Math 3a Practice for TEST 1

Name ______________________________

Section #____________________________

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1. Evaluate each limit or say that it does not exist.

$\lim\limits_{x\to 2}\dfrac{4-x^2}{2-x}$	$\lim\limits_{x\to 0}\dfrac{\sqrt{x^2+9}-3}{x^2}$
$\lim\limits_{x\to\infty}\dfrac{7x^2+2x-3}{2x^2+9}$	$\lim\limits_{x\to\infty} e^{-x}\sin(x)$
$\lim\limits_{x\to 0} x^2\sin(1/x)$	$\lim\limits_{x\to 0}\dfrac{\sin 7x}{3x}$
$\lim\limits_{x\to 1}\begin{cases} x^3-7x & x\geq 1 \\ -9x+5 & x<1 \end{cases}$	$\lim\limits_{x\to 4}\dfrac{x-4}{\sqrt{x}-2}$

2. For each function listed, decide if the function is continuous at x=1. Explain your answer using the definition of continuity .

$f(x)=\begin{cases}3x & x<1\\ x^2+2x & x>1\end{cases}$

$g(x)=\begin{cases}-3x & x\le 1\\ x^2-4x & x>1\end{cases}$

$h(x)=\begin{cases}\dfrac{x-1}{|x-1|} & x\ne 1\\ 1 & x=1\end{cases}$

3. Find a value for K so that $f(x)=\begin{cases}x^2-1 & x\le 1\\ 2x+K & x>1\end{cases}$ is continuous at x= 1.

4. Let $g(x)=\dfrac{2}{x}$. Sketch g(x) on a coordinate system and sketch the line that is tangent to g(x) at x=1.

Use the formal definition of the derivative to find $g'(x)$.

What is the equation of the tangent line that you sketched ?

5. Use the Intermediate Value Theorem to estimate a solution for $x^3 + 3x = 8$.

6 a. Find the equation of the line with positive slope that is tangent to the graph of $f(x) = 3x^2 + 12$ and that passes through the origin.

6 b. Find the equation of the tangent line to the graph of $g(x) = 3x^2$ that passes through the point (1,-9).

7. Use the definition of the derivative to show that f(x)=| x| is not differentiable when x=0.

8. Let f(x) =ax^2+bx+c. This is a quadratic function. What is the slope of this function at its vertex ? Use this fact and the definition of the derivative to show that the x coordinate of the vertex of this function is –b/2a.

Hint : write $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = 0$, and then solve for x in this equation .

Math 3A Practice for Test 1

1. Evaluate each limit or say that it does not exist.

   - $\lim\limits_{x\to 2}\dfrac{4-x^2}{2-x} = \lim\limits_{x\to 2}\dfrac{(2-x)(2+x)}{2-x} = \lim\limits_{x\to 2} 2+x = 2+2 = 4$
   - $\lim\limits_{x\to 0}\dfrac{\sqrt{x^2+9}-3}{x^2} = \lim\limits_{x\to 0}\dfrac{\sqrt{x^2+9}-3}{x^2}\cdot\dfrac{\sqrt{x^2+9}+3}{\sqrt{x^2+9}+3}$
     $= \lim\limits_{x\to 0}\dfrac{x^2+9-9}{x^2(\sqrt{x^2+9}+3)} = \lim\limits_{x\to 0}\dfrac{x^2}{x^2(\sqrt{x^2+9}+3)}$
     $= \lim\limits_{x\to 0}\dfrac{1}{\sqrt{x^2+9}+3} = \dfrac{1}{\sqrt{0^2+9}+3} = \dfrac{1}{6}$
   - $\lim\limits_{x\to\infty}\dfrac{7x^2+2x-3}{2x^2+9} = \dfrac{7}{2}$
   - $\lim\limits_{x\to\infty} e^{-x}\sin(x) = \lim\limits_{x\to\infty}\dfrac{\sin(x)}{e^x} = 0$ USE sandwich Thm here
   - $\lim\limits_{x\to 0} x^2\sin(\dfrac{1}{x})$. First, note that $-1 \le \sin(\frac{1}{x}) \le 1$ for all $x \neq 0$. So, since $x^2 > 0$ for all $x \neq 0$, $-x^2 \le x^2\sin(\frac{1}{x}) \le x^2$. We know $\lim\limits_{x\to 0} - x^2 = \lim\limits_{x\to 0} x^2 = 0$, so $\lim\limits_{x\to 0} x^2\sin(\dfrac{1}{x}) = 0$.
   - $\lim\limits_{x\to 0}\dfrac{\sin 7x}{3x}$. Let $z = 7x$ and note that $z \to 0$ as $x \to 0$ and vice versa. So this is equal to $\lim\limits_{z\to 0}\dfrac{\sin z}{3z/7} = \dfrac{7}{3}\lim\limits_{z\to 0}\dfrac{sinz}{z} = \dfrac{7}{3}\cdot 1 = \dfrac{7}{3}$.
   - $\lim\limits_{x\to 1}\begin{cases} x^3-7x & x \ge 1 \\ -9x+5 & x < 1 \end{cases}$
     $\lim\limits_{x\to 1^+} -9x+5 = -4$
     $\lim\limits_{x\to 1^-} x^3-7x = -6$
     So the limit *does not exist.*
   - $\lim\limits_{x\to 4}\dfrac{x-4}{\sqrt{x}-2} = \lim\limits_{x\to 4}\dfrac{(\sqrt{x}-2)(\sqrt{x}+2)}{\sqrt{x}-2} = \lim\limits_{x\to 4}\sqrt{x}+2 = \sqrt{4}+2 = 4$

