Completeness of S_n

A group G is called *complete* if every automorphism of G is inner. Almost all symmetric groups are complete.

Lemma 1: *If $\alpha : S_n \longrightarrow S_n$ is an automorphism that preserves transpositions (i.e. α sends any transposition to another transposition), then α is inner.*

Proof: For convenience we use exponential notation for functions; π^γ will denote $\gamma(\pi)$.

We shall prove by induction on t that there exist conjugations $\beta_2, \ldots, \beta_t$ such that $\beta_t^{-1} \ldots \beta_2^{-1}\alpha$ fixes $(1\ 2), \ldots, (1\ t)$. Now $(1\ 2)^\alpha = (i\ j)$; define β_2 to be conjugation by $(1\ i)(2\ j)$ (where $(1\ i) =$ identity if $i = 1$). Then $(1\ 2)^\alpha = (1\ 2)^{\beta_2}$, and so $\beta_2^{-1}\alpha$ fixes $(1\ 2)$.

Let $\gamma = \beta_2^{-1} \ldots \beta_t^{-1}\alpha$ be given by the inductive hypothesis. As γ preserves transpositions, $(1\ t{+}1)^\gamma = (l\ k)$. Moreover, $\{1,2\} \cap \{l,k\} \neq \emptyset$, otherwise $((1\ 2)(1\ t{+}1))^\gamma = (1\ 2)(l\ k)$ has order 2 while $((1\ 2)(1\ t+1))$ has order 3, contradicting the fact that automorphisms preserve orders. For notational convenience, assume $(1\ t+1)^\gamma = (1\ k)$. We must have $k > t$ lest $(1\ k)$ and $(1\ t+1)$ have the same image under γ. Define β_{t+1} to be conjugation by $(k\ t+1)$. Now β_{t+1} fixes $(1\ 2), \ldots, (1\ t)$, and sends $(1\ t+1)^{\beta_{t+1}} = (1\ t+1)^\gamma$, so that $\beta_{t+1}^{-1}\gamma = \beta_{t+1}^{-1}\beta_t^{-1} \ldots \beta_2^{-1}\alpha$ fixes $(1\ 2), \ldots, (1\ t+1)$. This finishes the inductive step.

It follows that $\beta_n^{-1} \ldots \beta_2^{-1}\alpha$ fixes $(1\ 2), \ldots, (1\ n)$. Since these $n-1$ transpositions generate S_n, this automorphism is the identity, from whence $\alpha = \beta_2 \ldots \beta_n$ is inner.

Theorem: *S_n is complete if $n \neq 2$ and $n \neq 6$.*

Proof: We remark that S_2 is not complete because it is abelian. We presume that $n \geq 3$.

Let C_1 be the conjugacy class of transpositions of S_n. Let C_k be the conjugacy class of products of k disjoint transpositions. Since any automorphism takes C_1 to a conjugacy class of elements of order 2, we must have C_1 taken to C_k for some k. Morevover, Lemma 1 says that any auto. taking C_1 to C_1 is inner. We are finished if we can show that, for $n \neq 6$, the order of C_1 cannot be equal to the size of C_k for $k \neq 1$.

One sees that the size of C_1 is $n(n-1)/2$, whereas the size of C_k is

$$\frac{n(n-1)}{2} \cdot \frac{(n-2)(n-3)}{2} \cdots \frac{(n-2k+2)(n-2k+1)}{2} \cdot \frac{1}{k!}$$

The last factor is present since there are $k!$ ways of permuting k disjoint transpositions without changing the element. Thus, the size of C_k is

$$\frac{1}{k!}\frac{1}{2^k}\ n(n-1)\cdots(n-2k+1)$$

This last is the size of C_1 if and only if the following equation holds:

(*) $$(n-2)(n-3)\cdots(n-2k+1) = k!2^{k-1}$$

Since the left side is positive, we must have $n \geq 2k$. Therefore, for fixed n,

$$\text{left side} \geq (2k-2)(2k-3)\cdots(2k-2k+1) = (2k-2)!$$

An easy induction shows that if $k \geq 4$ then

$$(2k-2)! > k!2^{k-1}$$

Then (*) can hold only if $k < 4$, regardless of the value of n. We may now assume that $k = 2$ or $k = 3$. It is easy to see that (*) never holds for $k = 2$ (a product of successive integers is seldom a power of 2). Only the case $k = 3$ remains. Now, since $n \geq 2k$, we must have $n \geq 6$. If $n > 6$, then the left side of (*) is bigger than $5 \cdot 4 \cdot 3 \cdot 2 = 120$ while the right side is $3!2^2 = 24$. If $n = 6$ and $k = 3$, then (*) does hold. There is an auto. of S_6 taking transpositions to products of three transpositions. But if $n \neq 6$, the size of C_1 is distinct from the size of any other C_k.