Last time: $w \in A^*$, A totally ordered alphabet $A = \{1 < 2 < \cdots < n\}$

Goal: find length of longest nondecreasing subword of w.

Schensted's insertion   word $w \to$ tableau $P(w)$

ex 13214 $1 \to 13 \to \begin{array}{ll} 1 & 2 \\ 3 & \end{array} \to \begin{array}{ll} 1 & 1 \\ 2 & \\ 3 & \end{array} \to \begin{array}{lll} 1 & 1 & 4 \\ 2 & & \\ 3 & & \end{array}$

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ex 13214 $\begin{array}{ccccc} & & & & 4 \\ & & & 1 & \\ & & 2 & & \\ & 3 & & & \\ 1 & & & & \end{array} \to \begin{array}{cccc} & & 1 & 4 \\ & & 2 & \\ & 3 & & \\ 1 & & & \end{array} \to \begin{array}{ccc} & 1 & 4 \\ & 2 & \\ 1 & 3 & \end{array} \to \begin{array}{ccc} & 1 & 4 \\ 2 & & \\ 1 & 3 & \end{array} \to \begin{array}{ccc} & 1 & 4 \\ & 2 & \\ 1 & 3 & \end{array} \to \begin{array}{ccc} & 1 & 4 \\ 1 & 2 & \\ 3 & & \end{array} \to \begin{array}{ccc} 1 & 1 & 4 \\ 2 & & \\ 3 & & \end{array}$

Thm 1 - The maximal length of a nondecreasing subword of w is the length of the first row of $P(w)$

- The maximal length of a decreasing subword of w is the length of the first column of $P(w)$

We will prove something stronger

let $\ell_k(w)$ be the maximum of the length of k disjoint nondecreasing subwords of w.

$\ell_1(w) =$ maximal length of a nondecreasin subword of w

($\ell'_k(w)$ same for decreasing subwords)

ex 13214 $\ell_1(w) = 3$, $\ell_2(w) = 4$, $\ell_3(w) = 5$, $\ell_4(w) = 5$

Thm 2 (Greene) For $w \in A^*$, If $P(w)$ has shape $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ & $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_s)$ then for $k = 1, \ldots, r$ $\lambda_k = \ell_k(w) - \ell_{k-1}(w)$

$k = 1, \ldots, s$ $\lambda'_k = \ell'_k(w) - \ell'_{k-1}(w)$

ex $w = 13254$, $P(w) = \begin{array}{lll} 1 & 2 & 4 \\ 3 & 5 & \end{array}$, $\lambda = (3,2)$, $\lambda' = (2,2,1)$, $\ell_1 = 3$, $\ell_2 = 3+2$, $\ell_3 = 5$, $\ell'_1 = 2$, $\ell'_2 = 4$, $\ell'_3 = 5$.

Thm 2 implies Thm 1.

To prove thm. 2 we will look at the equivalence relation in A^*

$$u \sim v \quad \text{iff} \quad P(u) = P(v)$$

ex $n=2$: $12 \to \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}$; $21 \to \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}$; $11 \to \begin{array}{|c|}\hline 1 \\ \hline 1 \\ \hline\end{array}$

so $u \sim v$ iff $P(u) = P(v)$

$n=3$: $123 \to \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}$; $213 \to \begin{array}{cc} 1 & 3 \\ 2 & \end{array}$; $132 \to \begin{array}{cc} 1 & 2 \\ 3 & \end{array}$

$231 \to \begin{array}{cc} 1 & 3 \\ 2 & \end{array}$; $312 \to \begin{array}{cc} 1 & 2 \\ 3 & \end{array}$; $321 \to \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}$

123, 321, $213 \sim 231$, $132 \sim 312$

* that's it! The congruence in A^* is generated by relations:

$$yxz \sim yzx \quad (x < y \le z)$$
$$xzy \sim zxy \quad (x \le y < z)$$

} Knuth transformations

This motivates the definition of the following monoid:

Definition The <u>plactic monoid</u> on A is $Pl(A) = A^*/\equiv$ where

$$yxz \equiv yzx \quad (x<y\le z) \qquad (*)$$
$$xzy \equiv zxy \quad (x \le y < z) \qquad (**)$$

<u>product</u> $u, v \in Pl(A)$ concatenation $u \cdot v = uv$ modulo $\equiv$

Prop 1 For $w \in A^*$, $w \equiv P(w)$

ex $132541 \equiv P(132541) = \begin{array}{ccc} 1 & 1 & 4 \\ 2 & 5 & \\ 3 & & \end{array} = 325114$

Prop 2 If $w \equiv w'$ then $\ell_k(w) = \ell_k(w')$ for all k.

