EMSS $\Rightarrow$ $H^*(\Omega X;\mathbb{Z}/2)$.

Geometric construction $\Rightarrow$

~~Example~~ Steenrod operations in two (only cohomology)

Example Two little A-modules:

$$\tilde H^*(\mathbb{CP}^2;\mathbb{Z}/2) \cong \begin{array}{cc} \mathbb{Z}/2 & 4 \\ \uparrow Sq^2 & \\ \mathbb{Z}/2 & 2 \end{array}$$

$$\tilde H^*(\mathbb{HP}^2;\mathbb{Z}/2) \cong \begin{array}{cc} & 8 \\ \uparrow Sq^4 & \\ \mathbb{Z}/2 & 4 \end{array}$$

Q. Could these be "stacked up"? Does there exist X with

$$\tilde H^*(X;\mathbb{Z}/2) = \begin{array}{cc} \mathbb{Z}/2 & 8 \\ \uparrow Sq^4 & \\ \mathbb{Z}/2 & 4 \\ \uparrow Sq^2 & \\ \mathbb{Z}/2 & 2 \end{array} \quad ?$$

A. No. "obvious" cup product problems.

Make the problem harder by suspending the whole:

Q. $\exists$? X with

$$\tilde H^*(X;\mathbb{Z}/2) \cong \begin{array}{cc} \mathbb{Z}/2 & 9 \\ \uparrow Sq^4 & \\ \mathbb{Z}/2 & 5 \\ \uparrow Sq^2 & \\ \mathbb{Z}/2 & 3 \end{array}$$

A. Still no. Give a proof due to Lionel Schwartz using EMSS.

Strategy: Suppose $\exists X$. Partially compute $H^*(\Omega X;\mathbb{Z}/2)$ using EMSS and show that there is no compatible action of $\mathcal{A}$ with cup products.

$\mathrm{Tor}_{H^*(X;\mathbb{Z}/2)}(\mathbb{Z}/2, \mathbb{Z}/2) = ?$

$H^*(X;\mathbb{Z}/2)$ is a "trivial" $\mathbb{Z}/2$ alg $\mathbb{Z}/2 \oplus M$ in pos. deg. $M \otimes M \xrightarrow{0} M$

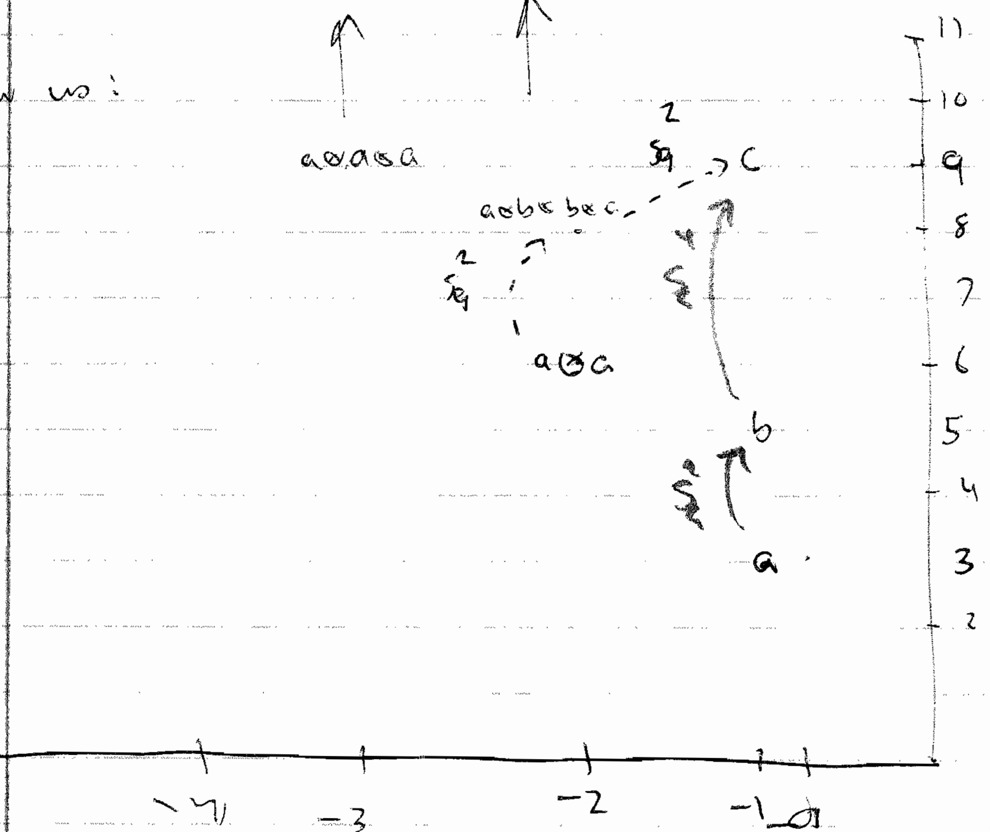
$\mathrm{Tor}_{\mathbb{F}_2 \oplus M}(\mathbb{F}_2, \mathbb{F}_2) \simeq T(M)$, with "shuffle product".

$\simeq \mathbb{F} \oplus M \oplus (M \otimes M) \oplus (M \otimes M \otimes M) \oplus \cdots$

$x, y \in M \quad x \cdot y = x \otimes y + y \otimes x$

$(x_1 \otimes \cdots \otimes x_i) \cdot (y_1 \otimes \cdots \otimes y_j) = \sum_{(i,j)\text{ shuffles}}$ ——

E_2 for us:



$\exists\ \alpha, \beta, \gamma \in H(\Omega X) \longleftrightarrow a, b, c$

$\exists\ \delta \in H^4 \longleftrightarrow a \otimes a$

$Sq^2(a \otimes a) = a \otimes b + b \otimes a = a \cdot b \Rightarrow Sq^2(\delta) = \alpha \cup \beta.$

$Sq^2 Sq^2(\delta) = Sq^2(\alpha \cup \beta) = \beta \cup \beta = Sq^4 \beta = \gamma$. But: $Sq^2 Sq^2 = Sq^3 Sq^1$.