \sqcup A_k$ $\left\{\begin{array}{l} A_i \cap A_j = \emptyset \\ \cup A_i = X \end{array}\right.$

▷ height of $x \underbrace{:=}_{\max} |C|$, C – chain in P

$C = \{x_1 \prec \dots \prec x_\ell = x\}$

$1 \leq \text{height}(x) \leq k$

Let $A_i := \{x \in X, \text{heigth}(x) = i\}$, $1 \leq i \leq k$

observe: A_i ← antichain

$\left[\begin{array}{l} x \prec y, \quad x, y \in A_i \Rightarrow \text{height}(x) < \text{height}(y) \quad \otimes \end{array}\right.$

$\Rightarrow X = A_1 \sqcup \dots \sqcup A_k$ ← desired □

Q: What about partitions into chains?

Th [Dilworth, 1950]

$P = (X, <)$, $\mathrm{width}(P) = k$. *Then*

P can be *partitioned* *into* k *chains*

$$X = C_1 \sqcup \ldots \sqcup C_k \ , \quad \begin{cases} C_i \cap C_j = \emptyset & \forall i, j \\ \bigcup C_i = X \end{cases}$$

Robert P. Dilworth

(1914–93)

b. Riverside Cnty

students: Al Hales

Curtis Greene

(Caltech)

L2 Chains & Antichains (part 2)

Oct 5, 2020

Last lecture: $P = (X, <)$

chains — $x_1 < x_2 < \dots < x_\ell$

antichains — $A \subset X$, $x \not< y \quad \forall x, y \in A$

height — max chain size

width — max antichain size

Prop $P = (X, <)$, $\text{height}(P) = k$

Then P can be partitioned into k antichains

Th [Dilworth, 1950]

$P = (X, <)$, $\text{width}(P) = k$. Then

P can be partitioned into k chains

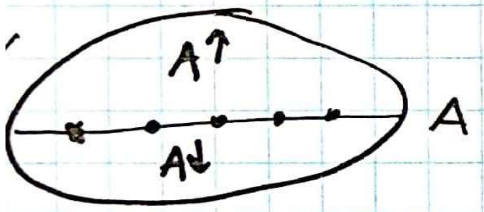
$X = C_1 \sqcup \ldots \sqcup C_k$, $\begin{cases} C_i \cap C_j = \emptyset & \forall i,j \\ \cup C_i = X \end{cases}$

Proof [Perles, 1963] Use induction on $n = |X|$ Base ✓ $n=1$ [West, 12.1.8]

$n>1$ $A \subset X$ ← largest antichain

$A^{\uparrow} = \{x \in X,\ x \preceq a \text{ some } a \in A\}$

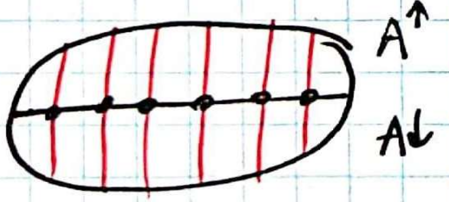
$A^{\downarrow} = \{x \in X,\ x \succeq a \text{ some } a \in A\}$



Obs $X = A^{\uparrow} \cup A^{\downarrow}$ /oth A not largest/

$A^{\uparrow} \cap A^{\downarrow} = A$

Case 1 $|A^{\uparrow}|, |A^{\downarrow}| < n$



← k chains

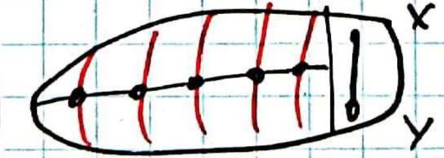
By ind. ✓

Case 2 $\left.\begin{array}{l} x \leftarrow \text{min elt} \\ y \leftarrow \text{max elt} \end{array}\right\}$ in P

$\forall A$ either $|A^{\uparrow}| = n$ or $|A^{\downarrow}| = n$

$\Rightarrow \text{width}(P') \leq k-1$

$P' := P - x - y$



✓ ▨

⑥

Cor 1 $\forall \sigma \in S_n : LIS(\sigma) \cdot LDS(\sigma) \geq n$

[Erdös-Szekeres theorem, 1935]

Cor 2 $\forall P = (X, \prec) : \text{width}(P)\ \text{height}(P) \geq n$

where $n = |X|$

D (Cor 2) By Dilworth thm $\exists\, C_1 \sqcup \dots \sqcup C_k = X$

$$\Rightarrow n = |X| \leq \text{width}(P) \cdot \max |C_i|$$
$$\leq \text{width}(P)\ \text{height}(P) \quad ▨$$

/ Also follows from Prop /

Now Cor 2 ⇒ Cor 1 for $P = P_\sigma$

since $\text{height}(P_\sigma) = LIS(\sigma)$

$\text{width}(P_\sigma) = LDS(\sigma)$

Cor 3 [= Hall's marriage thm, 1935]

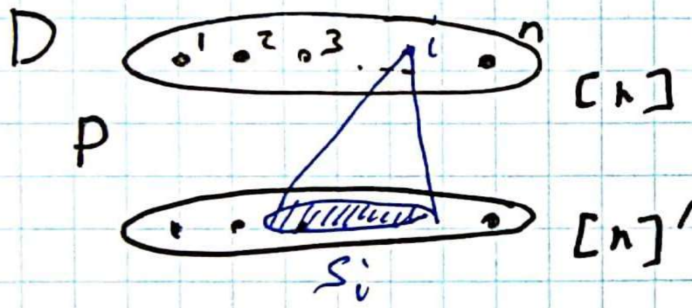
[Jukna, p. 98]

Let $S_1 \dots S_n \subseteq [n] = \{1 \dots n\}$

Suppose $\forall\, I \subseteq [n]$ we have:

$$\left| \bigcup_{i \in I} S_i \right| \geq |I|$$

Then $\exists$ bij $\pi : [n] \to [n]$ s.t. $\pi(i) \in S_i \;\forall i$

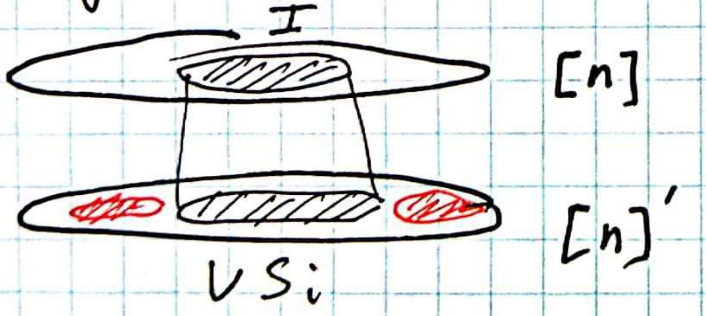


Γ – corresp. bipartite graph

Claim $\Leftarrow$ $\exists$ perfect matching in Γ

Obs width(P) $\geq n$ ✓

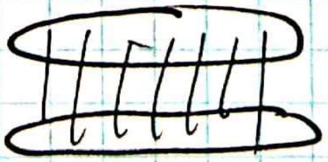
In fact width(P) $= n$



$|I| = k \Rightarrow |A| \leq k$

$I = A \cap [n]$ $\quad + \; n - k = n$

Now Dilworth Thm



$\Rightarrow \exists$ PM

▨

④

Ex Boolean lattice $B_n = (X, \leq)$

$X = 2^{[n]}$ ← all subsets of $[n] = \{1 .. n\}$

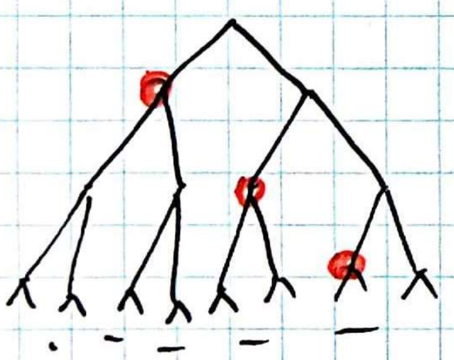
$\leq$ ← inclusion

$\text{height}(B_n) = n+1$

$\text{width}(B_n) \geq \binom{n}{\lfloor \frac{n}{2} \rfloor}$ / in fact $\textcircled{=}$ /

Non-ex (infinite posets) X – countable

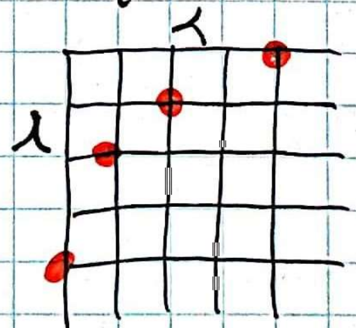
(1) inf. binary tree T_2



∃ ∞ anti-chain

$\text{width}(T_2) = \infty$

(2) quadrant $\mathbb{N}^2$



width = ∞

∄ ∞ anti chain

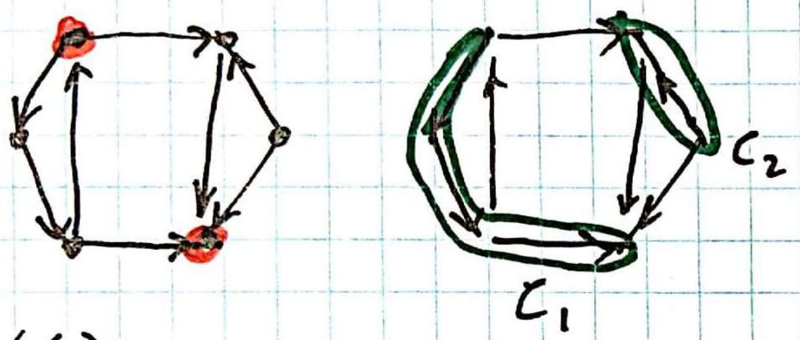
/ Hilbert's Basis Thm /

⑤

Graph theory generalization

[West, p.540]

$G=(V,E)$ — directed graph



$\alpha(G)$ = max size of indep set in G

$V = C_1 \sqcup \ldots \sqcup C_m$

directed path part'n

/also path cover/

$\alpha(G)=2$

Th [Gallai-Milgram, 1960]
$\forall$ directed $G=(V,E)$ $\exists$ directed path partition into $\leq \alpha(G)$ paths.

Obs: G-M thm $\Rightarrow$ Dilworth thm.

D $P=(X,<)$, take $G=(X,E)$
where $E=\{(x,y),\ x,y \in X,\ x<y\}$. ▨

/ comparability graph /

⑥

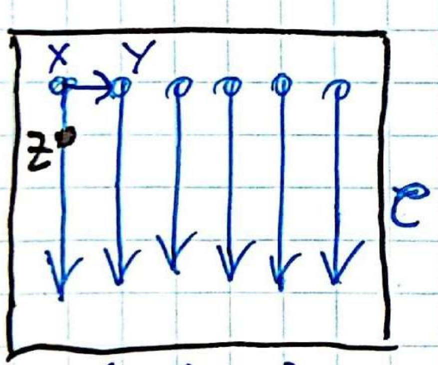
Proof of G-M thm (by ind.)

[West, p. 540]

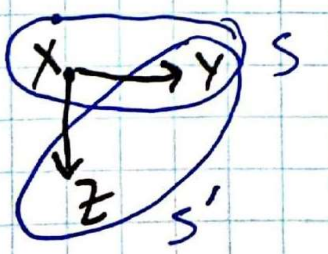
⊛
$\forall\ \mathcal{C} = \{C_1, \ldots, C_k\}$ ← directed path part'n
$\forall\ k > \alpha(G)$ $\quad S \leftarrow \{s(C_1) \ldots s(C_k)\}$
set of sources of $\mathcal{C}$
$\exists\ \mathcal{C}' = \{C_1' \ldots C_{k-1}'\}$ ← directed path part'n.
s.t. $\{s(C_1'), \ldots s(C_{k-1}')\} \subset S$.

/stronger ind. assumption/

$n=1$ ✓ | $n>1$, $k > \alpha(G) \Rightarrow \exists (x\ y) \in E$
$x, y \in S$



G=(V,E)



Let $\mathcal{C}' = \{z\to, y\to, \ldots\}$
$S' = \{z, y, \ldots\}$, $|S'| = k$
dir. path part'n for $G' = G - x$

By ⊛ $\exists\ \mathcal{C}''$ w/ $S'' \subset S'$ and
$|S''| < k \Leftarrow \alpha(G') \le \alpha(G)$

① y or $z \in S''$
→ attach $(x y)$ or $(x z)$ ✓

② $y, z \notin S'' \Rightarrow |S''| \le k-2$
add (x) to $\mathcal{C}''$ ▨

⑦

Cor $G = (V, E) \leftarrow$ tournament (directed K_n)

Then G has a Hamiltonian path

D $\alpha(G) = 1$ ▨

⑧

L3 Chains & Antichains (cont'd)

Oct 7, 2020
206A

Recap: Th [Dilworth] $P = (X, \le)$, width $(P) = k$

$\Rightarrow \exists$ partition of P into k chains

Th [Gallai-Milgram] $G = (V, E)$ digraph

$\alpha = \alpha(G)$ – max size independent set

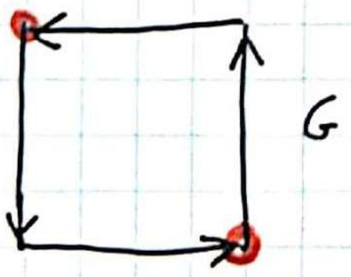
$\Rightarrow \exists$ directed path partion of G
into $\le \alpha$ path

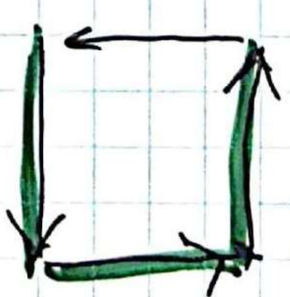
Cor G - tournament (orientation of K_n)

$\Rightarrow G$ has a Hamiltonian path

Note: $\alpha(G)$ is NOT always tight

Ex:





$\alpha(G) = 2$

①

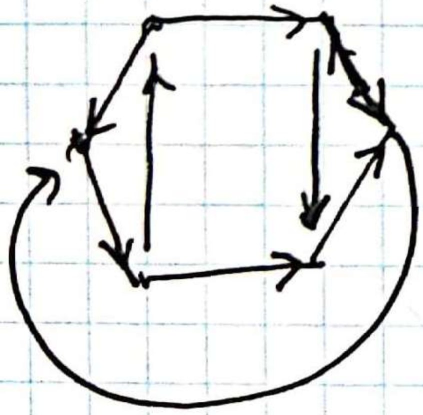
Application : [Gallai's Conj., 1963]

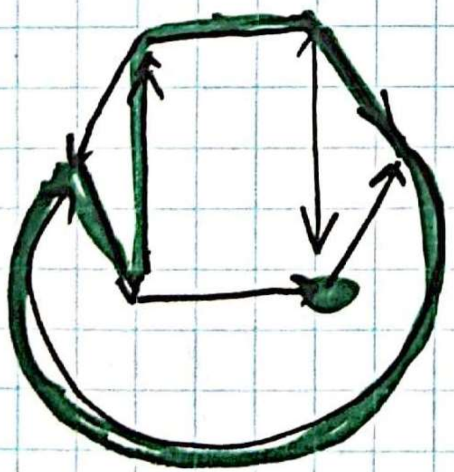
Th [Bessy - Thomassé, 2007]

$G=(V,E)$ ← strongly conn digraph, $k=\alpha(G)$

∃ directed circuit cover w/ k circuits:

$G = C_1 \cup \dots \cup C_k$ /← not necess. disjoint/





Proof : G-M ⊕ 3pp.

/see add. reading/

Cor Every strongly conn. tournament has a Ham. cycle

Def $\forall\ P=(X,\prec)$ let

$G=(X,E)$, $E=\{(x,y),\ x\prec y \mid x,y\in X\}$

comparability graph

$H=(X,E')$, $E'=\left\{\begin{array}{l}(x,y),\ x\prec y \mid x,y,z\in X \\ \text{s.t. no } x\prec z\prec y\end{array}\right\}$

Hasse diagram

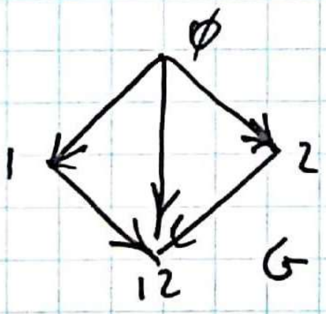
$D=(X,E'')$, $E''=\left\{\begin{array}{l}(x,y),\ x,y\in X \\ \text{s.t. } x\not\prec y,\ y\not\prec x\end{array}\right\}$

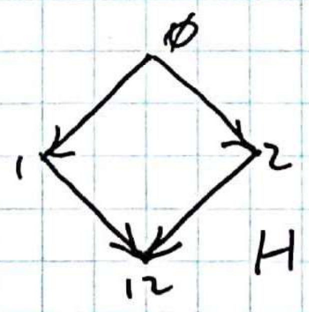
incomparability graph

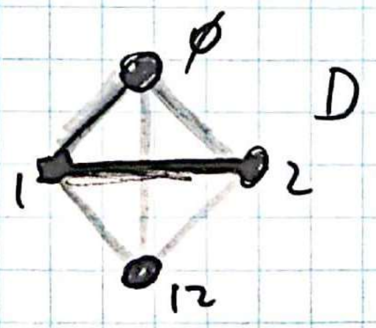
Note: $G, H \leftarrow$ graphs, digraphs /not D/

$D=\overline{G} \leftarrow K_n = G \sqcup D$, $n=|X|$

Ex $P=B_2$







GM ⇒ Dil.

Chains & Antichains in Boolean Lattice

$B_n = (2^{[n]}, \prec)$

$2^{[n]} \leftarrow$ all subsets of $[n]$ $\{1, \ldots, n\}$

$\prec = \subseteq$

OBS

1) maximal chains := chains in B_n which cannot be enlarged

maximal chains in P $\longleftrightarrow$ largest chains in B_n

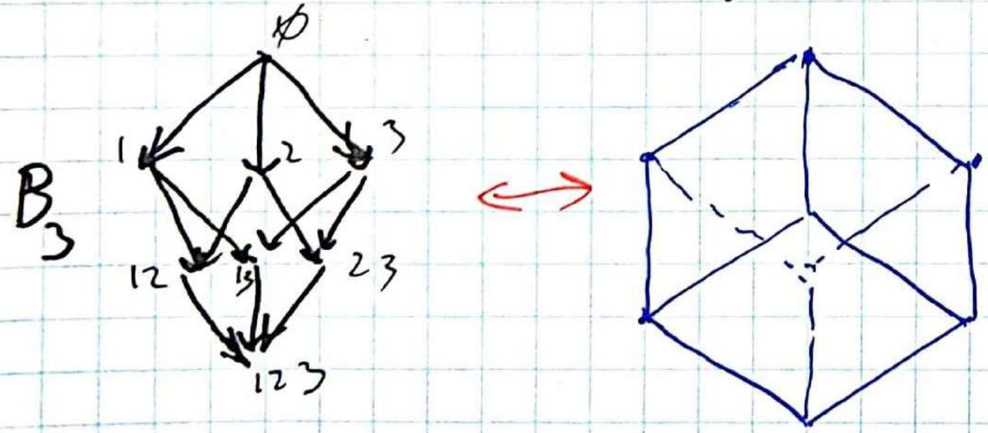
$\longleftrightarrow$ longest paths in Hasse diag

2) max chains in B_n $\longleftrightarrow$ S_n, so that

$$\sigma = (\sigma(1), \sigma(2) \ldots \sigma(n)) \Longleftrightarrow \left[\emptyset \to \{\sigma(1)\} \to \{\sigma(1), \sigma(2)\} \to \cdots \to [n]\right]$$

max chains in $B_n = n!$

3) Hasse diag of $B_n \simeq$ graph of a hypercube $[0,1]^n$



Q: # chains = ??

/all chains/

2, 4, 14, 76

Chains in $\mathcal{B}_n \leftrightarrow \left[\emptyset \to S_1 \to S_2 \to \cdots \to S_\ell = [n] \right]$

$$S_i \subseteq [n], \quad S_i \subsetneq S_{i+1}$$

$$\leftrightarrow (Y_1, Y_2, \dots, Y_\ell), \quad Y_i \subseteq [n]$$

$$Y_i \cap Y_j = \emptyset, \quad Y_1 \cup Y_2 \cup \dots = [n]$$

$Y_i = S_i \setminus S_{i-1}$

Obs $\forall\ m_1 + \dots + m_\ell = n \qquad \# \bar{Y} = \binom{n}{m_1 \dots m_\ell}$

$$|Y_i| = m_i, \quad \bar{Y} = (Y_1 \dots Y_\ell)$$

$$\Rightarrow a_n := \#\text{ chains in } B_n = \sum_{m_1 + \dots + m_\ell = n} \binom{n}{m_1 \dots m_\ell}$$

Obs $\forall\ P = (X, \prec), \quad |X| = N, \quad \text{height}(X) = h$

$$\#\text{ chains in } P \le \#\text{ max chains} \cdot 2^h$$

$$\Rightarrow P = B_n, \quad N = 2^n, \quad \#\text{ max chains} = n!$$

$$h = n+1$$

$$n! \le a_n \le n!\, 2^{n+1}$$

$$\left.\begin{aligned} a_n &\le \left(\frac{n}{e}\right)^n 2^n\, \Theta(\sqrt{n}) \\ &\ge \left(\frac{n}{e}\right)^n \Theta(\sqrt{n}) \end{aligned}\right\} \Rightarrow \begin{aligned} a_n &= e^{n \log n + O(n)} \\ \log a_n &= n \log n + O(n) \end{aligned}$$

⑤

$$\frac{a_n - 1}{n!} = \sum_{m_1 + \dots + m_\ell = n} \frac{1}{m_1!\, m_2! \dots m_\ell!}$$

$n>0,\ \ell>0$

$\boxed{m_i \geq 1\ \forall i}$

non-empty chains

$$= [t^n] \sum_{\ell=0}^{n} \prod_{i=1}^{\ell} \left(\frac{t}{1!} + \frac{t^2}{2!} + \dots \right)$$

$$= [t^n] \sum_{\ell=0}^{\infty} \left(e^t - 1\right)^\ell$$

$$= [t^n] \frac{1}{1-(e^t-1)} = [t^n] \frac{1}{2-e^t}$$

$$A(t) = \sum \frac{a_n t^n}{n!} = \frac{1}{2-e^t}$$

e.g.f

$\Rightarrow$ comp. anal. $\quad a_n \sim \dfrac{n!}{2(\log 2)^{n+1}}$

$$a_n \sim \frac{1}{2\log 2}\, n!\, \underbrace{(\log_2 e)}_{\approx 1.44}{}^{n}$$

Fubini numbers

1, 3, 13, 75, 541, 4683

$\boxed{A' = 2A^2 - A}$ $\leftrightarrow$ $a_n = \sum_{k\geq 1}^{n} \binom{n}{k} a_{n-k}$

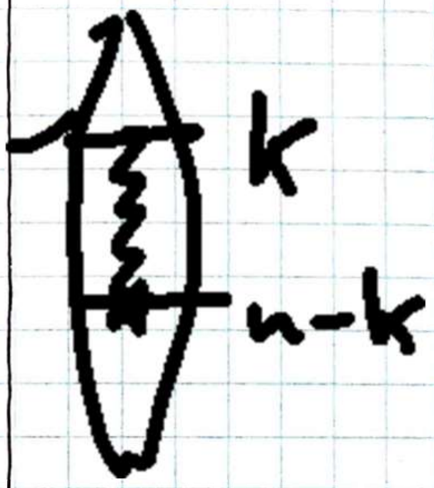
⑥

Prop width$(B_n) = \binom{n}{\lfloor \frac{n}{2} \rfloor}$ ⇐ LYM ineq.

Def $P = (X, <)$, $C = [\,x_1 < x_2 < \ldots < x_\ell\,]$ chain

$C \leftarrow$ saturated if $C \leftarrow$ max between x_1 & x_ℓ

$\Leftrightarrow$ $C \leftarrow$ directed path in Hasse diag H of P



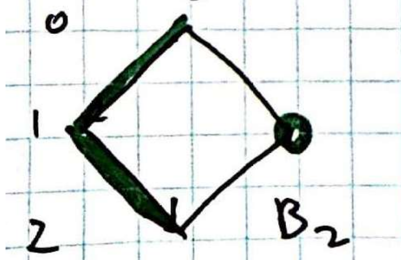
G-M Thm: $\forall P$ can be partitioned into $\leq$ width(P) sat. chains ← NOT

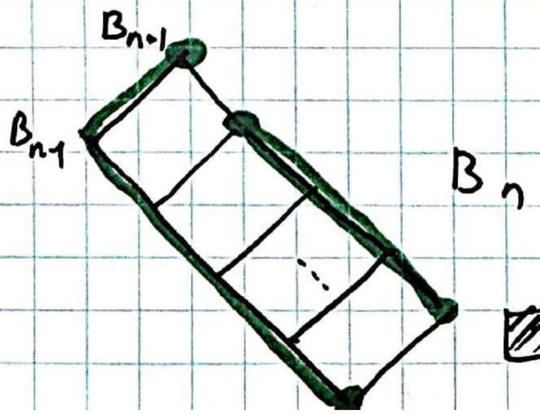
width$(P) \boxed{\leq}$ $\alpha(H)$ not always

Th [Greene-Kleitman, 1976] B_n can be part'd into $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ symmetric saturated chains

symmetric ← from height k to $n-k$

Proof: By ind.





▨

$\Rightarrow$ Prop

$\square$ width $\leq (\ )$ by GK

$\geq (\ )$ via middle level ▨

⑦

L4 Chains & Antichains (cont'd)

206A

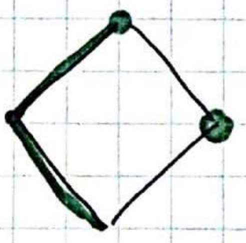
Oct 9, 2020

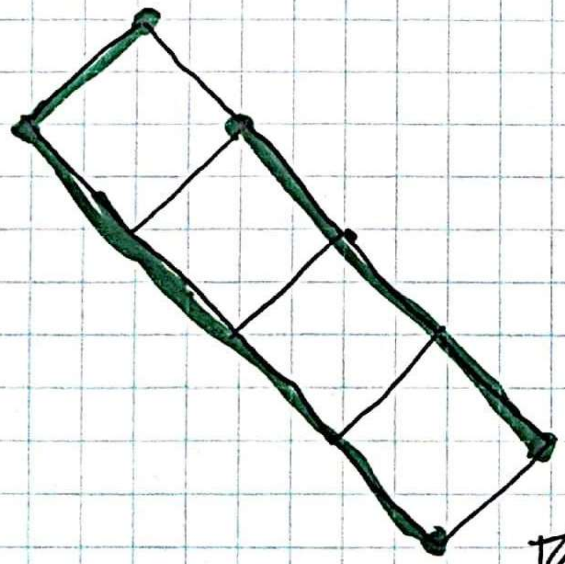
Last time:

Th [de Bruijn-Tengbergen-Kruyswijk, 1951] [West, p.546]

Boolean Lattice $B_n = (2^{[n]}, \subset)$ has a symmetric saturated chain decomposition

Proof: By induction





▨

$\Rightarrow$ width $(B_n) \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$

Q: can this be made efficient?

"effective"?

"poly-time computable?

①

Formal Q: can one compute $S \to S'$ $\forall S \subset 2^{[n]}$ in poly time?

Th [Greene–Kleitman, 1976] yes

[GK, p 30]

D $S = \{1\ 3\ 4\ 7\ 8\} \subset 2^{[11]}$, $n=11$

$$\begin{array}{ccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
\hline
) & (&) &) & (& (&) &) & (& (& (
\end{array} \leftarrow S$$

$$)\ \underbrace{(\ \ )}\ )\ \underbrace{(\ \underbrace{(\ \ )}\ )}\ (\ (\ ($$

$$C \left\{ \begin{array}{lllllll}
(& & (& & (& (& (\\
) & & (& & (& (& (\\
) & -//- &) & -//- & (& (& (\\
) & &) & &) & (& (\\
) & &) & &) &) & (\\
) & &) & &) &) &)
\end{array} \right.
\begin{array}{l}
\\
\\
\leftarrow S \\
\leftarrow S' \\
\\
\\
\end{array}$$

▨

②

Cor Binomial coeff are unimodal:

$$\binom{n}{0} \le \binom{n}{1} \le \binom{n}{2} < \ldots \le \binom{n}{\lfloor \frac{n}{2} \rfloor} \ge \ldots \ge \binom{n}{n}$$

Def $P=(X,<)$ is ranked if

$\forall\, x, y \in X$, $(xy) \in H$, $\text{height}(x) = \text{height}(y) - 1$

/ usually $rk(\hat{0}) = 0$, so $rk(x) := \text{height}(x) - 1$ /

Def $P=(X,<)$ has Sperner property

if $\max rk = \text{width}(P)$

/ $\iff$ $|\{x \in X,\ rk(x) = k\}| = \text{width}(P)$ some k

$\iff$ $\{x \in X,\ rk(x) = k\}$ – largest antichain /

Def $P=(X,<)$ ranked has strong Sperner property

if [later] .

③

Def $P=(X,<)$ - ranked poset, mz [redacted]

$P_k := \{x \in X, rk(x)=k\}$ level set

$m = \max_k |P_k|$ middle rank

Def $P \leftarrow$ symmetric chain order if

$\exists\ P = C_1 \sqcup C_2 \sqcup \dots \sqcup C_d$, $d = width(P)$

$C_i \leftarrow$ symmetric saturated chain

$C_i = \{x_0 < x_1 < \dots < x_\ell\}$ s.t.

$rk(x_0) = \underline{m} - k$, $rk(x_\ell) = \underline{m} + k$
some k

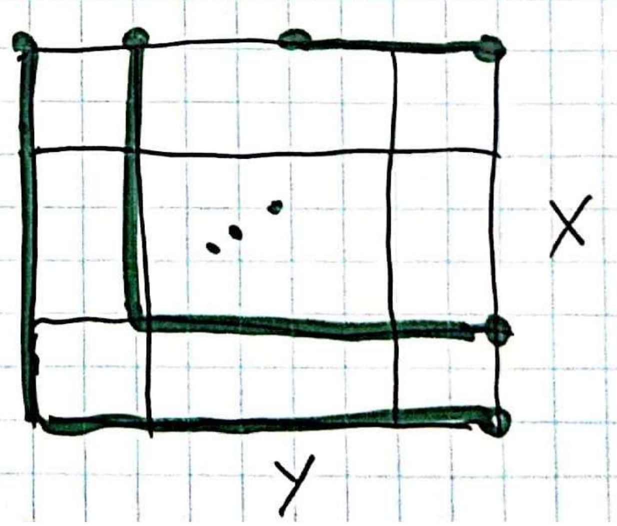
Th $P, Q \leftarrow$ symmetric chain orders
[Katona '72] $\Rightarrow$ so is $P \times Q$.

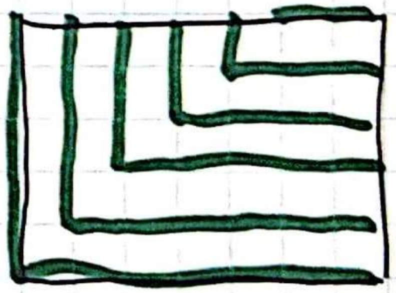
$P = (X, <)$, $Q = (Y, <')$

$P \times Q := (X \times Y, \underbrace{< \times <'}_{\triangleleft})$

$(x,y) \triangleleft (x',y')$

$\begin{cases} \text{if } x < x' \\ \quad y <' y' \end{cases}$





④

Cor $\forall\ m_1 \dots m_\ell$ $\quad \prod_{i=1}^{\ell} \underbrace{(1+t+\dots+t^{m_i-1})}_{\shortparallel\ [m_i]_t}$ is unim

[<u>unimodal</u>: $\quad f(t) = a_0 + a_1 t + a_2 t^2 + \dots + a_n t^n$

$$a_0 \le a_1 \le \dots \le a_\ell \geq a_{\ell+1} \geq \dots \geq a_n$$

Cor $(n!)_q = \prod_{i=1}^{n} (1+q+\dots+q^{i-1}) = \sum_{\sigma \in S_n} q^{inv(\sigma)}$

is unimodal

Exc - Generalize GK bracket sequences

- find a <u>combin.</u> <u>interpretation</u>

for $\left[\begin{array}{l} \#\{\sigma \in S_n,\ inv(\sigma) = k+1\} \\ - \#\{\sigma \in S_n,\ inv(\sigma) = k\} \end{array}\right.$ $\quad k \le \frac{1}{2}\binom{n}{2}$

Def int $a \in \mathbb{N}$ has a <u>comb. interpretation</u>

if $\exists\ \mathcal{A},\ |\mathcal{A}| = a,\ \mathcal{A} \subseteq \{0,1\}^N$ ← effective

---

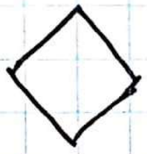
Def int. seq. $\{a_n\}$ —//— $\left.\begin{array}{l} \\ \text{if } \exists\ \mathcal{A}_n,\ |\mathcal{A}_n| = a_n \end{array}\right\}$ $\oplus$ membership $x \in \mathcal{A}_n$ can be done in poly-time

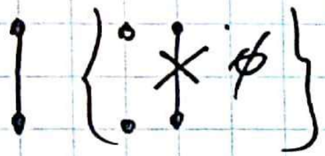
⑤

Counting Antichains in B_n

$a_n :=$ # antichains in $B_n = (2^{[n]}, \subset)$

---

n = 0 $\{\cdot \ \emptyset\}$ $a_0 = 2$ n = 2  $a_2 = 6$

n = 1  $a_1 = 3$

n - even

Obs $a_n \le 2^{2^n}$, $a_n \ge 2^{\binom{n}{n/2}}$

$\log a_n \le 2^n (\log 2)$, $\log a_n \ge \frac{2^n}{\sqrt{n}} \Theta(1)$

Prop $(\log a_n) = 2^n O\left(\frac{\log n}{\sqrt{n}}\right)$

▷ By Dilworth th $\exists\ B_n = \bigsqcup_{i=1}^{N} C_i$, $N = \binom{n}{n/2}$

each chain C_i has $|C_i|+1$ choices

$\Rightarrow a_n \le \prod_{i=1}^{N} \left(|C_i|+1\right) \le (n+2)^{\binom{n}{n/2}}$

$\log a_n \le \binom{n}{n/2} \log(n+2) = 2^n O\left(\frac{\log n}{\sqrt{n}}\right)$ ▨

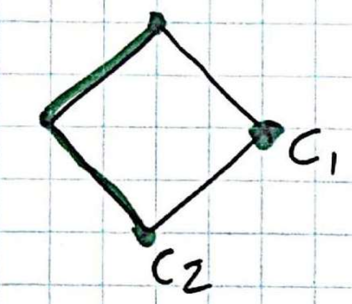
⑥

Th [Hansel, 1966]

$a_n \le 3^{\binom{n}{\lfloor n/2 \rfloor}}$ | $\Rightarrow \log a_n = 2^n O(\frac{1}{\sqrt{n}})$

D

B_2



2 choices for C_1

3 choices for C_2

$\underline{a_2 \le 6}$
$< 3^2$ ✓

In general $a_n \le 3 a_{n-1}$

Take GK bracket seq for B_n

$\boxed{C}$ $(\; (\; (\ldots (\;\to\;) \; (\; (\ldots (\;\to\;)) \; (\ldots (\;\to \ldots \to\;)) \ldots)$

$\boxed{C^*}$ $\underbrace{(\;)} \; (\ldots (\;\to\; \underbrace{(\;)} \;) \ldots (\to \ldots$

$\boxed{C^{**}}$ $(\; (\; (\quad \underbrace{(\;)} \;\to\;) \; (\; (- \underbrace{(\;)} \to \_\_\_$

Known choices for C^*, C^{**} $\Rightarrow$ at most 3 choices for C

∎

⑦

L5 Chains & Antichains (cont'd)

206 A

Oct 12, 2020

Greene-Kleitman bracket sequences

$S = \{1\,3\,4\,7\,8\} \subset [11]$, $n = 11$

1	2	3	4	5	6	7	8	9	10	11	
)	(	)	)	(	(	)	)	(	(	(	← S

$S \in C$ ← chain

(	(	)	(	(	(	)	)	(	(	(	
)	(	)	(	(	(	)	)	(	(	(	
)	(	)	)	(	(	)	)	(	(	(	← S
)	(	)	)	(	(	)	)	)	(	(	
)	(	)	)	(	(	)	)	)	)	(	
)	(	)	)	(	(	)	)	)	)	)	

①

L [G-K] Chains formed by Bracket sequences give a symm. sat. chain partition of B_n

D (sketch)

1) every $S \in$ some chain
2) no two chains intersect
3) every chain is sym & sat ▨

Prop Antichains in B_n are in bijection w/ $f: B_n \to \{0,1\}$ s.t. $f(X) \leq f(Y)$ $\forall X \prec Y$

D $\forall f$ as -//- let $A_f := \{ X \in B_n$ s.t. $f(X)=0,\ f(Y)=1\ \forall X \prec Y \}$

then $A_f \leftarrow$ antichain.