<u>Proof of Thm 2</u> By Prop 1 & Prop 2 enough to show that

$$\ell_k(t) = \lambda_1 + \lambda_2 + \cdots + \lambda_k \quad k = 1, \ldots, r \quad \text{for tableau } t \text{ shape } \lambda$$

$(w \equiv P(w),\ \ell_k(w) = \ell_k(P(w)))$

* take k first rows as subwords so $\lambda_1 + \lambda_2 + \cdots + \lambda_k \le \ell_k(w)$

* conversely, a nondecreasing subword w of t uses at most one letter from each column (column is a decreasing subword).
So k disjoint nondecreasing subwords use at most $\lambda_1 + \lambda_2 + \cdots + \lambda_k$ letters.
$\lambda_1 + \lambda_2 + \cdots + \lambda_k \ge \ell_k(w)$ so they are $=$. $\boxtimes$

ex. 68 4556 223357 1112444 $\longleftrightarrow$

$$\begin{array}{ccccccc} 1&1&1&2&4&4&4\\ 2&2&3&3&5&7&\\ 4&5&5&6&&&\\ 6&8&&&&& \end{array}$$

<u>Proof Prop 1</u> Induction on $|w|$, true for $|w|=3$

assume $P(w) \equiv w$ need to show $P(wx) \equiv wx \equiv P(w)\cdot x$

enough to show the case w is a row

- if wx is a row $P(wx) = wx$ ✓
- o/w $P(wx) = y\cdot w'$, y leftmost element in w, $y > x$

$w = \underbrace{\quad}_{u}\, y\, \underbrace{\quad}_{v}$ then

$$wx = \underbrace{\quad}_{u}\, y\, \underbrace{\quad}_{v}\, x \equiv \underbrace{\quad}_{u}\, y x\, \underbrace{\quad}_{v} \quad (\text{by } (*))$$

$$\equiv y\, \underbrace{\quad}_{u}\, x\, \underbrace{\quad}_{v} \quad [\text{by } (**)]$$

$\boxtimes$

<u>Proof of Prop 2</u> say w & w' differ only by a Knuth transformation

$w = uxzyv$ $\qquad w' = uzxyv$ $\qquad (x \le y < z)$

* nondecreasing subwords of w' are nondecreasing subwords of w

$$\ell_k(w) \geq \ell_k(w')$$

* let $w_1, w_2, \dots, w_k$ nondecr. subwords of w

w_i not nondecr. subword of w' if $w_i = u'xzv'$

- if y does not occur in other w_j's, let $w_i' = u'xyv'$ is nondecr. subword w'
- if $w_j = u''yv''$ then let $w'_i = u'xyv''$, $w'_j = u''zv'$ are nondecr. subwords of w'

So $\ell_k(w) \leq \ell_k(w') \implies \ell_k(w) = \ell_k(w')$ $\boxtimes$

Moreover, each equivalence class in $Pl(A)$ has exactly one tableau.

<u>Thm 3</u> The equivalence $\sim$ coincides with plactic congruence.
(i.e. $w \sim w'$ $(P(w) = P(w'))$ iff $w \equiv w'$.)

<u>Pf</u> $\Rightarrow$ If $w \sim w'$ then by Prop 1 $w \equiv P(w) \equiv P(w') \equiv w'$.

$\Leftarrow$ If $w \equiv w'$, by Prop 2 then $\ell_k(w) = \ell_k(w')$ for all k
so $P(w)$ and $P(w')$ have the same shape.

let z be greatest letter in w and w' $\quad w = u z v$, $w' = u' z v'$ $\quad$ (no z in v or v')

Claim $\quad uv \equiv u'v'$

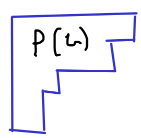
assume w and w' differ by one Knuth transformation
- if it does not involve z either $u \equiv u'$, $v = v'$ or $u = u'$, $v \equiv v'$
- if it involves z, erasing z in (*) or (**) gives $yx = yx$, $xy = xy$

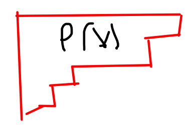
so $uv = u'v'$ ✓

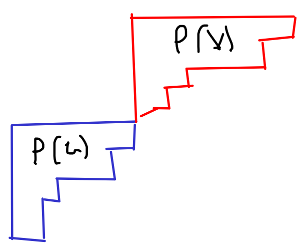
by induction on $|w|$ we have $P(uv) = P(u'v')$
now from Schensted's algorithm deleting z in $P(uzv)$ gives $P(uv)$
we know where z is since we know the shape of $P(w)$
so $P(w) = P(w')$ $\boxtimes$

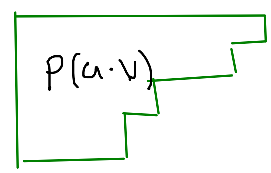
Prop 2 Each equivalence class in $P\ell(A)$ contains a unique tableau

consequence (product revisited)

$u, v \in P\ell(A)$ $\qquad u \equiv P(u) =$ 

, $\quad v = P(v) =$ 

$u \cdot v \equiv$ 

$\rightarrow$ 

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example $\quad u = 132 \quad v = 54 \qquad P(u) = \begin{matrix} 1 & 2 \\ 3 & \end{matrix} \qquad P(v) = \begin{matrix} 4 \\ 5 \end{matrix}$

$$\begin{matrix} & & 4 \\ & & 5 \\ 1 & 2 & \\ 3 & & \end{matrix} \;\rightarrow\; \begin{matrix} & & 4 \\ 1 & 2 & 5 \\ 3 & & \end{matrix} \;\rightarrow\; \begin{matrix} & 2 & 4 \\ 1 & 5 & \\ 3 & & \end{matrix} \;\rightarrow\; \begin{matrix} 1 & 2 & 5 \\ 3 & 5 & \end{matrix} = P(132 \cdot 45)$$