Applications of Derivatives – Related Rates Problems

Steps for Solving Related Rates Problems

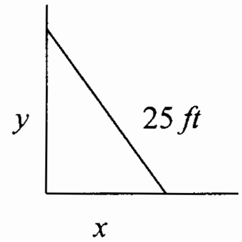
A. Read the problem carefully and identify all the quantities that are changing (either increasing or decreasing).
B. Use appropriate variables to represent the quantities involved in the problem. Write down all equations that relate these variables and other given constants. These equations might involve area or volume formulas, the Pythagorean theorem, or similar triangles from geometry, or other formulas from chemistry or physics.
C. Write a single equation that involves only the quantities whose rates of change are given and the one quantity whose rate of change is unknown.
D. Assume that all the variables in this equation are functions of time, usually given implicitly, and differentiate both sides of the equation with respect to time *t*; ie., $\frac{d}{dt}$. This usually involves implicit differentiation.
E. In the resulting related rate equation, substitute the given numerical quantities and rates. Remember that a rate of change of a quantity is a positive number if that quantity is increasing, and is a negative number if that quantity is decreasing.
F. Solve for the unknown rate of change. Be sure to include the appropriate units of measurement with your answer.

Example 1: Air is being pumped into a spherical balloon at a rate of 20 in^3/min. How fast is the radius of the balloon increasing when the radius is 6 *in*?

$V = \frac{4}{3}\pi r^3 \Rightarrow \frac{d}{dt}\left\{V = \frac{4}{3}\pi r^3\right\} \Rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$. Substituting $\frac{dV}{dt} = 20\ in^3/min$ and $r = 6\ in$ into the last equation gives $20\ in^3/min = 4\pi\,(6\ in)^2 \frac{dr}{dt}$. Thus, $\frac{dr}{dt} =$

$\frac{20 \cdot in^3/\text{min}}{144\pi \cdot in^2} = \frac{5}{36\pi}\ in/min \approx .0442\ in/min.$

Example 2: A 25-foot ladder is leaning against a vertical wall. The floor is slightly slippery and the foot of the ladder slips away from the wall at the rate of 0.2 *in/sec*. How fast is the top of the ladder sliding down the wall when the top is 20 feet above the floor?

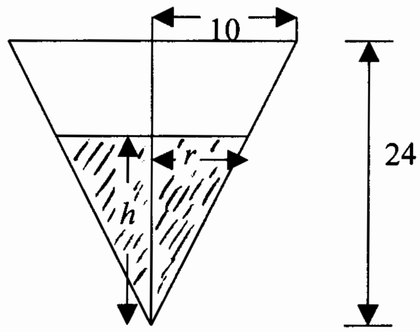


By the Pythagorean Theorem, $x^2 + y^2 = 25^2 \Rightarrow$ $\frac{d}{dt}\left(x^2 + y^2 = 625\right) \Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$. When $y = 20\,ft$, then $x^2 + 20^2 = 25^2 \Rightarrow x^2 = 225 \Rightarrow$

$x = 15\,ft$. Substituting gives $2(15\,ft)(0.2 in/\sec) + 2(20\,ft)\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} =$

$\frac{-6\,ft \cdot in/\sec}{40\,ft} = -0.15$ *in/sec* and the negative sign indicates that y is decreasing.

Example 3: A conical water tank with vertex down has a radius of 10 *ft* at the top and is 24 *ft* high. If water flows out of the tank at a rate of 20 ft^3/min, how fast is the depth of the water decreasing when the water is 16 *ft* deep?



The water is in the shape of a cone, so $V = \frac{1}{3}\pi r^2 h$. By similar triangles,

$\frac{10}{24} = \frac{r}{h} \Rightarrow 10h = 24r \Rightarrow r = \frac{5}{12}h \Rightarrow$

$V = \frac{1}{3}\pi\left(\frac{5}{12}h\right)^2 h = \frac{25\pi}{432}h^3 \Rightarrow$

$\frac{d}{dt}\left\{V = \frac{25\pi}{432}h^3\right\} \Rightarrow \frac{dV}{dt} = \frac{25\pi}{144}h^2\frac{dh}{dt}.$

Substituting gives $-20\,ft^3/min = \frac{25\pi}{144}(16\,ft)^2\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{-20(144)\,ft^3/\min}{25\pi(256\,ft^2)} =$

$-\frac{9}{20\pi}$ *ft/min* ≈ -0.143 *ft/min* and the negative sign indicates that the depth is

decreasing.