2. For each function listened, decide if the function is continuous at $x = 1$. Explain your answer using the definition of continuity.

- $f(x) = \begin{cases} 3x & x < 1 \\ x^2 + 2x & x > 1 \end{cases}$

  f is not defined at 1, so f is not continuous at $x = 1$.

- $g(x) = \begin{cases} -3x & x \leq 1 \\ x^2 - 4x & x > 1 \end{cases}$

  $g(1) = -3(1) = -3$.

  $\lim_{x \to 1^+} g(x) = \lim_{x \to 1^+} x^2 - 4x = -3$

  $\lim_{x \to 1^-} g(x) = \lim_{x \to 1^+} -3x = -3$

  So, since $\lim_{x \to 1^+} g(x) = \lim_{x \to 1^-} g(x)$, the limit of g as $x \to 1$ exists, and $\lim_{x \to 1} g(x) = -3 = g(1)$.

  Thus, g is continuous at 1.

- $h(x) = \begin{cases} \frac{x-1}{|x-1|} & x \neq 1 \\ 1 & x = 1 \end{cases}$

  $\lim_{x \to 1^-} h(x) = \lim_{x \to 1^-} \frac{x-1}{|x-1|} = \lim_{x \to 1^-} \frac{x-1}{-(x-1)} = \lim_{x \to 1^-} -1 = -1$

  $\lim_{x \to 1^+} h(x) = \lim_{x \to 1^+} \frac{x-1}{|x-1|} = \lim_{x \to 1^+} \frac{x-1}{x-1} = \lim_{x \to 1^+} 1 = 1$

  So, since the limit of h as $x \to 1$ does not exist, h is not continuous at 1.

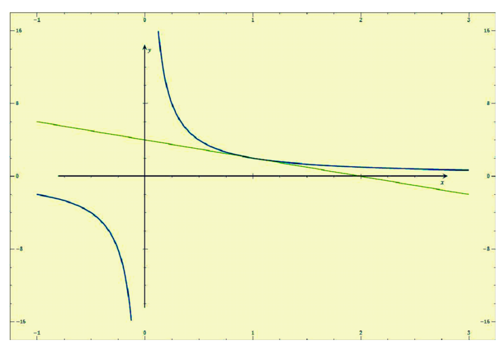
3. Find a value for K so that $f(x) = \begin{cases} x^2 - 1 & x \leq 1 \\ 2x + K & x > 1 \end{cases}$ is continuous at $x = 1$.

   $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 2x + K = 2 + K$

   $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^2 - 1 = 0$

   So $K = -2$ will make the limit of f as $x \to 1$ exist and be equal to $f(1)$, hence, making f continuous at $x = 1$.

4. Let $g(x) = \frac{2}{x}$. Sketch $g(x)$ on a coordinate system and sketch the line that is tangent to $g(x)$ at $x = 1$.

   the

   Use the formal definition of hte derivative to find $g'(x)$.