$\forall A \leftarrow$ antichain in B_n let $f: B_n \to \{0,1\}$

$f(X) = 0$ if $\exists Y \in A$, $X \preccurlyeq Y$

$f(X) = 1$ oth. ▨

③

Note $\begin{matrix} x = & * & * & --- & (& -- & * \\ y = & * & * & --- &) & --- & * \end{matrix} \Big\}$ $xy \in H_n$ Hasse diag of B_n

---

Th [Hansel, 1966]

$a_n :=$ # antichains in $B_n \leq 3^{\binom{n}{\lfloor n/2 \rfloor}}$

[West, Thm 12.2.14]

D (by induction) Base ✓

n > 1 $\boxed{a_n \leq 3 a_{n-1}}$

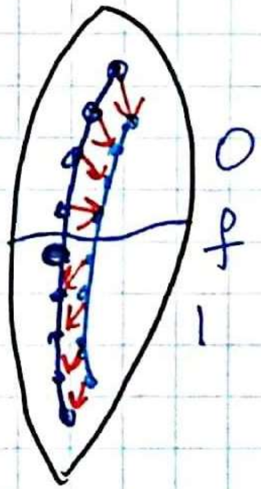
Claim ∀ chain C in GK bracket construction there are ≤ 3 choices of f given all ~~f~~ values of f on smaller chains C'

D (by ind)

$C = \{) \ ) \ ...) \ ) \ (\ (.... (\ (\}$

$C' = \{) \ ) ...) \ \underbrace{(\ )}, (\ ... (\ (\}$

f on C' det half of f on C



LYM property

[L = Lubell, 1966]
[Y = Yamamoto, 1954]
[M = Meshalkin, 1963]

Th $\mathcal{A} \subset B_n$ antichain

then $\sum_{A \in \mathcal{A}} \binom{n}{|A|}^{-1} \leq 1$ ← LYM inequality

---

D $\forall A \in \mathcal{A}$ let $\mathcal{C}_A = \{ \underset{A \subset C}{C} \leftarrow \text{max chain in } B_n \}$

clearly $\forall A, A' \in \mathcal{A}$ $\quad \mathcal{C}_A \cap \mathcal{C}_{A'} = \phi$

$| \mathcal{C}_A | =$ [illegible] $|A|! \, (n - |A|)!$

$\Rightarrow \sum_{A \in \mathcal{A}} |A|! \, (n - |A|)! \leq n!$ ▨

Obs LYM Th $\Rightarrow$ $\text{width}(B_n) \leq \binom{n}{\lfloor n/2 \rfloor}$

D $\binom{n}{|A|} \leq \binom{n}{\lfloor n/2 \rfloor} \Rightarrow 1 \geq \sum_{A} \binom{n}{|A|}^{-1} \geq \text{width}(B_n) \binom{n}{\lfloor n/2 \rfloor}^{-1}$

▨

④

Cor width$(B_n) = \binom{n}{\lfloor n/2 \rfloor}$ ✓ Sperner property

Lattice of subspaces of $V = \mathbb{F}_q^n$

$\mathcal{F}_n := \left(\{ W \subseteq V \} , \subset \right)$

Th $\mathcal{F}_n$ has Sperner property

width $= \# \{ W \subset V , \dim W = \lfloor \frac{n}{2} \rfloor \}$

Prop1 # k-subspaces of $V = \binom{n}{k}_q = \dfrac{(n!)_q}{(k!)_q (n-k)!_q}$ [Stanley, §1.7]

where $(n!)_q = (1)_q (2)_q \cdots (n)_q$

and $(m)_q = \dfrac{q^m - 1}{q - 1} = 1 + q + \dots + q^{m-1}$

Prequel: # k-subsets of $[n] = \{1 .. n\} = \binom{n}{k}$

▷ $S_k \times S_{n-k}$ act on $S_n = \{ \sigma = (\sigma(1), \dots, \sigma(n)) \}$

action ← free, orbits ⟷ k-subsets ⇒ # $\dfrac{|S_n|}{|S_k \times S_{n-k}|}$ ▨

⑤

Proof of Prop1 $G_n = GL(n, \mathbb{F}_q)$

$$|G_n| = (q^n-1)(q^n-q)\cdots(q^n-q^{n-1})$$

$$\begin{array}{|cccc|l}
\hline
* & * & - & * & v_1 \\
\hline
* & * & -- & * & v_2 \\
\hline
-- & & & - & \\
\hline
* & * & -- & * & v_n \\
\hline
\end{array}$$

$|G_n| =$ # n-subspaces w/ $\underbrace{\text{marked basis}}_{\{v_1 \cdots v_n\}}$

$G_n(k) :=$ k-subspaces w/ marked basis

$$|G_n(k)| = (q^n-1)(q^n-q)\cdots(q^n-q^{k-1})$$

G_k acts freely on $G_n(k)$

orbits $\longleftrightarrow$ k-subspaces of V

$$\Rightarrow \text{\# k-subspaces} = \frac{|G_n(k)|}{|G_k|} = \frac{(q^n-1)(q^n-q)\cdots(q^n-q^{k-1})}{(q^k-1)(q^k-q)\cdots(q^k-q^{k-1})}$$

$$= \frac{(q^n-1)\cdots(q^n-q^{k-1})\ (q^n-q^k)\cdots(q^n-q^{n-1})}{(q^k-1)\cdots(q^k-q^{k-1})\quad —\ (1 —}$$

$$= \frac{(n!)_q}{(k)_q\ (n-k)_q} = \binom{n}{k}_q$$

⑧

Prop 2 ∀ antichain $\mathcal{A} \subset \mathcal{F}_n$

$$\sum_{A\in\mathcal{A}} \binom{n}{|A|}_q^{-1} \le 1$$

D max chains ⟷ complete flags of subspaces $V_0 \subset V_1 \subset V_2 \cdots \subset V$

$\#$ max chains $= (n!)_q$

$\#$ max chains containing W, $\dim W = k$

$$= (k!)_q \, (n-k)!_q$$

⟹ q-LYM $$\sum_{A\in\mathcal{A}} |A|!_q \, (n-|A|)!_q \le (n!)_q$$

☒

Prop 3 $\forall\, q \geq 2$

$$\binom{n}{0}_q \le \binom{n}{1}_q \le \dots \le \binom{n}{\lfloor n/2 \rfloor}_q$$

D $$\binom{n}{k}_q = \binom{n}{k-1}_q \cdot \frac{(n-k)_q}{(k)_q} \geq \binom{n}{k\,2}_q$$ ☒

⑦

L6 # Applications of Chains & Antichains

206A

Oct 14, 2020

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① Gray codes & Universal sequences

Def A cyclic seq $\bar{a} = (a_1, a_2 \dots a_N)$, $a_i \in \{0,1\}$ is a n-Gray code if $\forall (\alpha_1, \dots \alpha_n) \in \{0,1\}^n$ $\bar{a}$ contains $(\alpha_1, \dots \alpha_n)$

c.f. de Bruijn sequnces

Obs 1) trivial Gray code has length $N = n2^n$

2) every Gray code — // — $N \geq 2^n$

Th/Prop $\exists$ Gray code of length $N = 2^n$

D $\Gamma_n = (V, E)$, $V = \{0,1\}^{n-1}$, $E = \{(v, v')\}$

$v = (\alpha_1, \alpha_2 \dots \alpha_{n-1})$, $v' = (\alpha_2 \dots \alpha_{n-1}, \alpha_n) \in V$

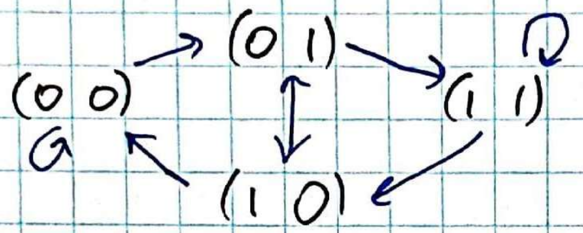
Obs1 $\text{indeg}(v) = \text{outdeg}(v) = 2 \quad \forall v \in V$

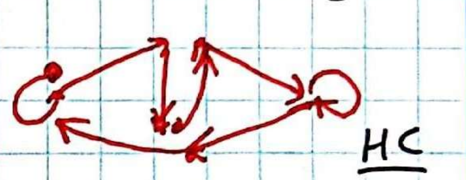
Euler th $\Rightarrow$ Γ_n has a Hamiltonian circuit

ⓡ

Obs 2 Ham. circuits in Γ_n $\leftrightarrow$ n-Gray code of length 2^n

Ex $n=3$ Γ_2 :





(comp. aspects)

3-Gray code $\underbrace{\overbrace{0\,0\,0}\,1}\,0\,1\,1\,1$ ▨

Note Using BEST thm #nGray codes = product formula

Def Seq $\bar{a} = (a_1 \dots a_N)$ is n-universal if $a_i \in [n] = \{1 \dots n\}$ and $\bar{a}$ contains every subset $X \in 2^{[n]}$

[Jukna §9.1.2]

Ex $\bar{a} = (1\,2\,3\,4\,5\,1\,2\,\underbrace{4\,1\,3}\,5\,2\,4)$ is 5-universal

$N = 13$, e.g. $X = \{1,3,4\} \in 2^{[5]}$

Obs 1) $N \geq \binom{n}{\lfloor n/2 \rfloor}$ since $\lfloor \frac{n}{2} \rfloor$-elt subsets must start at diff place $\Big| \sim \sqrt{\frac{2}{\pi n}}\, 2^n$

2) $N = 2^{n-1} n$ works ✓

$$\sum_{k=0}^{n} k \binom{n}{k} = n\, 2^{n-1}$$

②

Th [Lipski, 1978]

$\exists$ n-univ. sequence of length $N \leq \left(\frac{2}{\pi}\right) 2^n$

D $\underline{n = 2k}$ $[n] = S \sqcup T$, $S = \{1 .. k\}$, $T = \{k+1 ... 2k\}$

$B_S = (2^S, \subset)$, $B_T = (2^T, \subset)$ $\leftarrow$ Boolean lattices

$B_S = C_1 \sqcup .. - \sqcup C_m$, $B_T = D_1 \sqcup .. - \sqcup D_m$, $m = \binom{k}{\lfloor k/2 \rfloor}$

s.t. C_i, D_j $\leftarrow$ sym, saturated. $\leftarrow$ BKT (or GK)

$C_i : \{X_1\, X_2 .. X_a\} \to \{X_1, X_2 .. X_{a+1}\} \to ... \to \{X_1\, X_2 ... X_b\}$

$\to$ $\overrightarrow{C_i} := (X_1 X_2 ... X_b)$ seq in $[S]^*$

$D_j : \{Y_1\, Y_2 .. Y_a\} \to \{Y_1\, Y_2 .. Y_{a+1}\} \to .. \to \{Y_1\, Y_2 .. Y_b\}$

$\to$ $\overleftarrow{D_j} := (Y_b ... Y_2\, Y_1)$ seq in $[T]^*$

Let $\bar{a} := \overleftarrow{D_1}\, \overrightarrow{C_1}\; \overleftarrow{D_1}\, \overrightarrow{C_2} \;...\; \overleftarrow{D_j}\, \overrightarrow{C_i} \;...\; \overleftarrow{D_m}\, \overrightarrow{C_m}$

Obs every $X \cup Y$, $X \subseteq 2^S$, $Y \subseteq 2^T$ is in some $\overleftarrow{D_j}\, \overrightarrow{C_i}$. Also $|\bar{a}| \leq k\, m^2 \sim \left(\frac{2}{\pi}\right) 2^n$ ▨

③

② Extremal combinatorics

Th [Littlewood - Offord '1943, Kleitman '1970]

$v_1 \ldots v_n \in \mathbb{R}^d$, $\|v_i\| \geq 1$ ← vectors

$K \subset \mathbb{R}^d$ ← open region s.t. diam $K < 2$

Then $\# \{ \varepsilon_1 v_1 + \ldots + \varepsilon_n v_n \in K, \ \varepsilon_i \in \{0,1\} \} \leq \binom{n}{\lfloor n/2 \rfloor}$

[Aigner - Ziegler, Proof from the Book]

cf. random subproducts [Math 285, Spring '2

OBs $K = \{a\}$, $n = 2k$, $v_1 = \ldots = v_n = 1$, $a > 0$, $d = 1$

$\Rightarrow \#\{ \ \} = \binom{n}{n/2} \Rightarrow$ Th is tight.

Prop Th holds for $K = \{a\}$, $\forall d \geq 1$ $a, w_i > 0$

▷

OBs $\{ \bar{\varepsilon} = (\varepsilon_1 \ldots \varepsilon_n) \text{ s.t. } \varepsilon_1 w_1 + \ldots + \varepsilon_n w_n = 0 \}$

is an antichain in $B_n \Rightarrow \#\{ \ \} \leq \binom{n}{\lfloor n/2 \rfloor}$ ▨

Note $\|V_i\| \geq 1$ & diam $K < 2$ is a geom. generalization of width $B_n = \binom{n}{\lfloor n/2 \rfloor}$

Th [Bollobás, 1965]

Let $A_1 \ldots A_m$, $B_1 \ldots B_m \subseteq [n] = \{1 \ldots n\}$

s.t. $A_i \cap B_j = \emptyset$ if and only if $i = j$

Then $\sum_{i=1}^{m} \binom{|A_i| + |B_i|}{|A_i|}^{-1} \leq 1$ ⊛

Cor $|A_i| = a$, $|B_j| = b \Rightarrow m \leq \binom{a+b}{a}$

Note Cor has LA proof even under weaker assumpt: $A_i \cap B_i = \emptyset \ \forall i$, $A_i \cap B_j \neq \emptyset$ $\forall i < j$

[Babai-Frankl Book]

D (Bollobás thm) $a_i := |A_i|$, $b_j := |B_j|$

[Jukna, §9.2.2]

induction on n $n = 1$ ✓

n > 1 $\mathcal{F}_k \leftarrow \{(A_i, B_i - k), k \in A_i, i \in [m]\} \subsetneq [n]$

$\Rightarrow \forall k$ we have ⊛. Sum them over $[n]$.

$$\Rightarrow \sum_k \sum_i = \sum_i \underbrace{(n - a_i - b_i)\binom{a_i + b_i}{a_i}^{-1}}_{k \notin A_i \cup B_i} + \sum_i b_i \underbrace{\binom{a_i + b_i - 1}{a_i}^{-}}_{k \in B_i} \leq n$$

⑤

$$\Rightarrow \sum_{K}\sum_{i} = \sum_{i} n \binom{a_i+b_i}{a_i}^{-1} \leq n \Rightarrow \text{thm.} \quad ▨$$

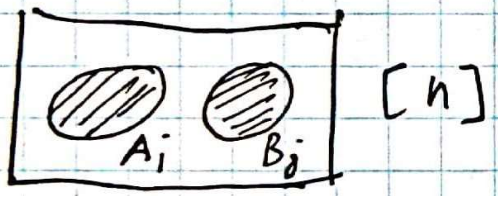
Obs [Tuza, 1984]

Bollobás thm $\Rightarrow$ LYM ineq.

D $\mathcal{A} = \{A_1, \dots, A_m\}$, $A_i \in 2^{[n]}$ antichain

Let $B_i := [n] \setminus A_i$.

$$\Rightarrow A_i \cap B_i = \emptyset, \quad A_i \cap B_j \neq \emptyset \quad \forall i \neq j$$



$$\Rightarrow \sum_{i=1}^{m} \binom{a_i+b_i}{a_i}^{-1} = \sum_{i=1}^{m} \binom{n}{a_i}^{-1} \leq 1 \quad ▨$$

⑥

L7 Perfect graphs

206A
Oct 16, 2020

Def $G=(V,E)$ simple graph (undirected, no loops, multiple edges)

$\chi(G)$ ← chromatic number $:=$ min # of colors of proper coloring

$\omega(G)$ ← clique number $:=$ max size of a clique K_ℓ in G

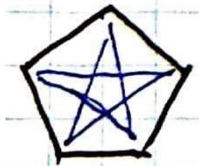
$\alpha(G)$ ← independence number $:=$ max size of an empty O_ℓ in G

Obs $\omega(G) = \alpha(\overline{G})$, where $\overline{G} = (V, 2^{\binom{V}{2}} \setminus E)$ complement graph

---

Ex 1 $\overline{K_5} = O_5$, $\chi(K_5)=5$, $\omega(K_5)=5$
$\alpha(K_5)=1$

Ex 2 $G:=C_5$ ← 5-cycle, $\omega(C_5)=2$



$\alpha(C_5) = 2$ since $\overline{C_5} = C_5$

$\chi(C_5) = 3$

Prop $\forall G=(V,E)$

$\chi(G) \geqslant \omega(G)$

D K_ℓ in G
$\Rightarrow \chi \geqslant \ell$ ▨

①

Exc $\forall k \ \exists G$ s.t. $\omega(G)=2$, $\chi(G)>k$

Def $G=(V,E)$ is *perfect* if $\chi(H)=\omega(H)$ $\forall$ *induced subgraph* H of G

$/ \ H=(V',E')$, s.t. $V'\subseteq V$, $E'=E|_{V'} \ /$

Ex G - Bipartite $\Rightarrow$ G - perfect $/\chi=\omega=2/$

Prop1 $P=(X,<)$ - poset, $G=\big(X, \{(xy), x<y\}\big)$ *comparibility graph*

$\Rightarrow$ G is perfect

D $h=$ height of P $\Rightarrow$ $\underbrace{\omega(G)=h}_{def}$, $\underbrace{\chi(G)=h}_{\text{antichain partition}}$ ▨

Prop 2 $P=(X,<)$ - poset, $G=\big(X, \{(xy), x\not<y, y\not<x\}\big)$ *incomparibility graph*

$\Rightarrow$ G is perfect.

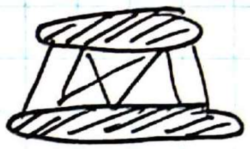
D $w=$ width of P $\Rightarrow$ $\underbrace{\omega(G)=w}_{def}$, $\underbrace{\chi(G)=w}_{\text{chain part, Dilworth Thm}}$ ▨

Th1 [Lovász, 1972] ← Weak perfect graph conjecture

[Berge, 1961]

G is perfect $\Longleftrightarrow$ $\bar{G}$ is perfect

Cor G - Bipartite $\Rightarrow$ $\bar{G}$ is perfect



[König th.]

Cor ∃ antichain part in P $\Rightarrow$ ∃ chain part in P [Dilworth Th]

/ WPGC = Lovász's Th $\Rightarrow$ Dilworth Th /

Th2 [Chudnovsky-Robertson-Seymour-Thomas, 2003]

[Berge, 1961]

/← Strong perfect graph conjecture /

/150 pp. proof/

G is perfect $\Longleftrightarrow$ G contains no $C_{2\ell+1}$, $\overline{C_{2\ell+1}}$ as induced subgraphs, $\ell \geqslant 2$

Note SPGC $\Rightarrow$ WPGC

Cor G is chordal $\Rightarrow$ G is perfect

[Dirac, 1961]

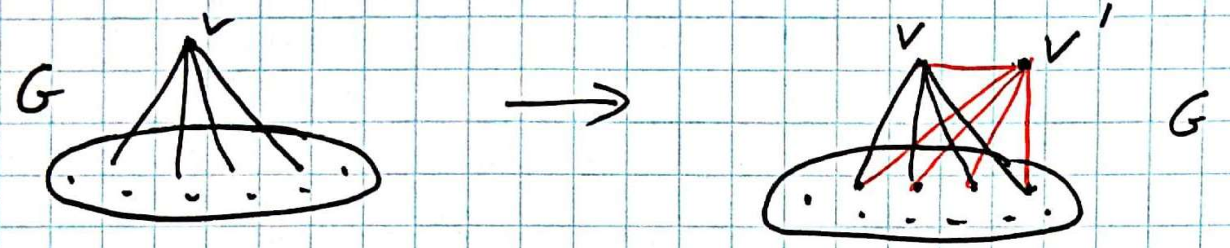
/ chordal $\Longleftrightarrow$ no induced C_K , $K \geqslant 4$ /

③

Proof of Weak Perfect Graph Theorem

[Diestel, §5.5]

Def $G=(V,E)$, $G'=\left(V+v',\ E \uplus \{v'w : vw \in E\} + (vv')\right)$



/replication/

Replication Lemma [Lovász, 1972]

G is perfect $\Rightarrow$ G' is perfect

D (induction on $n=|V|$) $n=1$ ✓ / $G=K_1$, $G'=K_2$ /

$n>1$ G-perfect $\Rightarrow$ $\chi(H)=\omega(H)$ $\forall$ induced H in G

$\boxed{\Rightarrow}$ suffices to show $\chi(G')=\omega(G')$

since every induced H' in G' is either in G or replication of some H in G

Now induction gives $\boxed{\Rightarrow}$

Obs $\omega(G') \in \{\omega, \omega+1\}$, where $\omega=\omega(G)$

④

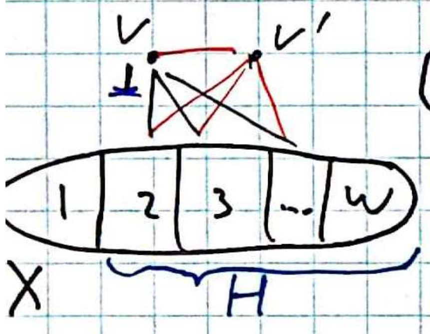
Case 1 $\omega(G') = \omega + 1 \underset{\text{Prop}}{\Rightarrow} \chi(G') \geq \omega + 1$

OTOH $\chi(G') \leq \chi(G) + 1 = \omega(G) + 1 = \omega + 1$ ✓

Case 2 $\omega(G') = \omega \Rightarrow \boxed{v \notin K_\omega \text{ in } G}$

/otherwise $K_\omega + v' = K_{\omega+1}$ /

Fix $f : V \to [\omega] = \{1 .. \omega\}$ proper coloring of G



ⓕ $f(v) = 1$, $X = f^{-1}(1) - v$

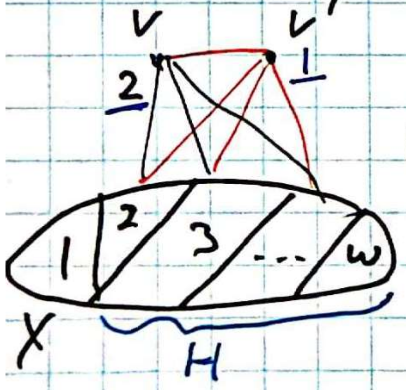
$H := G - X$ / $v \in H$ /

G - perfect $\Rightarrow \omega(H) = \chi(H)$

Obs: every K_ω in G intersects X ← by □

$\Rightarrow \omega(H) \leq \omega - 1 \Rightarrow \chi(H) \leq \omega - 1$

ⓖ



Fix $g : V \setminus X \to \{2 ... \omega\}$ prop. coloring

Since $X + v' \Leftarrow X + v$ - indep sets

let $\hat{g} : V + v'$ $\begin{cases} \hat{g}(x) = g(x), & x \in H \\ g(x) = 1, & x \in X \vee x = v' \end{cases}$ ▨

⑤

Replication Lemma ⇒ WPGC

▷ (by induction on $n = |V|$) $\quad n=1$ ✓

$n>1$ $\quad \alpha := \alpha(G)$, $\quad \mathcal{A} = \{$ indep sets A of size $\alpha \}$

G-perfect ⇒ $\chi(H) = \omega(H)$ ∀ induced H in G

⇒ H perfect ∀ induced H w/ $\le n-1$ vert

⇒ $\bar{H}$ perfect ∀ ind $\bar{H}$ of $\bar{G}$ —//—

Thus we need $\chi(\bar{G}) = \omega(\bar{G}) = \alpha$

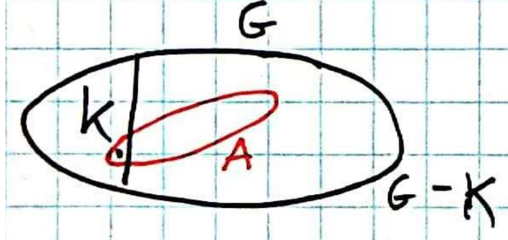
Claim ∃ clique K s.t. $K \cap A \neq \emptyset$ ∀ $A \in \mathcal{A}$

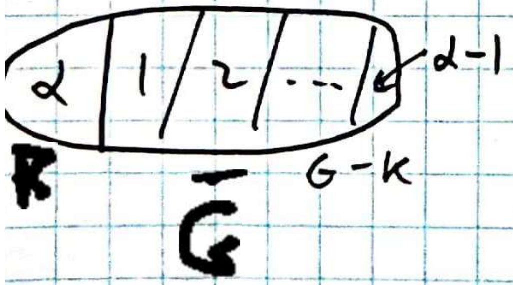
claim ⇒ $\omega(\bar{G} - K) = \alpha(G - K) < \alpha = \omega(\bar{G})$

⇒ $\chi(\bar{G}) \le \chi(\bar{G} - K) + 1$ $\quad$ ⊜ ind

$\le \omega(\bar{G} - K) + 1$

$\le \omega(\bar{G}) = \alpha$





⑥

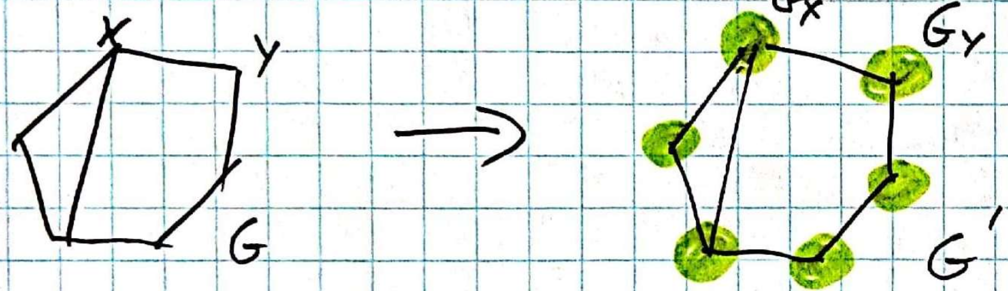
Proof of Claim (by contr.)

$\nexists K \Rightarrow \forall$ clique $K \quad \exists A_K \in \mathcal{A}, \; K \cap A_K = \emptyset$

Replace $\forall$ vertex $x \in V$ w/ clique $G_x = K_{k(x)}$

$k(X) = \#\{K$ cliques in G s.t. $X \in A_K\}$

by repeated replication $\longrightarrow$ graph G'



By RL $\Rightarrow$ $\boxed{\chi(G') = \omega(G')}$

Note/Obs: every max clique in G'

$= \bigcup_{x \in X} G_x$, X some clique in G

$$\Rightarrow \omega(G') = \sum_{x \in X} k(x) = \#\left\{ (x, K), \; x \in X, \; K\text{-clique in } G, \; x \in A_K \right\}$$

$$= \sum_{K \text{ cliques in } G} |X \cap A_K| \leq (\#\text{ cliques in } G) - 1$$

⑦

Here $\textcircled{\scriptsize$\leq$} \Leftarrow |X \cap A_K| \leq 1 \quad \forall K, X$ cliques

and $|X \cap A_X| = 0$ by def of A_X

OTOH $|G'| = \sum_{x \in V} k(x) = \# \left\{ \begin{matrix} (x, K), & x \in V \\ K\text{-clique}, & x \in A_K \end{matrix} \right\}$

$$= \sum_{K\text{-clique in } G} |A_K| = (\#\text{ cliques in } G)\alpha$$

Since $\alpha(G') \leq \alpha$ by def of G'

$$\Rightarrow \chi(G') \geq \frac{|G'|}{\alpha(G')} \geq \frac{|G'|}{\alpha} = \begin{pmatrix} \#\text{ cliques} \\ \text{in } G \end{pmatrix}$$

We conclude

$$\omega(G') < (\#\text{ cliques in } G) \leq \chi(G')$$

a contradiction w/ $\omega(G') = \chi(G')$

▨

⑧

L8

Linear Algebra Methods

206A

Oct 19, 2020

① Perfect graphs

Recall $G=(V,E)$ simple graph, $\bar{G}$ - compl.

$\chi(G)$ ← chromatic number

$\omega(G)$ ← clique number

$\alpha(G) = \omega(\bar{G})$ ← indep number

Prop $\chi(G) \geq \omega(G)$

Def G - perfect $\iff$ $\chi(H)=\omega(H)$ $\forall$ induced H of G

Th [Lovász] / = Weak Perfect Graph Conj /

G is perfect $\iff$ $\bar{G}$ is perfect

Ex 1) G - bipartite

2) G - comparability graph of $P=(X,<)$

3) $\bar{G}$ - incomparability — // — — // —

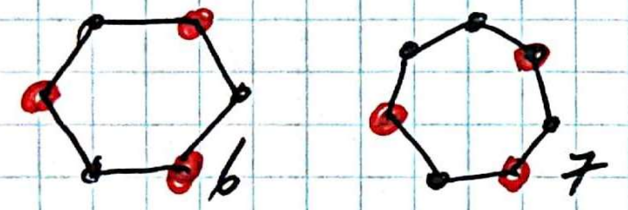
Th [Lovász, 1972] /⇒ WPG Conj/

[Diestel, Th 5.5.6]

G - perfect $\iff$ $\boxed{|H| \le \alpha(H)\ \omega(H)}$ $\forall$ induced H of G

Ex H - 6-cycle , $\alpha(H) = 3$, $\omega(H) = 2$ ✓

H - 7-cycle , $\alpha(H) = 3$, $\omega(H) = 2$ ✗



Note: used heavily in the proof of strong perfect Graph Conj

Proof (⇒) $\forall$ H induced $\chi(H) = \omega(H)$ ← perfect

⇒ ∃ coloring w/ $\omega(H)$ colors

each class of coloring has $\le \alpha(G)$

⇒ $|H| \le \omega(H)\ \alpha(H)$ ✓

(⇐) Use induction on $n = |G|$

If □ holds $\forall H$ ⇒ by ind.

all H s.t. $|H| < n$ are perfect.

ssume □ is false

⇒ suffices to show $\boxed{\chi(G) = \omega(G)}$

②

Obs1 $\boxed{\chi(G-U) \overset{①}{=} \omega(G-U) \overset{②}{=} \omega}$, $\omega = \omega(G)$

$\forall$ indep. set $U \subset V$, $|U| \geq 1$

Indeed ① ← $G-U$ is perfect

② $\leq$ ← ✓

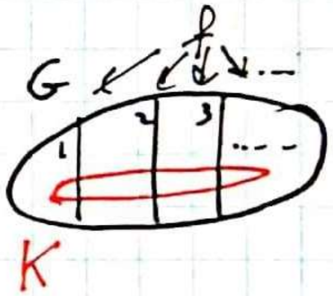
② $<$ ⇒ $\chi(G-U) < \omega \Rightarrow \chi(G) \leq \omega$

⇒ G - perfect ✗ ▨

Obs2 $U = \{u\}$, $u \in V$, f ← ω-coloring of $G-u$

K ← ω-clique in G

Then
- $u \in K \Rightarrow_{\boxed{1}}$ K intersects every color class of f except one
- $u \notin K \Rightarrow_{\boxed{2}}$ K intersects every color class of f



Let $A_0 = \{u_1 \dots u_\alpha\}$ ← α-indep. set in G

$\alpha = \alpha(G)$

- $A_1 \dots A_\omega$ ← color classes of ω-col. of $G-u_1$
- $A_{\omega+1} \dots A_{2\omega}$ ← —//— —//— $G-u_2$
- ⋮

$\Rightarrow$ we have $A_0\ A_1\ A_2\ \dots\ A_{\alpha\omega}$ $\leftarrow$ indep sets in G

Let K_i $\leftarrow$ ω-clique in G corr by $\square$ to A_i
(in obs 1)

Summary: $\forall$ ω-clique K in G we have

$$K \cap A_i = \emptyset \quad \text{for exactly one } i \in \{0 .. \alpha\omega\}$$

Indeed
$$\begin{cases} K \cap A_0 = \emptyset \Rightarrow K \cap A_i \neq \emptyset & \text{by } \boxed{2} \\ K \cap A_0 = \{u_j\} \Rightarrow K \cap A_i \neq \emptyset \ \ \forall\, i \neq j & \text{by } \boxed{1} \end{cases}$$

Now LA:

$$A := \underbrace{\begin{pmatrix} A_0 \\ \hline A_1 \\ \hline \vdots \\ \ \end{pmatrix}}_{n} \alpha\omega+1 \qquad B := \underbrace{\left(\begin{array}{c|c|c} K_0 & K_1 & \cdots \end{array}\right)}_{\alpha\omega+1} n$$

$n = |V|$

Summary $\Rightarrow$

$$AB = \underbrace{\begin{pmatrix} 0 & & 1 \\ & 0 & \\ & 0 & \\ 1 & & 0 \\ & & & 0 \end{pmatrix}}_{\alpha\omega+1} \alpha\omega+1 \quad =: J$$

$\det J \neq 0 \Rightarrow \operatorname{rk} A = \alpha\omega+1$

$\Rightarrow n \geq \alpha\omega+1$ ✗

▨

④

② Sets with many equal subset sums

Recall: Prop1 [weak Littlewood-Offord]

$$\forall A \subset \mathbb{R}_+ , \quad |A| = n , \quad \forall k > 0$$

$$\#\{ S \subseteq A : \sum_{a \in S} a = k \} \le \binom{n}{\lfloor n/2 \rfloor}$$

Prop2 $\forall A \subset \mathbb{R} , \quad |A| = n , \quad \forall k$

—//—

D $A = \{ a_1 \le \dots \le a_\ell < 0 \le a_{\ell+1} \le \dots \le a_n \}$

$$\sum_{a_i \in S} a = \sum_{i=1}^{\ell} \varepsilon_i a_i + \sum_{i=\ell+1}^{n} \varepsilon_i a_i , \quad \varepsilon_i \in \{0,1\}$$

$$\pi : (\varepsilon_1 \dots \varepsilon_\ell \; \varepsilon_{\ell+1} \dots \varepsilon_n) \to (1-\varepsilon_1, \dots 1-\varepsilon_\ell \; \varepsilon_{\ell+1} \dots \varepsilon_n)$$

$$\overset{\wedge}{B_n}$$

Obs $\sum_{i=1}^{n} \varepsilon_i a_i = \sum_{i=1}^{n} \varepsilon_i' a_i$

$\Rightarrow \pi(\bar{\varepsilon})$ and $\pi(\bar{\varepsilon}')$ are indep in B_n

$\Rightarrow \#\{..\} \le \text{width}(B_n) = \binom{n}{\lfloor n/2 \rfloor}$ ▨

Prop2 $\Leftrightarrow$ worst case $A = \{1 \ldots 1\}$, $k = \lfloor \frac{n}{2} \rfloor$ $\leftarrow$ multiset

when $\# \{ S \subseteq A : \sum_{a \in S} a = k \} = \binom{n}{\lfloor n/2 \rfloor}$

Th [Stanley, 1980] /= Erdős–Moser Conj /

Let $c(n) := \# \left\{ S \subseteq [n] : \sum_{s \in S} s = \left\lfloor \frac{1}{2} \binom{n+1}{2} \right\rfloor \right\}$

Then $\forall A \subset \mathbb{R}_+$ set $A = \{a_1 < a_2 < \ldots < a_n\}$, $\forall x$

$$\# \{ S \subseteq A : \sum_{s \in S} s = x \} \le c(n)$$

$\Leftrightarrow$ set $\{1 \ldots n\}$ is the worst!

Note: Stanley's original proof uses Hard Lefschetz theorem

We present: LA proof of [Proctor, 1982]

---

⑥

L9 Proof of Erdős-Moser Conjecture

206A

Oct 21, 2020

Th [Stanley, 1980] /= Erdős-Moser Conj /

Let $c(n) := \#\left\{ S \subseteq [n] : \sum_{s \in S} s = \left\lfloor \frac{1}{2} \binom{n+1}{2} \right\rfloor \right\}$

Then $\forall A \subset \mathbb{R}_+$ set $A = \{a_1 < a_2 < \ldots < a_n\}$, $\forall x$

$$\#\left\{ S \subseteq A : \sum_{s \in S} s = x \right\} \leq c(n)$$

Note: Stanley's original proof uses Hard Lefschetz theorem

We present: LA proof of [Proctor, 1982]

Proof ① E-M Conj $\Leftrightarrow$ Sperner property of some poset M_n

② Proof of Sperner property of M_n

Def $M_n = (X_n , \leq)$

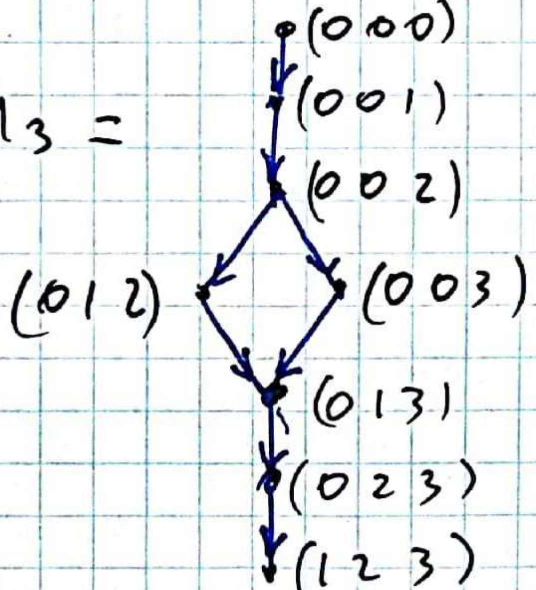
$X_n = \{ (b_1 \, b_2 \ldots b_n)$ s.t. $0 \leq b_i \leq n \quad \forall i$

and $0 = b_1 = \ldots = b_\ell < b_{\ell+1} < \ldots < b_n \leq n \}$

Ex $n = 3$

$X_3 = \{ (000), (001), (002), (003), (012), (013), (023), (123) \}$

$M_3 =$



Bijection $\pi : 2^{[n]} \rightarrow X_n$

$S = \{ s_1 < \ldots < s_r \} \subset [n] = \{1 \ldots n\}$

$\pi(S) := (0 0 \ldots 0 \, s_1 \ldots s_r)$

$\Rightarrow \ |X_n| = 2^n$

[Lindström, 1970]

L $\Sigma \varepsilon_i a_i = \Sigma \varepsilon_i' a_i \Rightarrow$ $\pi(\bar{\varepsilon})$ & $\pi(\bar{\varepsilon}')$ indep

D $(\ldots — \ 1 0 .. 0) = \bar{\varepsilon}$ ← max, etc.

$(\ldots — \ 1 0 .. 0) = \bar{\varepsilon}'$ ▨

②

L $\Rightarrow$ $\#\{S \subseteq A : \sum_{s \in A} s = x\} \leqslant \text{width}(M_n)$

Sperner of M_n $\Rightarrow$ $\text{width}(M_n) = \text{max rk size of } M_n$

③

Note: $f_n := \sum r_i t^i = \prod_{k=1}^{n} (1+t^k)$

$r_i = |i\text{-th rank in } M_n|$

$= \#\{ \bar{b} = (b_1 .. b_n) : \sum b_j = i \} = R_n(i)$

rank generating function

Sperner property of M_n vs.

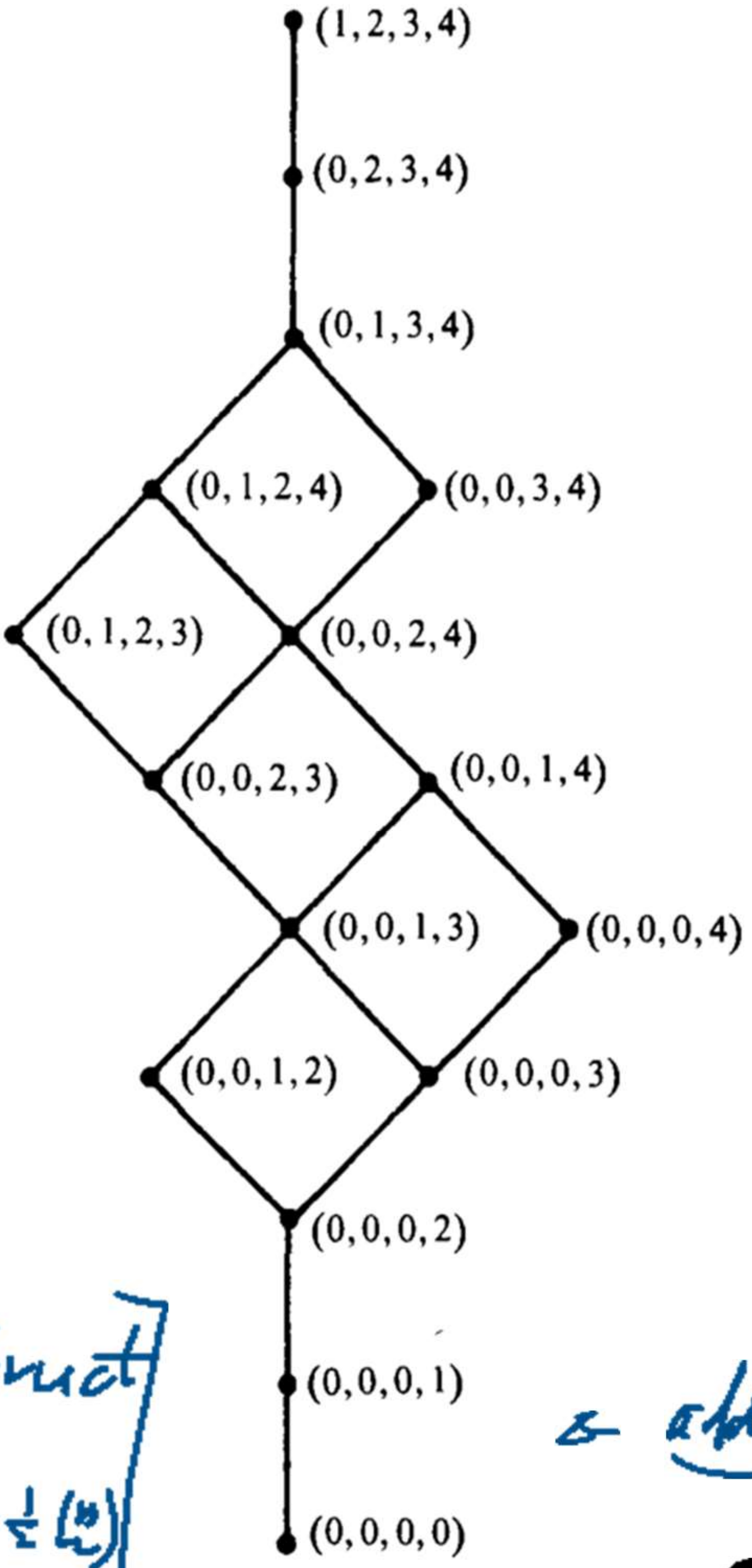
$(r_0\ r_1 \ldots r_{\binom{n}{2}})$ is symmetric & unimodal

$r_0 \le r_1 \le r_2 \le \ldots \le r_{\binom{n}{2}/2} \ge \ldots \ge r_{\binom{n}{2}}$

Note: No injection is known!