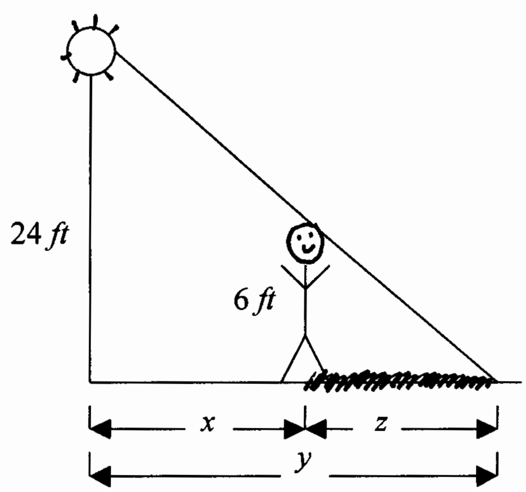
Example 4: On a dark night, a thin man 6 feet tall walks away from a lamp post 24 feet high at the rate of 5 *mph*. How fast is the end of his shadow moving? How fast is the shadow lengthening?

To answer the first question, let x be the distance of the man from the lamp post and y be the distance from the tip of his shadow to the lamp post. By similar triangles,

$\frac{24}{6} = \frac{y}{y-x} \Rightarrow 24y - 24x = 6y \Rightarrow$

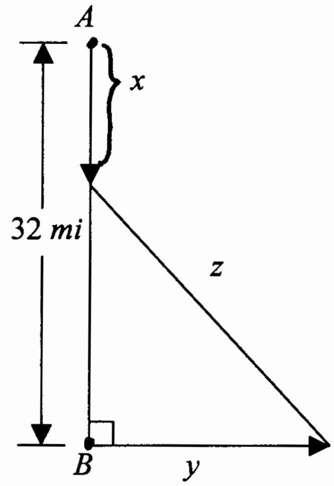
$18y = 24x \Rightarrow 3y = 4x \Rightarrow$

$\frac{d}{dt}\{3y = 4x\} \Rightarrow 3\frac{dy}{dt} = 4\frac{dx}{dt}.$



Substituting gives $3\frac{dy}{dt} = 4(5mph) \Rightarrow \frac{dy}{dt} = 6\frac{2}{3}$ *mph.* To solve the second question, let z be the length of his shadow. By similar triangles, $\frac{24}{6} = \frac{x+z}{z} \Rightarrow 24z = 6x + 6z$ $\Rightarrow 18z = 6x \Rightarrow 3z = x \Rightarrow \frac{d}{dt}\{3z = x\} \Rightarrow 3\frac{dz}{dt} = \frac{dx}{dt} \Rightarrow \frac{dz}{dt} = \frac{1}{3}\frac{dx}{dt} = \frac{1}{3}(5mph) = 1\frac{2}{3}\,mph.$

Example 5: Ship A is 32 miles north of ship B and is sailing due south at 16 *mph.* Ship B is sailing due east at 12 *mph.* At what rate is the distance between them changing at the end of 1 hour? Is the distance increasing or decreasing?



By the Pythagorean Theorem, $(32-x)^2 + y^2 = z^2 \Rightarrow \frac{d}{dt}\{(32-x)^2 + y^2 = z^2\} \Rightarrow$ $2(32-x)\left(-\frac{dx}{dt}\right) + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$. At the end of 1 hour, $x = 16$ *mi*, $y = 12$ *mi*, and $z = 20$ *mi.* Also, $\frac{dx}{dt} = 16\,mph$ and $\frac{dy}{dt} = 12$ *mph.* Thus, $2(32mi - 16mi)(-16mph) +$ $2(12mi)(12mph) = 2(20mi)\frac{dz}{dt} \Rightarrow -512\ mi^2/hr +$ $288\ mi^2/hr = (40\ mi)\frac{dz}{dt} \Rightarrow \frac{dz}{dt} = \frac{-224mi^2/hr}{40mi} = -5.6$ *mph* and so the distance is decreasing.

Example 6: Two sides of a triangle have lengths of 12*m* and 15*m*. The angle between them is increasing at a rate of $2°/min$. How fast is the area of the triangle increasing when the angle between the sides of fixed length is $60°$?

Let A be the area of the triangle and θ be the angle between the two sides. Then, $A = \frac{1}{2}(12m)(15m)\sin\theta = 90m^2\sin\theta \Rightarrow \frac{dA}{dt} = 90m^2\cos\theta\frac{d\theta}{dt}$. $2°/min \approx$ $0.035rad/min \Rightarrow \frac{dA}{dt} = (90m^2)(\cos 60°)(0.035rad/\min) \approx 1.575m^2/\min$.