   $g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \to 0} \frac{\frac{2}{x+h} - \frac{2}{x}}{h} = \lim_{h \to 0} \frac{2x - 2(x+h)}{hx(x+h)} = \lim_{h \to 0} \frac{-2h}{hx(x+h)} = \lim_{h \to 0} \frac{-2}{x(x+h)} = \frac{-2}{x^2}$

   the

   What is the equation of hte tangent line that you sketched?

   $g(1) = \frac{2}{1} = 2$, so the equation of the tangent line is $y - 2 = -2(x - 1)$, that is, $y = -2x + 4$.

5. Use the Intermediate Value Theorem to estimate a solution for $x^3 + 3x = 8$.

   Let $f(x) = x^3 + 3x - 8$. Note that x is a solution to $x^3 + 3x = 8$ if and only if $f(x) = 0$. Further note that $f(1) = -4$ and $f(2) = 6$. Thus, by the Intermediate Value Theorem, there is c in $(1, 2)$ such that $f(c) = 0$. So c is approximately 1.5.

6. (a) Find the equation of the line with positive slope that is tangent to hte graph of $f(x) = 3x^2 + 12$ and that passes through the origin.

   $g'(x) = 6x$, so we want to find the point (x_1, y_1) such that the slope of the line through (x_1, y_1) and the origin $(0, 0)$ is $6x_1$. That is, $\frac{y_1 - 0}{x_1 - 0} = 6x_1$, so $y_1 = 6x_1^2$. In particular, we need this point to be on the curve g, so $y_1 = 3x_1^2 + 12$. Thus, $3x_1^2 + 12 = 6x_1^2$. Hence, we must have $x_1 = \pm 2$. Note that $g'(-2) = -12$ and $g'(2) = 12$, so for positive slope, we use $x_1 = 2$. Note that $g(2) = 3(2)^2 + 12 = 24$, so the equation of the tangent line is $y - 24 = 12(x - 2)$, that is, $y = 12x$.

   (b) Find the equation of the tangent line to the graph of $g(x) = 3x^2$ that passes through the point $(1, -9)$.

   $g'(x) = 6x$, so we want to find the point (x_1, y_1) such that the slope of the line through (x_1, y_1) and $(1, -9)$ is $6x_1$. That is, $\frac{y_1 + 9}{x_1 - 1} = 6x_1$, so $y_1 = 6x_1^2 - 6x_1 - 9$. In particular, we need this point to be on the curve g, so $y_1 = 3x_1^2$. Thus, $3x_1^2 = 6x_1^2 - 6x_1 - 9$, so $x_1^2 - 2x_1 - 3 = 0$, thus, $(x_1 - 3)(x_1 + 1) = 0$. Hence, $x_1 = 3$ or $x_1 = -1$.

   We see that the point $(-1, g(-1)) = (-1, 3)$ works: $g'(-1) = -6$, so the equation of the tangent line at $(-1, 3)$ is $y - 3 = -6(x + 1)$, that is, $y = -6x - 3$. Note that $-9 = -6(1) - 3$, so this is a tangent line of g which passes through the point $(1, -9)$.

7. Use the definition of the derivative to show that $f(x) = |x|$ is not differentiable when $x = 0$.

   $$\lim_{h \to 0^+} \frac{|0 + h| - |0|}{h} = \lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} 1 = 1$$
   $$\lim_{h \to 0^-} \frac{|0 + h| - |0|}{h} = \lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^+} -1 = -1$$

   So the limit, and thus, the derivative does not exist since the left and right limits are not equal.

8. Let $f(x) = ax^2 + bx + c$. This is a quadratic function. What is the slope of this function at its vertex? Use this fact and the definition of hte derivative to show that the x coordinate of the vertex of this function is $-\frac{b}{2a}$.

   Hint: write $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x + h) - f(x)}{h} = 0$, and then solve for x in this equation.

   $$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{a(x + h)^2 + b(x + h) + c - (ax^2 + bx + c)}{h}$$
   $$= \lim_{h \to 0} \frac{ax^2 + 2axh + ah^2 + bx + bh + c - ax^2 - bx - c}{h} = \lim_{h \to 0} \frac{2axh + ah^2 + bh}{h}$$
   $$= \lim_{h \to 0} 2ax + ah + b = 2ax + b.$$

   So if $f'(x) = 2ax + b = 0$, then $x = -\frac{b}{2a}$.