$R_n(i-1) \to R_n(i)$

[Note: We will not construct sym. sat. chain decomp, just sat chains cont $R_n^{\frac{1}{2}\binom{n}{2}}$]



& added

FIG. 5. $M(4)$.

④

$\underline{L} \Rightarrow \quad \#\{S \subseteq A : \sum_{s \in A} s = x\} \le \text{width}(M_n)$

$\underline{\text{Sperner of } M_n} \Rightarrow \quad \text{width}(M_n) = \text{max rk size of } M_n$

Main Lemma rk sizes of M_n are symmetric & unimodal.

$$r_0 \le r_1 \le \dots \le r_N \ge \dots \ge r_{\binom{n}{2}} \quad , \quad r_i = r_{\binom{n}{2}-i}$$

where $\quad r_i = \#\{\bar{b} \in X_n,\ b_1 + \dots + b_n = i\}$

Summary

$$\underbrace{ML + \text{Sperner}}_{\text{Now!}} + \underset{\checkmark}{\underline{L}} \Rightarrow \underset{\underset{\text{E-M conj}}{\parallel}}{\text{Stanley Th}} \quad ① \quad \checkmark$$

③

Linear Algebra Approach

[Stanley-slides]

$V_n := \bigoplus V_n^{(i)}$, $V_n^{(i)} = \mathbb{C}\langle R_n(i)\rangle$, $\dim V_n^{(i)} = r_i$

$U : V_n^{(i-1)} \to V_n^{(i)}$, $U\bar{b} := \sum_{\bar{b}' > \bar{b},\ \bar{b}' \in R_n^{i+1}} \bar{b}'$

raising operator

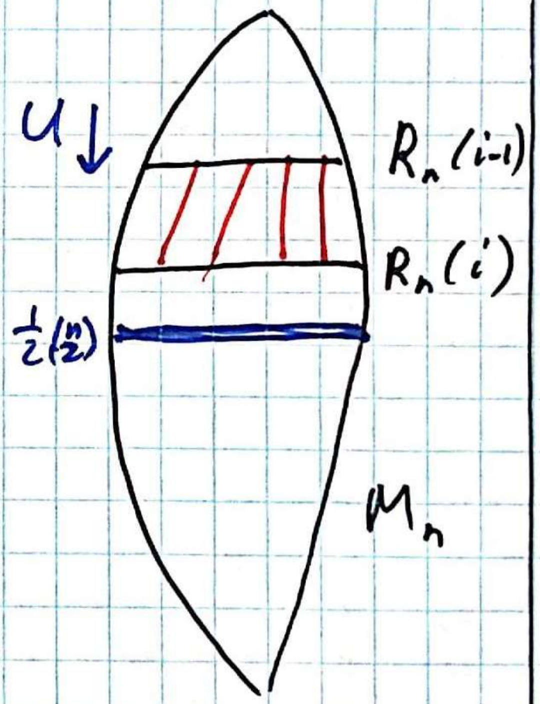
Suppose: U injective $\forall\ i \le \frac{1}{2}\binom{n}{2}$

$\Rightarrow\ r_0 \le r_1 \le r_2 \le \dots \le r_{\frac{1}{2}\binom{n}{2}}$

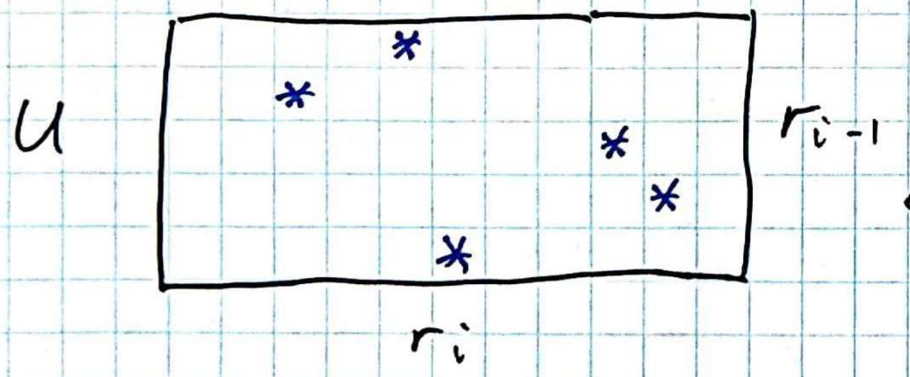
Claim [$\Rightarrow \exists$ matching P]

$P \leftarrow$ perfect matching in $[M_n|_{i,i-1}]$

graph $(R_n(i-1) \cup R_n(i), E_<)$



Proof of claim



$* \neq 0$

← nonzero minor

▨

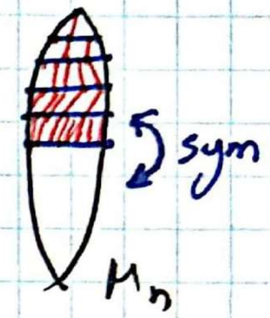
⑤

L1 Raising operator U is injective $\forall\, i < \frac{1}{2}\binom{n}{2}$

L + Claim $\Rightarrow$ ~~symmetric~~ sat. chain decomposition

$\Rightarrow$ M_n has Sperner, rank-unim.

$\Rightarrow$ E-M conj



Def Lowering operator $D : V_n^{(i+1)} \to V_n^{(i)}$

Let $\bar{b} \in R_n(i)$, $\bar{b}' \in R_n(i+1)$, $\bar{b} < \bar{b}'$

$k :=$ unique index s.t. $b_k < b'_k$

$$D\bar{b}' := \sum_{\bar{b} < \bar{b}',\ b \in R_n(i)} \underbrace{(n-b_k)(n-b_k+1)}_{c(\bar{b},\bar{b}')}\ \bar{b}$$

L2 $D_{i+1} U_i - U_{i-1} D_i = \left[\binom{n+1}{2} - 2i\right] I_i$

I = Identity

/ index ← level at which operators act /

Proof of L2 ← Exc ← [Proctor]

Proof of L1

Obs1 $A: V \to W$, $B: W \to V$ linear op.

$\Rightarrow \lambda^{\dim W} \det(1-\lambda BA) = \lambda^{\dim V} \det(1-\lambda BA)$ ← check this

$\Rightarrow$ AB & BA have same non-zero eig's.

Obs2 $D_1 U_0 = (1)$ ← positive eig's

[stanley-slides]

$$\begin{cases} D_{i+1} U_i - U_{i-1} D_i = \left[\binom{n+1}{2} - 2i\right] I_i \\ \text{Obs1} \Rightarrow U_{i-1} D_i \text{ has non-negative eig's.} \end{cases}$$

$\Rightarrow \forall\, i$ s.t. $[\ ] > 0$

eigs $(D_{i+1} U_i) \geq$ eigs $(U_{i-1} D_i)$ by $[\ ]^{*}$

$\Rightarrow$ eig $(D_{i+1} U_i)$ are > 0

$\Rightarrow$ U_i injective ▨

What is going on???

Recall $sl_2(\mathbb{C})$ *Lie algebra* $\Leftrightarrow M \in Mat_2(\mathbb{C})$
$tr(M)=0$

$e=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $f=\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $h=\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$[e,f]=h$, $[h,f]=-2f$, $[h,e]=2e$

Irreps of $sl_2(\mathbb{C})$: $-\lambda \xrightarrow{e} \cdot \xrightarrow{e} \cdot \xrightarrow{e} \cdot - - \xrightarrow{e} \cdot \xrightarrow{e} \lambda$

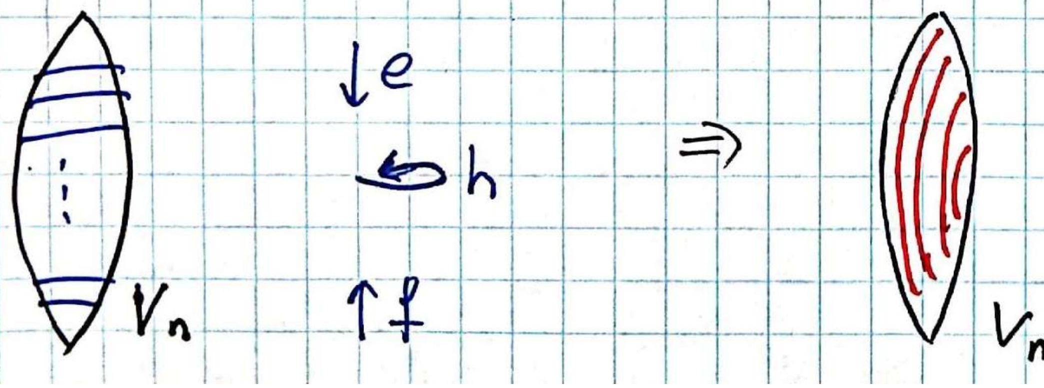
/characterized by the heighest weight/

We have $e \leftarrow U$, $f \leftarrow D$

$h \leftarrow \left[\binom{n+1}{2} - 2i \right] I_i$

L2 $\leftarrow [e,f]=h$, other rel. similar

Thus $\langle e\ f\ h \rangle$ define $sl_2(\mathbb{C})$-rep on V_n



$V_n = \oplus$ irreps

L10 Combinatorial Optimization Methods

206A
Oct 23, 2020

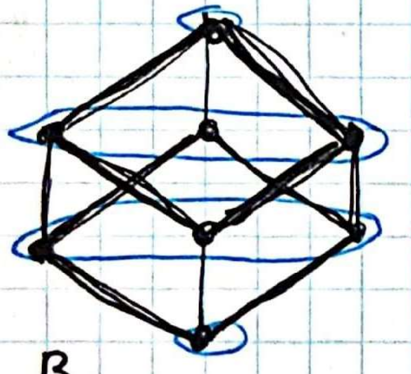
Def $P = (X, <)$ fin. poset

$a_k(P) := \max |A_1 \cup \dots \cup A_k|$ union of k disjoint antichains

$b_k(P) := \max |C_1 \cup \dots \cup C_k|$ union of k disjoint chains

so $a_1(P) = \text{width}(P)$, $b_1(P) = \text{height}(P)$

Ex



B_3

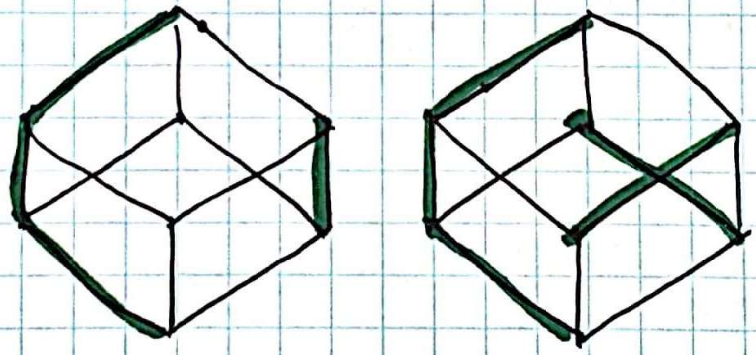
$a_1 = 3$

$a_2 = 6$

$a_3 = 7$

$a_4 = 8$

$b_1 = 4$ $b_2 = 6$ $b_3 = 8$



Def $\alpha(P) := (\alpha_1 \geq \alpha_2 \geq \dots)$ $\alpha_i = a_i(P) - a_{i-1}(P)$

$\beta(P) := (\beta_1 \geq \beta_2 \geq \dots)$ $\beta_i = b_i(P) - b_{i-1}(P)$

Ex $\alpha = (3311) \vdash 8$, $\beta = (422) \vdash 8$

①

Greene - Kleitman Theory (1976)

Th1 (antichains) $\forall\ P=(X,<)$, $k \le \text{height}(P)$

$$a_k(P) = \min_{\underbrace{\mathcal{C}=C_1\sqcup C_2\sqcup\cdots}_{\text{partition of } P \text{ into chains}}} \sum_i \min\{k, |C_i|\}$$

Th2 (chains) $\forall\ P=(X,<)$, $k \le \text{width}(P)$

$$b_k(P) = \min_{\underbrace{\mathcal{A}=A_1\sqcup A_2\sqcup\cdots}_{\text{partition of } P \text{ into antichains}}} \sum_i \min\{k, |A_i|\}$$

Th3 $\forall\ P=(X,<)$, $\alpha(P) = \beta(P)'$ conjugate partitions

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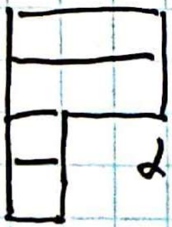
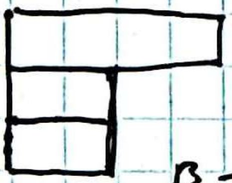
Ex



$\alpha = (3311)$



$\beta = (422)$

Note $k=1$

Th1 ← Dilworth Th

Th2 ← Prop

②

Permutation Posets

$\sigma \in S_n$, $P_\sigma = ([n], \prec) \leftarrow i \prec j \Leftrightarrow \underline{i<j,\ \sigma(i)<\sigma(j)}$

$\begin{cases} a_k(P_\sigma) = \text{max size of } k \text{ increasing subs} \\ b_k(P_\sigma) = \text{— // — // — decreasing —//—} \end{cases}$

Th [Greene, 1974] $\alpha(P_\sigma) = \beta(P_\sigma)' = \alpha(P_{\overleftarrow{\sigma}}) = \lambda$

where λ is given by RSK: $S_n \xleftrightarrow{RSK} \bigcup_{\lambda \vdash n} SYT(\lambda)^2$

$RSK(\sigma) = (A, B)$, $shape(A) = shape(B) = \lambda$

RSK = Robinson-Schensted (-Knuth) corresp.

Ex $n = 9$, $\sigma = (5\,2\,7\,3\,6\,1\,9\,4\,8)$

RSK

$$5 \to \begin{array}{l} 2 \\ 5 \end{array} \to \begin{array}{ll} 2 & 7 \\ 5 \end{array} \to \begin{array}{ll} 2 & 3 \\ 5 & 7 \end{array} \to \begin{array}{lll} 2 & 3 & 6 \\ 5 & 7 \end{array} \to \begin{array}{lll} 1 & 3 & 6 \\ 2 & 7 \\ 5 \end{array}$$

$$\to \begin{array}{llll} 1 & 3 & 6 & 9 \\ 2 & 7 \\ 5 \end{array} \to \begin{array}{llll} 1 & 3 & 4 & 9 \\ 2 & 6 \\ 5 & 7 \end{array} \to \boxed{\begin{array}{llll} 1 & 3 & 4 & 8 \\ 2 & 6 & 9 \\ 5 & 7 \end{array}} \ A$$

$$\boxed{\begin{array}{llll} 1 & 3 & 5 & 7 \\ 2 & 4 & 9 \\ 6 & 8 \end{array}} \ B \qquad \underline{\lambda = (4\,3\,2)}$$

Note $a_1 = \lambda_1 = LIS(\sigma)$

[Schensted, 1961]

③

Combinatorial Optimization Proof of Th 1

[Schrijver, §14.6]

/ proofs by Fomin (1974) and Frank (1980) /

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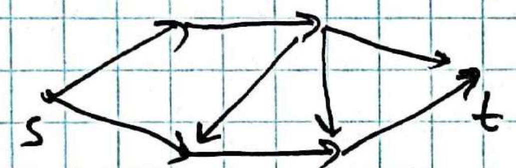
① min-cost circulation problem

Def $D = (V, E)$ digraph $\xi : E \to \mathbb{R}$ cost function

$\forall f : E \to \mathbb{R}$

$$\boxed{\mathrm{cost}(f) := \sum_{e \in E} \xi(e)\, f(e)}$$

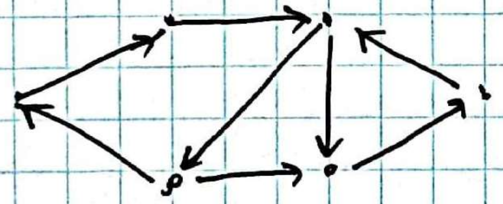
I [min-cost s-t flow problem]



cost function ξ
capacity function c

Want: min cost(f), $f \le c$, s-t flow f

II [min-cost circulation problem]



cost function ξ
capacity function c
demand function d

Want: min cost(f), $d \le f \le c$, feasible circulation f

Note: $I \subset II$

$d(ts) = 1$

$d = 0$ oth.

④

Note min flow ← [Ford-Fulkerson, 1956]

min circulation ← [Dinitz, 1970]

[Edmonds–Karp, 1972]

RAND, Santa Monica

MSU, Moscow

U Waterlo, Berkeley

Def $D=(V,E)$ digraph, $\cancel{\xi} f: E \to \mathbb{R}$

$D_f := (V, E_f)$ residual digraph

$E_f := \{e \in E,\ f(e) < c(e)\} \cup \{e^{-1},\ e \in E,\ f(e) > d(e)\}$

$e=(uv) \Leftrightarrow e^{-1}=(vu),\ u,v \in V,\ \xi(e^{-1}) := -\xi(e)$

Def C ← directed circuit in D_f

$$\chi^C(e) := \begin{cases} 1, & e \in C \\ -1, & e^{-1} \in C \\ 0, & \text{oth.} \end{cases}$$

Th $D=(V,E)$ digraph, $d, c, \xi : E \to \mathbb{R}$

[Schrijver, §12.2]

f ← feasible circulation. Then

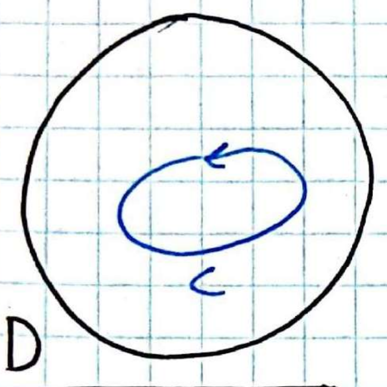
$$f\text{ - mincost} \Leftrightarrow \mathrm{cost}(\chi^C) \geq 0 \quad \forall C \subseteq D_f$$

⑤

Proof $\boxed{\Rightarrow}$ By contradiction, supsose $\exists\, C \subseteq D_f$ s.t. $\text{cost}(\chi^C) < 0$

Let $f' := f + \varepsilon \chi^C$. Then

f' - circulation, feasible, $\text{cost}(f') < \text{cost}(f)$ ▨



$\boxed{\Leftarrow}$ Suppose $\forall\, C \subset D_f$ directed circuit $\text{cost}(\chi^C) \geq 0$. Let $f' \leftarrow$ feasible circulation

Then $f' - f \leftarrow$ circulation

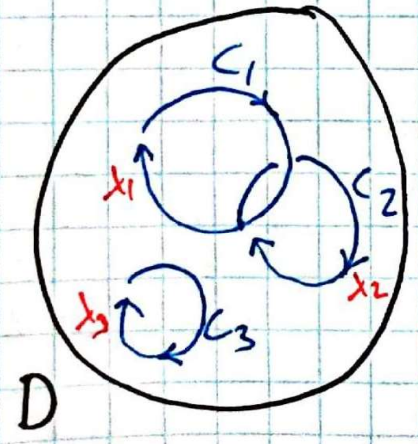
$$f' - f = \sum_{i=1}^{m} \lambda_i \chi^{C_i}, \quad \lambda_i > 0 \ \forall i$$

$$\text{cost}(f') - \text{cost}(f) = \text{cost}(f' - f)$$

$$= \sum_{i=1}^{m} \lambda_i \, \text{cost}(\chi^{C_i}),$$

$$= \sum_{i=1}^{m} \lambda_i \sum_{e \in C_i} \chi^{C_i}(e) \geq 0$$

▨



⑥

Next time: Proof of Th1 (antichains) ⊕

Th2 (chains) ← analogous ⊖

Th3 (conjugation) ← long accounting ⊖

RSK ← last year, next year

Greene Th ← —//—

L10 Greene-Kleitman Theory

206A
Oct 28, 2020

Th [G-K, 1976] (antichains) $\forall\ P=(X,<)$

$$a_k(P) := \max |A_1 \cup \dots \cup A_k|$$

~~$A_i \cap A_j = \emptyset$ ← antichains $(i \neq j)$~~ (!)

Then

$$a_k(P) = \min_{C_1 \sqcup C_2 \sqcup \dots = X} \sum_i \min\{k, |C_i|\}$$

$C_i \cap C_j = \emptyset$ ← chains

Last time: $D=(V,E)$ digraph, $f: E \to \mathbb{R}$

$c, d: E \to \mathbb{R}$ capacity, demand } functions

$D_f := (V, E_f)$ residual digraph

$E_f := \{e \in E,\ f(e) < c\} \cup \{e^*,\ f(e) > d(e)\}$

Th $\forall\ D=(V,E)$, $d, c, \xi : E \to \mathbb{R}$, $d \leq f \leq c$ circulation

f - min cost $/\sum \xi(e) f(e) = \text{cost}(f)/$ $\Longleftrightarrow$ $\left\{\begin{array}{l}\text{cost of every directed} \\ \text{circuit in } D_f \text{ is } \geq 0\end{array}\right.$

①

Thm $D=(V,E)$ acyclic digraph, $B \subseteq E$, $k \geq 1$

Then $\max \left| B \cap \left[\bigcup C_i\right]\right| = \min \left[\left| B \setminus \left[\bigcup P_i\right]\right| + k\,|\mathcal{P}| \right]$

where $\mathcal{C} = \{C_i\} \leftarrow$ at most k directed cuts in D

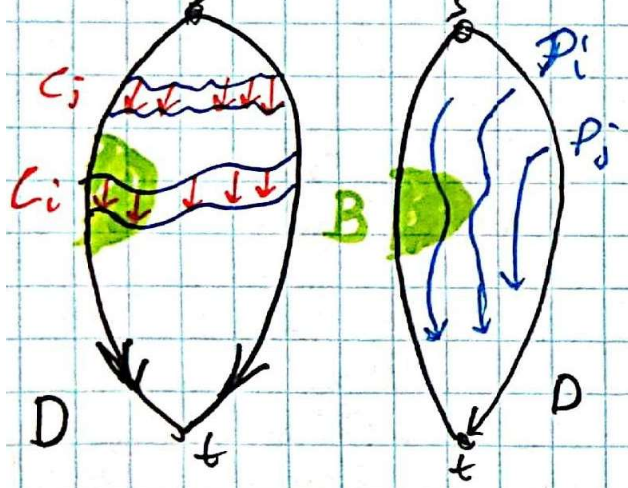
$\mathcal{P} = \{P_i\} \leftarrow$ directed paths /can $\cap$/

Proof of Thm ≤ $\Gamma := \bigcup C_i$, $\Pi := \bigcup P_i$ Then

$$|B \cap \Gamma| \leq |B \setminus \Pi| + |\Gamma \cap \Pi|$$

$$\leq |B \setminus \Pi| + k\,|\mathcal{P}|$$

/since every $|P_i \cap C_j| \leq 1$/



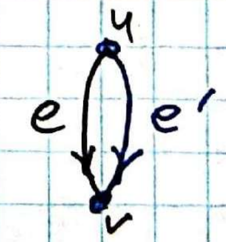
② ≥ Assume unique source s & sink t } /add them otherwise/

Define $\tilde{D} = (V, \tilde{E})$

$\tilde{E} = E \cup E'$

$\begin{cases} c(e) = \infty \\ \xi(e) = 0 \end{cases}$ $\forall e \in E$

$\begin{cases} c(e') = 1 \\ \xi(e') = -1 \end{cases}$ $\forall e' \in B$



(+) $\begin{cases} c(ts) = \infty \\ \xi(ts) = k \end{cases}$

②

Let $f : \tilde{E} \to \mathbb{Z}$, $0 = d \leq f \leq c$

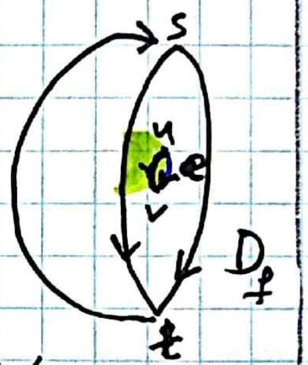
/non-negative feasible circulation/

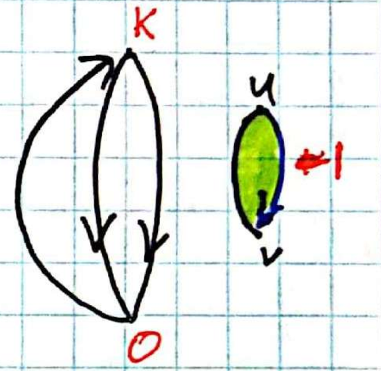
By **Th1** $\Rightarrow D_f$ has no negative cost circuits /easy part of $\Leftrightarrow$/

$\Rightarrow \exists\, p : V \to \mathbb{Z}$ s.t. $\underline{\forall e = (uv) \in E}$

$p(v) \leq p(u)$ w/ ⊜ if $f(e) \geq 1$, **and**

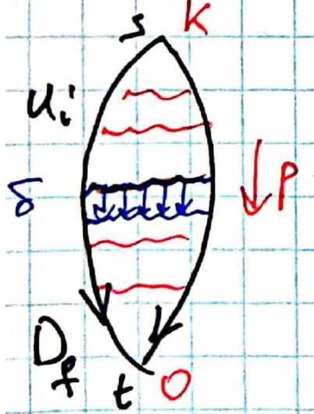
$\begin{cases} p(v) \leq p(u) - 1 & f(e') = 0 \\ p(v) \geq p(u) - 1 & f(e') = 1 \end{cases}$ $\underline{\forall\, e = (uv) \in B}$





$\begin{cases} p(t) = 0 \\ p(s) \leq p(t) + k \end{cases}$ $\Rightarrow$ $0 \leq p(s) \leq k$

/ $p(s) = k \Leftrightarrow f(ts) \geq 1$ ✓/



$U_i := \{ v \in V : p(v) \geq i \}$

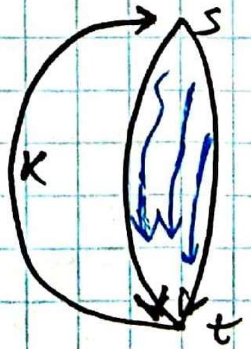
$\Rightarrow \delta^{out}_E(U_i) \leftarrow$ s-t cut /$s \in U_i$, $t \notin U_i$/

$C_i := \delta^{out}_E(U_i)$, [illegible] $:= C_1 \cup C_2 \cup \ldots \cup C_K$

$\mathcal{C} := \{C_1, C_2 \ldots\}$

$f = f_1 + f_2 + \ldots$ sum over circuits

remove $(ts) \Rightarrow P_1, P_2, \ldots$



$\underline{\mathcal{J} = \cup P_i}$

③

Obs $\Pi := \bigcup P_i \Rightarrow B \setminus \Pi = (B \cap \Gamma) \setminus \Pi$ /all action is on Π and Γ/

Bookeeping:

/balance eq./

D $e = (u,v) \in B \setminus \Pi \Rightarrow f(e) = 0 \Rightarrow p(v) \leq p(u) - 1$

$\Rightarrow e \in \delta^{out}(u_i), \; i = p(u) \Rightarrow e \in \Gamma$ $\boxtimes$

$$k \cdot |\mathcal{J}| = [p(s) - p(t)]\, f(ts)$$

$$= \sum_{e=(uv)\in E} [p(u) - p(v)]\, f(e) + \sum_{e'=(uv)\in B} [p(u) - p(v)]\, f(e')$$

$$= \sum_{e'} [p(u) - p(v)]\, f(e')$$ /either $f(e) = 0$ or $p(u) = p(v)$/

$$= |B \cap \Gamma \cap \Pi|$$

$$\Rightarrow |B \setminus \cancel{B}\Pi| + k\,|\mathcal{J}| = |B \setminus \Pi| + |B \cap \Gamma \cap \Pi|$$

$$= |B \cap \Gamma \setminus \Pi| + |B \cap \Gamma \cap \Pi|$$

$$= |B \cap \Gamma|$$ $\boxtimes$

/cf. $\textcircled{\leq}$ in reverse /

④

Proof of G-K Th

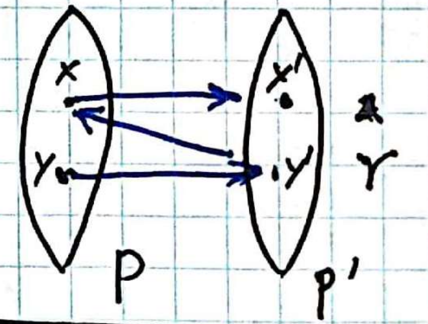
$P=(X,\leq)$, $k\in\mathbb{N}$. We want:

$$\max_{\mathcal{A}} |A_1\cup\dots\cup A_k| = \min_{\mathcal{C}} \sum_i \min\{k, |C_i|\}$$

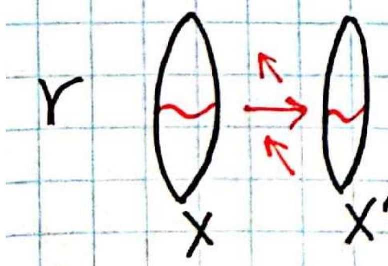
$X=\{x\}$, $X':=\{x'\}$ copy of X. $\boxed{D:=(V,E)}$

$$\begin{cases} V:=X\cup X' \\ E:=\{(x,x')\}\cup\{(y',x),\ y<x\} \end{cases}$$

$B:=\{(x,x')\}$



Cuts in D



max antichains A_i

$B\cap\left[\bigcup \text{cuts}\right] = \bigcup A_i$ union of max antichains

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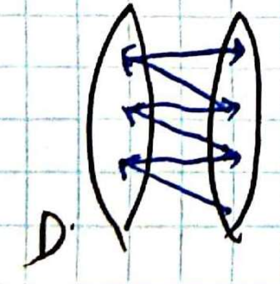
Paths in D ⟷ chains in P

$(x,x') \leftrightarrow C=\{x\}$



Exc check ⟹ ▨

⑤

L12 Poset Theory

Oct 30, 2020

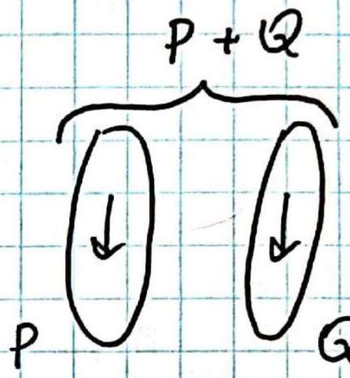
Operations on posets

[Trotter survey]

(1) $P=(X,<)$, $Q=(Y,<')$

$P+Q := (X \sqcup Y, \lhd)$

s.t. $\begin{cases} x \lhd x' & \forall\ x<x',\ x,x' \in X \\ y \lhd y' & \forall\ y<'y',\ y,y' \in Y \\ x \ntriangleleft y & \forall\ x \in X \quad y \in Y \end{cases}$

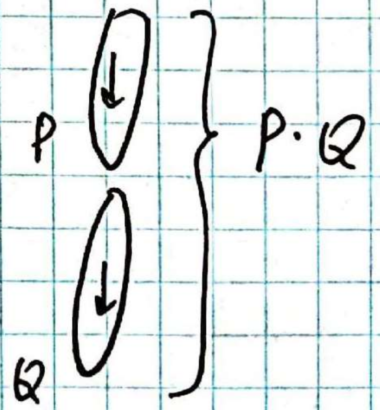


sum $P+Q$

(2) $P=(X,<)$, $Q=(Y,<')$

$P\cdot Q := (X \sqcup Y, \lhd)$ s.t.

$\begin{cases} x \lhd x' & \forall\ x<x',\ x,x' \in X \\ y \lhd y' & \forall\ y<'y',\ y,y' \in Y \\ x \lhd y & \forall\ x \in X,\ y \in Y \end{cases}$

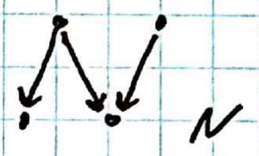


①

Def Posets obtained from $\bullet = P_1$ using sum & product operations are *series-parallel*

Th [HAI; Problem V]

P - series-parallel $\iff$ P is N-free



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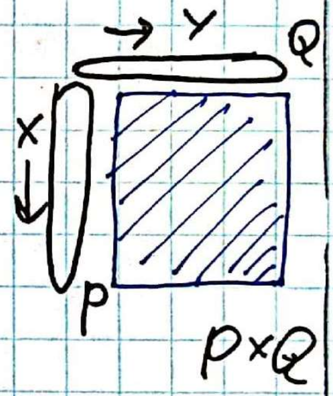
(3) $P = (X, \prec)$, $Q = (Y, \prec')$

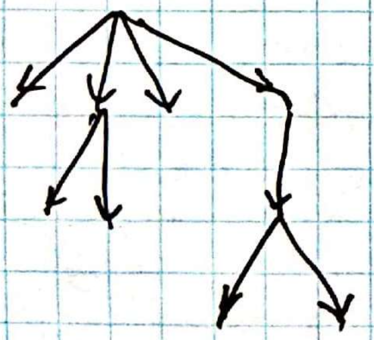
$P \times Q := (X \times Y, \lhd)$ s.t.

$(x,y) \lhd (x',y') \iff x \prec x' , y \prec' y'$

cartesian product



Ex tree posets are series-parallel (by induction)



Boolean lattice $B_n = \underbrace{P_2 \times P_2 \times \dots P_2}_{n}$

where $P_2 =$ 

②

(4) $P = (X, \prec)$, $Q = (Y, \prec')$, $|X| = m$, $|Y| = n$

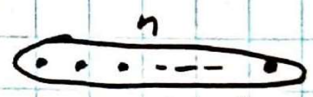
$Q^P := (F, \lhd)$ where

$$\begin{cases} F = \{ f : X \to Y \mid f(x) \preceq' f(x') \ \forall x \preceq x',\ x, x' \in X \} \\ \text{and} \quad f \unlhd g \iff f(x) \preceq' g(x) \quad \forall x \in X \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad f, g \in F \end{cases}$$

power poset

Exc/Prop $P \times (Q + R) = (P \times Q) + (P \times R)$

$P^{Q+R} = P^Q \times P^R$

Ex $B_n = P_2^{A_n}$, $P_2 =$  , $A_n =$ 

2-chain

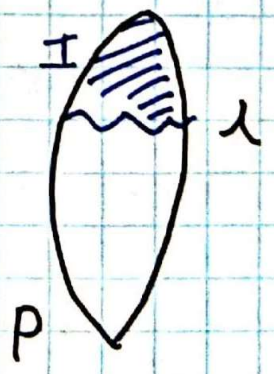
n-antichain

---

(5) $P = (X, \prec)$

$J(P) := (\mathcal{I}(P), \lhd)$ where

$$\begin{cases} \mathcal{I}(P) := \{ I \subseteq X \text{ order ideal} \} \\ \qquad\qquad \text{s.t. } \forall x \prec x',\ x' \in I \Rightarrow x \in I \\ \text{and} \quad I \lhd I' \iff I \subset I' \end{cases}$$



Ex $B_n = J(A_n)$

③

Lattices

[Stanley, §3.3]

$P=(X,\preceq)$, $x,y\in X$

Def u - upper bound for x, y $\iff$ $u\succeq x, \ u\succeq y$

v - lower bound $\iff$ $v\preceq x, \ v\preceq y$

---

u - least upper bound (join) $\iff$ $\begin{cases} u\succeq x, \ u\succeq y \quad \text{and} \\ \forall u'\succeq x, \ u'\succeq y \Rightarrow u'\succeq u \end{cases}$

$\boxed{u = x\vee y}$

v - greatest lower bound (meet) $\iff$ $\begin{cases} v\preceq x, \ v\preceq y \quad \text{and} \\ \forall v'\preceq x, \ v'\preceq y \Rightarrow v'\preceq v \end{cases}$

$\boxed{v = x\wedge y}$

Def $P=(X,\preceq)$ s.t. meet & join are well-defined. is called a *lattice*

Exc/Prop

$\forall \ P=(X,\preceq) \Rightarrow J(P)$ is a lattice s.t.

$I\wedge I' := I\cap I'$, $\quad I\vee I' := I\cup I'$

Ex B_n is a lattice $\boxed{\vee\leftarrow\cup, \ \wedge\leftarrow\cap}$

$\mathcal{F}_n(q) = (\{\text{substpaces of } \mathbb{F}_q^n\}, \subseteq)$ is a lattice

④

Exc/Prop $P = (X, \leq)$ ← lattice

$$\Rightarrow \begin{cases} x \vee (y \vee z) = (x \vee y) \vee z \\ x \wedge (y \wedge z) = (x \wedge y) \wedge z \end{cases} \quad \forall\, x\, y\, z \in X$$

associative law

Def $P = (X, \leq)$ ← distributive lattice

if P ← lattice and

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \qquad \forall\, x, y, z \in X$$
$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

[Stanley, §3.4]

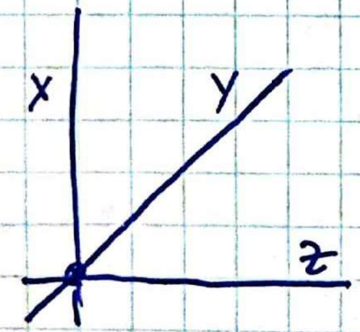
distributive law

Exc/Prop $\forall\, P = (X, \leq) \Rightarrow J(P)$ is a distributive lattice

---

Ex $B_n = J(A_n)$ is distributive

$\mathcal{F}_n(q)$ is NOT distributive



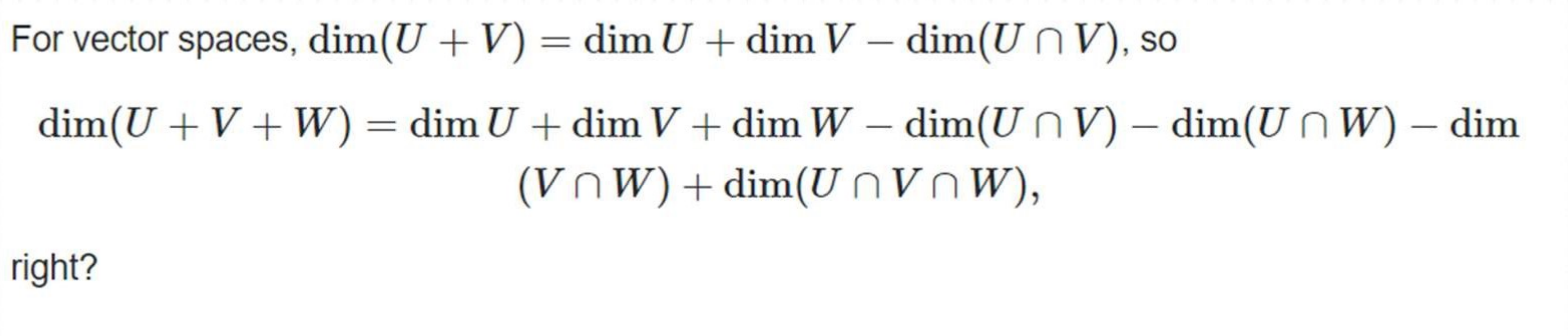
$x, y, z \in \{$ lines in $V = \mathbb{F}_q^2 \}$

$(x \vee y) = (x \vee z) = V$, $y \wedge z = \{0\}$

$x \vee (y \wedge z) = x \neq (x \vee y) \wedge (x \vee z) = V$

⑤

Question:



For vector spaces, $\dim(U+V) = \dim U + \dim V - \dim(U \cap V)$, so

$$\dim(U+V+W) = \dim U + \dim V + \dim W - \dim(U \cap V) - \dim(U \cap W) - \dim (V \cap W) + \dim(U \cap V \cap W),$$

right?

