The alternating group

Definition: A permutation $\theta \in S_n$ is said to be *even* if it can be represented as a product of an even number of transpositions. If θ is a product of an odd number of transpositions, it is said to be *odd*.

The definition above is slightly fraudulent, since it is not clear whether a permutation can be both even and odd. The aim of this note is to show that this does not happen – once a permutation is a product of an even number of transpositions, it cannot be presented as a product of an odd number of transpositions. In order to prove this, it is unfortunately necessary to introduce an auxilliary polynomial.

Consider the polynomial in n-variables

$$p(x_1, x_2, \ldots, x_n) = \prod_{i<j} (x_i - x_j)$$

For short, we will write p for $p(x_1, x_2, \ldots, x_n)$. If $\theta \in S_n$, let θ act on p via

$$\theta(p) = \prod_{i<j} (x_{\theta(i)} - x_{\theta(j)})$$

It is clear that every term in p is in its original order, or the negative of its original, so $\theta(p) = \pm p$. For instance, let $\theta = (1\ 2\ 3)$, $\phi = (1\ 2)$, and $n = 3$. Then $p = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$ and

$$\theta(p) = (x_2 - x_3)(x_2 - x_1)(x_3 - x_1) = p$$

since two terms got inverted. Similarly, $\phi(p) = -p$ (check).

In order to prove what we want, we will discuss how a transposition acts on p. In fact we will show that every transposition changes the sign of p. This is enough, because of the following: Recall that every $\theta \in S_n$ is a product of transpositions. If θ is a product of t transpositions, then we would conclude $\theta(p) = (-1)^t p$ (as sign gets changed t times). This means that $\theta(p) = p$ if and only if θ is a product of an even number of transpostions, and our "definition " passes the test of being well-defined. Thus, it suffices to assume that $\theta = (k\ l)$, and then show that $\theta(p) = -p$. We may also assume that $k < l$.

So what does $(k\ l)$ do to a typical term $(x_i - x_j)$, $i < j$? A great preponderence of them are not even moved. If i, j does not involve k or l, then $(x_i - x_j)$ is fixed, and so does not change sign. Thus we consider the cases:

$$(x_i - x_k) \to (x_i - x_l) \tag{1}$$
$$(x_k - x_i) \to (x_l - x_i) \tag{2}$$
$$(x_i - x_l) \to (x_i - x_k) \tag{3}$$
$$(x_l - x_i) \to (x_k - x_i) \tag{4}$$

There are no sign changes in case (1) since $i < k < l$, and none also in case (4) since $k < l < j$. There is a sign change in case (2) when $k < i < l$, and in case (3) under exactly the same circumstances. Thus, every sign change attributed to case (2) is cancelled by a sign change in case (3). But there is exactly one term left which changes sign, the term $(x_k - x_l) \to (x_l - x_k)$. As nothing is left to cancel this sign change, we conclude $\theta(p) = -p$ for $\theta = (k\ l)$, and this ends the proof.

Let A_n denote the elements of S_n which are a product of an even number of transpositions. You should have no problem showing that A_n is a subgroup of S_n, called the alternating group on n letters. We investigate the nature of A_n a little closer.

Clearly, any element of S_n is in A_n if and only if its inverse is. The rules for multiplying elements, by the above, amount to:

$$\text{even} \times \text{even} = \text{even},\ \text{even} \times \text{odd} = \text{odd},\ \text{odd} \times \text{even} = \text{odd},\ \text{odd} \times \text{odd} = \text{even}$$

These rules say that $\theta A_n \theta^{-1} \subset A_n$ for all $\theta \in A_n$, so A_n is a normal subgroup of S_n. Furthermore, if a and b are odd permutations, then ab^{-1} is even, so $aA_n = bA_n$. It follows that there are two cosets: A_n and aA_n where $a = (1\ 2)$, so A_n has index 2 in S_n for $n \geq 2$. It follows that A_n is a subgroup of S_n whose order is $n!/2$.

Here are some more rules of computation: An r-cycle is even if and only if r is odd (this is a potential point of confusion). The odd elements do not form a group (as they are a non-trivial coset). We will show later that A_n is the only normal subgroup of S_n when $n > 4$.