Practice Sheet – Related Rates Problems

1. Gas is escaping from a spherical balloon at the rate of 2 ft^3/min. How fast is the surface area changing when the radius is 12 *ft*?

2. Sand falling from a chute forms a conical pile whose height is always equal to $\frac{4}{3}$ the radius of the base. How fast is the volume changing when the radius of the base is 3 *ft* and is increasing at the rate of 3 *in/min*?

3. Two parallel sides of a rectangle are being lengthened at the rate of 2 *in/sec*, while the other two sides are shortened in such a way that the figure remains a rectangle with constant area of 50 in^2. What is the rate of change of the perimeter when the length of an increasing side is 5 *in*? Is the perimeter increasing or decreasing?

4. A man 5 feet tall walks at the rate 4 *ft/sec* directly away from a street light which is 20 feet above the street. At what rate is the length of his shadow changing? Is the length increasing or decreasing?

5. Water is being withdrawn from a conical reservoir 3 feet in radius and 10 feet deep at a rate of 4 ft^3/min. How fast is the surface falling when the depth of the water is 6 feet? How fast is the area of the surface decreasing at this instant?

6. A train, starting at 11am, travels east at 45 *mph* while another, starting at noon from the same point, travels south at 60 *mph*. How fast are they separating at 3pm?

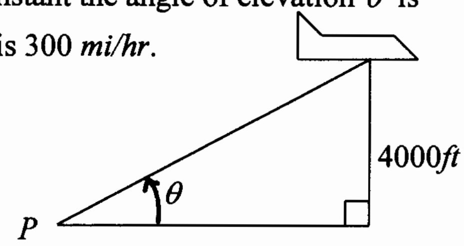
7. A young child is flying a kite horizontally 120 *ft* above the ground. The child lets out 2.5 feet of string per second. If we assume that there is no sag in the string, at what speed is the kite moving when there is 130 *ft* of string out?

8. Oil spilled from a ruptured tanker spreads out in a circle whose area increases at a constant rate of 6 mi^2/hr. How fast is the radius of the spill increasing when the area is 9 mi^2?

9. Ship A is 15 miles east of P and is moving west at 20 *mph*; ship B is 60 miles south of P and is moving north at 15 *mph*. At what rate is the distance between them changing after 1 hour? Is the distance increasing or decreasing?

10. The radius of a right circular cylinder is increasing at a rate of 2 *in/min* and the height is decreasing at a rate of 3 *in/min*. At what rate is the volume changing when the radius is 8 *in* and the height is 12 *in*? Is the volume increasing or decreasing?

11. An aircraft is flying horizontally at a constant height of 4000 *ft* above a fixed observation point *P* (see diagram). At a certain instant the angle of elevation θ is 30° and decreasing, and the speed of the aircraft is 300 *mi/hr*.
(a) How fast is θ decreasing at this instant?
(b) How fast is the distance between the aircraft and the observation point changing at this instant?



12. A beacon that makes one revolution every 10 seconds is located on a ship anchored 4 *km* from a straight shoreline. How fast is the beam moving along the shoreline when it makes an angle of 45° with the shore?

Solution Key for Related Rates Problems

1. $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2 \Rightarrow \frac{dS}{dt} = -\frac{1}{3}\, ft^2/min$

2. $V = \frac{1}{3}\pi r^2 h$ and $h = \frac{4}{3}r \Rightarrow \frac{dV}{dt} = 3\pi\, ft^3/min$

3. $A = lw$ and $P = 2w + 2l \Rightarrow \frac{dP}{dt} = -4 in/\sec \Rightarrow$ perimeter is decreasing

4. $1\frac{1}{3}\, ft/\sec \Rightarrow$ length of the shadow is increasing

5. $V = \frac{1}{3}\pi r^2 h$ and $A = \pi r^2 \Rightarrow \frac{dh}{dt} = -\frac{100}{81\pi}\, ft/min$ and $\frac{dA}{dt} = -\frac{4}{3}\, ft^2/min$

6. 74.25 *mph*

7. 6.5 *ft/sec*

8. $\frac{dr}{dt} = \frac{1}{\sqrt{\pi}}\, mi/hr$

9. – 12.7 *mph*, decreasing

10. $\frac{dV}{dt} = 192\pi\, in^3/min$, increasing

11. (a) Use 300 *mph* = 440 *ft/sec* $\Rightarrow$ angle is decreasing at 0.0275 *rad/sec*
(b) 260 *mph*

12. $\frac{8\pi}{5}$ *km/sec*