Answer: No, take 3 lines in the plane as in the Example above.

Th [= Fundamental Theorem for Distributive Lattices] [Stanley, §3.4]

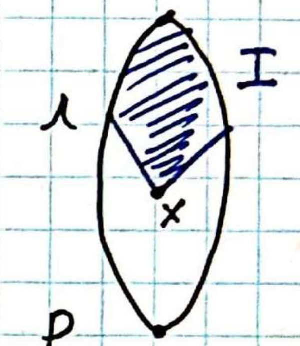
$\forall L$ finite distributive lattice

$\exists P = (X, \prec)$ s.t. $L = J(P)$

Def $P = (X, \prec)$, $I \in \mathcal{I}(P)$ order ideal

$I \leftarrow$ principal if $I = \{ y \preceq x \,,\, y \in X \}$ for some $x \in X$

$\iff \exists x$ s.t. $\{ y \in X : y \prec x \} = I$



Def $L = (Y, \lhd)$ lattice

$y \in Y \leftarrow$ join irreducible if

$\nexists s, t \in Y$ s.t. $\begin{cases} s, t \lhd y \,,\ s \neq y,\ t \neq y \\ \text{and} \quad s \wedge t = y \end{cases}$

Sketch of proof of FTDL $L = (Y, \lhd)$

(1) $X :=$ set of join-irreducible elts, $P := (X, \lhd)$

(2) check that $J(P) = L$ ∎

⑥

L13

Linear Extensions

206 A

Nov 2, 2020

Last time: $P = (X, \prec)$

$J(P) := (\mathcal{I}(P), \trianglelefteq)$, where

$\mathcal{I}(P) = \{ I \subseteq X$ order ideal (closed under $\prec$) $\}$, $\trianglelefteq = $ "$\subseteq$"

Prop: $J(P)$ is a distributive lattice

$I \wedge I' := I \cap I'$, $I \vee I' := I \cup I'$

Th [FTFDL = Fundamental Th Finite Distr. Lattices]

$\forall L \in$ finite distr. lattice

$\exists P = (X, \prec)$ s.t. $L = J(P)$

Proof idea: look at join irreducibles

they form desired P

①

Def [Linear Extensions]

$P = (X, <)$, $|X| = n$

$\mathcal{L}(P) := \left\{ f: X \to [n] = \{1 \ldots n\} \text{ s.t. } f\text{-bijective and } f(x) < f(y) \ \forall x < y \in X \right\}$

$e(P) := |\mathcal{L}(P)|$ number of linear extensions

Ex

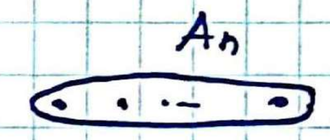
1) $P = A_n$ ← antichain $\Rightarrow$ $e(P) = n!$

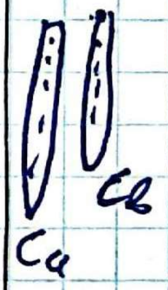
2) $P = C_n$ ← chain $\Rightarrow$ $e(P) = 1$

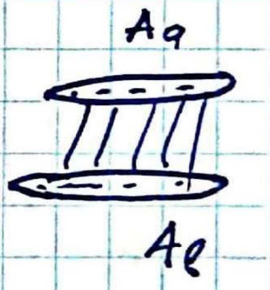
---

3) $P = C_a + C_b$ $\Rightarrow$ $e(P) = \binom{a+b}{a}$

4) $P = A_a \cdot A_b$ $\Rightarrow$ $e(P) = a!\, b!$







Prop 1 $e(P \cdot Q) = e(P)\, e(Q)$, $\forall P = (X, <)$, $Q = (Y, <')$

Prop 2 $e(P + Q) = e(P)\, e(Q) \binom{a+b}{a}$, $|X| = a$, $|Y| = b$

$\Rightarrow$ $e(P)$ can be computed in poly time $\forall P = (X, <)$
$P \leftarrow$ series-parallel.

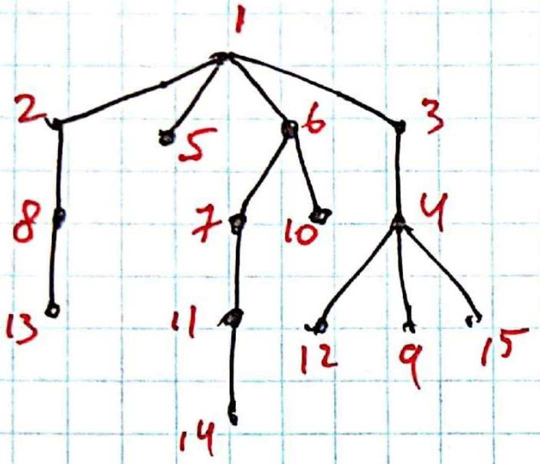
②

EX/EXC P ← tree poset (on tree T)

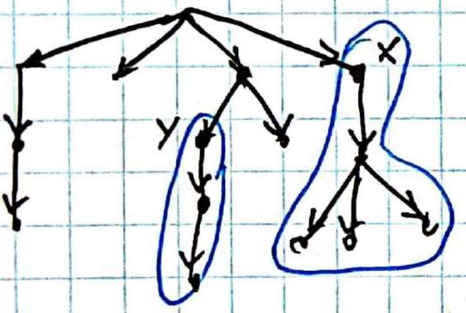
$\mathcal{L}(P)$ = increasing trees of shape T

↓

$e(P)$ = # — // —



tree T, $n = 15$



Th/Exc P ← tree poset

Then $\boxed{e(P) = n! \prod_{x \in T} \frac{1}{b(x)}}$

where $b(x) = |I^*(x)|$

size of the principal order ideal in reverse poset P^*

Proof idea: recall that trees are series-parallel

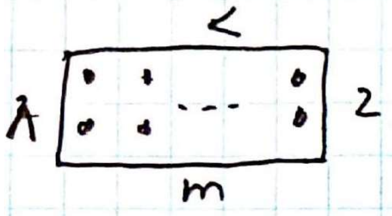
OBS $\mathcal{L}(P) = \{ I_0 \subset I_1 \subset \dots \subset I_n$ ← order ideals in P }
$|I_j| = j$

= maximal chains in $J(P)$

③

Ex $P = (X, <)$ ← 2-dim poset, $X = \{(1\,i), (2\,i) \mid 1 \le i \le m\}$

$n = |X| = 2m$

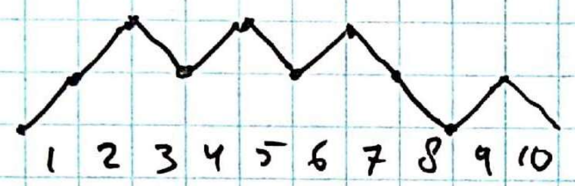


Prop $e(P) = \frac{1}{m+1}\binom{2m}{m}$ Catalan number.

1	2	4	6	9
3	5	7	8	10

$m = 5$

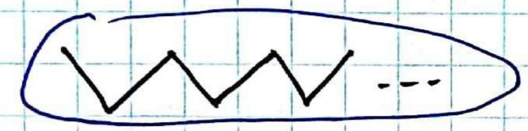
D (by Bijection)



$f(1\,i) = k \Rightarrow$ k-th step **UP**

$f(2\,i) = k \Rightarrow$ k-step **Down** ▨

Ex $P =$



zigzag poset Z_n

$\Rightarrow e(P) = \#\{\sigma \in S_n : \sigma(1) < \sigma(2) > \sigma(3) < \sigma(4) > \dots\}$

$a_n := e(Z_n)$

n	1	2	3	4	5	6	7
a_n	1	1	2	5	16	61	272
C_n	1	2	5	14	42	132	429

Prop / Exc

$C_n \leqslant a_{n+1} \quad \forall n$

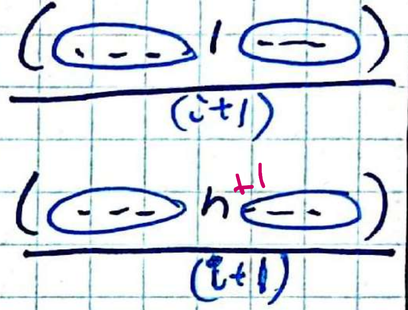
Th $a_n \sim \frac{4}{\pi}\, n! \left(\frac{2}{\pi}\right)^n$ | $C_n \sim \frac{1}{\sqrt{\pi}\, n^{3/2}}\, 4^n$

L $\mathcal{A}(t) := \sum_{n=0}^{\infty} a_n \frac{t^n}{n!}$ ← EGF , $a_0 = 1$

Then $\mathcal{A}(t) = \sec(t) + \tan(t) = \dfrac{1+\sin(t)}{\cos(t)}$

D $a_{n+1} = \sum_{i-\text{even}} \binom{n}{i} a_i \, a_{n-i}$

$= \sum_{i-\text{odd}} \binom{n}{i} a_i \, a_{n-i}$



$\Rightarrow 2a_{n+1} = \sum_{i=0}^{n} \binom{n}{i} a_i \, a_{n-i}$

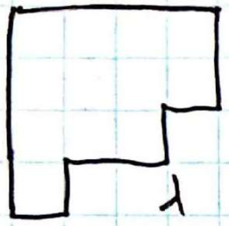
$\Rightarrow 2\mathcal{A}' = 1 + \mathcal{A}^2$, $\mathcal{A}(0) = 1$ ← ODE

$\Rightarrow \mathcal{A} = \dfrac{1+\sin(t)}{\cos(t)}$ $\blacksquare$

L + Complex analysis ⇒ Th

OBs ⇒ a_n can be computed in poly time.

Young diagrams



$\lambda = (4431)$

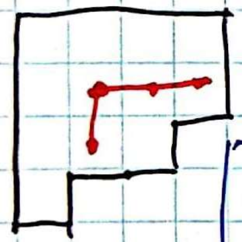
$\lambda \vdash 12$

1	2	4	8
3	5	7	10
6	9	12	
11			

$<$ $\wedge$

$A \in SYT(\lambda)$

Hook numbers (lengths)



$h(22) = 4$

$$\boxed{h(i,j) := \lambda_i + \lambda'_j - i - j + 1}$$

m+1		4	3	2
m	..	3	2	1

$\lambda = (m,m)$

$\lambda = (\lambda_1\ \lambda_2 \ldots)$ integer part.

$\lambda_1 + \lambda_2 + \ldots = n$, $\lambda_1 \geqslant \lambda_2 \geqslant \ldots > 0$

$\lambda \vdash n$ ← notation

$\mathcal{P}_\lambda$ ← partition poset

$\mathcal{P}_\lambda = ([\lambda] , <)$ ← 2-dim poset on Young diag $[\lambda]$

$\mathcal{L}(P_\lambda) \leftrightarrow SYT(\lambda)$

standard Young tableaux of shape λ

$e(\lambda) := e(P_\lambda) = |SYT(\lambda)|$

f^λ in [Stanley] [Sagan]

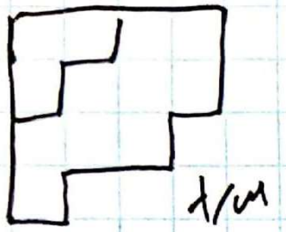
Th [hook-length formula] HLF

$$e(\lambda) = n! \prod_{(i,j)\in\lambda} \frac{1}{h(i,j)}$$

$\Rightarrow$ $e(\lambda)$ can be computed in poly time

⑥

Skew Young diagrams



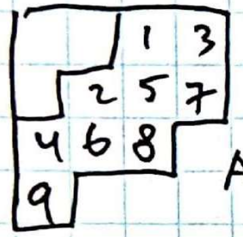
λ/μ ← skew Young diag, $\lambda \geq \mu$

⟺ $\lambda_i \geq \mu_i \quad \forall i$

Th [Feit] $|\lambda/\mu| = n \; /= |\lambda| - |\mu|/$

Then $$e(\lambda/\mu) = n! \det\left(\frac{1}{(\lambda_i - \mu_j - i + j)!}\right)$$

⟹ $e(\lambda/\mu)$ can be computed in poly-time



$A \in SYT(\lambda/\mu)$

$\lambda = (4431)$
$\mu = (21)$

⑦

L14

Counting Linear Extensions

206A
Nov 4, 2020

Recall $P = (X, \prec)$ finite poset, $|X| = n$.

$\mathcal{L}(P) = \{ f : X \to [n]$ order preserving bij$\}$

linear extensions

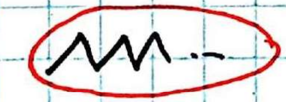
$e(P) = |\mathcal{L}(P)|$ ← number of linear ext.

Ex P – series parallel ⇒ $e(P)$ has a product formula

/e.g. tree poset P_T/

P_λ – poset of squares of Young diagram λ ⇒ — // —

$P_{\lambda/\mu}$ – — // — skew λ/μ ⇒ $e(P)$ has a det formula

λ/μ ← zigzag  ⇒ EGF

Today: more on $e(P)$ in special cases

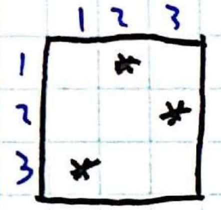
estimates on $e(P)$

①

Permutation & 2-dim posets

$\sigma \in S_n$, $P_\sigma = ([n], \prec)$, $i \prec j \Leftrightarrow i < j , \sigma(i) < \sigma(j)$

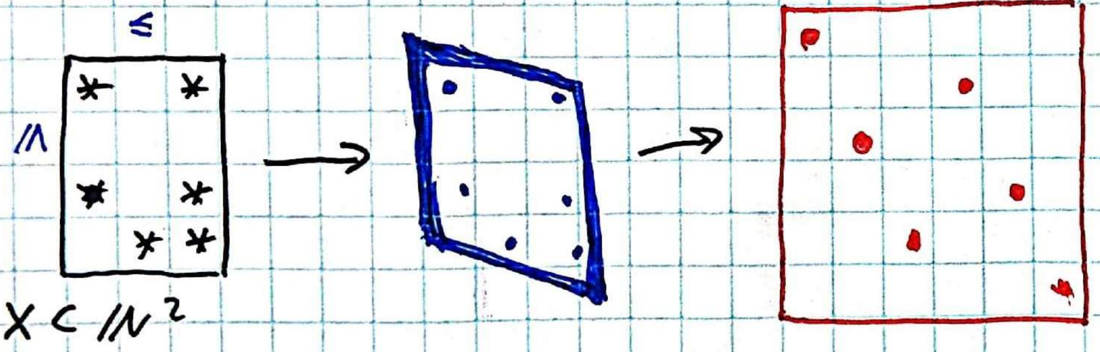
Obs Every P_σ is 2-dim.



$\sigma = (2\,3\,1)$ so $1 \prec 2 \leftrightarrow \sigma(1) < \sigma(2)$

Prop Every 2-dim poset $\simeq P_\sigma$ some $\sigma \in S_n$

▷



$\sigma \in S_n$

$n = |X|$

▨

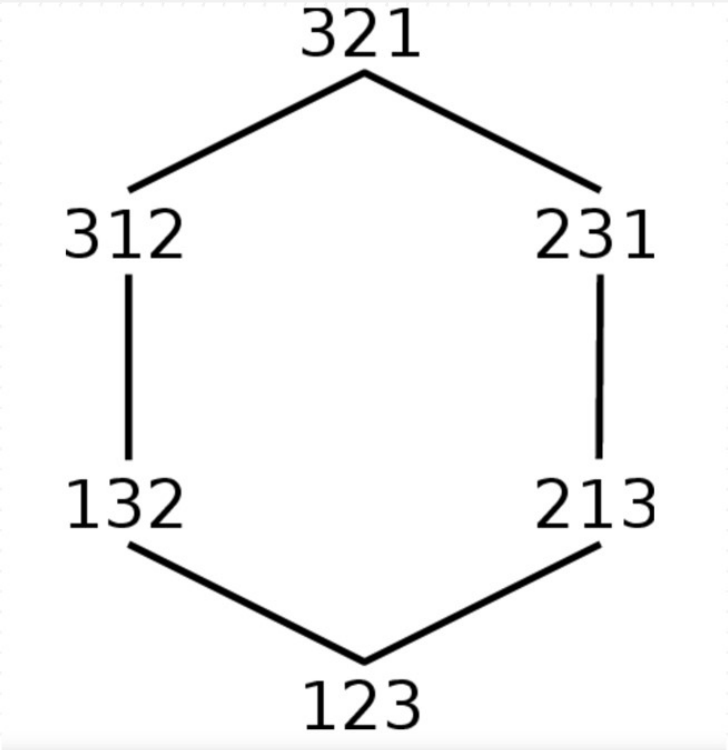
Def Bruhat order on S_n (weak B.O.)

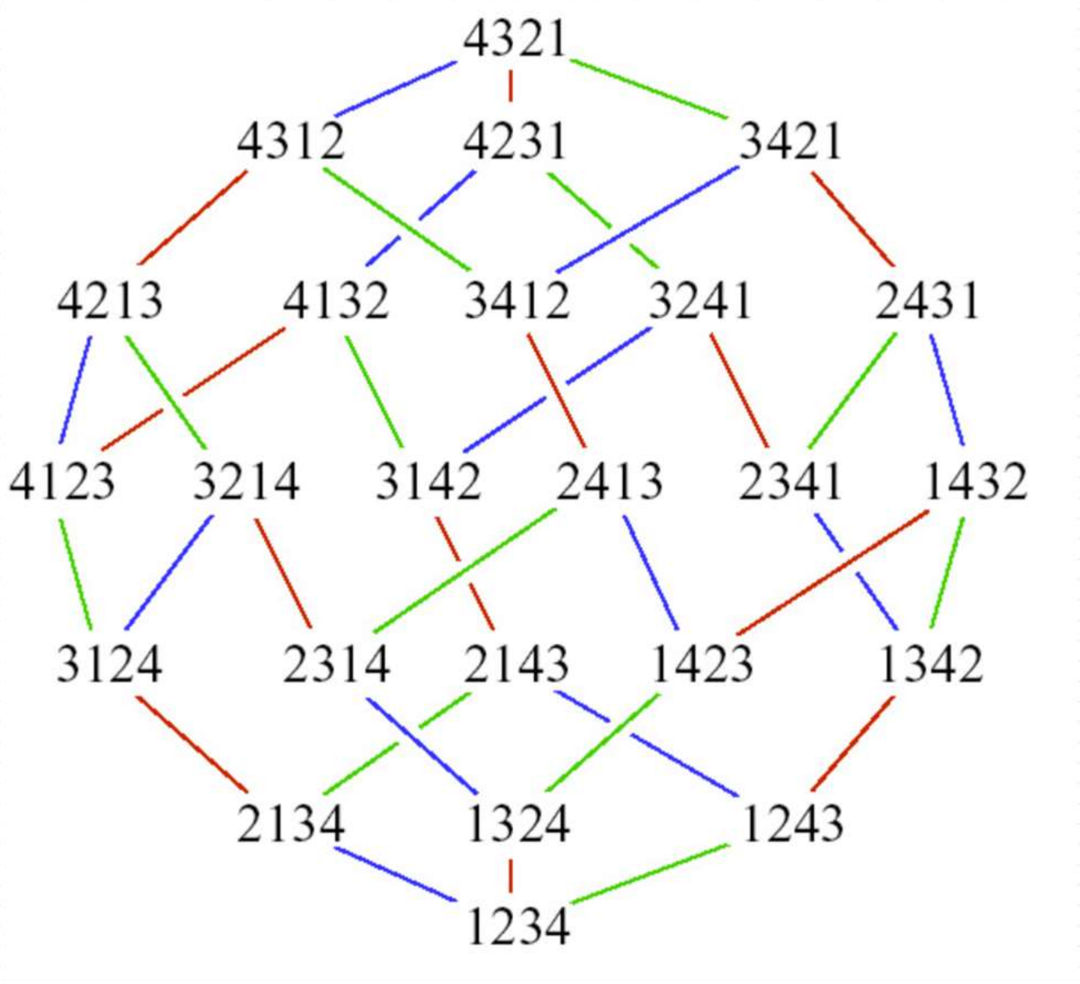
$\text{Bruhat}_n := (S_n, <_B)$ s.t. $\sigma <_B \tau$ iff

$\tau = \sigma\,(i_1\ i_1{+}1)\,(i_2\ i_2{+}1) \ldots (i_\ell\ i_\ell{+}1)$

and $\text{inv}(\tau) = \text{inv}(\sigma) + \ell$

Weak Bruhat order examples (upside down)





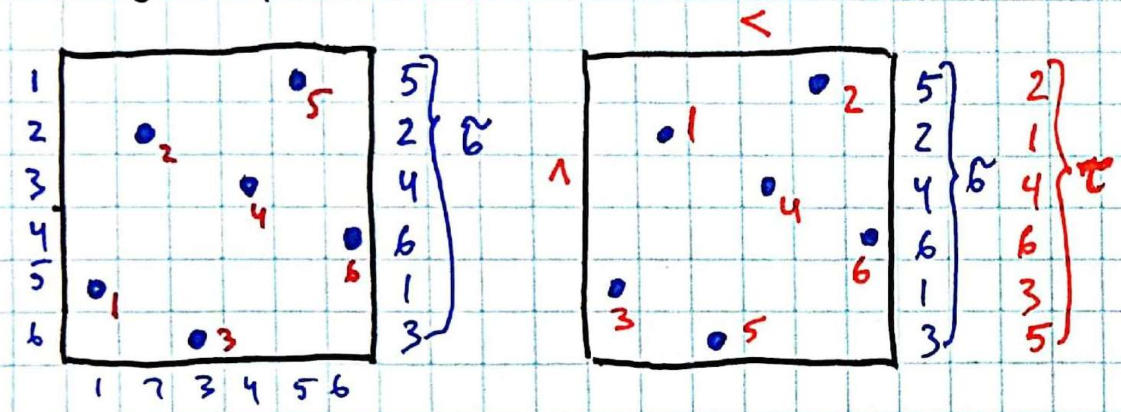
$\mathrm{inv}(\sigma) = \#\{ (i\,j),\ i<j,\ \sigma(i) > \sigma(j) \}$

e.g. $\mathrm{inv}(2\,4\,1\,3) = 3$

Th $\forall \sigma \in S_n \quad e(P_\sigma) = \#\{ \tau \in S_n,\ \tau \leq_B \sigma \}$

/size of the principal order ideal in Bruhat/

▷ $\sigma = (5\,2\,4\,6\,1\,3)$ $\qquad (2\,1\,4\,6\,3\,5) = \tau$



$\boxed{\text{rest} \Leftarrow \text{Exc}}$

$(\underbrace{3\,1}\ \underbrace{5\,4\,2}\ 6) = \tau'$

▨

Th [Brightwell-Winkler, 1991]

Computing $e(P)$ is #P-complete

Th [Dittmer-P., 2018]

Computing $e(P_\sigma)$ is #P-c /⇔ $\dim(P) = 2$/

Th [-‖-]

Computing $e(P)$ is #P-c, $\text{height}(P) = 2$

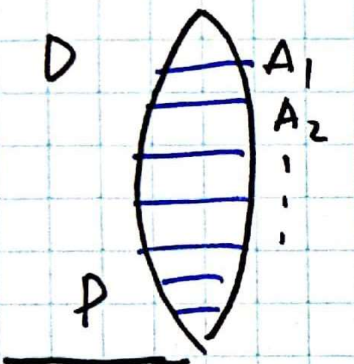
③

Bounds on e(P)

Prop 1 $P = (X, <)$, $\mathcal{A} = (A_1\ A_2 \ldots)$ antichain partition

Suppose $A_i < A_j \quad \forall\, i<j$

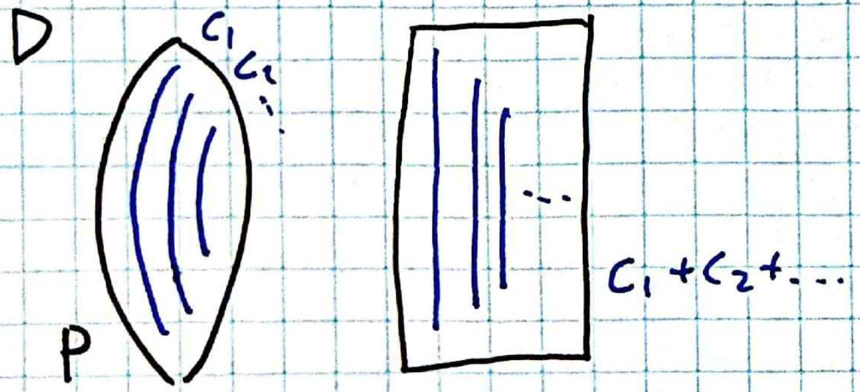
Then $e(P) \geq |A_1|!\ |A_2|! \cdots$



all permutations of A_i can be concatinated into LE of P ▨

Prop 2 $P = (X, <)$, $\mathcal{C} = (C_1, C_2, \ldots)$ chain partition

Then
$$e(P) \leq \frac{n!}{|C_1|!\ |C_2|! \cdots}$$



$$e(P) \leq e(C_1 + C_2 + \ldots) \leq \frac{n!}{|C_1|!\ |C_2| \cdots -}$$ ▨

/series-parallel poset/

④

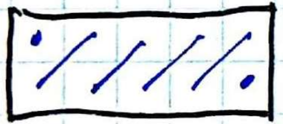
Th [Bochkov - Petrov, 2019]

$P = (X, <)$, $|X| = n$, $\lambda = (\lambda_1, \lambda_2 \dots) \leftarrow$ GK part

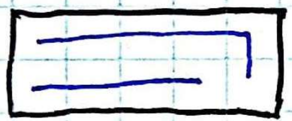
Then $\boxed{e(P) \leq \frac{n!}{\lambda_1! \, \lambda_2! \dots}}$, $\boxed{e(P) \geq (\lambda_1')! \, (\lambda_2')! \dots}$

and these ineq are stronger than Prop1, Prop2

Ex $P_{(m,m)} \leftarrow$ Catalan poset , $n = 2m$, $e(P) = \frac{1}{m+1}\binom{2m}{m}$



$\lambda = (m,m)$



$LB = 2^{m-1}$, $UB = \frac{2m!}{(m+1)!\,(m-1)!}$

$e(P) \sim c \, \frac{4^m}{m^{3/2}}$

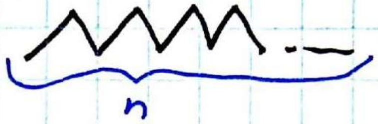
Prop 3 $e(P) \leq \text{width}(P)^n$

▷ By induction, $\exists \leq w =$ width of P ways to place n / ($\Leftrightarrow$ assign $f^{-1}(n)$)/

▤

Ex For P_{mm} Catalan poset $\Rightarrow e(P) \leq 2^n = 4^m$

⑤

Ex $P = Z_n$ zigzag poset 

$e(Z_n) \sim c\; n!\, \boxed{\left(\frac{2}{\sqrt{\pi}}\right)}^n$ Euler numbers $n = 2m$

LB: Prop1 $e(Z_n) \geqslant \left(\frac{n}{2}\right)!^2 = m!^2 = n! \binom{n}{n/2}^{-1} \approx n!\, \textcircled{2}^{-n}\, \Theta(\sqrt{n})$

UB Prop2 $\Rightarrow$ $e(Z_n) \leqslant n!\; 2^{-m} = n!\, \boxed{(\sqrt{2})}^{-n}$

Prop3 $\Rightarrow$ $e(Z_n) \leqslant m^n = \left(\frac{n}{2}\right)^n$, $n! \leqslant \left(\frac{n}{2}\right)^n$

---

⑥

L15 Counting Linear Extensions

206 A

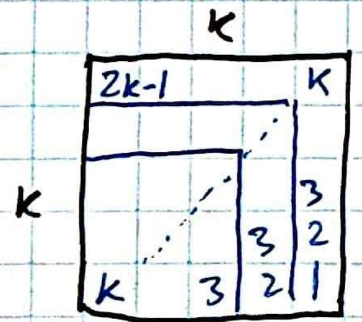
Nov 6, 2020

Last time: $P = (X, <)$, $|X| = n$

$$P = C_1 \cup C_2 \cup \cdots \Rightarrow e(P) \leqslant \frac{n!}{|C_1|!\,|C_2|! \cdots}$$

$$P = A_1 \cup A_2 \cup \cdots \Rightarrow e(P) \geqslant |A_1|!\,|A_2|! \cdots$$

Ex $\lambda = (k^k) = \underbrace{(k, \ldots k)}_{k}$, $n = k^2$



$$\text{HLF} \Rightarrow e(\mathcal{P}_\lambda) = \frac{n!}{\prod h(i j)} = \frac{n!}{(1 \cdot 2^2 \cdot 3^3 \cdots k^k) \times (k+1)^{k-1} \cdots (2k-1)}$$

$$= \boxed{\frac{n!\ \Phi(k-1)^2}{\Phi(2k-1)}}$$

where $\Phi(m) = 1! \cdot 2! \cdots m!$ superfactorial

Stirling $\log n! = n \log n - n + O(\log n)$

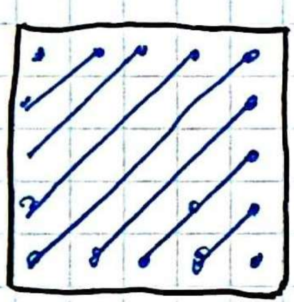
Barnes $\log \Phi(m) = m^2 \log m - \frac{3}{4} m^2 + O(m \log m)$

$\Rightarrow$ log HLF $\qquad \log e(k^k) = \log n! \dfrac{\Phi(k-1)^2}{\Phi(2k-1)}$

$$= \boxed{\tfrac{1}{2} n \log n} + \boxed{\left(\tfrac{1}{2} - 2\log 2\right)} n \qquad -0.8863$$

$$+ O(\sqrt{n} \log n)$$

LB

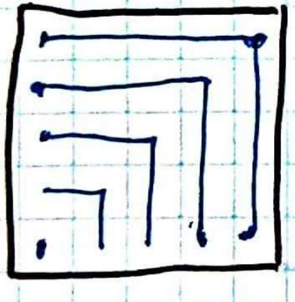


$$e(k^k) \geqslant \Phi(k)\, \Phi(k-1)$$

$$\log e(k^k) \geq \tfrac{1}{2} n \log n - \tfrac{3}{2} n + O(\sqrt{n} \log n)$$

-1.5

UB



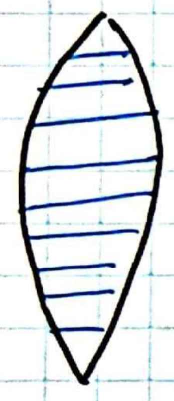
$$e(k^k) \leq \binom{n}{2k-1,\ 2k-3,\ \ldots,\ 1} = \frac{n!}{(2k-1)!\,(2k-3)!\cdots}$$

$$\log e(k) \leq \tfrac{1}{2} n \log n + \left(\tfrac{1}{2} - \log 2\right) n \qquad -0.1931$$

$$+ O(\sqrt{n} \log n)$$

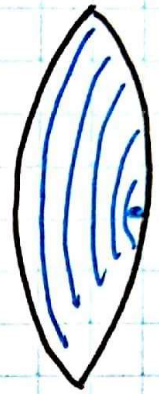
②

Boolean Lattice example



$$e(B_n) \geq \prod_{k=0}^{n} \binom{n}{k}! \quad \leftarrow LB$$

$$e(B_n) \leq n! \left[\prod_{k=1}^{m} (2k-1)!^{\binom{n}{2k}-\binom{n}{2k-1}} \right]^{-1}$$



$n = 2m-1$

$$\Rightarrow \log e(B_n) = (n+1)\, 2^n \log 2 - 2^{n-1} \log(2\pi n) + \underbrace{\Theta(1)}_{c}\, 2^n + o(1)$$

[Kleitman-Sha] 1987

where $LB \Rightarrow c \geq -\frac{3}{2}$ $\quad$ $UB \Rightarrow c \leq -\frac{1}{2}$

Th [Brightwell-Tetali, 2003]

$$\boxed{c = -\frac{3}{2}}$$

Moreover [Kahn-Kim, 1995] + [B-T] $\Rightarrow$

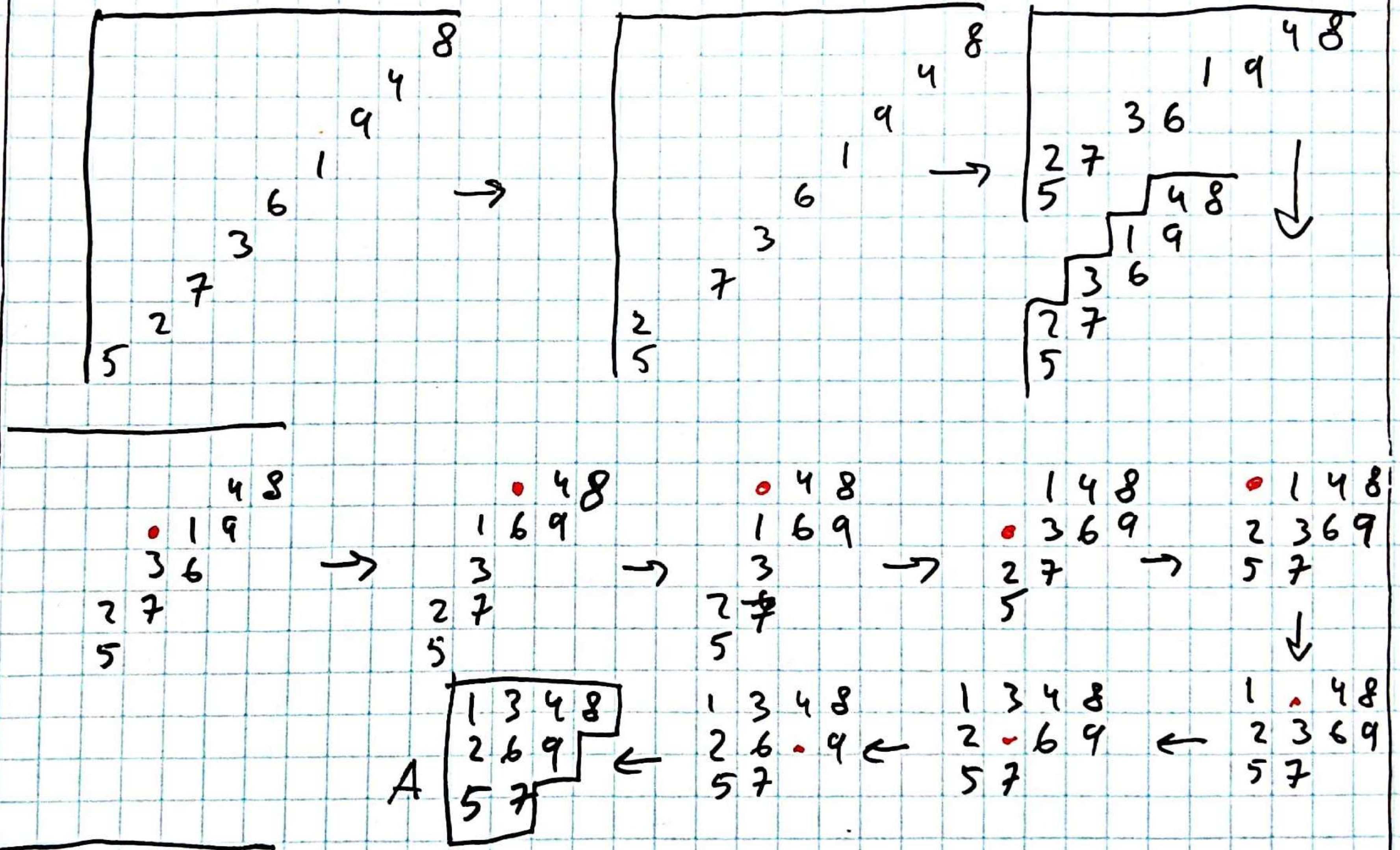
$e(P) \leq \prod r_i^{r_i}$ $\quad \forall P$ w/ LYM and $|R_i| = r_i$

③

Schützenberger's Promotion

(1) Jeu-de-taquin

$\sigma = (5\,2\,7\,3\,6\,1\,9\,4\,8)$



Th [Schützenberger, 1980] $RSK(\sigma) = (A, B)$

Permutation Posets

L10

$\sigma \in S_n$, $\quad P_\sigma = ([n], \prec) \leftarrow i \prec j \iff \underline{i<j,\ \sigma(i)<\sigma(j)}$

$$\begin{cases} a_k(P_\sigma) = \text{max size of } k \text{ increasing subs} \\ b_k(P_\sigma) = \text{—//— //— decreasing —//—} \end{cases}$$

Th [Greene, 1974] $\alpha(P_\sigma) = \beta(P_\sigma)' = \alpha(P_{\sigma^{-1}}) = \lambda$

where λ is given by RSK: $S_n \overset{RSK}{\longleftrightarrow} \bigcup_{\lambda \vdash n} SYT(\lambda)^2$

RSK = Robinson-Schensted (-Knuth) corresp.

