Automorphism Groups Are Perfect

Suppose G is a non-abelian finite simple group. We have (since G has no center) $G \subset \mathrm{Aut}(G)$. We want to prove:

Theorem. $\mathrm{Aut}(G)$ *is complete.*

We must show first that $\mathrm{Aut}(G)$ has no center. In fact this can be strengthened to:

1) The centralizer of G in $H = \mathrm{Aut}(G)$ is $\{1\}$.

Proof: Assume that $\sigma \in H$ commutes with all i_x. Then for all $y \in G$, $(\sigma \cdot i_x)(y) = (i_x \cdot \sigma)(y)$, and so $\sigma(x)\sigma(y)\sigma(x)^{-1} = x\sigma(y)x^{-1}$. It follows that $x^{-1}\sigma(x) \in Z(G)$ and so $x^{-1}\sigma(x) = e \Rightarrow \sigma = 1$.

2) G is characteristic in H.

Proof: Let $\sigma \in \mathrm{Aut}(H)$. Now G and $\sigma(G)$ are normal subgroups of H; if they are not equal, then they are disjoint and elementwise commute. By 1), this is impossible.

Suppose then $\sigma \in \mathrm{Aut}(H)$ is arbitrary. σ acts on G as an element of $\mathrm{Aut}(G)$ by 2). Multiplying by the inverse of that automorphism, we may suppose that $\sigma|G = 1$. Then it remains to show:

3) If $\sigma \in \mathrm{Aut}(H)$ acts trivially on G, then $\sigma = 1$

Proof: Take $h \in \mathrm{Aut}(G)$, $x \in G$. Then σ fixes $i_{h(x)} = hi_xh^{-1}$, as well as i_x. So

$$\sigma(h) \cdot i_x \cdot \sigma(h)^{-1} = h \cdot i_x \cdot h^{-1}$$

The upshot is that $h^{-1}\sigma(h)$ commutes with all i_x. By 1), $h^{-1}\sigma(h) = 1$, i.e. $\sigma(h) = h$ as required.