Groups of order $p^a q$ are solvable

Suppose G is a group of order $p^a q$ where p and q are primes. Our aim is to show G is solvable. By induction, it is clearly enough to show G has a non-trivial normal subgroup. Our proof only works for orders of this form – divisible by two primes and one to the first power.

If a p-Sylow subgroup of G is normal, there is nothing more to prove. We assume then G has more than one p-Sylow subgroup. For all choices of p-Sylow subgroups P_1 and P_2, choose a pair so that $D = P_1 \cap P_2$ has maximal possible order. D will retain this meaning in everything that comes below. Note that D is necessarily a p-group but is not p-Sylow.

Lemma 1: *The map $P \to P \cap N_G(D)$ describes a 1-1 correspondence between p-Sylow subgroups of G WHICH CONTAIN D and p-Sylow subgroups of $N_G(D)$.*

Proof: To show the map is surjective, let Q be a p-Sylow of $N_G(D)$. As Q normalizes D, QD threatens to be a p-group in $N_G(D)$. Thus $QD = Q$, and so $D \subset Q$. Enlarge Q to P which is p-Sylow in G. Then $P \cap N_G(D)$ is a p-group containing the Sylow subgroup Q, so as required $P \cap N_G(D) = Q$.

Suppose then Q_1 and Q_2 are p-Sylow containing D and $Q_1 \cap N_G(D) = Q_2 \cap N_G(D)$. As $D \neq Q_1$, there is a subgroup H of Q_1 properly containing D in which D is normal. Thus $H \subset Q_1 \cap N_G(D)$. Then $H \subset Q_2 \cap N_G(D)$, and now $Q_1 \cap Q_2 \supset H > D$, a contradiction.

Finally, suppose $P \supset D$ is p-Sylow in G. Let $S = P \cap N_G(D)$. Note that S contains D. Enlarge S to Q which is p-Sylow in G, and suppose $S \neq Q$. By Lemma 1, there is a Sylow M of G with $M \cap N_G(D) = Q$. $Q \neq S$ implies $M \neq P$. But then M and P each contain D, so $M \cap P = D$. However, M and P each contain S. It must follow that $S = D$. This is absurd, since clearly P contains elements that normalize D which are not in D (i.e. $P \cap N_G(D) \supset N_P(D) > D$).

Note that the above still holds in any finite group. Now to finish because the order of G is $p^a q$.

How many p-Sylows has $N_G(D)$? It certainly has more than one (by Lemma 1), since P_1 and P_2 must contract to distinct Sylow subgroups of $N_G(D)$. From the hypotheses, $N_G(D)$ must have q Sylow p-subgroups. This is the same number that G has (else there is a normal p-Sylow). The correspondence in Lemma 1 is then a 1-1 correspondence among all p-Sylow subgroups. As D is in all Sylows of $N_G(D)$, it follows that D is in all Sylows of G. Thus the intersection of all p-Sylows is a normal subgroup of G containing D, and non-trivial as it is a p-group.

One must do the numerics for the last case when all intersections are trivial.