$RSK(\sigma) = (A, B)$, shape(A) = shape(B) = λ

Ex $n = 9$, $\sigma = (5\,2\,7\,3\,6\,1\,9\,4\,8)$

RSK

$$5 \to \begin{array}{l} 2 \\ 5 \end{array} \to \begin{array}{ll} 2 & 7 \\ 5 \end{array} \to \begin{array}{ll} 2 & 3 \\ 5 & 7 \end{array} \to \begin{array}{lll} 2 & 3 & 6 \\ 5 & 7 \end{array} \to \begin{array}{lll} 1 & 3 & 6 \\ 2 & 7 \\ 5 \end{array}$$

$$\to \begin{array}{llll} 1 & 3 & 6 & 9 \\ 2 & 7 \\ 5 \end{array} \to \begin{array}{llll} 1 & 3 & 4 & 9 \\ 2 & 6 \\ 5 & 7 \end{array} \to \boxed{\begin{array}{llll} 1 & 3 & 4 & 8 \\ 2 & 6 & 9 \\ 5 & 7 \end{array}}\ A$$

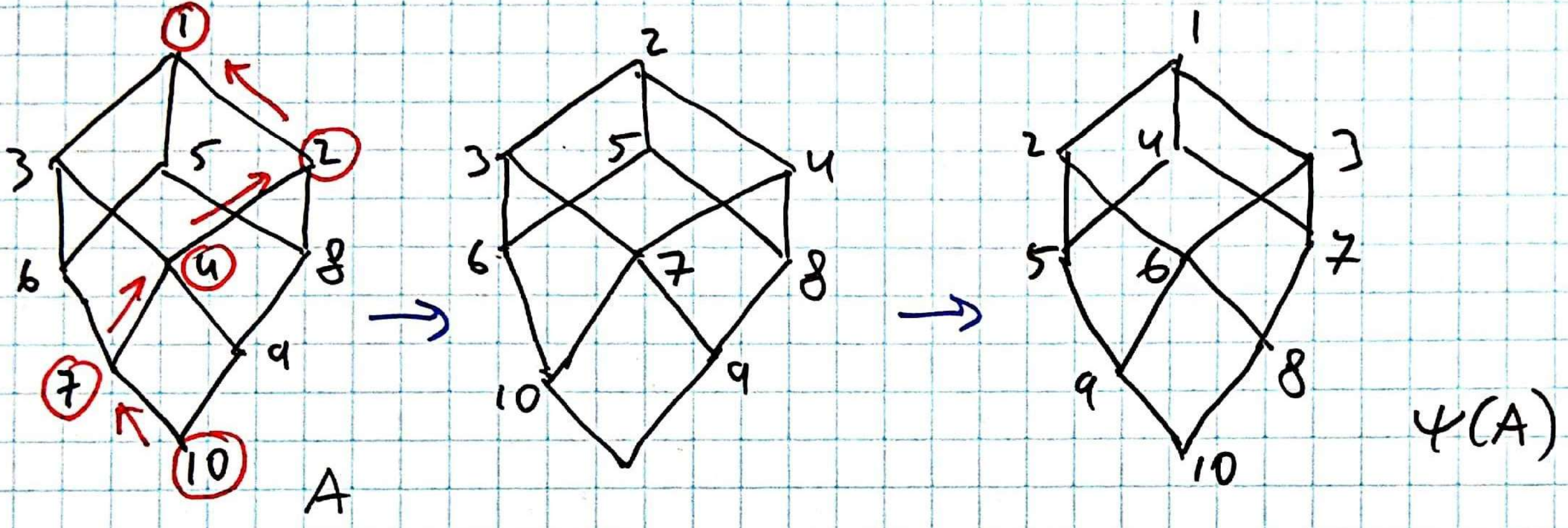
$$\boxed{\begin{array}{llll} 1 & 3 & 5 & 7 \\ 2 & 4 & 9 \\ 6 & 8 \end{array}}\ B \qquad \underline{\lambda = (4\,3\,2)}$$

Note $a_1 = \lambda_1 = LIS(\sigma)$

[Schensted, 1961]

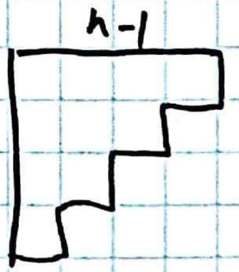
(2) General LE of posets

Promotion [Schützenberger, 1968] Ψ



Th [Lascoux-Schützenberger]

$\#\left[\text{max chains in } \text{Bruhat}_n\right] = \#\text{SYT of}$



[Stanley, 1980]

$= \#$ reduced factorizations

of $(n\ n-1 \ldots\ 2\ 1) = (i_1\ i_1+1)\,(i_2\ i_2+1) \ldots$
$\cdot\,(i_N\ i_N+1)$

$N = \binom{n}{2} = \text{inv}(n\ n-1 \ldots 1)$

$$A = \begin{array}{lll} 1 & 2 & 6 \\ 3 & 5 & \\ 4 & & \end{array} \qquad \psi(A) = \begin{array}{lll} 2 & 5 & 6 \\ 3 & \sqcup & \\ 4 & & \end{array} \rightarrow \begin{array}{lll} 1 & 4 & 5 \\ 2 & 6 & \\ 3 & & \end{array} \qquad ②$$

$$\begin{array}{lll} 1 & 4 & 5 \\ 2 & 6 & \\ 3 & & \end{array} \rightarrow \begin{array}{lll} 2 & 4 & 5 \\ 3 & 6 & \\ \sqcup & & \end{array} \rightarrow \begin{array}{lll} 1 & 3 & 4 \\ 2 & 5 & \\ 6 & & \end{array} \qquad ①$$

$$\begin{array}{lll} 1 & 3 & 4 \\ 2 & 5 & \\ 6 & & \end{array} \rightarrow \begin{array}{lll} 2 & 3 & 4 \\ 5 & \sqcup & \\ 6 & & \end{array} \rightarrow \begin{array}{lll} 1 & 2 & 3 \\ 4 & 6 & \\ 5 & & \end{array} \qquad ②$$

$$\begin{array}{lll} 1 & 2 & 3 \\ 4 & 6 & \\ 5 & & \end{array} \rightarrow \begin{array}{lll} 2 & 3 & \sqcup \\ 4 & 6 & \\ 5 & & \end{array} \rightarrow \begin{array}{lll} 1 & 2 & 6 \\ 3 & 5 & \\ 4 & & \end{array} \qquad ③$$

$$\begin{array}{lll} 1 & 2 & 6 \\ 3 & 5 & \\ 4 & & \end{array} \rightarrow \begin{array}{lll} 2 & 5 & 6 \\ 3 & \sqcup & \\ 4 & & \end{array} \rightarrow \begin{array}{lll} 1 & 4 & 5 \\ 2 & 6 & \\ 3 & & \end{array} \qquad ②$$

$$\begin{array}{lll} 1 & 4 & 5 \\ 2 & 6 & \\ 3 & & \end{array} \to \begin{array}{lll} 2 & 4 & 5 \\ 3 & 6 & \\ 4 & & \end{array} \to \begin{array}{lll} 1 & 3 & 4 \\ 2 & 5 & \\ 6 & & \end{array} \qquad ①$$

$\Rightarrow$ reduced factorization

$$\underline{212\ 321} \longrightarrow \underbrace{(23)(12)(23)}(34)(23)(12)$$

$$= (32145)(34)(23)(12)$$

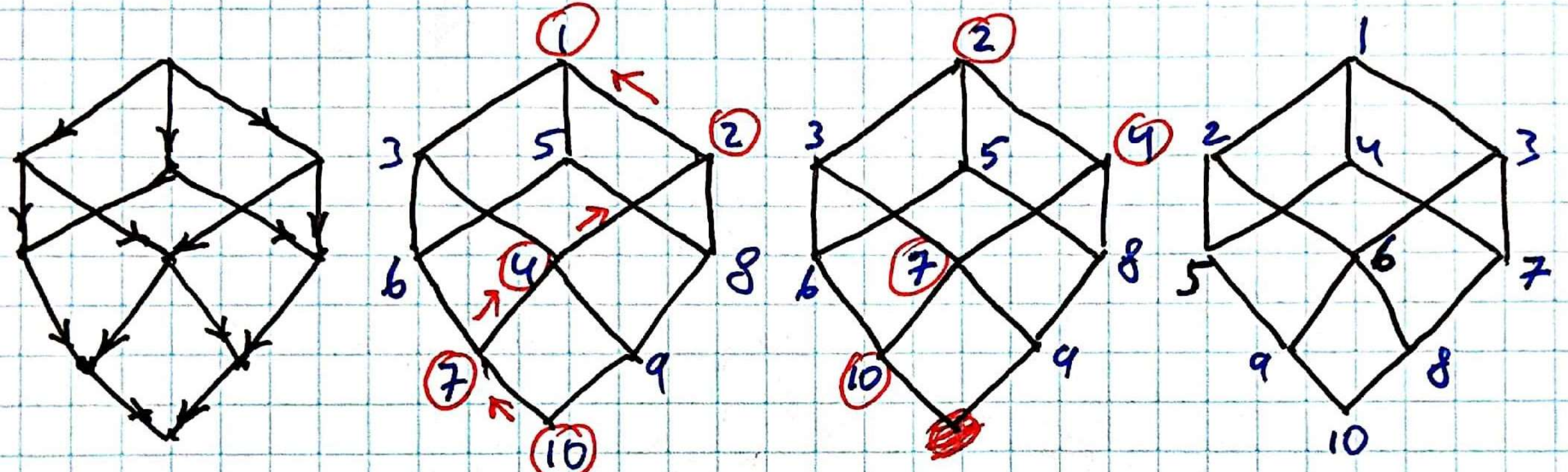
$$= (54321)$$

✓

L16 Linear Extensions & Actions

206A
Nov 9, 2020

Recall: promotion [Schützenberger, 1972] Ψ



$P = (X, \prec)$, $\psi : LE(P) \to LE(P)$ Bij

Last time: iterated promotion was used to obtain a bijection of $SYT(n-1, n-2, \ldots 2, 1)$ and max chains in Bruhat_n

[See Ex in L15
/no proof/

/Lascoux-Schützenberger bijection/

[Edelman-Greene
[Garsia]

①

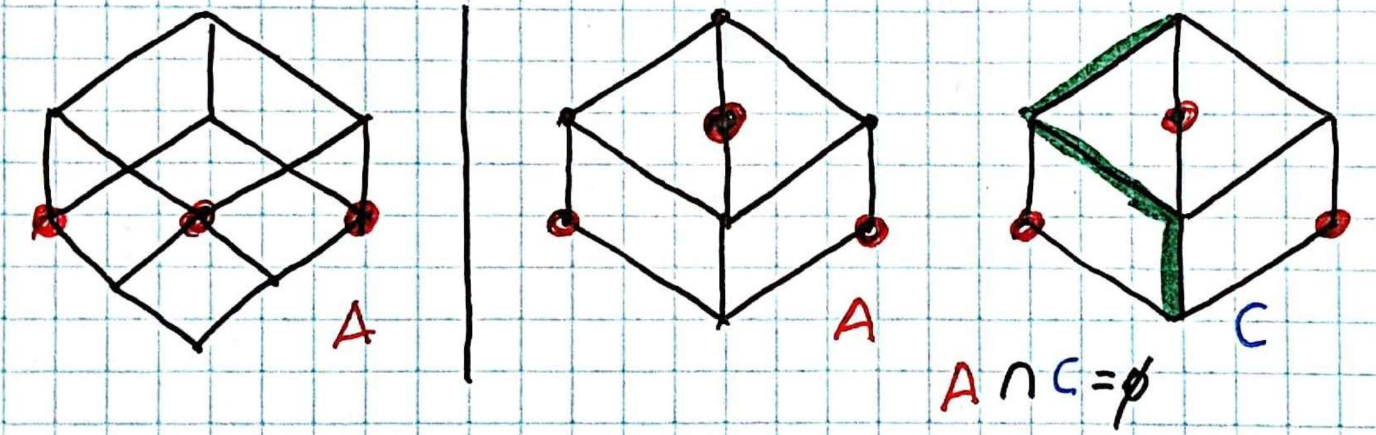
Th [Edelman-Hibi-Stanley, 1989]

Let $P=(X,\prec)$, $A \subset X$ max antichain

Further, suppose A intersects every max chain in P. Then

$$e(P) = \sum_{x\in A} e(P-x)$$

Ex



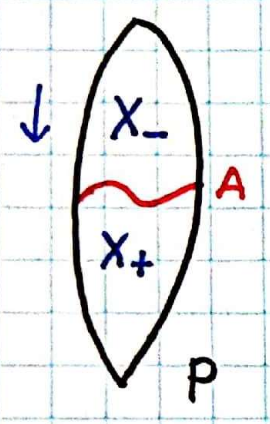
Obs $A = \{\text{min elts in } P\}$

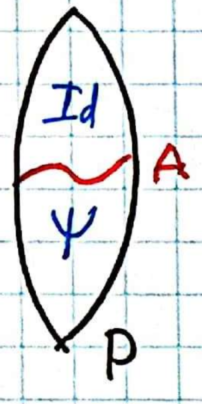
$\Rightarrow$ A intersect every max chain C in P

$\Rightarrow$ $e(P) = \sum_{x\in A} e(P-x)$ ← obvious

Proof of E-H-S Thm

Construct a bijection $\boxed{\Phi : LE(P) \leftarrow \bigcup_{x\in A} LE(P-x)}$





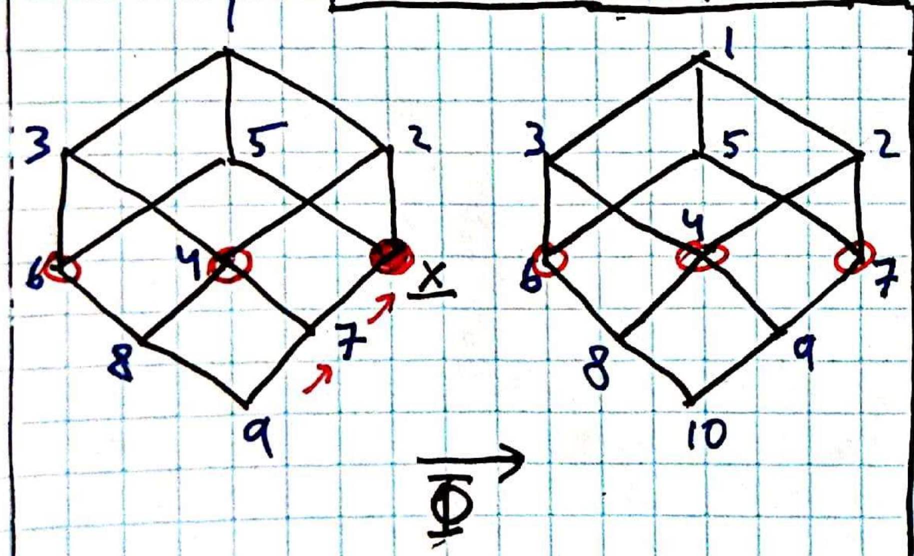
$X = X_- \sqcup X_+$, where

$X_- = \{x \in X : x < a,\ a \in A\}$

$X_+ = \{x \in X : x > a,\ a \in A\}$

Let $Q \in LE(P-x)$, $x \in A$

$\Phi(Q) := Q|_{X_-} + \tilde{\Psi}(Q|_{X_+ +x})$ ← no relabeling



Claim 1) Φ is valid

2) Φ is invertible

For 2) use $\tilde{\Psi}^{-1}|_{X_+ \cup A}$ —//—

▨

$x \in A$

Note: algorithmically useless

③

Th $P = (X, \leq)$, $P' = (X', \leq')$,
$\Gamma = \mathrm{Com}(P)$, $\Gamma' = \mathrm{Com}(P')$ comparability graphs

Then $\Gamma \simeq \Gamma' \Rightarrow e(P) = e(P')$

D In Γ we have:

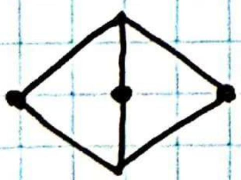
antichains $\longleftrightarrow$ indep sets
chains $\longleftrightarrow$ cliques
$\Big\}$ max

$\Rightarrow$ assumptions in EHS Th are graph theoretic
/depend only on Γ/

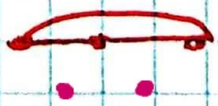
Obs: $\mathrm{Com}(P - x) = \mathrm{Com}(P)|_{X-x}$
/induced subgraph/

$\Rightarrow$ Th follows by induction ▨

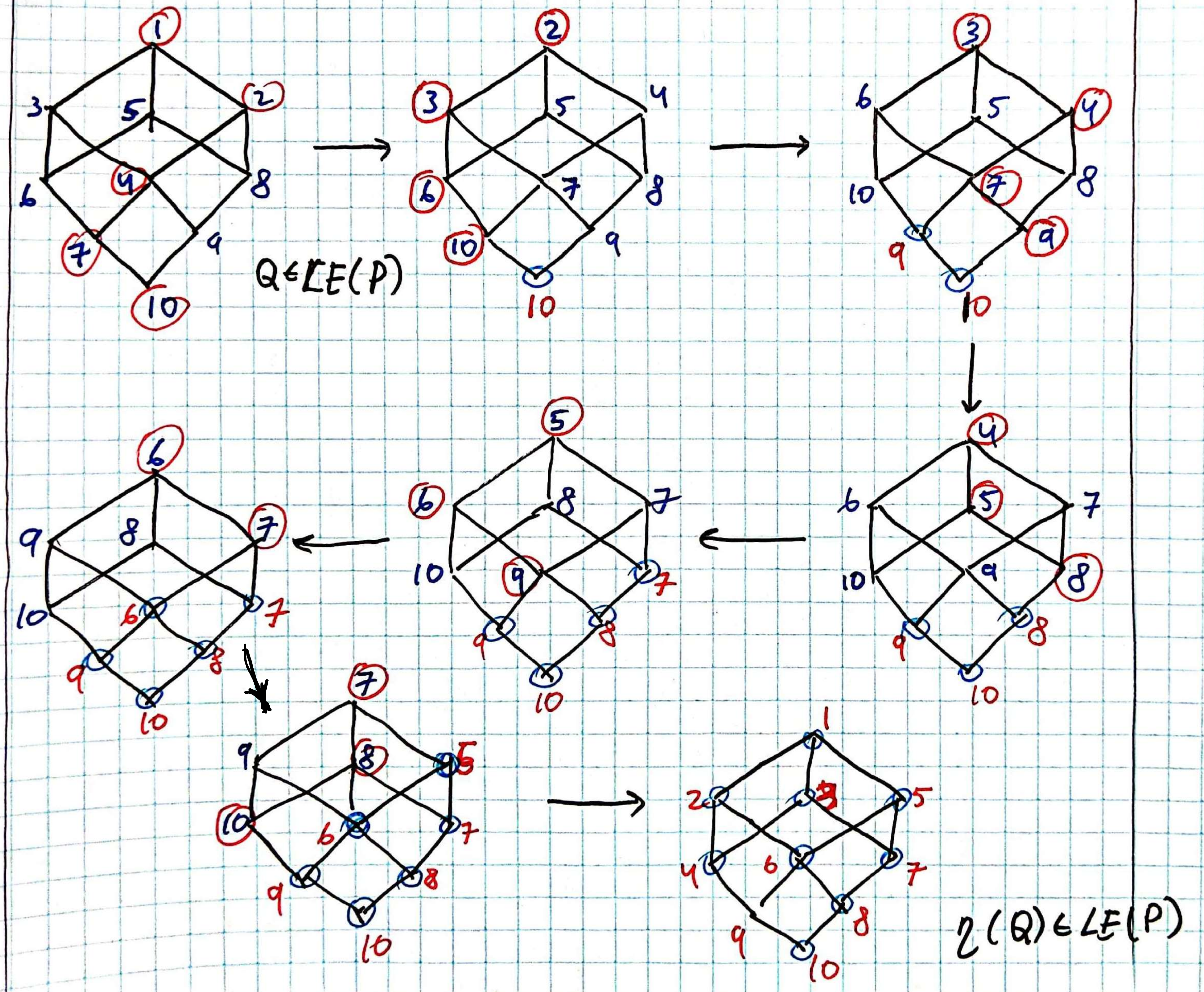
Ex



$\overline{\Gamma}$ = incomp of P, P'

=  $\simeq K_3 + K_1 + K_1$,

Evacuation [Schützenberger, 1972] η



⑤

Th [Schützenberger, 1972] $P = (X, *)$, $|X| = n$

[stanley survey]

Let ψ – promotion on P

ψ^* – dual promotion on $P \leftarrow [\psi(P^*)]^*$

η – evacuation on P

η^* – dual evacuation on $P \leftarrow [\eta(P^*)]^*$

Then

1) $\psi^* = \psi^{-1}$
2) $\eta^2 = (\eta^*)^2 = 1$
3) $\psi^n = \eta\,\eta^*$
4) $\psi\eta = \eta\psi^{-1}$

Proof idea

$S_n = \langle \tau_1 \ldots \tau_{n-1} \rangle$

$\tau_i = (i, i+1)$

$\delta := \tau_1 \ldots \tau_{n-1}$

/Coxeter element/

Note

$S_n = G / \langle (\tau_i \tau_{i+1})^3 = 1 \;\forall i \rangle$

$\gamma := (\tau_1 \tau_2 \ldots \tau_{n-1})(\tau_1 \tau_2 \ldots \tau_{n-2}) \ldots (\tau_1 \tau_2)\, \tau_1$

$\gamma^* := (\tau_{n-1} \ldots \tau_2 \tau_1)(\tau_{n-2} \ldots \tau_3 \tau_2) \ldots (\tau_{n-1} \tau_{n-2})\, \tau_{n-1}$

$G = \langle \tau_1 \ldots \tau_{n-1} \rangle / \tau_i^2 = (\tau_i \tau_j)^2 = 1, \quad |i-j| > 1$

Coxeter group

⑥

$\underline{L}$ [Haiman, Malvenuto-Reutenauer]

In G <u>we</u> <u>have</u>:

(a) $\gamma^2 = (\gamma^*)^2 = 1$

(b) $\delta^n = \gamma\gamma^*$

(c) $\delta\gamma = \gamma\delta^{-1}$

D $\gamma^2 = (\tau_1 \tau_2 \tau_3 \tau_1 \cancel{\tau_2} \cancel{\tau_1})(\cancel{\tau_1} \cancel{\tau_2} \tau_3 \tau_1 \tau_2 \tau_1)$

$= \tau_1 \tau_2 \overset{\curvearrowright}{\tau_3 \tau_1} \qquad \tau_3 \tau_1 \tau_2 \tau_1$

$= \tau_1 \tau_2 \cancel{\tau_1} \cancel{\tau_3} \qquad \cancel{\tau_3} \cancel{\tau_1} \tau_2 \tau_1$

$= \tau_1 \tau_2 \qquad \tau_2 \tau_1$

$= 1$ $\qquad$ $/n=4/$

(+) more of the same ▨

D (of Sch. Thm) $\underline{\tau_i : LE(P) \to LE(P)}$, $X=[n]$

$\tau_i(Q) = Q'$, where $Q' = Q|_{[n]\setminus i,i+1}$ $\boxed{\tau_i : \ i \leftrightarrow i+1}$ if possible ☑

⑦

L17

Domino Tableaux

206A
11/13/2020

Recall: $G_n = \langle \tau_1 \ldots \tau_{n-1} \rangle / \left(\tau_i^2 = (\tau_i \tau_j)^2 = 1 \ \ \forall \ |i-j| \geqslant 2 \right)$

[Schützenberger]
[Stanley]

$\pi : G_n \longrightarrow S_n$, $\tau_i \to (i \ \ i+1)$ /Coxeter group/

$\delta := \tau_1 \cdot \ldots \cdot \tau_{n-1}$ $\pi : \delta \to (1 2 \ldots n)$ Coxeter el't

$\gamma := (\tau_1 \tau_2 \ldots \tau_{n-1})(\tau_1 \tau_2 \cdots \tau_{n-2}) \cdot \ldots \cdot (\tau_2 \tau_1) \tau_1$

$\gamma^* := (\tau_{n-1} \ldots \tau_2 \tau_1)(\tau_{n-1} \ldots \tau_3 \tau_2) \cdot \ldots \cdot (\tau_{n-1} \tau_{n-2}) \tau_{n-1}$

$\pi(\gamma) = \pi(\delta^*) = (n \ n-1 \ldots 2 \ 1)$ max el't

---

G_n acts on $LE(P)$, $P = (X, \prec)$, $|X| = n$

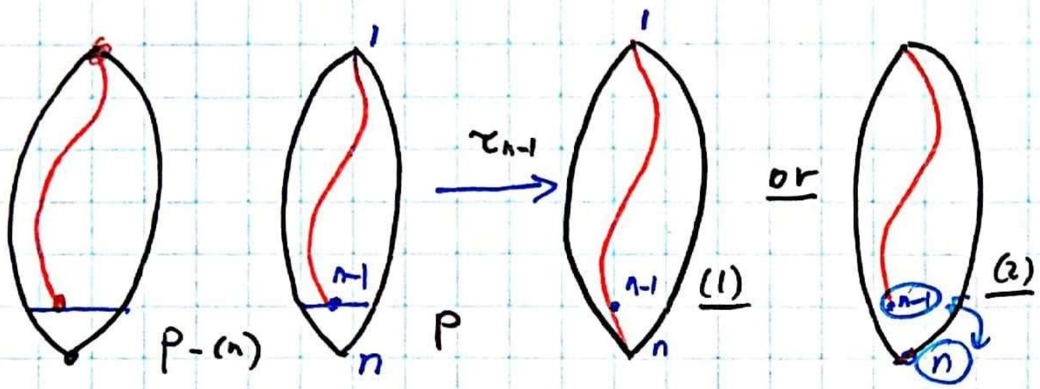
$\tau_i : Q \to Q'$, $Q' \leftarrow$ switch (i) and $(i+1)$ in Q if possible, $Q, Q' \in LE(P)$

L [Haiman, Malvenuto-Reutenauer] $\forall$ $P = (X, \prec)$, $|X| = n$

δ, δ^{-1} act on $LE(P)$ as promotion, dual prom. $\underline{\psi, \psi^*}$

γ, γ^* act on $LE(P)$ as evacuation, dual evac. $\underline{2, 2^*}$

Proof idea (by induction)



2 cases

(1) $(n-1) \prec (n)$

(2) $(n-1) \not\prec (n)$

$\checkmark \begin{cases} \delta_{n-1}\ (12\ldots n-1) \leftarrow \text{promotion w/o relabeling on } P-(n) \ /\text{by ind}/ \\ \Rightarrow \delta_n\ (12\ldots n) \leftarrow \text{— || — on } P \end{cases}$

+ same for evac, dual evac. ▨

Th [Schützenberger, 1972] $P=(X,\prec)$, $|X|=n$, ψ, ψ^* **prom**; η, η^* **evac**

Then $\psi^* = \psi^{-1}$, $\eta^2 = (\eta^*)^2 = 1$

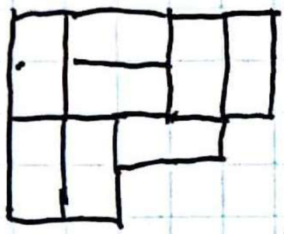
$\psi^n = \eta\eta^*$, $\psi\eta = \eta\psi^{-1}$

D L ⇒ Th via identities in S_n ▨

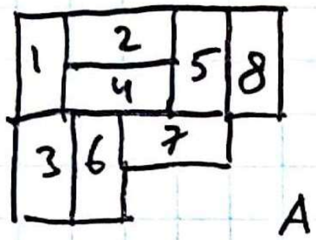
Note/Exc $(\tau_i \tau_{i+1})^6 = 1$ $\forall 1 \le i \le n-1$ [Berenstein-Kirillov 2000]

②

Domino tilings and domino tableaux



λ



$A \in DT(\lambda)$

$\lambda \vdash n = 2k$

$A \leftarrow$ increasing labeling $1 .. k$

Q1: $\# DT(\lambda) = ??$

Q2: What are $DT(P)$, $P = (X, <)$?

Def $Q \in LE(P)$, $|X| = 2k$ is a P-domino (tableau)

iff Q-label $(1) < (2)$, Q-label $(3) < (4)$, - - -

Th $LE_2(P) = \{P\text{-dominoes}\}$, $e_2(P) := \# LE_2(P)$

Then $e_2(P) = \# \{ Q \in LE(P) : \eta(Q) = Q \}$

Note Th $\leftarrow$ [Stanley], direct generalization of [B-K], [Stembridge] and [van Leeuwen, 1996]

$$\left\{ \begin{array}{l} SYT \rightarrow LE \\ DT \rightarrow LE_2 \end{array} \right\}$$

L $\delta_i := \tau_1 \tau_2 \cdots \tau_i$, $\delta_i^* = \tau_i \cdots \tau_2 \tau_1$, $u, v \in S_n$

Then $u \tau_1 \tau_3 \cdots \tau_{2j-1} = v \iff$

$$u(\delta_1^* \delta_3^* \cdots \delta_{2j-1}^*) = v(\delta_1^* \delta_3^* \cdots \delta_{2j-1}^*)\underbrace{(\delta_{2j-1} \cdots \delta_2 \delta_1)}_{\gamma}$$

D (sketch) [By ind]

$j=1$ $u \tau_1 = v \tau_1 \tau_1 \iff u \tau_1 = v \leftarrow \tau_1^2 = 1$

$j=2$

$$u \tau_1 (\tau_3 \underbrace{\tau_2 \tau_1}) = v \cancel{\tau_1} \cancel{(\tau_3 \tau_2 \tau_1)} \cancel{(\tau_1 \tau_2 \tau_3)} (\cancel{\tau_1} \tau_2) \tau_1$$

$$= v \underbrace{\tau_2 \tau_1}$$

$\iff u \tau_1 \tau_3 = v$ ▨

Proof of Th $w \in LE(P)$, $\eta(w) = w$

$\iff w = w(\tau_1 \tau_2 \cdots \tau_{n-1})(\tau_1 \cdots \tau_{n-2}) \cdots (\tau_2 \tau_1) \tau_1$

$\boxed{L \Rightarrow (u = v = w)}$ $w \tau_{2k-1} \tau_{2k-3} \cdots \tau_3 \tau_1 = w$

$u \in LE_2(P)$ $\underset{L}{\Rightarrow} \tilde{w} := w(\tau_1)(\tau_3 \tau_2 \tau_1) \cdots (\tau_n \cdots \tau_1)$

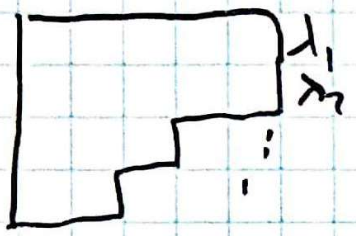
$\tilde{w} \gamma = \tilde{w}$

$\Rightarrow$ $w \to \tilde{w}$ is a Bij ▨

④

Young Lattice $\mathbb{Y} = (\{\lambda\}, \subseteq)$

$\lambda \subset \mathbb{N}^2$ partition $\lambda = (\lambda_1 \lambda_2 \dots)$



$\lambda \lessdot \mu \Leftarrow \lambda \subset \mu$

$P_\lambda \leftarrow$ pricipal order ideal in $\mathbb{N}^2$

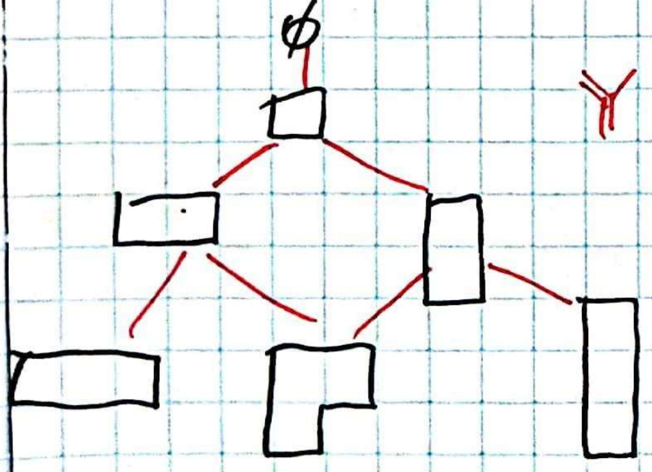
$e(P_\lambda) = \# SYT(\lambda) = \#$ max chains $\emptyset \to \lambda$ in $\mathbb{Y}$

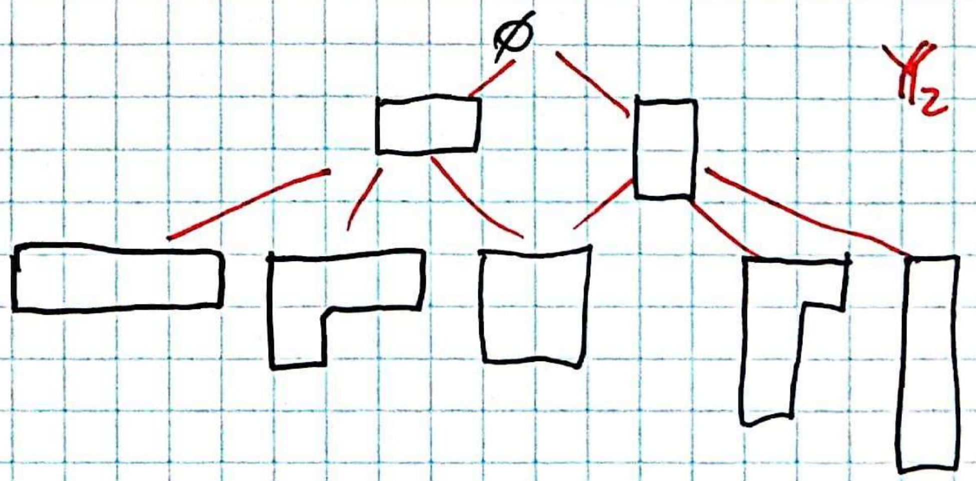
$e_2(P_\lambda) = \# DT(\lambda)$ domino tableaux

$= \# LE_2(P_\lambda)$

$= \#$ chains $\emptyset \to \mu^{(1)} \to \mu^{(2)} \to \dots \to \mu^{(k)} = \lambda$

s.t. $\mu^i \vdash 2i$ and $\mu^{(i)} \setminus \mu^{(i-1)}$ = domino





Th [Fomin-Stanton, 1997] / also [Stanton-White 1985]

$\mathbb{Y}_2 = \mathbb{Y} \times \mathbb{Y}$

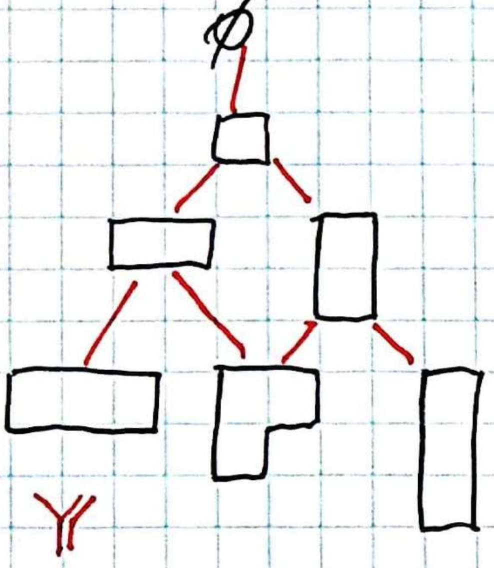
$\Longleftrightarrow \forall \lambda \in \mathbb{Y}_2 \quad \exists \mu, \nu \in \mathbb{Y}$

$$DT(\lambda) = \binom{|\mu|+|\nu|}{|\mu|} \underbrace{\#SYT(\mu)}_{HLF} \underbrace{\#SYT(\nu)}_{HLF}$$

⑥

L18 Domino tableaux & the HLF

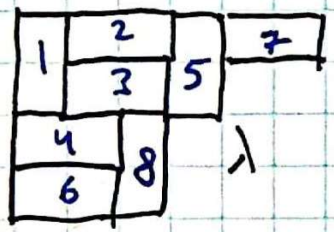
206A
Nov 16, 2020



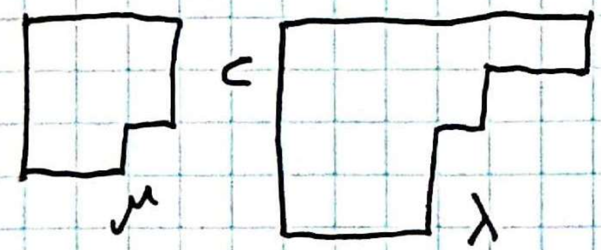
Recall: Young Lattice $\mathbb{Y}$

$\mathbb{Y} = (\{\lambda, |\lambda| \geq 0\}, \subset)$

$\mathbb{Y} = J(\mathbb{N}^2)$



Domino tableaux – domino tiling of λ w/ labeling increasing ↓ and →



Domino Lattice $\mathbb{D} = \mathbb{Y}_2 \subset \mathbb{Y}$

$\mathbb{D} = \left(\left\{ \lambda \text{ - tileable w/ dominoes}, \begin{matrix} |\lambda| \geq 0 \\ \text{even} \end{matrix} \right\}, \subset \right)$

Note $\mu \subset \lambda$, $\mu, \lambda \in \mathbb{D}$

$\Rightarrow$ λ/μ is tileable w/ dominoes

Th [Fomin-Stanton, 1997]

$\mathbb{D} = \mathbb{Y}^2$

$e_2(\lambda) := \# LE_{\mathbb{D}}(\lambda) \in FP$

①

Summary: $\mathbb{Y} = J(\mathbb{N}^2)$, $\mathbb{D} = \mathbb{Y}^2 \Rightarrow \mathbb{D} = J(\mathbb{N}^2 + \mathbb{N}^2)$

$e_2(\lambda) = \# DT(\lambda) = \# LE(\underline{I_\lambda} \text{ in } \mathbb{D})$

principal order ideal

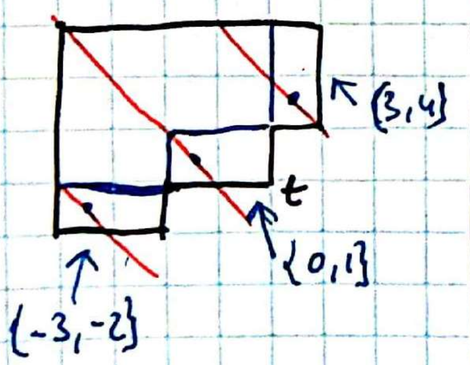
Prop $P = Q + R$ ← posets of size a and b

$\Rightarrow e(P) = \binom{a+b}{a} e(Q)\, e(R)$ | **Proof** ← Exc ■

Proof of F-S Thm $\mathbb{D} = \{\lambda \leftarrow$ tileable w/ dominoes$\}$

Bijection $\phi : \mathbb{D} \to \mathbb{Y} \times \mathbb{Y}$

(By induction) $\phi : \emptyset \to (\emptyset, \emptyset)$ ✓



$\lambda \in \mathbb{D}$, $t =$ domino, $\lambda - t \in \mathbb{D}$

$\phi(\lambda - t) = (\mu, \nu)$

$$\phi(\lambda) := \begin{cases} (\mu + \square_i, \nu) \\ (\mu, \nu + \square_i) \end{cases} \quad \text{where}$$

$$\begin{cases} t \in \{(2i),(2i+1) \text{ diagonals in } \lambda\} \\ t \in \{(2i-1),(2i) \text{ diagonals in } \lambda\} \end{cases}$$

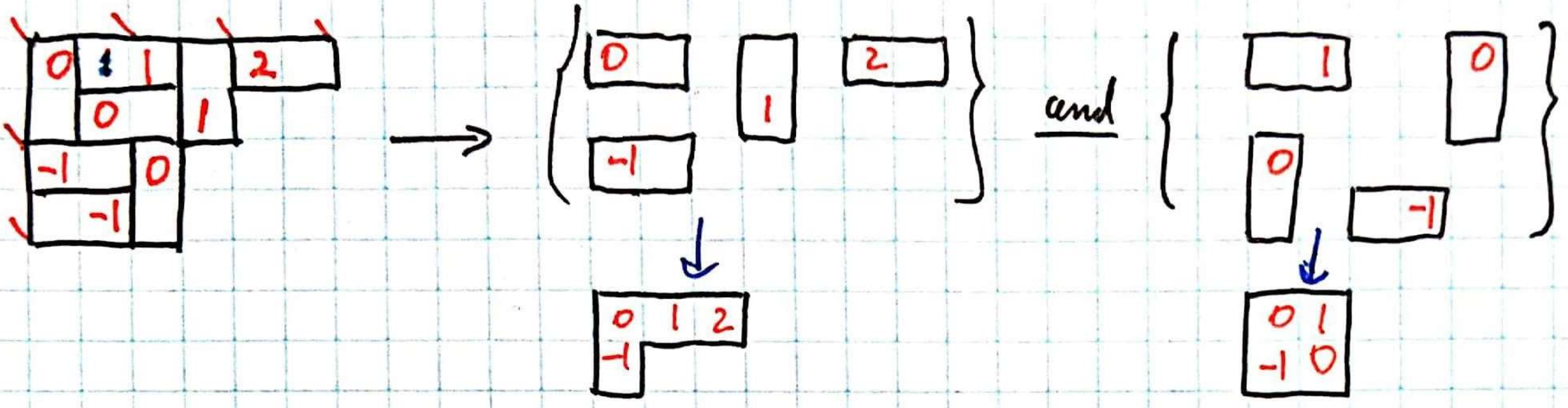
▨

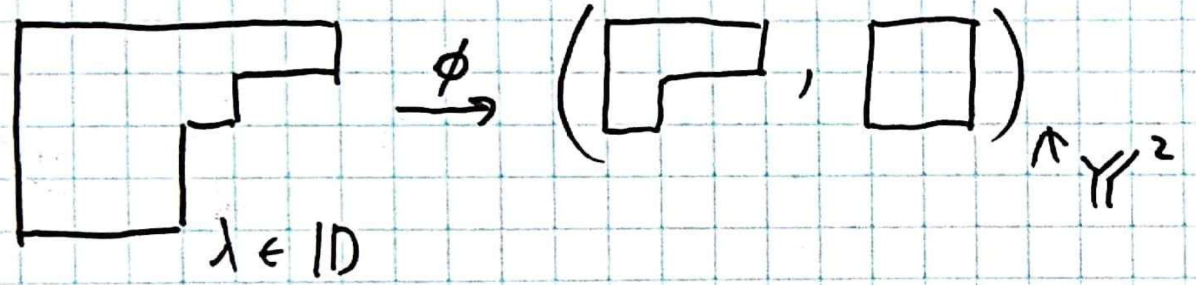
$$\begin{Bmatrix} (\mu, \nu + \square_2) \\ (\mu + \square_0, \nu) \\ (\mu, \nu + \square_{-1}) \end{Bmatrix}$$

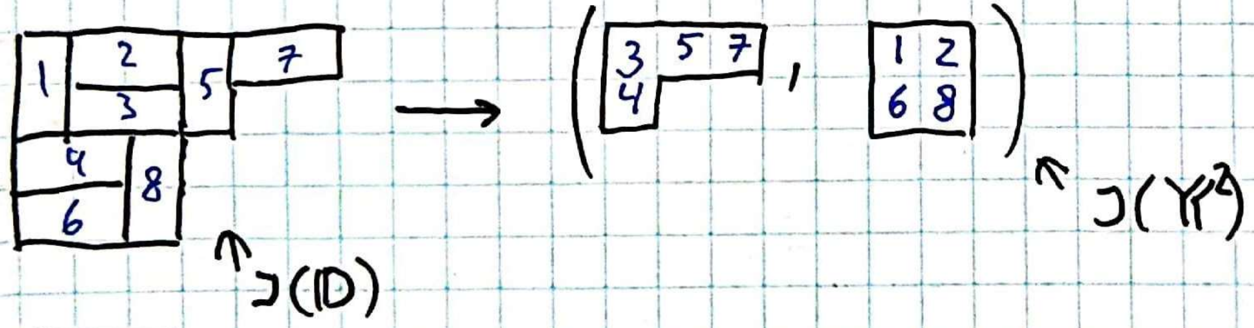
i-th diagonal $= \{(x,y) : x - y = i\}$

②

Ex





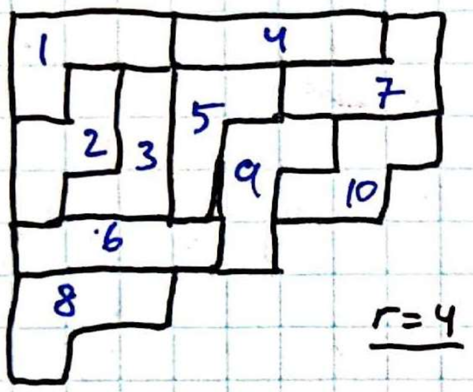


Cor $\phi(\lambda) = (\mu, \nu) \Rightarrow e_2(\lambda) = \binom{a+b}{a} e(\mu)\, e(\nu)$

$\Leftrightarrow \#DT(\lambda) = \binom{a+b}{a} \#SYT(\mu) \cdot \#SYT(\nu)$

③

Note:



More generally **F-S Th** proves that Lattice of r-ribbon tileable diagrams $\simeq \mathbb{Y}^r$

[Fomin–Stanton]
[Pak]

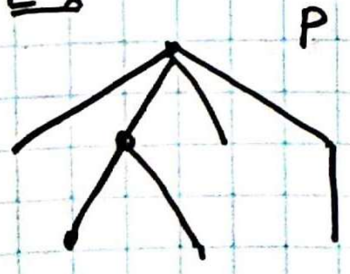
Proof: same idea, mod r

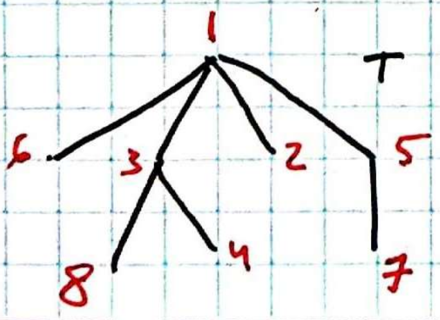
Poset Sorting

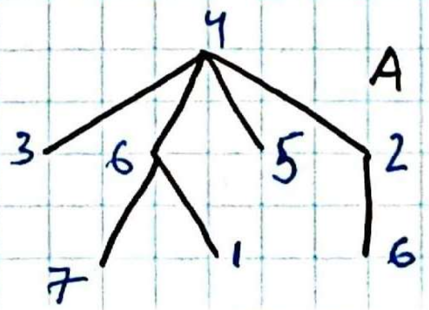
$P = (X, \prec)$ finite poset, $T \in \mathcal{LE}(P)$ fixed labeling

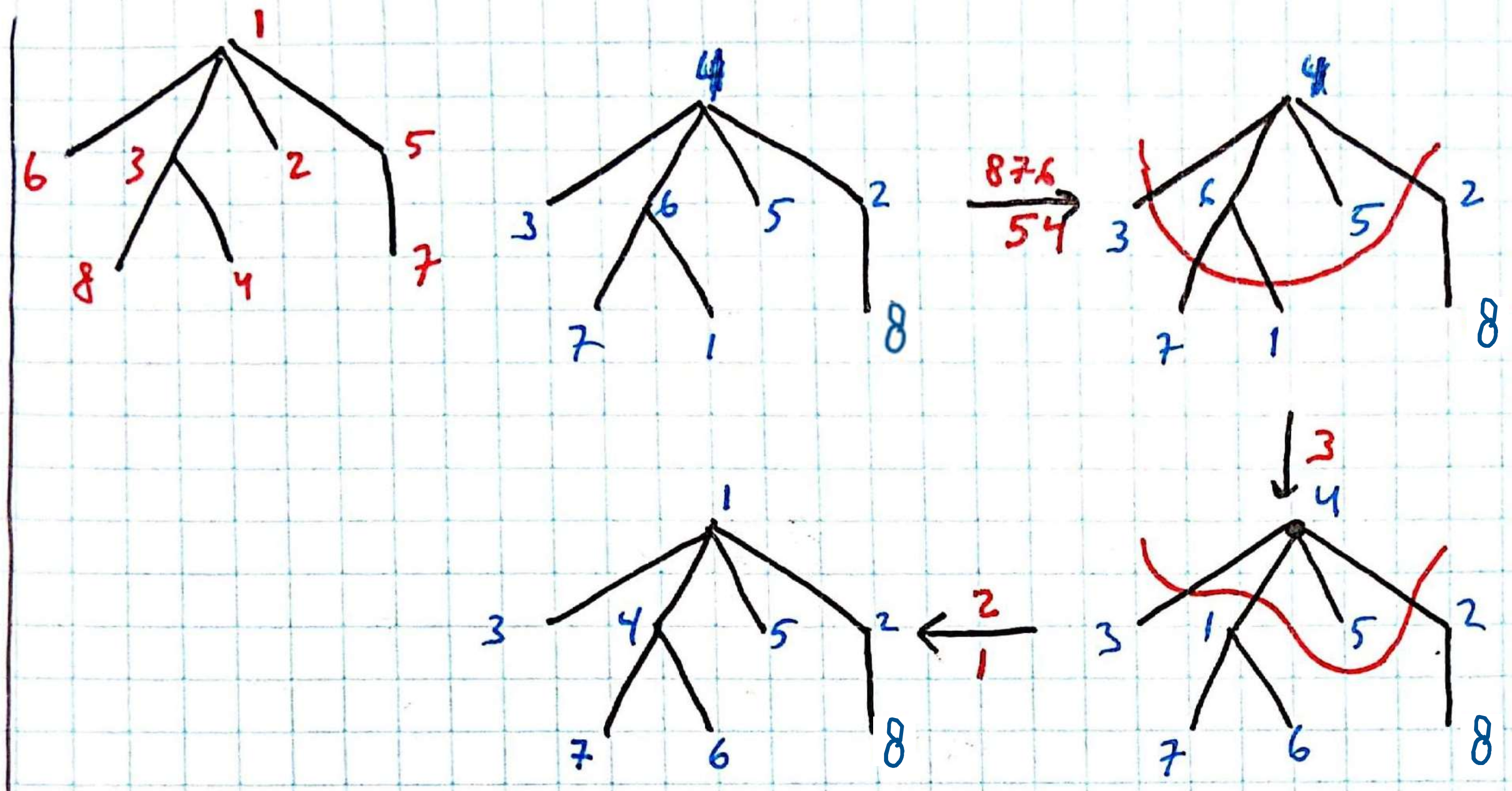
```
Alg  $P = (X,\prec)$, $|X| = n$, $A \in \Sigma(P)$, $B := A$
For $i = n \ldots 1$
    Demote-Promote in B elements in $\{T^{-1}(n), \ldots, T^{-1}(i)\}$
Output B
```

Ex

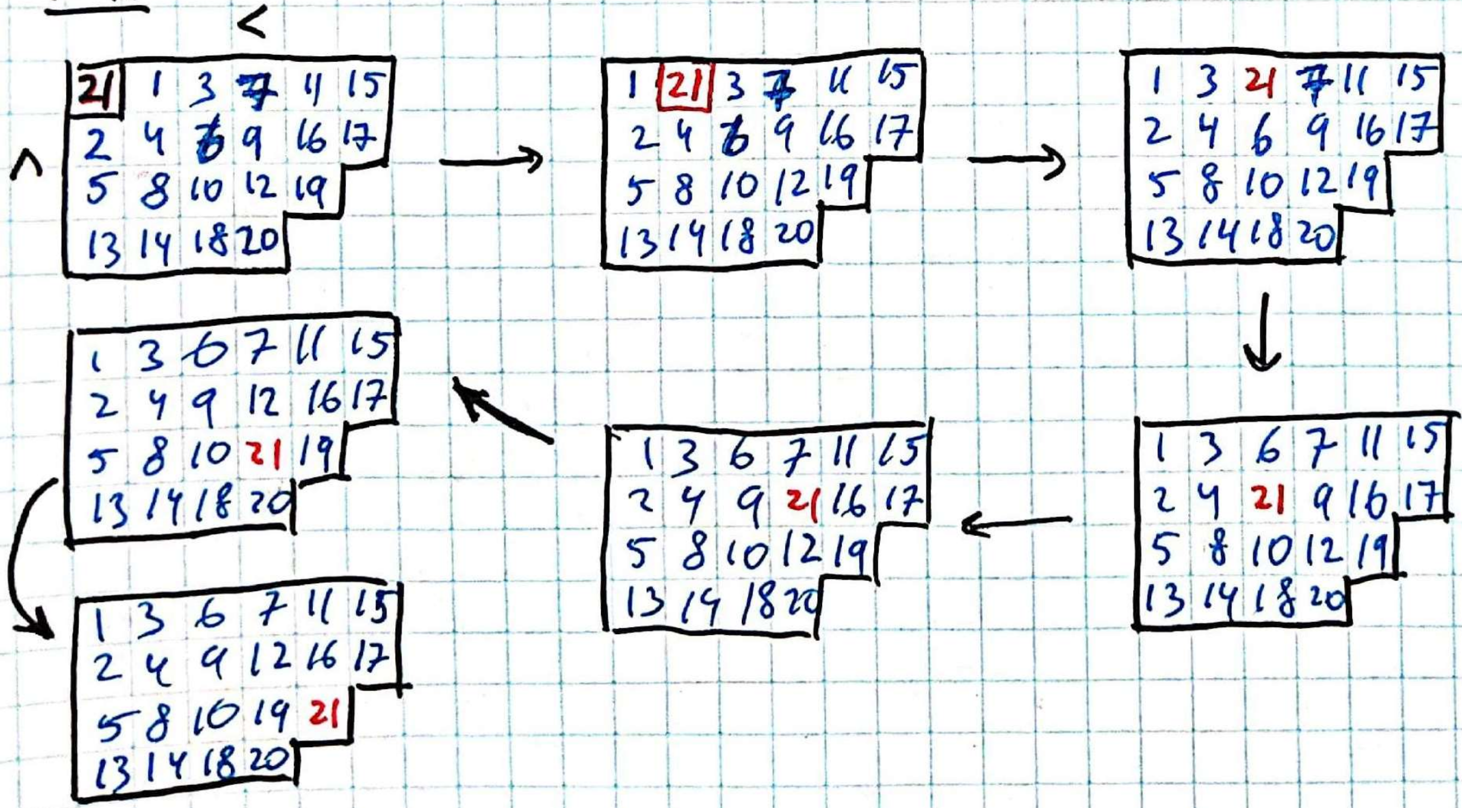








Ex



21	1	3	7	11	15
2	4	6	9	16	17
5	8	10	12	19	
13	14	18	20		

1	21	3	7	11	15
2	4	6	9	16	17
5	8	10	12	19	
13	14	18	20		

1	3	21	7	11	15
2	4	6	9	16	17
5	8	10	12	19	
13	14	18	20		

1	3	6	7	11	15
2	4	21	9	16	17
5	8	10	12	19	
13	14	18	20		

1	3	6	7	11	15
2	4	9	21	16	17
5	8	10	12	19	
13	14	18	20		

1	3	6	7	11	15
2	4	9	12	16	17
5	8	10	21	19	
13	14	18	20		

1	3	6	7	11	15
2	4	9	12	16	17
5	8	10	19	21	
13	14	18	20		

Th Alg is equinumerous for

(1) partitions λ, $T \in SYT(\lambda)$ column ordering

(2) shifted partitions λ, —||—

(3) tree posets, $\forall T$

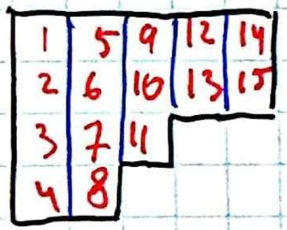
[P.-Stoyanovsky 1994]

[Fischer 2001]

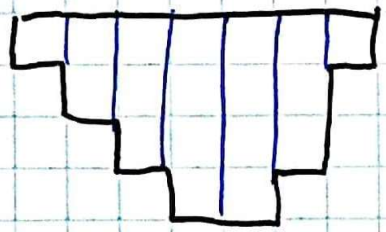
[Beata, 2012]

Def $\pi : X \to Y$ equinumerous

if $\forall\, y, y' \in Y$ $\quad |\pi^{-1}(y)| = |\pi^{-1}(y')|$



column ordering



shifted partition

Note: (3) ← Exc

(2) ← hard extension of (1)

(1) ← proof via Bijection

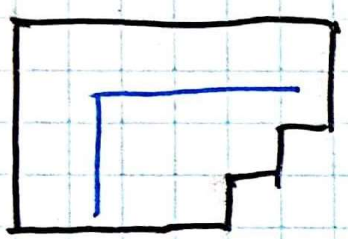
Bij $\varphi : S_n \longrightarrow SYT(\lambda) \times H(\lambda)$

$$\# H(\lambda) = \prod_{(i,j)\in\lambda} h(i,j)$$

Bijective Proof of the HLF

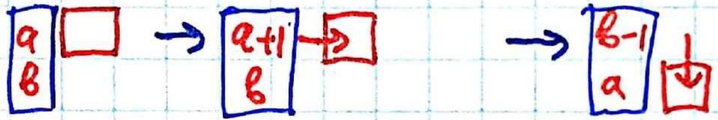
$$\varphi: \underset{\overset{\uparrow}{S_n}}{\sigma} \to \big(\underset{\overset{\uparrow}{SYT(\lambda)}}{B}, \underset{\overset{\uparrow}{H(\lambda)}}{g}\big), \quad g = (g(i,j))_{(i,j)\in\lambda}$$

$$\boxed{j - \lambda_j' \le g(i,j) \le \lambda_i - i}$$

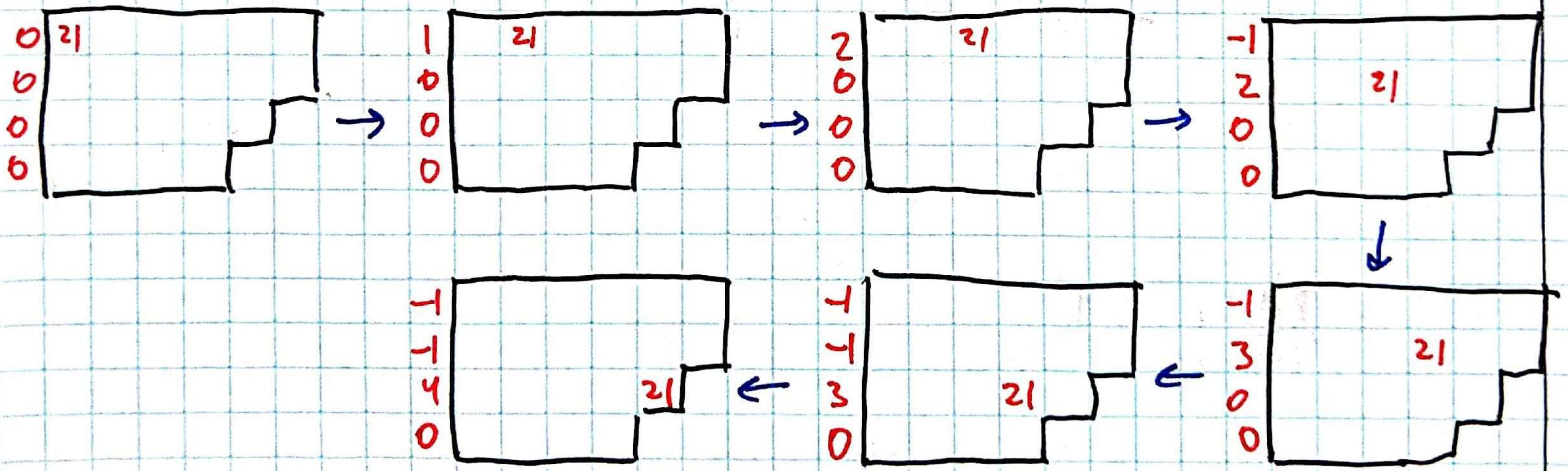


$h(2,2) = 7$ $\Rightarrow$ $\textcircled{-2} = 2-4 \le g(2,2) \le 6-2 = \textcircled{4}$

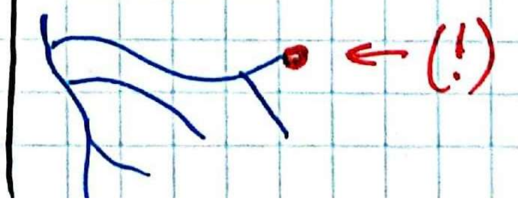
Idea



Ex



Proof idea: backtrack!



⑦

L19

P- partitions

206A
11/18/2020

Q: Why do LE of posets play major role?

A1: Max chains in corr distributive lattices

A2: P-partition theory

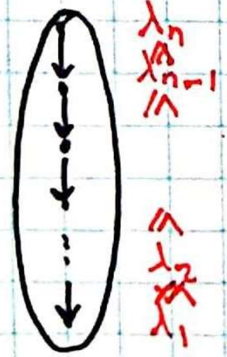
Def $P = (X, \prec)$, $|X| = n$

$f : X \to \mathbb{N} = \{0,1,2,..\}$ is called P-partition $\Big\}\ \mathbb{N}^P$

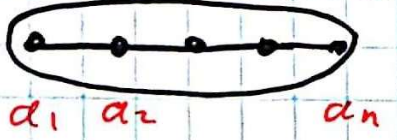
if $f(x) \leq f(x')$ $\forall\ x \prec x'$, $x, x' \in X$

$$\boxed{\mathcal{F}_P(t) := \sum_{f \in \mathbb{N}^P} t^{|f|}} \quad , \quad \boxed{|f| := \sum_{x \in X} f(x)}$$

Ex (1) $P = C_n \Rightarrow \mathcal{F}_P(t) = \sum_{\substack{(\lambda_1 \ldots \lambda_n) \\ \lambda_1 \geq \ldots \geq \lambda_n}} t^{|\lambda|} = \prod_{i=1}^{n} \frac{1}{1-t^i}$



(2) $P = A_n \Rightarrow \mathcal{F}_P(t) = \frac{1}{(1-t^n)}$



①

Th [Stanley, 1968]

$\forall\ P = (X, \prec)$ we have

$$\mathcal{F}_P(t) = \frac{\mathcal{L}_P(t)}{(1-t)(1-t^2)\cdots(1-t^n)}$$ where

$$\boxed{\mathcal{L}_P(t) := \sum_{A \in LE(P)} t^{\alpha(A)}}$$ some explicit α

$\alpha : LE(P) \to \mathbb{N}$

Ex (1) $P = C_n$, $e(C_n) = 1$, $\alpha(A) = 0$, $LE(C_n) = \{A\}$

so $\mathcal{F}_P = \prod_{i=1}^{n} \frac{1}{(1-t^i)}$

---

(2) $P = A_n$, $e(A_n) = n!$, $LE(A_n) = S_n$

so $\mathcal{F}_P = \prod_{i=1}^{n} \frac{1}{1-t} = \frac{\mathcal{L}_P(t)}{\prod_{i=1}^{n}(1-t^i)}$ where

$$\mathcal{L}_P(t) = \prod_{i=1}^{n} \frac{1-t^i}{1-t} = \prod_{i=1}^{n}(1+t+\ldots+t^{i-1})$$

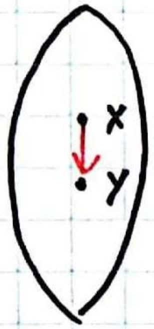
$$= \sum_{\sigma \in S_n} t^{\mathrm{inv}(\sigma)}$$

②

Prop $\mathcal{F}_P(t) = \sum_{N=0}^{\infty} a_P(N)\, t^N$

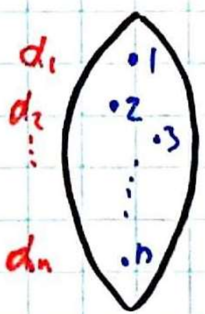
$$\Rightarrow \quad a_P(N) \sim e(P)\, \frac{n \cdot N^{n-1}}{(n!)^2}$$

▷ $f \in \mathbb{N}^P$, $f(X) \le M$ random, $M \to \infty$



$\Rightarrow f(x) < f(y) \quad \forall\, x \prec y \quad$ whp

Same for $|f| \le N$ random, $N \to \infty$



$X = \{1 \dots n\} \Rightarrow$ such f define $A_f \in \mathcal{LE}(P)$

$$\Rightarrow \quad a_P(N) \sim e(P)\ a_{C_n}(N)$$

$$\mathcal{F}_{C_n} = \prod_{i=1}^{n} \frac{1}{1-t^i} = \sum_{N=0}^{\infty} \left[\underbrace{\overline{\sum_{\substack{\alpha = (\alpha_1 \dots \alpha_n) \\ 0 \le \alpha_1 \le \dots \le \alpha_n \\ \alpha_1 + \dots + \alpha_n = N}} \quad}}_{} \; t^N \right]$$

$a_{C_n}(0) + a_{C_n}(1) + \dots$

Cone $\mathcal{C}_n = \{ 0 \le \alpha_1 \le \dots \le \alpha_n \}$

Simplex $\Delta_n = \mathcal{C}_n \cap \sum \alpha_i = 1$

$$\Rightarrow \operatorname{vol} \Delta_n = \frac{1}{n!} \det \begin{vmatrix} \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ 0 & \frac{1}{n-1} & \cdots & \frac{1}{n-1} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{vmatrix} = \frac{1}{(n!)^2}$$

$$\Rightarrow + a_{C_n}(N) \sim \frac{N^n}{(n!)^2}$$

▨

③

Proof of Stanley's Thm

[Stanley, EC1, §3.15]

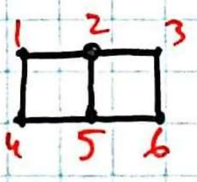
Fix $L \in LE(P)$ ← poset labeling [n]

We have $\mathbb{N}^P = \bigcup_{A \in LE(P)} \underbrace{\mathbb{N}^{C_n}}_{\shortparallel}$

$$\mathcal{C}_A := \left\{ f \in \mathbb{N}^P : \begin{array}{l} f(x) \le f(y) \\ \forall\ A(x) \le A(y) \end{array} \right\}$$

We want $\mathbb{N}^P = \bigsqcup_{A \in LE(P)} \underbrace{\omega_A}_{\uparrow} + \mathbb{N}^{C_n}$

$\mathbb{R}^P_0$ ← vector space of $f: P \to \mathbb{R}$

Ex $P =$ 

$L \in LE(P)$, $n = 6$

$L = \begin{smallmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{smallmatrix}$ $\quad x_1 \le x_2 \le x_3 \le x_4 \le x_5 \le x_6$ $\quad \alpha(L) = 0$

$A_1 = \begin{smallmatrix} 1 & 2 & 4 \\ 3 & 5 & 6 \end{smallmatrix}$ $\quad x_1 \le x_2 \le x_4 \le x_3 \le x_5 \le x_6$ $\quad \alpha(A_1) = 3$

$\textcircled{<}\ 1\ 1\ 1$

$A_2 = \begin{smallmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{smallmatrix}$ $\quad x_1 \le x_2 \le x_4 \le x_5 \le x_3 \le x_6$ $\quad \alpha(A_2) = 2$

$\textcircled{<}\ 1\ 1$

$A_3 = \begin{smallmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{smallmatrix}$ $\quad x_1 \le x_4 \le x_2 \le x_5 \le x_3 \le x_6$ $\quad \alpha(A_3) = 6$

$\textcircled{<}\ 1\ 1\ \textcircled{<}\ 2\ 2$

$A_4 = \begin{smallmatrix} 1 & 3 & 4 \\ 2 & 5 & 6 \end{smallmatrix}$

$x_1 \le x_4 \le x_2 \le x_3 \le x_5 \le x_6$

$\textcircled{<}\ 1\ 1\ 1\ 1$

$\alpha(A_4) = 4$

$$\boxed{\alpha_P = 1 + t^2 + t^3 + t^4 + t^6}$$

④

In general

$L \leftrightarrow (1\,2 \cdots n)$, $\omega_L = (0 \cdots 0)$, $\alpha(L) = 0$

$A \leftrightarrow \sigma \in S_n$, $\sigma = (\sigma_1\, \sigma_2 \cdots \sigma_n)$

$\omega_A = (s_1\, s_2 \cdots s_n)$ where

$s_i = \# \{ j : j < i \,,\, \sigma(j) > \sigma(j+1) \}$.

$\alpha(A) = s_1 + \cdots + s_n$

$$\Rightarrow \left(\mathcal{C}_A + \omega_A\right) \cap \left(\mathcal{C}_B + \omega_B\right) = \emptyset \quad \begin{cases} \forall\ A, B \in \mathcal{L}E(P) \\ A \neq B \end{cases}$$

and $\mathbb{N}^P = \displaystyle\bigsqcup_{B \in \mathcal{L}E(P)} \mathcal{C}_B + \omega_B$

$$\Rightarrow \mathcal{F}_P(t) = \sum_{B \in \mathcal{L}E(P)} t^{|\omega_B|} \sum_{f \in \mathcal{C}_B} t^{|f|}$$

$$= \left[\sum_{B \in \mathcal{L}E(P)} t^{\alpha(B)} \right] \frac{1}{(1-t)(1-t^2) \cdots (1-t^n)}$$

$$= \mathcal{L}_P(t) \prod_{i=1}^{n} \frac{1}{1-t^i}$$ ▨

⑤

Cor [MacMahon, 1915]

$$\mathrm{maj}(\sigma) := \sum_{i=1}^{n-1} \begin{cases} i & \sigma(i) > \sigma(i+1) \\ 0 & , \text{oth.} \end{cases}$$ Major index

$\forall \sigma \in S_n$

Then $$\sum_{\sigma \in S_n} t^{\mathrm{maj}(\sigma)} = \sum_{\sigma \in S_n} t^{\mathrm{inv}(\sigma)}$$

$$= \prod_{i=1}^{n} (1+t+\dots+t^{i-1})$$

D $P = A_n$ antichain. Rest ⇐ Exc. ▨

Th [Stanley, 1972]

$P := P_\lambda$, $\lambda \vdash n$ partition poset

$$\Rightarrow \quad \mathcal{F}_P = \prod_{(i,j)\in\lambda} \frac{1}{1-t^{h(i,j)}} = t^{-n(\lambda)} \underline{s_\lambda(1,t,t^2,\dots)}$$ Schur function

where $n(\lambda) := \sum_{(i,j)\in\lambda} (i-1)$

$\Rightarrow$ HLF (Bijective proof in [Hillman-Grassl] 1976)

⑥

L20

Poset Polytopes

206A

Nov 20, 2020

[Stanley, Two poset polytopes] 1986

(I) Order polytope

Def $P = (X, \prec)$, $|X| = n$

$$O_P := \left\{ f : X \to \mathbb{R} \ \underline{\text{s.t.}} \ 0 \le f(x) \le 1 \ \forall x \in X \ \underline{\text{and}} \ f(x) \le f(y) \ \forall x \prec y \right\}$$

$O_P \subset \mathbb{R}^n$, $\dim O_P = n$, $O_P \simeq \underbrace{[0\,1]}_{\text{subposet of } \mathbb{R}}{}^P$

Note: suffices

$$\begin{cases} f(x) \geq 0 & \forall \ x \leftarrow \text{min elt in } P \\ f(y) \le 1 & \forall \ x \leftarrow \text{max elt in } P \\ f(x) \le f(y) & \forall \ x \prec y, \ (x\,y) \in \Gamma(P) \end{cases}$$

← Hasse diag

Prop1 Facets of O_P / ← faces of dim $= n-1$ /

are

$$\begin{cases} f(x) = 0, & x \in \min(P) \\ f(y) = 1, & y \in \max(P) \\ f(x) = f(y), & (x\,y) \in \Gamma(P) \end{cases}$$

▨

①

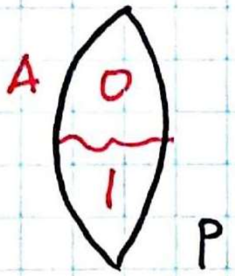
Prop2 Vertices of $\mathcal{O}_P$ are $\{1-\chi_A\}$ <u>where</u>

A − order ideal in P / ⇐ $a \in A,\ b < a \Rightarrow b \in A$ /

$$\chi_A(a) = \begin{cases} 1, & a \in A \\ 0, & \text{oth.} \end{cases}$$

D $g \in \mathcal{O}_P$, g − vertex ⇒ $g \in \cap$ of $\geq n$ facets

⇒ $g(x) \in \{0, 1\}$ / otherwise extra facet can be added /



$$\Rightarrow \begin{cases} g(x) = 0 \text{ on some } A, \text{ i.e } \forall x \in A \\ \quad A\text{-order ideal} \\ g(x) = 1, \quad x \notin A \end{cases}$$ ▨

<u>Th</u> $\text{vol}\, \mathcal{O}_P = \dfrac{e(P)}{n!}$

D $\mathcal{O}_P = \bigsqcup_{Q \in LE(P)} \Delta_Q$ <u>where</u> $\Delta_Q \subseteq \mathcal{O}_P$ simplex

$$\Delta_Q := \left\{ \begin{array}{l} 0 \leq f(x_1) \leq f(x_2) \leq \dots \leq f(x_n) \leq 1 \\ \text{where } f(x_i) = i \end{array} \right\}$$

$$\boxed{\text{vol}\, \Delta_Q = \frac{1}{n!}}$$

$$\Rightarrow \text{vol}\, \mathcal{O}_P = \sum_{Q \in LE(P)} \frac{1}{n!} = \frac{e(P)}{n!}$$ ▨

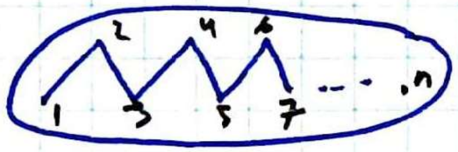
Ex (1) $P = A_n$ antichain $\Rightarrow P = [01]^n$

$$\mathrm{vol}\, \mathcal{O}_P = \frac{e(A_n)}{n!} = \frac{n!}{n!} = 1 \quad ✓$$

(2) $P = C_n$ chain $\Rightarrow \mathcal{O}_P = \Delta_n \subset \mathbb{R}^n$

$$\mathrm{vol}\, \mathcal{O}_P = \frac{1}{n!} \quad ✓$$

(3) $P = Z_n$ zigzag poset, $\mathcal{O}_P \subset \mathbb{R}^n$



$$\mathcal{O}_P = \{ x_1 \leq x_2 \geq x_3 \leq \dots , \ 0 \leq x_i \leq 1 \}$$

Vertices of $\mathcal{O}_P$ are $\{$ 0-1 seq $x_1 \leq x_2 \geq x_3 \leq \dots \}$

$\Leftrightarrow \{$ 0-1 seq $y_1 \leq 1-y_2 \geq y_3 \leq 1-y_4 \geq \dots \}$

$= \{$ 0-1 seq $(y_1, y_2, \dots)$, $y_i + y_{i+1} \leq 1 \}$

$=$ Fib. sequences

$\Rightarrow |V(\mathcal{O}_P)| = F_n \sim c\,\phi^n$, $\phi = \frac{1+\sqrt{5}}{2}$

$$\mathrm{vol}\, \mathcal{O}_P = \frac{e(Z_n)}{n!} \sim c' \left(\frac{2}{\pi}\right)^n$$

since $e(Z_n) = \#\{ \sigma(1) < \sigma(2) > \sigma(3) < \dots \in S_n \}$

- number of alt. permutations

③

Ⅱ Chain Polytope

Def $P = (X, \prec)$, $|X| = n$

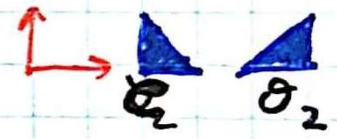
$$\mathcal{C}_P := \left\{ \begin{array}{l} f : X \to \mathbb{R} \quad \text{s.t.} \quad f(x) \geq 0 \quad \forall x \in X \\ \text{and} \quad f(x_1) + \cdots + f(x_k) \leq 1 \quad \forall \text{ chain } \{x_1 \ldots x_k\} \text{ in } P \end{array} \right\}$$

Ex (1) $P = A_n \Rightarrow \mathcal{C}_P = \mathcal{O}_P = [0\,1]^n$

(2) $P = C_n \Rightarrow \mathcal{C}_P = [0\,1]^n \cap \{x_1 + \cdots + x_n \leq 1\}$

(3) $P = Z_n \Rightarrow \mathcal{C}_P = \{y_i + y_{i+1} \leq 1 , \quad y_i \geq 0\}$

$\mathcal{C}_P \simeq \mathcal{O}_P$ in this case



Note: suffices to take $\{ f(x_1) + \cdots + f(x_k) \leq 1 , f(x) \geq 0$ $\forall$ max chains in $P \}$

Prop 3 vertices $V(\mathcal{C}_P) = \{ \chi_A , A\text{-antichain in } P \}$

Thm $\mathrm{vol}\, \mathcal{C}_P = \dfrac{e(P)}{n!}$

▷ **Def** $\Phi : \mathcal{O}_P \to \mathcal{C}_P$ **transfer map**

$[\Phi f](\cancel{Y}) := \min\{ f(Y) - f(X),\ X \lessdot Y,\ x \in X\}$

$\forall f \in \mathcal{O}_P$ **Obs**: Thm follows from:

∠ Φ is a continuous, piecewise-linear vol-preserving bijection.

▷ 1) Φ – continuous ✓ / by def /

---

2) $\mathcal{O}_P = \bigsqcup_{Q \in LE(P)} \Delta_Q$, Φ is linear on Δ_Q $\forall Q$

$\Rightarrow \Phi$ is PL ✓

---

3) $X = [n]$, $P = C_n \Rightarrow \mathcal{O}_P = \Delta_n \Rightarrow \Phi = \begin{pmatrix} 1 & -1 & & 0 \\ & 1 & \ddots & \\ & & \ddots & -1 \\ 0 & & & 1 \end{pmatrix}$

$\Rightarrow \Phi$ is vol-preserving on each Δ_Q

$\Rightarrow \Phi$ is vol-preserving ✓

---

4) $\Psi : \mathcal{C}_P \to \mathcal{O}_P$ def by $[\Psi g](X) := \max_{Y_1 \lessdot \ldots \lessdot Y_k = X} \{ g(Y_1) + \ldots + g(Y_k) \}$

Obs $\Phi\Psi = \Psi\Phi = I \Rightarrow \Phi$ is a bij ✓ ▨ ▨

⑤

Cor 1 $e(P)$ depends only on the comparability graph Com(P)

D By def of $\mathcal{C}_P$ *or* Prop 3

polytope $\mathcal{C}_P$ depends only on Com(P)

since $e(P)=(\mathrm{vol}\,\mathcal{C}_P)\,n!$ $\Rightarrow$ claim ▨

Note Cor 1 was first proved via promotion

L21 Poset Polytopes

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11/23/2020

Recall $P=(X,\prec)$, $|X|=n$

$\mathcal{O}_P := [0\,1]^P_{\le} = \{ f: X\to\mathbb{R} \,,\, 0\le f(x)\le 1 \,,\, f(x)\le f(y)\ \forall x,y\in X \}$ Order polytope

$\mathcal{C}_P := \{ g: X\to\mathbb{R} \,,\, g(x)\ge 0\ \forall x\in X \,,\, \sum_{x\in C} f(x)\le 1\ \forall \text{chain } C \text{ in } P \}$ Chain polytope

Th $\operatorname{vol}\mathcal{O}_P = \operatorname{vol}\mathcal{C}_P = \dfrac{e(P)}{n!}$

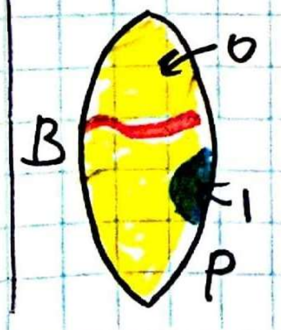
Cor $e(P)$ depends only on $\operatorname{Com}(P) \leftarrow$ comparibility graph

Prop Vertices $V(\mathcal{C}_P) = \{ \chi_A \,,\, A \leftarrow \text{antichain in } P \}$

D $\supseteq$ All $\chi_A \in \mathcal{C}_P$ and $\chi_A \in V([0\,1]^n) \Rightarrow$ ✓

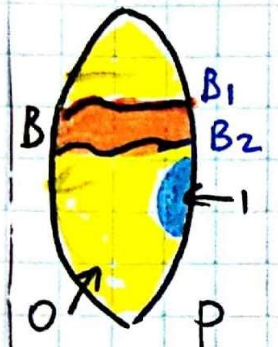
$\subseteq$ Fix $g\in V(\mathcal{C}_P)$, $B := \{ x\in X : 0 < g(x) < 1 \}$

$\varepsilon := \min\{ g(x) \,,\, 1-g(x) \,,\, x\in B \} \leftarrow \boxed{\varepsilon \le g(x) \le 1-\varepsilon}$



Case 1 B - antichain $\Rightarrow g = \frac{1}{2}(g_1+g_2)$ where

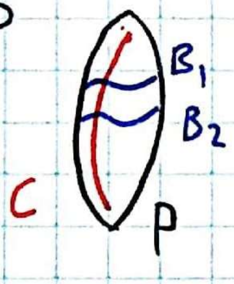
$g_1 = g+\varepsilon\chi_B$, $g_2 = g-\varepsilon\chi_B$ ✗



Case 2 B – not antichain

$B_1 := \{\min \text{ elt's in } B\}$

$B_2 := \{\min \text{ elt's in } B \setminus B_1\}$ ↙ ↖ two antichains in B



$$\left.\begin{aligned} g_1 &:= g + \varepsilon\chi_{B_1} - \varepsilon\chi_{B_2} \\ g_2 &:= g - \varepsilon\chi_{B_1} + \varepsilon\chi_{B_2} \end{aligned}\right\} \in \mathcal{C}_P$$

$\Rightarrow \quad g = \frac{1}{2}(g_1 + g_2) \notin V(\mathcal{C}_P) \quad \times \quad$ ▨

Plan for today: using geometry to understand combinatorics of posets

Let $Q \subset \mathbb{R}^n$ convex polytope, $\dim(Q) = n$

Q is integral if $V(Q) \subset \mathbb{Z}^n$

Th [Ehrhart, Macdonald '71]

$Q \subset \mathbb{R}^n$ integral, $\dim(Q) = n$. Then $L_Q(N) := |NQ \cap \mathbb{Z}^n|$ is a polynomial $\in \mathbb{Q}[N]$ w/ $\deg = n$ and lead coeff $= \mathrm{vol}(Q)$

②

Ex $Q=[0\,1]^n \Rightarrow L_Q(N) = (N+1)^n$ ✓ vol $Q = 1$

$Q = \{0 \le x_1 \le \dots \le x_n \le 1\} = \Delta \Rightarrow L_Q(N) = \binom{N+1+n}{n}$ ✓

vol $Q = \frac{1}{n!}$

Def $P = (X, \prec)$, $|X| = n$

$a_P(m) := \#[m]^P = \#\left\{ \begin{array}{l} f: X \to \{1 \dots m\} \text{ s.t. } 1 \le f(x) \le m \\ \text{and } \ f(x) \le f(y) \ \forall x \prec y \end{array} \right\}$

Prop [Stanley, 1970] $a_P(m)$ is a polynomial in m

$a_P(m) \in \mathbb{Q}[m]$, lead coeff $= e(P)/(m-1)!$

D **Obs** $a_P(m+1) = |\mathcal{O}_P \cdot m \cap \mathbb{Z}^n| = L_{Q_P}(m)$ ▨

Th [Stanley, 1986] $a_P(m)$ depends only on $\mathrm{Com}(P)$ [Stanley, Two poset polytopes]

D **Recall** $\boxed{\Phi: \mathcal{O}_P \to \mathcal{C}_P}$ transfer map ∈ PL, vol-pres cont, bij

$\Rightarrow m\Phi: m\mathcal{O}_P \cap \mathbb{Z}^n \to m\mathcal{C}_P \cap \mathbb{Z}^n$ is a bijection

$\Rightarrow a_P(m+1) = |m\mathcal{O}_P \cap \mathbb{Z}^n| = \underbrace{|m\mathcal{C}_P \cap \mathbb{Z}^n|}$ depends only on Com (P) ▨

③

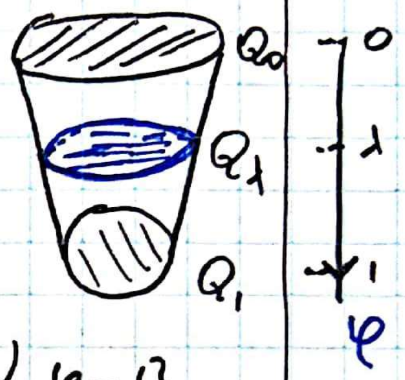
Alexandrov-Fenchel inequalities

$Q_0, Q_1 \subset \mathbb{R}^n$ convex polytopes

$Q := \mathrm{conv}\{Q_0, Q_1\}$, $\varphi : \mathbb{R}^n \to \mathbb{R}$ linear

s.t. $\mathrm{dist}\{\varphi = 0, \varphi = 1\} = 1$

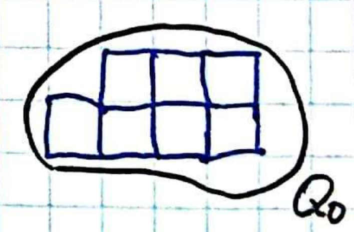
$Q_\lambda := Q \cap \{\varphi = \lambda\}$, $Q_0 \subset \{\varphi = 0\}$, $Q_1 \subseteq \{\varphi = 1\}$



Prop $$\mathrm{vol}_{n-1}(Q_\lambda) = \sum_{i=0}^{n-1} \binom{n-1}{i} \underbrace{V_i(Q_0, Q_1)}_{\text{mixed volume}} \lambda^i (1-\lambda)^{n-i-1}$$
$\forall\ 0 \leq \lambda \leq 1$

/ i.e. $\exists\, V_i(\ ) \geq 0$ s.t. —||— /

Proof idea: use additivity & continuity via partitioning Q_0, Q_i into boxes



Th [A'37,'38, F'36]

$\forall\, Q_0, Q_1 \subset \mathbb{R}^n$ as above, $1 \leq i \leq n-1$

$$\Rightarrow V_i(Q_0 Q_1)^2 \geq V_{i-1}(Q_0 Q_1)\, V_{i+1}(Q_0, Q_1)$$

special case of general A-F

Proof idea ← hard use of convexity / additivity fails /

④

A-F for order polytopes

$P = (X, \prec)$, $|X| = n$, $\mathcal{O}_P \subset \mathbb{R}^n$

$x \in X$ fixed el't

$\alpha_j(x) := \# \{ A \in LE(P) : A(x) = j \}$

Th [Stanley, 1981] $\forall\, P = (X, \prec)$, $\forall\, x \in X$, $|X| = n$

$$\alpha_j(x)^2 \geqslant \alpha_{j-1}(x)\, \alpha_{j+1}(x) \, , \quad 2 \leq j \leq n-1$$

[Stanley, Two applications of A-F inequalities]

D $\mathcal{O}_P = \{ f : X \to \mathbb{R} ,\ 0 \leq f(z) \leq 1 ,\ f(z) \leq f(y) \ \forall\, z \prec y \}$

Def

$Q_\lambda := \mathcal{O}_P \cap \{ f(x) = \lambda \} \qquad \forall\, \lambda \in [0,1]$

$\Delta_A := \{ 0 \leq f(x_1) \leq f(x_2) \leq \ldots \leq f(x_n) \leq 1 \}$ where

$x_i = A^{-1}(i)$, $A : X \to \{1 \ldots n\} \in LE(P)$

$$\Rightarrow \ \mathrm{vol}_{n-1} \Delta_A \cap \{ f(x_i) = \lambda \} = \mathrm{vol} \left\{ \begin{array}{l} 0 \leq f(x_1) \leq \ldots \leq f(x_i) = \lambda \leq \\ \leq f(x_{i+1}) \leq \ldots \leq f(x_n) \end{array} \right\}$$

$$\Rightarrow \ \mathrm{vol}_{n-1} \Delta_A \cap \{ f(x_i) = \lambda \} = \begin{cases} \dfrac{\lambda^{i-1} (1-\lambda)^{n-i}}{(i-1)! \ (n-i)!} , & A(i) = x_i \ /{=}x/ \\ 0 , & \text{oth.} \end{cases}$$

⑤

$$\Rightarrow \mathrm{vol}_{n-1} Q_\lambda = \sum_{A \in \mathcal{L}E(P)} \mathrm{vol}_{n-1}(\Delta_A \cap Q_\lambda)$$

$$= \frac{1}{(n-1)!} \sum_{i=0}^{n-1} \alpha_{i+1}(x) \binom{n-1}{i} \lambda^i (1-\lambda)^{n-1-i}$$

$$\Rightarrow \alpha_{i+1}(x) = (n-1)!\, V_i(Q_0, Q_1)$$

$$\Rightarrow \alpha_{i+1}(x)^2 \geq \alpha_i(x)\, \alpha_{i+2}(x) \quad \forall\, 0 \leq i \leq n-1$$

▨

Cor [Rivest, Chung-Fishburn-Graham conjecture]

[Stanley '81] $\{\alpha_i(x)\} \leftarrow$ unimodal

$/\ 0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_k \geq \alpha_{k+1} \geq \cdots \geq \alpha_n \geq 0\ /$

D log-concavity $\Rightarrow$ unimodality $\forall$ seq ▨

Th [Kahn-Saks, 1984] $P = (X, \prec)$, $|X| = n$

similar proof

$\beta_i(u,v) := \#\{A \in \mathcal{L}E(P) : A(u) - A(v) = i\}$.

Then $\{\beta_i(u,v),\ 1 \leq i \leq n-1\}$ is log-concave $\forall\, u,v \in X$

⑥

L22 Applications of Poset Polytopes

206A
11/25/2020

Recall: $P=(X,\prec)$, $|X|=n$

$\mathcal{O}_P := \{ f : X \to \mathbb{R} ,\ 0 \le f(x) \le 1\ \forall x \in X ,\ f(x) \le f(y)\ \forall x \prec y \}$ order polytope

$\mathcal{C}_P := \{ g : X \to \mathbb{R} ,\ g(x) \geq 0\ \forall x \in X ,\ \sum_{x\in C} g(x) \le 1\ \forall C \text{ in } P \}$ chains polytope

---

Th [Stanley] $\operatorname{vol} \mathcal{O}_P = \operatorname{vol} \mathcal{C}_P = e(P)/n!$

Prop [-ll-] Vertices $V(\mathcal{C}_P) = \{ \chi_A ,\ A \text{ - antichain in } P \}$

Th [Stanley] $\forall x \in X$ fixed : $\alpha_i(x)^2 \geq \alpha_{i-1}(x)\, \alpha_{i+1}(x)$

$\forall\ 2 \le i \le n-1$, **where** $\alpha_i(x) := \# \{ A \in \mathcal{LE}(P) ,\ A(x) = i \}$

Th [Kahn-Saks] $\forall x,y \in X$ fixed: $\beta_i(x,y)^2 \geq \beta_{i-1}(x,y)\, \beta_{i+1}(x,y)$

$\forall\ 1 \le i \le n-1$, **where** $\beta_i(x,y) := \# \{ A \in \mathcal{LE}(P) ,\ A(x) - A(y) = i \}$

Cor Both $\{ \alpha_i(x) \}$ and $\{ \beta_i(x,y) \}$ are unimodal

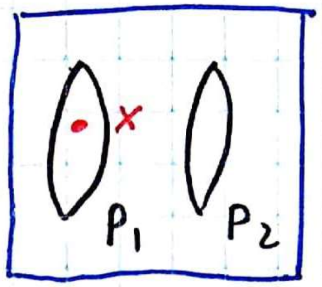
---

Ex (1) $P = A_n \Rightarrow \alpha_i = \frac{1}{n} \underbrace{n!}_{\overset{\shortparallel}{e(A_n)}}$ ✓ | $\beta_i = \frac{(n-i)}{n(n-1)} \underset{\overset{\shortparallel}{e(A_n)}}{n!}$ ✓

①

Ex (cont'd)

(2) $\widehat{P} = P_1 + P_2$, $P_1 = (X_1, \alpha)$, $P_2 = (X_2, \alpha)$



$$\widehat{\alpha}_i(\cancel{X}) = \sum_k \alpha_k(x) \binom{i-1}{k} e(P_2)$$

$$\frac{\widehat{\alpha}_i(x)}{(i-1)!} = \sum_k \frac{\alpha_k(x)}{k!} \frac{1}{(i-k-1)!} e(P_2)$$

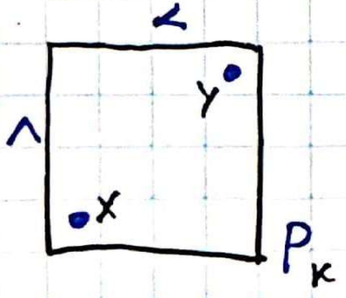
Def $\{a_j,\ 0 \le j \le n\}$ is ultra-log-concave if $\{a_j / \binom{n}{j}\}$ is log-concave.

Th [Liggett, 1997] $\{a_i\}$, $\{b_r\} \in$ ultra-log-concave

$\Rightarrow$ so is $\{c_n\}$, $c_n = \sum_k \binom{n}{k} a_k b_{n-k}$ (convolution)

Cor $P \leftarrow$ series parallel $\Rightarrow \{\alpha_{i-1}(x)\}$ ultra-log concave

Ex (3) $P_k := [k \times k] \leftarrow$ 2-dim poset w/ $n = k^2$ elt's



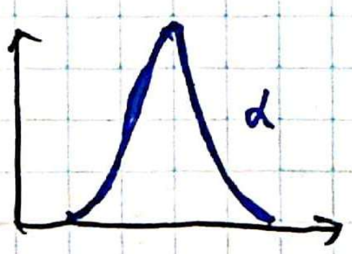
$\alpha_i(x) =$ # SYT A s.t. $A(k,1) = i$, $k \le i \le k^2 - k$

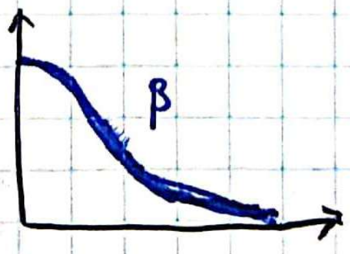
$\beta_i(x,y) =$ # SYT B s.t. $B(k,1) - B(1,k) = i$

$0 \le i \le k^2 - 2k$

$x = (k,1)$

$y = (1,k)$





Th [Brightwell- Tetali, 2003] $P=(X,\prec)$, $|X|=n$

Let $h: X \to \mathbb{R}_+$ s.t. $\sum_{x\in A} h(x) \le 1$ ∀ antichain A in P

Then $e(P) \le \prod_{x\in X} \frac{1}{h(x)}$

D Let $Q_h := \{ g: X \to \mathbb{R}_+ \text{ s.t. } g\cdot h := \sum_{x\in X} g(x)h(x) \le 1 \}$

L $\mathcal{C}_P \subseteq Q_h$ ∀ h as in the Th.

D ∀ $g \in \mathcal{C}_P$ by **Prop** /on vertices of $\mathcal{C}_P$/

$$g = \sum_{k=1}^{n+1} w_k \chi_{A_k}, \quad \sum w_k = 1, \quad A_k \in \text{antichains in } P$$

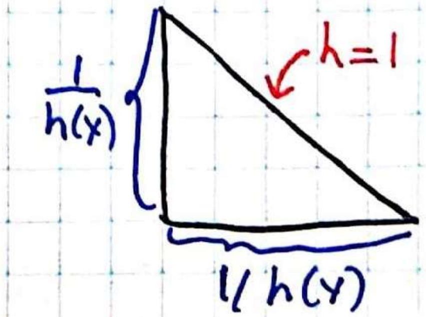
$$\Rightarrow g\cdot h = \sum_k (\chi_{A_k}\cdot h)\, w_k \le \sum_k w_k = 1$$

$$\Rightarrow g h \in Q_h \quad ▨$$

By **L** $\Rightarrow$ vol $\mathcal{C}_P \le$ vol Q_h.

Obs vol $Q_h = (n!)^{-1} \prod_{x\in X} \frac{1}{h(x)}$

since vol $\mathcal{C}_P = e(P)/n!$ $\Rightarrow$ ✓



③

LYM property $P=(X,<)$ ranked w/ $r_j \in$ rank numbers (See L15)

Def (LYM) suppose $\forall$ antichain A in P, $\hbar = \text{height}(P)$

$$\sum_{x \in A} \frac{1}{r(x)} \le 1, \quad r(x) = r_i \text{ if } rk(x) = i$$

Th [Kahn-Kim, 1995] $\forall$ ranked P w/ LYM

$$e(P) \le \prod_{k=0}^{\hbar} (r_k)^{r_k}$$

D In B-T Thm take $h(x) := \frac{1}{r(x)}$

$$\Rightarrow e(P) \le \prod_{x \in X} r(x) = \prod_{k=1}^{\hbar} (r_k)^{r_k} \quad ▨$$

Ex [Boolean Lattice] $P = B_n = (2^{[n]}, \subseteq)$

$$\Rightarrow e(B_n) \le \prod_{k=0}^{n} \binom{n}{k}^{\binom{n}{k}} \Rightarrow \text{upper bounds in L15}$$

Ex $[\underbrace{\text{Subspaces of } \mathbb{F}_q^n}_{L}]$ By q-LYM $\Rightarrow$ upper bound on $e(L)$

④

Back to Perfect Graphs

$G = (V, E)$, $\mathcal{K}_G := \{\text{cliques in } G\}$, $|V| = n$

$S_G := \{\text{conv hull of } \chi_K ,\ K \in \mathcal{K}_G\} \subset \mathbb{R}^n$

stable set polytope

$\mathcal{F}S_G := \{ f : V \to \mathbb{R}_+ \ \text{s.t.} \sum_{v \in K} f(v) \le 1 \ \ \forall K \in \mathcal{K}_{\bar G} \}$

fractional stable set polytope

Ex $P = (X, \prec)$, $|X| = n$, $G := \overline{Com(P)}$ incomp graph

Then $S_G = \mathcal{C}_P$ (by Prop)

$\mathcal{F}S_G = S_G = \mathcal{C}_P$ by def of $\mathcal{C}_P$ / cliques in $\bar G$ = chains in P /

Th [Lovász] $S_G = \mathcal{F}S_G$ if and only if G - perfect

Obs $S_G \subseteq \mathcal{F}S_G$ $\forall G$ since $|K \cap A| \le 1$

Ex $G = C_5 \Rightarrow \mathcal{F}S_G$ contains $(\frac{1}{2}, \ldots, \frac{1}{2}) \leftarrow$ not

in convex hull of $(0 \ldots 1 1 \ldots 0) \in \mathbb{R}^5$ and $(0 \ldots 0)$

⑤

L23 Correlation Inequalities

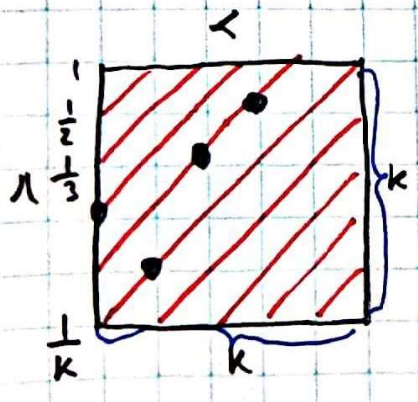
206A
11/30/2020

Last time:

Th [B-T] $P = (X, <)$, $f : X \to \mathbb{R}_+$ s.t. $\sum_{a \in A} f(a) \le 1$
$\forall$ antichain $A \Rightarrow e(P) \le \prod_{x \in X} \frac{1}{f(x)}$

Th [Kahn-Kim] P – ranked w/ rank sizes $r_1 r_2 \ldots$
Suppose P has F-LYM w/ $f(x) = \frac{1}{rk(x)}$
$\Rightarrow e(P) \le \prod_{i=1}^{h(P)} r_i^{\,r_i}$

Ex $P_k = [k \times k]$ ← 2-dim poset $F = (1\,2 \ldots k{-}1\,k\,k{-}1 \ldots 1)$



OBS/Exc P_k has LYM w/ $f(x) = \frac{1}{rk(x)}$

$$\Rightarrow e(P_k) \le \left[1^1\, 2^2 \cdots (k-1)^{k-1}\right]^2 k^k$$

Compare w/ $e(P_k) \ge \left[1!\; 2! \ldots (k-1)!\right]^2 k!$

$UB/LB = \exp O(n)$, $n = k^2$

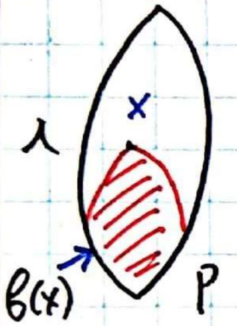
①

Note: $P_k \leftarrow$ 2dim is unimportant here

For general posets [k-k] define entropy

w/ similar $\exp O(n)$ approx

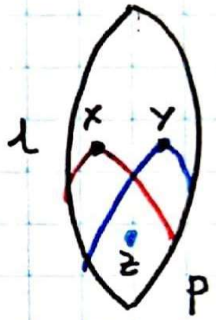
Th [Hammett-Pittel, 2008]



$P = (X, \prec)$, $|X| = n$

Then $e(P) \geqslant \dfrac{n!}{\prod\limits_{x\in X} b(x)}$

where $b(x) = \#\{y \succcurlyeq x \,,\, y \in X\}$

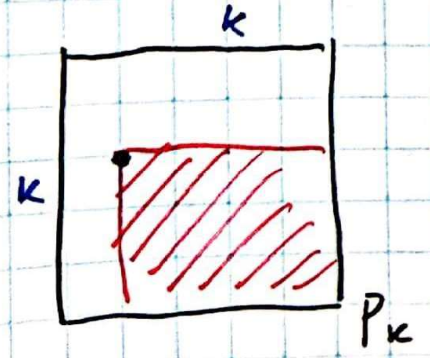


$P\big(\sigma(x) \le \sigma(\succ x)\big)$
$\geqslant P\big(-\shortparallel- \mid \sigma(y) \le \sigma(\succ y)\big)$
/positive corr./

Ex (1) ⊜ for trees

(2) ⊜ for series-parall poset /$\Rightarrow$(1)/

(3) weak for P_k



HP $\Rightarrow$ $e(P_k) = (n!)^{\varepsilon}$ for some $\varepsilon < \frac{1}{2}$

Hint: high correlation!

Classical Correlation Inequalities

Motivational Thm $G = (V, E)$, $V = [n]$

G-random graph w/ prob. of $e \in E = p > 0$

Then $P[G$-planar and Hamiltonian$]$

$\le P[G$-planar$] \cdot P[G$-Hamiltonian$]$

Hint: negative corr follows from

planarity ← down-closed property / $G \subseteq H$, H-planar $\Rightarrow$ G-planar /

Hamiltonicity ← up-closed propery / $G \supseteq H$, H-Hamiltonian $\Rightarrow$ G-Hamiltonian /

Th [Kleitman, 1966] $B_n = (2^{[n]}, \subset)$

$\mathcal{U}, \mathcal{D} \in 2^{B_n}$ s.t. $\forall A \in \mathcal{U}, B \supseteq A \Rightarrow B \in \mathcal{U}$

$\subseteq 2^{[n]}$ $\forall A \in \mathcal{D}, B \subseteq A \Rightarrow B \in \mathcal{D}$

Then $|\mathcal{U} \cap \mathcal{D}| \cdot 2^n \le |\mathcal{U}| \cdot |\mathcal{D}|$

③

Proof (by induction) $n=1$ ✓

$\boxed{n \Rightarrow (n+1)}$ $\quad a := (n+1)$

$\mathcal{U}_a := \{A \in \mathcal{U},\ a \in A\}$

$\mathcal{U}_a' := \{A \in \mathcal{U},\ a \notin A\}$

$\mathcal{D}_a := \{B \in \mathcal{D},\ a \in B\}$

$\mathcal{D}_a' := \{B \in \mathcal{D},\ a \notin B\}$

$\mathcal{U} = \mathcal{U}_a \sqcup \mathcal{U}_a'$

$\mathcal{D} = \mathcal{D}_a \sqcup \mathcal{D}_a'$

By ind. $\Rightarrow$ $|\mathcal{U}_a' \cap \mathcal{D}_a'| \cdot 2^n \leq |\mathcal{U}_a'| \cdot |\mathcal{D}_a'|$

Let $\mathcal{U}_a'' := \{A - a,\ A \in \mathcal{U}_a\}$, $\mathcal{D}_a'' := \{B - a,\ B \in \mathcal{D}_a\}$

since $(\mathcal{U}_a'', \mathcal{D}_a'')$ satisfy ind. assumption

$$\Rightarrow |\mathcal{U}_a \cap \mathcal{D}_a|\, 2^n = |\mathcal{U}_a'' \cap \mathcal{D}_a''|\, 2^k \leq |\mathcal{U}_a''| \cdot |\mathcal{D}_a''|$$

$$|\mathcal{U}_a''| \cdot \underline{|\mathcal{D}_a|}$$

$a \geq b$

$c \leq d$

$\left.\begin{array}{l} \Rightarrow ac + bd \\ \geq ad + bc \end{array}\right\}$

Thus

$$2^{n+1} |\mathcal{U} \cap \mathcal{D}| = 2^{n+1} \left[|\mathcal{U}_a \cap \mathcal{D}_a| + |\mathcal{U}_a' \cap \mathcal{D}_a'| \right]$$

$$\leq 2 \left[|\mathcal{U}_a| \cdot |\mathcal{D}_a| + |\mathcal{U}_a'| \cdot |\mathcal{D}_a'| \right]$$

$$\leq |\mathcal{U}_a| \cdot |\mathcal{D}_a| + \underline{|\mathcal{U}_a|\, |\mathcal{D}_a'|} + |\mathcal{U}_a'|\, |\mathcal{D}_a'|$$

$$+ \underline{|\mathcal{U}_a'| \cdot |\mathcal{D}_a|} = |\mathcal{U}| \cdot |\mathcal{D}| \quad \blacksquare$$

$|\mathcal{U}_a| \geq |\mathcal{U}_a'|$

$|\mathcal{D}_a| \leq |\mathcal{D}_a'|$

④

Th [Ahlswede-Daykin, 1978] Four funcions thm

[Alon-Spencer]
§6.1

Let $\alpha, \beta, \gamma, \delta : 2^{[n]} \to \mathbb{R}_+$ s.t.

$$\alpha(A)\,\beta(B) \le \gamma(A\cup B)\,\delta(A\cap B) \quad \forall A,B\subseteq [n]$$

Then $\alpha(\mathcal{A})\,\beta(\mathcal{B}) \le \gamma(\underline{\mathcal{A}\cup\mathcal{B}})\,\delta(\underline{\mathcal{A}\cap\mathcal{B}})$

$\forall\, \mathcal{A},\mathcal{B} \subseteq 2^{[n]}$ where

$$\underline{\mathcal{A}\cup\mathcal{B}} := \{A\cup B,\ A\in\mathcal{A},\ B\in\mathcal{B}\}$$
$$\underline{\mathcal{A}\cap\mathcal{B}} := \{A\cap B,\ A\in\mathcal{A},\ B\in\mathcal{B}\}$$

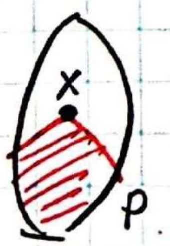
Proof ← essentially same induction as in KleitmanTh

Cor 1 L ← distributive lattice, $\alpha\,\beta\,\gamma\,\delta : X \to \mathbb{R}_+$

s.t. $\alpha(x)\,\beta(y) \le \gamma(x\vee y)\,\delta(x\wedge y) \quad \forall x\, y\in X$

Then $\forall\, A,B\subset X \quad \alpha(A)\,\beta(B) \le \gamma(A\vee B)\,\delta(A\wedge B)$

D



$L = J(P)$

▨

⑤

Cor 2 $L = (X, \prec)$ – distributive lattice

$A, B \subseteq X \Rightarrow |A| \cdot |B| \leq |A \vee B| \cdot |A \wedge B|$

⑥

L24 More correlation inequalities

206 A
12/2/2020

Last time: $\mathcal{A} \subseteq 2^{[n]}$, $\varphi : 2^{[n]} \to \mathbb{R}_+$, $\varphi(\mathcal{A}) := \sum_{A \in \mathcal{A}} \varphi(A)$

Def $\mathcal{A}, \mathcal{B} \subseteq 2^{[n]} \Rightarrow \mathcal{A} \underline{\cup} \mathcal{B} := \{ A \cup B,\ A \in \mathcal{A}, B \in \mathcal{B}\}$

$\mathcal{A} \underline{\cap} \mathcal{B} := \{ A \cap B,\ \text{—}//\text{—} \}$

Th [A-D, Four functions theorem]

Let $\alpha, \beta, \gamma, \delta : 2^{[n]} \to \mathbb{R}_+$ s.t. $\alpha(A)\,\beta(B) \le \gamma(A\cup B)\,\delta(A\cap B)$ $\forall A, B \in 2^{[n]}$

Then $\alpha(\mathcal{A}) \cdot \beta(\mathcal{B}) \le \gamma(\mathcal{A} \underline{\cup} \mathcal{B}) \cdot \delta(\mathcal{A} \underline{\cap} \mathcal{B})$ $\forall \mathcal{A}, \mathcal{B} \subseteq 2^{[n]}$

Def $L = (X, \wedge, \vee)$ ← distributive lattice

$A, B \subseteq X \Rightarrow A \vee B := \{ a \vee b,\ a \in A,\ b \in B\}$

$A \wedge B := \{ a \wedge b,\ \text{—}//\text{—} \}$

Th $L = (X, \wedge, \vee)$ distributive, $\alpha, \beta, \gamma, \delta : X \to \mathbb{R}_+$ s.t.

$\forall a, b \in X$: $\underline{\alpha(a)\,\beta(b) \le \gamma(a \vee b)\,\delta(a \wedge b)}$

Then $\forall A, B \subseteq X$

$$\boxed{\alpha(A)\ \beta(B) \le \gamma(A \vee B)\ \delta(A \wedge B)}$$

①

Cor 1 $\mathcal{L} = (X, \wedge, \vee)$ – distributive, $A, B \subseteq X$

$$\Rightarrow |A|\cdot|B| \le |A \vee B| \cdot |A \wedge B|$$

Ex (1) $\mathcal{L} \leftarrow$ lattice of subgraphs of K_n, $\mathcal{L} \simeq B_{\binom{n}{2}}$

$A = \{\text{forests}\}$, $B = \{\text{Ham subgraphs}\}$

$\Rightarrow A \vee B = B$, $A \wedge B = A$, $\textcircled{\le}$ holds trivially.

(2) $A = \{\text{planar subgraphs}\}$, $B = $ –//–. $\textcircled{\le} \leftarrow$ hard

---

Cor 2 $\mathcal{A} \subseteq 2^{[n]}$, $\mathcal{A} \setminus \mathcal{A} := \{A \setminus A',\ A, A' \in \mathcal{A}\}$

then $|\mathcal{A} \setminus \mathcal{A}| \ge |\mathcal{A}|$

D $\mathcal{L} := B_n$. By Cor 1 $\Rightarrow$ $\mathcal{B} := \{\bar{A},\ A \in \mathcal{A}\}$

$$|\mathcal{A}|^2 = |\mathcal{A}|\cdot|\mathcal{B}| \le_{\text{Cor 1}} \underbrace{|\mathcal{A} \cup \mathcal{B}|}_{\#\{A \cup \bar{A}'\}} \cdot \underbrace{|\mathcal{A} \cap \mathcal{B}|}_{\#\{A \cap \bar{A}'\}} = |\mathcal{A} \setminus \mathcal{A}|^2 \quad ▨$$

Note: Partly motivated by Arithmetic Combinatorics

/ Bounds on $|A+B|$, $|A-B|$, $|A \cdot B|$ via $|A|$, $|B|$, where $A, B \subset \mathbb{Z}$ /

②

FKG inequality

[A-S, § 6.2]

Def $L=(X,\wedge,\vee)$ – distributive lattice

$\mu: X \to \mathbb{R}_+$ is log-supermodular if

$$\mu(x)\,\mu(y) \le \mu(x\wedge y)\,\mu(x\vee y) \quad \forall x,y\in X$$

$f: X \to \mathbb{R}_+$ is increasing if $f(x)\le f(y)$ $\forall\, x\preceq y$

order-preserving

decreasing if $f(x)\le f(y)$ —//—

Th [Fortuin–Kasteleyn–Ginibre, 1971]

Let $L=(X,\wedge,\vee)$ – distributive, $\mu, f, g: X\to\mathbb{R}_+$

s.t. $\mu \leftarrow$ log-supermod, $f,g \leftarrow$ increasing

Then

$$\underbrace{\left(\sum_{x\in X}\mu(x)\,f(x)\right)}_{\langle\mu,f\rangle}\underbrace{\left(\sum_{x\in X}\mu(x)\,g(x)\right)}_{\langle\mu,g\rangle}\le\underbrace{\left(\sum_{x\in X}\mu(x)\,f(x)\,g(x)\right)}_{\langle\mu,fg\rangle}\underbrace{\left(\sum_{x\in X}\mu(x)\right)}_{\langle\mu,1\rangle}$$

D <u>Def</u> $\alpha\,\beta\,\gamma\,\delta : X \to \mathbb{R}_+$ <u>as follows</u>

$$\begin{cases} \alpha(X) = \mu(X)\, f(X) \\ \beta(X) = \mu(X)\, g(X) \end{cases} \qquad \begin{cases} \gamma(X) = \mu(X)\, f(X)\, g(X) \\ \delta(X) = \mu(X) \end{cases}$$

<u>4F assumption</u>: $\alpha(X)\,\beta(\cancel{X}) = \mu(X)\underset{\mu(Y)}{}\, f(X)\, g(\cancel{X}) \overset{\text{supermod}}{\leq} \mu(X\wedge Y)\,\mu(X\wedge Y)\, f(X)\, g(\cancel{X})$

$\underset{\text{incr}}{\leq} \mu(X\wedge Y)\,\mu(X\vee Y)\; \underline{f(X\vee Y)\, g(X\vee Y)} = \gamma(X\vee Y)\,\delta(X\wedge Y)$

$\Rightarrow \left\{\begin{matrix} \text{4F Thm} \\ A = B = X \end{matrix}\right\} \quad \alpha(X)\,\beta(X) \leq \gamma(X)\,\delta(X)$ ▨

<u>Th</u>' f - incr, g - decreasing $\Rightarrow$ (-‖-)(-‖-) $\geq$ (-‖-)(-‖-)

D $g' := N - g$ where N - suff. large ▨

<u>Cor</u> [Kleitman's Thm] $\mathcal{A}, \mathcal{B} \subseteq 2^{[n]}$

1) $\mathcal{A}$ - up-closed, $\mathcal{B}$ - down-closed $\Rightarrow |\mathcal{A}\cap\mathcal{B}|\cdot 2^n \leq |\mathcal{A}|\,|\mathcal{B}|$ <u>usual</u> <u>intersection</u>

2) $\mathcal{A}, \mathcal{B}$ - up-closed $\Rightarrow 2^n\,|\mathcal{A}\cap\mathcal{B}| \geq |\mathcal{A}|\,|\mathcal{B}|$

④

D **For 2)** $f: 2^{[n]} \to \mathbb{R}_+$, $g: 2^{[n]} \to \mathbb{R}_+$, $\mu := 1$

$$f(A) := \begin{cases} 1, & A \in \mathcal{A} \\ 0, & \text{oth} \end{cases} \qquad g(B) := \begin{cases} 1, & B \in \mathcal{B} \\ 0, & \text{oth.} \end{cases}$$

$\Rightarrow$ $\langle \mu\, 1 \rangle = 2^n$, $\langle \mu, f \rangle = |\mathcal{A}|$, $\langle \mu\, g \rangle = |\mathcal{B}|$

and $\langle \mu, fg \rangle = |\mathcal{A} \cap \mathcal{B}|$ $\Rightarrow$ ✓ ☒

Note: probab. interpretation of Cor = Kleitman th

$$\left.\begin{array}{ll} 1) & \mathbb{P}[\mathcal{A} \cap \mathcal{B}] \le \mathbb{P}[\mathcal{A}] \cdot \mathbb{P}[\mathcal{B}] \\ 2) & \quad -//- \quad \ge \quad -//- \end{array}\right\}$$ $\mathcal{A}, \mathcal{B}$ - events

---

By taking $\mu(A) := \prod_{i \in A} p_i \prod_{i \notin A} (1 - p_i)$

$X = [n]$, $\bar{p} = (p_1 \ldots p_n)$ ← probab of (i)

$$\Rightarrow \left\{\begin{array}{ll} 1) & \mathbb{P}_{\bar{p}}[\mathcal{A} \cap \mathcal{B}] \le \mathbb{P}_{\bar{p}}[\mathcal{A}] \cdot \mathbb{P}_{\bar{p}}[\mathcal{B}] \\ 2) & \quad -//- \quad \ge \quad -//- \end{array}\right\}$$ $\Rightarrow$ $\mathcal{A}$ - Ham, $\mathcal{B}$ - planar

⑤

Th [Shepp, 1982] ← **XYZ Theorem**

$P = (X, \prec)$, $|X| = n$

$x\,y\,z \in X$ ← incomparable elt's

Then $$\mathbb{P}_{LE}[A(x) \le A(y),\ A(x) \le A(z)] \ge \mathbb{P}_{LE}[A(x) \le A(y)] \cdot \mathbb{P}_{LE}[A(x) \le A(z)]$$

Note $\iff \mathbb{P}_{LE}[A(x) \le A(y)] \le \mathbb{P}_{LE}[A(x) \le A(y) \mid A(x) \le A(z)]$ conditional probab.

Th [Winkler, 1983]

Let $P = (X, \prec)$, $Q = (X, \prec')$ s.t. $\forall\, x\,y \in X$

$$\mathbb{P}_{A \in LE(P)}[A(x) \le A(y)] \le \mathbb{P}_{B \in LE(Q)}[B(x) \le B(y)]$$ universally correlated

Then Q can be obtained from P by adding valid $\prec$

$\iff$ Shepp's thm is optimal

L25 The XYZ Theorem

206A

Dec 4, 2020

Th [Shepp, 1982] ← XYZ theorem

$P = (X, \prec)$, $|X| = n$, $x\, y\, z \in X$ incomp.

Then $\mathbb{P}_{LE(P)}\big[A(x) < A(y),\ A(x) < A(z)\big]$

$\geq \mathbb{P}_{LE(P)}\big[A(x) < A(y)\big] \cdot \mathbb{P}_{LE(P)}\big[A(x) < A(z)\big]$

[A-S, § 6.4]

Proof Let $\mathcal{L}_N := \{ f : X \to [N]$ ~~s.t. $f(u) \leq f(v)$ $\forall\, u \prec v$~~ $\}$

$\mathcal{L}_N \neq [N]^P$ ← poset of order-preserving maps

Define $\boxed{\prec \text{ on } \mathcal{L}_N}$ **as follows**

$$f \preceq g \iff f(x) \geq g(x) \quad \text{and} \quad f(x') - f(x) \leq g(x') - g(x) \quad \forall\, x' \in X$$

Define $[f \wedge g](u) = \min\{ f(u) - f(x),\ g(u) - g(x) \} + \max\{f(x), g(x)\}$

$[f \vee g](u) = \max\{ \text{—}\,\text{''}\,\text{—} \quad \text{—}\,\text{''}\,\text{—} \} + \min\{\text{—}\,\text{''}\,\text{—}\}$

Main Lemma $\mathcal{L}_N$ is a distributive lattice

D (of ML) **Obs 1** L_N is a lattice

Indeed $\begin{cases} f\wedge g \leq f, g \leq f \vee g \\ f \vee (f \wedge g) = f, \quad f \wedge (f \vee g) = f \end{cases}$ $\forall f, g \in L_N$

Obs 2 $\min\{a, \max\{b, c\}\} = \max\{\min\{a, b\}, \min\{a, c\}\}$
$\forall\ a, b, c \in \mathbb{N}$ ④ **Obs 2'** min ⟷ max

/ ⟺ $([N], \leq) = ([N], \wedge = \min, \vee = \max)$ ← distributive /

Obs 3 $\forall f, g, h \in L_N$ we have $\boxed{\begin{array}{l} f \wedge (g \vee h) = \\ (f \wedge g) \vee (f \wedge h) \end{array}}$

Indeed $\forall u \in X$

$$[f \wedge (g \vee h)](u) = \min\{f(u) - f(x), [g \vee h](u) - [g \vee h](x)\} + \max\{f(x), [g \vee h](x)\}$$

$$= \min\{f(u) - f(x), \max\{g(u) - g(x), h(u) - h(x)\}\} + \max\{f(x), \min\{g(x), h(x)\}\}$$

Similarly

$$\left[(f\wedge g)\vee(f\wedge h)\right](u) = \max\left\{\begin{array}{l}[f\wedge g](u)-[f\wedge g](x), \\ {[f\wedge h](u)-[f\vee g](x)}\end{array}\right\} + \min\left\{[f\wedge g](x), [f\wedge h](x)\right\}$$

$$= \max\left\{\begin{array}{l}\min\{f(u)-f(x), g(u)-g(x)\} \\ \min\{f(u)-f(x), h(u)-g(x)\}\end{array}\right\} + \min\left\{\max\{f(x), g(x)\}, \max\{f(x), h(x)\}\right\}$$

Use Obs 2 w/ $\left|\begin{array}{l} a = f(u)-f(x) \\ b = g(u)-g(x) \\ c = h(u)-h(x)\end{array}\right.$

and Obs 2' w/ $\left|\begin{array}{l} a = f(x) \\ b = g(x) \\ c = h(x)\end{array}\right.$

$\Rightarrow$ L_N is distributive (Obs 3) ▨

Now use FKG

③

FKG inequality

[A-S, § 6.2]

Def $L = (X, \wedge, \vee)$ – distributive lattice

$\mu : X \to \mathbb{R}_+$ is *log-supermodular* if

$$\mu(x)\,\mu(y) \le \mu(x \wedge y)\,\mu(x \vee y) \quad \forall x, y \in X$$

$f : X \to \mathbb{R}_+$ is *increasing* if $f(x) \le f(y)$ $\forall x \le y$

order-preserving

decreasing if $f(x) \le f(y)$ —//—

Th [Fortuin-Kasteleyn-Ginibre, 1971]

Let $L = (X, \wedge, \vee)$ – distributive, $\mu, f, g : X \to \mathbb{R}_+$

s.t. $\mu \leftarrow$ log-supermod, $f, g \leftarrow$ increasing

Then

$$\underbrace{\left(\sum_{x \in X} \mu(x)\, f(x)\right)}_{\langle \mu, f\rangle} \underbrace{\left(\sum_{x \in X} \mu(x)\, g(x)\right)}_{\langle \mu, g\rangle} \le \underbrace{\left(\sum_{x \in X} \mu(x)\, f(x)\, g(x)\right)}_{\langle \mu, fg\rangle} \underbrace{\left(\sum_{x \in X} \mu(x)\right)}_{\langle \mu, 1\rangle}$$

Back to P $\mu : L_N \to \{0,1\}$

$$\mu(f) = \begin{cases} 1, & f \leftarrow \text{order-proserving} \\ 0, & \text{oth.} \end{cases}$$

OBS 4 $f, g \in [N]^P \Rightarrow f \vee g,\ f \wedge g \in [N]^P$

/ $\Rightarrow$ μ – log-supermodular, since

$\mu(f)=\mu(g)=1 \Rightarrow \mu(f \wedge g) = \mu(f \vee g) = 1$ /

Indeed $\forall\ u \precsim v$, $u, v \in X$,

suppose $f(u) \le f(v)$, $g(u) \le g(v)$

Then $[f \vee g](u) = \max\{f(u)-f(x),\ g(u)-g(x)\} + \min\{f(x),\ g(x)\}$

$\le \max\{f(v)-f(x),\ g(v)-g(x)\} + \min\{f(x),\ g(x)\} = [f \vee g](v)$

Same argument for $[f \wedge g](u) \le [f \wedge g](v)$

Back to FKG and XYZ

Def $F, G : L_N \to \{0,1\}$ $\underline{\forall f \in L_N}$

$$F(f) = \begin{cases} 1, & f(x) \le f(y) \\ 0, & \text{oth} \end{cases}$$

$$G(f) = \begin{cases} 1, & f(x) \le f(z) \\ 0, & \text{oth} \end{cases}$$

Obs 5 F, G are increasing

Indeed $\forall f, g \in L_N$ s.t. $f \preccurlyeq g$, $F(f) = 1$

$\Rightarrow 0 \le f(y) - f(x) \le g(y) - g(x) \Rightarrow F(g) = 1$ ▨

Conclusion of XYZ proof: $\Delta_N := [N]^P \subset L_N$

$$\mathbb{P}_{\Delta_N}[f(x) \le f(y), f(x) \le f(z)] \geq (\text{FKG})$$

$$\geq \mathbb{P}_{\Delta_N}[f(x) \le f(y)] \cdot \mathbb{P}_{\Delta_N}[f(x) \le f(z)]$$

Let $N \to \infty$, $\tilde{f} := \frac{1}{N} f \to$ unifom in O_P → unif $LE(P)$

$$\mathbb{P}_{O_P}[\tilde{f}(x) < \tilde{f}(y), \tilde{f}(x) \le \tilde{f}(z)] \geq \mathbb{P}_{O_P}[\text{-/-}] \cdot \mathbb{P}_{O_P}[\text{-/-}]$$ ▨

⑥

Note: $\textcircled{\geq} \to \textcircled{>}$ in XYZ th. [Fishburn, 1984]

[Brigtwel-Trotter, 2002] ← proof via counting, ≈ 12 pp.

[Swee-Hong Chan, 2020]: XYZ ⇒ Hammett-Pottel

$$e(P) \geq n! \prod_{x\in X} b(x)^{-1}$$

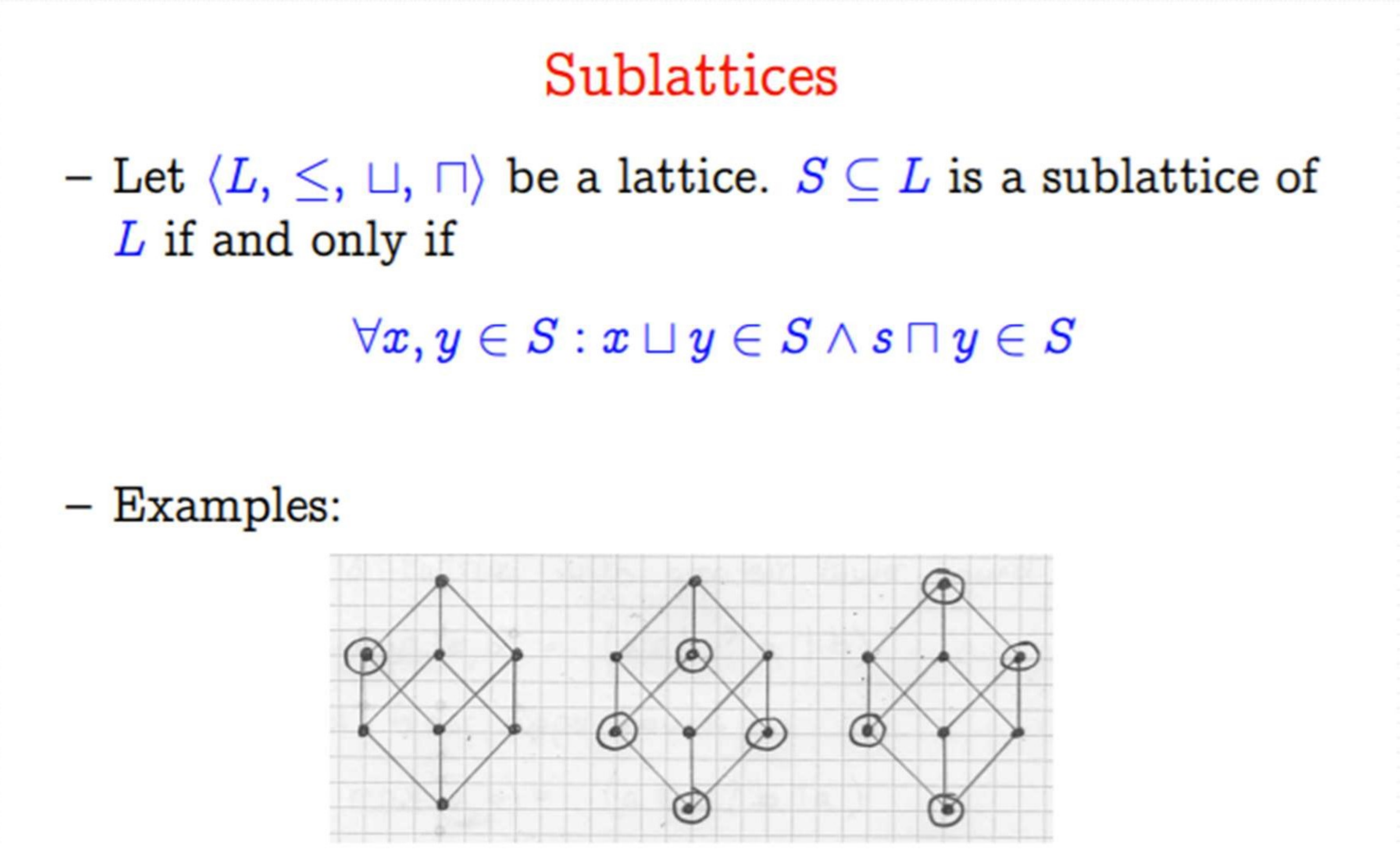
Winkler's canonical linear ordering [Social choice]

$$f_P(x) := \frac{1}{e(P)} \sum_{A\in LE(P)} A(x) \quad , \quad f_P : X \to \mathbb{R}_+$$

$P = (X, \prec)$ poset

$f_P \to A$ some LE , $A \in LE(P)$

Note: The sublattice $\Rightarrow$ lattice assertion is true indeed!



Sublattices

- Let $\langle L, \leq, \sqcup, \sqcap \rangle$ be a lattice. $S \subseteq L$ is a sublattice of L if and only if

$$\forall x, y \in S : x \sqcup y \in S \wedge s \sqcap y \in S$$

- Examples:

L26 Comparisons via linear extensions

206A
Dec 7, 2020

Th [Shepp, 1982] ← XYZ theorem

$P = (X, \prec)$, $|X| = n$, $x, y, z \in X$ ← incomp

$$\Rightarrow \mathbb{P}_{LE(P)}[A(x)<A(y), A(x)<A(z)] \geqslant \mathbb{P}_{LE(P)}[A(x)<A(y)] \cdot \mathbb{P}_{LE(P)}[A(x)<A(z)]$$

Def [Winkler, 1982] $h_P : X \to \mathbb{R}_+$ ← canonical ordering

$$h_P(x) := \frac{1}{e(P)} \sum_{A \in LE(P)} \underline{A(x)-1} = \mathbb{E}_{LE(P)}[A(x)] \underline{-1} \geqslant 0$$

Th [Winkler, 1982] $P = (X, \prec)$, $x, y \in X$ incomp

Let $P' := P \cup (x \succ y)$, $P'' := P \cup (x \prec y)$

Then $h_{P'}(x) \geqslant h_P(x)$. **Moreover** $\boxed{h_{P'}(x) \geqslant 1 + h_{P''}(x)}$

D Obs Let $p := \mathbb{P}_{LE}(A(x) > A(y))$

Then $h_P(x) = h_{P'}(x) \cdot p + h_{P''}(x) \cdot (1-p) \le h_{P'} \cdot p + (h_{P'} - 1)(1-p)$

$\le h_{P'}(p + (1-p)) - (1-p) \le h_{P'}$.

Moreover ⇒ Then

①

D (cont'd) Let $\xi_Y = \begin{cases} 1, & A(x) > A(y) \\ 0, & \text{oth.} \end{cases}$ ← r.v.

$$\Rightarrow \quad h_P(x) = \mathbb{E}_{LE}[A(x)-1] = \sum_{z \neq x} \mathbb{E}_{LE}[\xi_z] = \sum_{z \in X} \mathbb{P}_{LE}[A(x) > A(z)]$$

Similarly

$$h_{P'}(x) = \sum_{z \notin X} \mathbb{P}_{LE}[A(x) > A(z) \mid A(x) > A(y)]$$

$$h_{P''}(x) = \sum_{z \in X} \mathbb{P}_{LE}[A(x) > A(z) \mid A(x) < A(y)]$$

XYZ theorem $\Rightarrow \mathbb{P}_{LE(P)}[A(x) > A(z) \mid A(x) > A(y)]$

$$\geq \mathbb{P}_{LE(P)}[A(x) > A(z) \mid A(x) < A(y)]$$

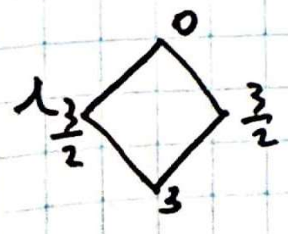
$$\Rightarrow \quad h_{P'}(x) = \sum_{z \neq x, y} \mathbb{P}[>|>] + \mathbb{P}\,\Big|_{z=y} + \mathbb{P}\,\Big|_{z=x}$$

$$\geq \sum_{z \notin xy} \mathbb{P}[>|<] + 1 + 0 = h_{P''}(x) + 1$$

▨

Note $h_P : X \to \mathbb{R}_+$ is not always linear

but at least it's well-defined!



②

Social Choice Def [preferential ordering]

$P=(X,\prec)$, $|X|=n$, $x,y\in X$ incomp.

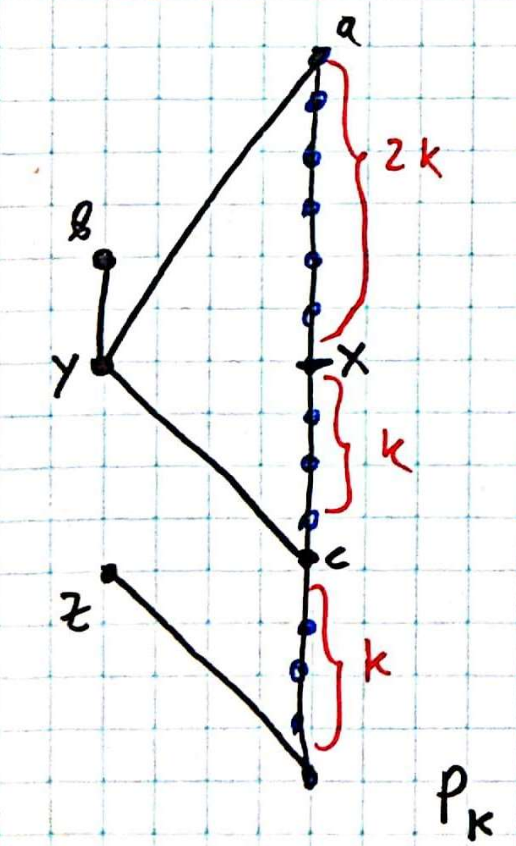
Let $x \lhd y$ if $\mathbb{P}_{LE(P)}[A(x)<A(y)] > \frac{1}{2}$

Th [Fishburn, 1974] $\exists\, P=(X,\prec)$ s.t.

$x \lhd y$, $y \lhd z$, $z \lhd x$ for some $x,y,z\in X$

D (sketch)

$$e(P_k) \sim c\binom{4k}{3} = \Theta(k^3)$$



$$\mathbb{P}[A(x)>A(y)] \sim \frac{5}{9}$$

$$\mathbb{P}[A(y)>A(z)] \sim \frac{1}{2}+$$

$$\mathbb{P}[A(z)>A(x)] \sim \frac{1}{2}+$$

▨

Note: $k=6$ works ✓

Q: Why?

③

Intransitive dice

Def Die A **beats** die B if $\quad / A \rhd B /$

$\mathbb{P}[A > B] > \frac{1}{2}$

Obs / Th $\exists$ dice A, B, C s.t. $A \rhd B,\ B \rhd C,\ C \rhd A$

D
$$\left.\begin{aligned} A &:= [2\ 2\ \ 4\ 4\ \ 9\ 9] \\ B &:= [1\ 1\ \ 6\ 6\ \ 8\ 8] \\ C &:= [3\ 3\ \ 5\ 5\ \ 7\ 7] \end{aligned}\right\} \Rightarrow \mathbb{P}[A > B] = \mathbb{P}[B > C] = \mathbb{P}[C > A] = \frac{5}{9}$$

▨

Ex [Efron's Dice] ← W. Buffett vs. Bill Gates

$A := [4\ 4\ 4\ 4\ 0\ 0]$ $\qquad C := [6\ 6\ \ 2\ 2\ 2\ 2]$ $\qquad \left\{\begin{array}{l} A \rhd B \rhd C \rhd D \rhd \\ D \rhd A \end{array}\right.$

$B := [3\ 3\ 3\ \ 3\ 3\ 3]$ $\qquad D := [5\ 5\ 5\ \ 1\ 1\ 1]$

Th [P. J. Polymath, 2017] $A, B, C \leftarrow$ random n-sided dice w/ sides $\in \mathbb{N}$, $\oplus = \binom{n}{2}$, **Then** $\mathbb{P}[A \rhd C \mid A \rhd B \rhd C] \sim \frac{1}{2}$

Th [Hazla - Mossel - Ross - Zheng, 2020] sides $\leftarrow \mathcal{N}(0,1)$, $\oplus = 0$

$\Rightarrow \mathbb{P}[\quad] = 0$ a.s.

④

$\frac{1}{3}-\frac{2}{3}$ Conjecture

Conj ($\frac{1}{3}-\frac{2}{3}$) $P=(X,\prec)$, $|X|=n$, $P\neq C_n$
$\exists\, x,y\in X$ s.t. $\mathbb{P}_{LE(P)}\left[A(x)<A(y)\right]\in\left[\frac{1}{3},\frac{2}{3}\right]$

History: Kislitsyn (1968), Fredman (1976)

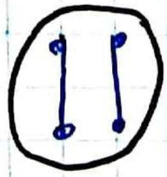
Motivation: Sorting w/ partial information

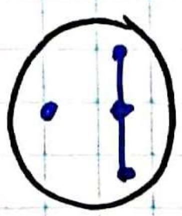
Let $f: X\to\mathbb{N}$ s.t. $f(x)<f(y)$ $\forall\, x\prec y$
and $f(x)\neq f(y)$ $\forall\, x,y\in X$

Find $LE(P)\ni A$ defined by f

Conj known for

1) width(P) ≤ 2
2) height(P) ≤ 2
3) series-parallel posets
4) $n\leq 11$
5) semiorders := posets w/ no

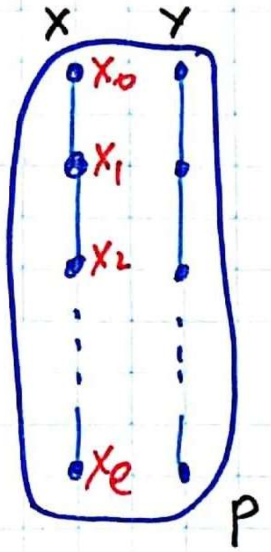




Th [Linial, 1984]

$\frac{1}{3}-\frac{2}{3}$ conj holds for posets of width 2

D $P=(X,<)$, $|X|=n$, $\exists$ partition $P=\underbrace{C_1 \sqcup C_2}_{\text{chains}}$



$x \leftarrow$ min in $C_1 := \{x=x_0, x_1, x_2, \dots x_e\}$
$y \leftarrow$ min in C_2

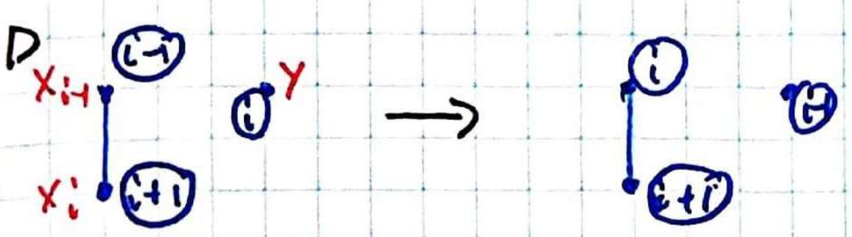
If $\mathbb{P}[A(x)<A(y)] \in [\frac{1}{3}\ \frac{2}{3}]$ ✓

Assume $\mathbb{P}[\quad] < \frac{1}{3}$ /relabel oth./

Let $q_0 := \mathbb{P}[A(y)<A(x_0)] < \frac{1}{3}$
$q_1 := \mathbb{P}[A(x_0)<A(y)<A(x_1)]$
$\vdots$
$q_i := \mathbb{P}[A(x_{i-1})<A(y)<A(x_i)]$

Obs 1 $q_0+q_1+q_2+\dots+q_e = 1$ ✓
Obs 2 $q_0 \geq q_1 \geq q_2 \geq \dots \geq q_e \geq 0$
$\Rightarrow$ some $q_0+\dots+q_i \in [\frac{1}{3}, \frac{2}{3}]$

D



$\Rightarrow q_i \leq q_{i-1}$ ▨

Next time

Th [Kahn - Saks, 1984] $\exists\ \varepsilon > 0$ $/= \frac{3}{11}/$

s.t. $\forall\ P = (X, \prec)$ $\exists\ x, y \in X$

$$\mathbb{P}_{LE(P)}\left[A(x) < A(y)\right] \in [\varepsilon, 1-\varepsilon]$$

Note: this can be made effective + fast [Cardinal et al., 2010]

⑦

L27 $\frac{1}{3}$-$\frac{2}{3}$ Conjecture

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Dec 9, 2020

Conj [$\frac{1}{3}$-$\frac{2}{3}$ Conj]

$P = (X, \prec)$, $|X| = n$. Then $\exists\, x, y \in X$ s.t.

$$\frac{1}{3} \le \mathbb{P}_{LE(P)}\left[A(x) < A(y) \right] \le \frac{2}{3}$$

Th [Linial, 1984]

$\frac{1}{3}$-$\frac{2}{3}$ conj holds for posets of width 2

Th [Kahn-Saks, 1984]

$P = (X, \prec)$, $|X| = n$. Then $\exists\, x, y \in X$ s.t.

$$\frac{3}{11} \le \mathbb{P}_{LE(P)}\left[A(x) < A(y) \right] \le \frac{8}{11}$$

Today: we prove $\mathbb{P}[-\shortparallel-] \in [\varepsilon, 1-\varepsilon]$

for some $\varepsilon > 0$.

①

Recall poset polytope $\mathcal{O}_P \subseteq [0,1]^n$ order polytope

$\mathcal{O}_P := \{ f : X \to [0\ 1]$ s.t. $f(x) \le f(y)\ \ \forall\, x \prec y,\ x y \in X \}$ ← def

Vertices of $\mathcal{O}_P$ are $\chi^{U}(x) = \begin{cases} 1, x \in U \\ 0, \text{oth} \end{cases}$ where ← vertices L20

$U \subset X$ upper. order ideal

$/\ x \in U \Rightarrow y \in U\ \forall\, y \succ x\ /$

$\mathrm{Vol}_n(\mathcal{O}_P) = \dfrac{e(P)}{n!}$ ← Vol L20

$h_P : X \to \mathbb{R}_+$, $h_P(x) := \dfrac{1}{e(P)} \displaystyle\sum_{A \in LE(P)} A(x)$ ← Winkler's canonical ordering L24.

$1 \le h_P(x) \le n \quad \forall\, x \in X$

L center of mass $\quad cm(\mathcal{O}_P) = \dfrac{1}{n+1}\, h_P$

P Recall $\quad \mathcal{O}_P = \displaystyle\bigsqcup_{A \in LE(P)} \Delta_A$

For $P = C_n$ we have $\mathcal{O}_P = \Delta = \mathrm{conv}\{(00\ldots 11\ldots 1)\}$

and $cm(\mathcal{O}_P) = cm(\Delta) \textcircled{=} \dfrac{1}{n+1}(1\ 2 \ldots n) = \dfrac{1}{n+1} h_{C_n}$ ← Exc

$\Rightarrow \forall P \quad cm(P) = \dfrac{1}{e(P)} \displaystyle\sum_{A \in LE(P)} cm(\Delta_A) = \dfrac{1}{n+1} \Big[\underbrace{\dfrac{1}{e(P)} \sum h_{A_i}}_{h_P} \Big]$ ▨

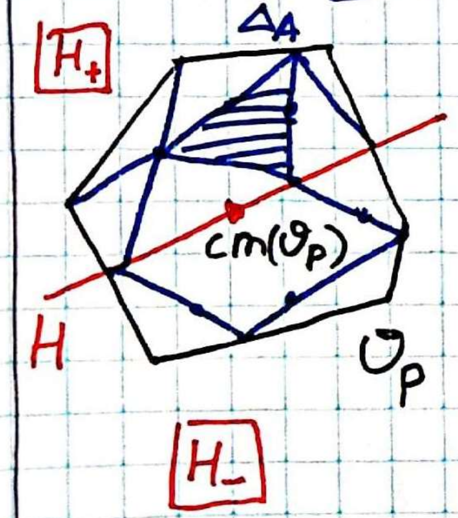
Obs $\exists\, x, y \in X$ s.t. $|h_p(x) - h_p(y)| \leq 1$

▷ **Indeed** $|X| = n$ but $h_p(x) \in [1 \dots n]$ $\forall x \in X$ $\blacksquare$

Main Lemma $\forall\, x, y \in X$ s.t. $|h_p(x) - h_p(y)| < 1$

$$\mathbb{P}_{LE(P)}\left[A(x) < A(y) \right] \in [\varepsilon, 1-\varepsilon]$$ /Some $\varepsilon > 0$ we make explicit/

Case $\boxed{h_p(x) = h_p(y)}$



$$\mathcal{O}_p = \bigsqcup_{A \in LE(P)} \Delta_A$$

$H := H_{xy} \subset \mathbb{R}^n$ hyperplane $:= \{ f(x) = f(y),\ f \in \mathcal{O}_p \}$

$\Rightarrow cm(\mathcal{O}_p) \in H$

Obs $\forall A \in LE(P)$

$$\begin{cases} \Delta_A \subseteq H_+ & \Leftrightarrow \quad A(x) > A(y) \\ \Delta_A \subseteq H_- & \Leftrightarrow \quad A(x) < A(y) \end{cases}$$

$$\Rightarrow \mathbb{P}_{LE(P)}\left[A(x) < A(y) \right] = \frac{\#\{ A \in LE(P) : A(x) < A(y) \}}{e(P)}$$

$$= \frac{vol(\mathcal{O}_p \cap H_-)}{vol(\mathcal{O}_p)}$$

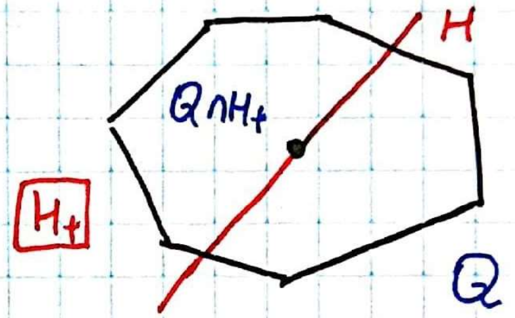
③

Th [Grünbaum, 1960]

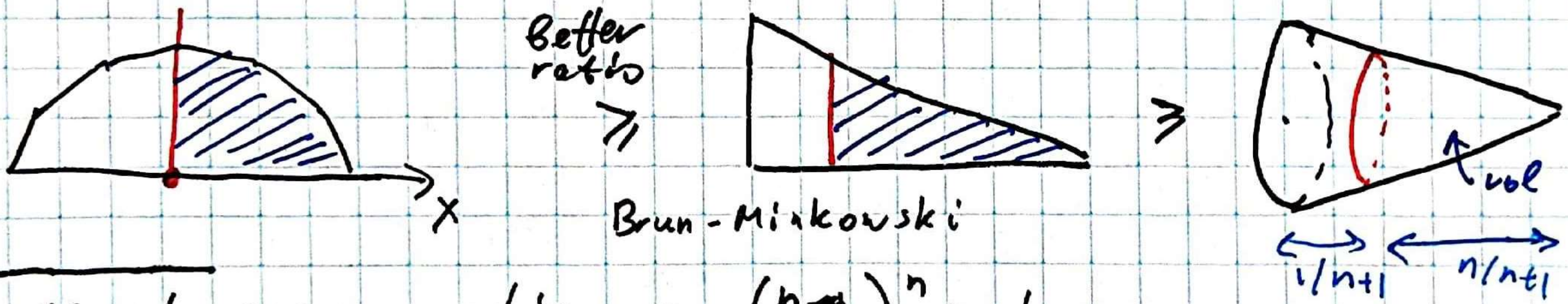
$Q \subset \mathbb{R}^n$ convex body, $\boxed{cm(Q) \in H}$

$H \subset \mathbb{R}^n$ hyperplane

then $\dfrac{vol(Q \cap H_+)}{vol(Q)} > \dfrac{1}{e}$



Proof idea: project onto x-coordinate ⊥ H



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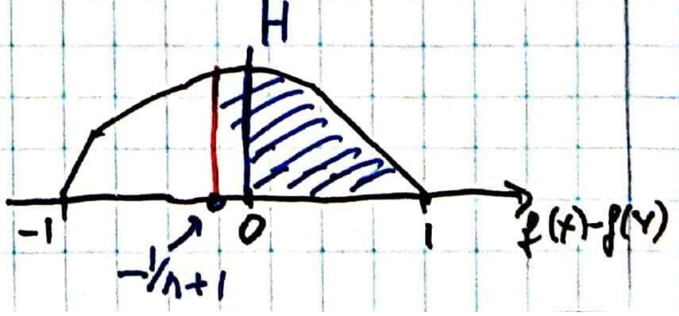
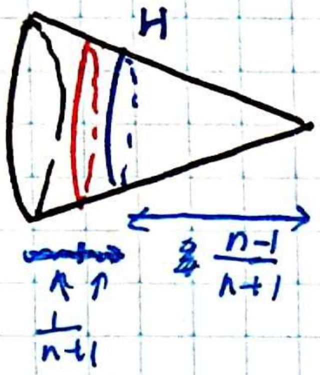
worst case ratio $= \left(\frac{n}{n+1}\right)^n > \frac{1}{e}$ ◁

$$\Rightarrow \quad \mathbb{P} = \frac{vol(O_p \cap H_+)}{vol(O_p)} \in \left[\frac{1}{e}, 1-\frac{1}{e}\right] \quad ✓$$

General Case: $cm(O_p) \notin H$



Now similar B-M-type argument



$$\Rightarrow \quad \frac{vol(Q\cap H_+)}{vol(Q)} > \left(1-\frac{2}{n+1}\right)^n > \frac{1}{e^2}$$

$$\Rightarrow \quad \mathbb{P} = \frac{vol(O_P\cap H_+)}{vol(O_P)} \in \left[\frac{1}{e^2}, 1-\frac{1}{e^2}\right] \qquad \checkmark$$

Best known: $\mathbb{P} \in \left[\underbrace{\frac{1}{2}-\frac{1}{2\sqrt{5}}}_{.2763}, \frac{1}{2}+\frac{1}{2\sqrt{5}}\right]$

[Brightwell - Felsner - Trotter, 1995]

Conj [Kahn-Saks] $\forall \varepsilon > 0 \ \exists k = k(\varepsilon)$

s.t. $\forall P = (X, <)$, width$(P) > k$

$$\mathbb{P}_{LE(P)}[A(x) < A(y)] \in \left[\frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon\right]$$

for some $x, y \in X$

⑤

L28 Final chapter

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Dec 11, 2020

What else is known about $\frac{1}{3}-\frac{2}{3}$ conj?

Conj [$\frac{1}{3}-\frac{2}{3}$] $\forall P=(X,\prec)$, $|X|=n$ $\exists x,y\in X$ s.t. $\frac{1}{3} \le \mathbb{P}_{LE(P)}[A(x)<A(y)] \le \frac{2}{3}$ / $P\ne C_n$ /

Def $\delta(P) := \min_{x y \in X} \left| \mathbb{P}_{LE}[A(x)<A(y)] - \mathbb{P}_{LE}[A(x)>A(y)] \right|$

Conj [Kahn-Saks, 1984]

$\forall \{P_n\}$, width$(P_n)\to\infty \Rightarrow \delta(P_n)\to 0$ as $n\to\infty$

Th [Komlós, 1990]

$\exists g: \mathbb{N}\to\mathbb{R}_+$, $g(n)\to\infty$ s.t. $\forall \{P_n\}$ with [# min elt's of $P_n > \frac{n}{g(n)}$] $\Rightarrow \delta(P_n)\to 0$ as $n\to\infty$

Note: Probaly works for [width$(P_n) > \frac{n}{g(n)}$]

①

Special Cases

Th [Chan-P.-Panova] Fix $d \geq 2$, $\varepsilon > 0$

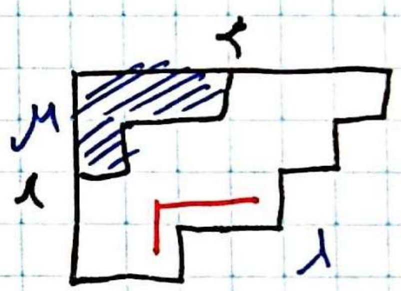
$\lambda = (\lambda_1, \ldots \lambda_d)$ ← partition of n s.t. $\lambda_d \geq \varepsilon n$.

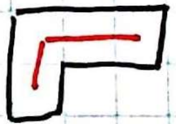
then $\delta(P_\lambda) = O(\frac{1}{\sqrt{n}})$ / i.e. $\leq \frac{C(d,\varepsilon)}{\sqrt{n}}$ /

Proof uses some ideas by Linial + asympotics + AC

Th [Olson-Sagan] (2018)

$\frac{1}{3}-\frac{2}{3}$ conj holds ∀ $P_{\lambda/\mu}$



Proof **idea**: use Linial th for  + case analysis

Th [Trotter-Gehrlein-Fishburn, 1992]

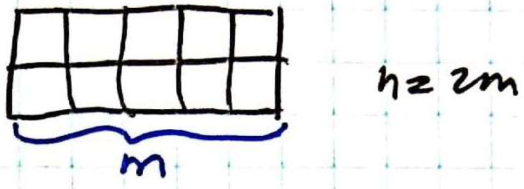
$\frac{1}{3}-\frac{2}{3}$ conj holds ∀ P w/ height(P) = 2

Proof **idea**: use Komlós th + case analysis.

Th [Chan – P. – Panova, 2020]

P_n = Catalan poset

$\Rightarrow \delta(P_n) = O\left(\frac{1}{n^{5/4}}\right)$



Proof idea: delicate asymptotic analysis

Conj: 5/4 cannot be improved.

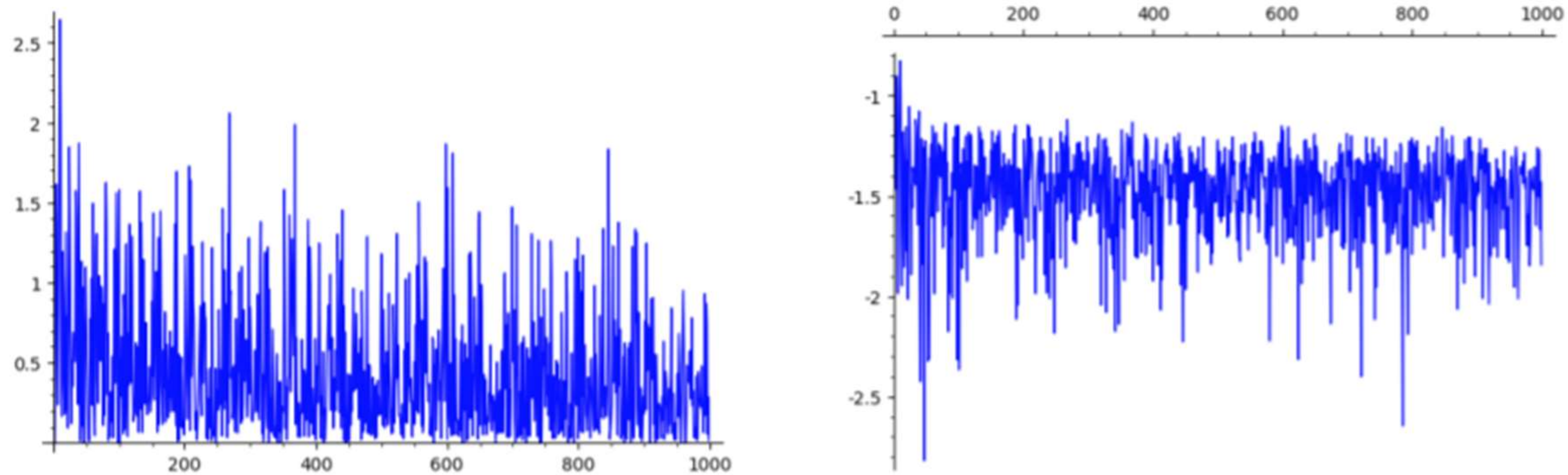
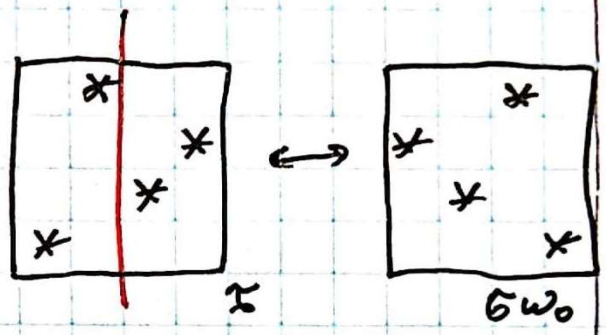


FIGURE 6. Graphs of $\delta(P_n)\,n^{5/4}$ and $\log_n \delta(P_n)$, for $3 \le n \le 1000$.

Def (duality) $P=(X,\prec)$, $P^*=(X,\prec')$ s.t.

$\mathrm{Com}(P^*) = \overline{\mathrm{Com}(P)}$

Obs: $P_\sigma^* = P_{\sigma w_0}$, $w_0 = (n\, n{-}1 \ldots 1)$



Th [Sidorenko, 1981]

$$e(P_\sigma)\, e(P_\sigma^*) \geqslant n!$$

First proof **idea**: use chain polytopes

Second —"— : special case of **Mahler Conj**

$$\mathrm{vol}(B)\, \mathrm{vol}(B^*) \geqslant \frac{4^n}{n!} \quad \forall \text{ c-s. convex body } B \subseteq \mathbb{R}^n$$

$[0,1]^n$ extreme example

where $B^* = \{ y \in \mathbb{R}^n : |y \cdot x| \leq 1 \quad \forall\, x \in B \}$

For B - corner + reflections = symmetric w.r.t. $(\pm x_i)$

this was proved by Saint-Raymond [1980)

Th [Bollobás - Brightwell - Sidorenko, 1999]

$$e(P_\sigma)\, e(P_\sigma^*) \leq c \cdot n! \left(\frac{\pi}{2}\right)^n$$

Proof: Santalo inequality

unit ball extreme ex

Complexity issues

Th [Brightwell-Winkler, 1991]

Computing $e(P)$ is #P-complete.

Note: No parsimoneous Bij i.e. no bij proof

$LE(P) \leftrightarrow 3SAT(F)$ / Because $e(P) \geq 1$ /

Th $Q \subset \mathbb{R}^n$ convex polytope given by facets

$\Rightarrow$ vol(Q) is #P-hard to compute

/ Linial obs + BW Th /

Proof of BW uses mod p argument + CRT

Th [Karzanov-Khachiyan, Matthews, 1991]

RW on Γ_P /← graph ~~of~~ on $LE(P)$ w/ 2-flips /

mixes in poly time.

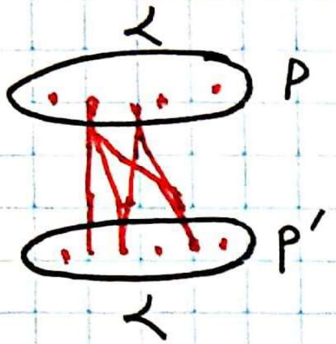
$\Rightarrow$ $e(P)$ can be approx $(1 \pm \varepsilon)\, e(P)$ in poly time.

⑤

Th [Dittmer-P, 2019]

$e(P)$ is #P-c for height$(P)=2$

D



$P = P'$ ← BW poset

+ CRT argument

+ Wilson's th $(p-1)! = -1 \pmod p$ ▨

Th [Dittmer-P, 2019]

$e(P_\sigma)$ is #P-c, $\sigma \in S_n$

D Again CRT + gadget constructions via heavy algebraic computation ▨