Steenrod Operations & Eilenberg-MacLane Spaces

Note Title 2/5/2009

<u>Def</u> / <u>Axioms</u> For each $i \geq 0$, there is a natural transformation

$$Sq^i : H^n(X,A;\mathbb{F}_2) \longrightarrow H^{n+i}(X,A;\mathbb{F}_2)$$

s.t.

1. $Sq^0 = Id$
2. If $i > |x|$, then $Sq^i x = 0$
3. If $i = |x|$, then $Sq^i x = x^2$
4. If δ is the conn. hom in the LES for the pair (X,A) then
$$\delta \, Sq^i = Sq^i \, \delta$$
5. (Cartan Formula) $\quad Sq^n(x \cdot y) = \sum_{i+j=n} Sq^i(x) \cdot Sq^j(y)$
6. (Adem Relation) If $a < 2b$, then
$$Sq^a Sq^b = \sum_{c=0}^{\lfloor a/2 \rfloor} \binom{b-c-1}{a-2c} Sq^{a+b-c} Sq^c$$

These are natural transformations, but we can tie this to our story.

<u>Prop 1</u> $H^n(X;G) = [X, K(G,n)]$

This is a basic consequence of obstruction theory. Quick primer:

If $g : X^{(n)} \to Y$, then for every $(n+1)$-cell σ, we have an element of $\pi_n(Y)$ given by $g \circ \partial(\sigma)$. This gives an $(n+1)$-cochain on X with coefficients in $\pi_n Y$.

Obvious fact: There is an extension g_{n+1} of g over $X^{(n+1)}$ iff this cochain vanishes.

The obstruction cochain is actually a cocycle & key story is understanding the corresponding cohomology class.

There is a fundamental class $i_n \in H^n(K(\pi,n);\pi)$ s.t.

$$f \in [X, K(\pi,n)] \longleftrightarrow f^* i_n$$

Since cohomology is representable, the Yoneda Lemma gives an equiv

$$\mathrm{Nat}\big(H^n(-;\pi),\, H^m(-;G)\big) = H^m(K(\pi,n);G).$$

Can easily understand this $\mathbb{Q}$ly.

① $K(\mathbb{Q},1) = S^1_{\mathbb{Q}}$ = mapping telescope $S^1 \xrightarrow{2} S^1 \xrightarrow{6} S^1 \to \dots \xrightarrow{n!} S^1 \to \dots$

Universal coefs ensures that $H^*(K(\mathbb{Q},1);\mathbb{Q}) = E(x_1)$.

② $K(\mathbb{Q},2) = \mathbb{C}P^\infty_{\mathbb{Q}}$ & $H^*(K(\mathbb{Q},2);\mathbb{Q}) = \mathbb{Q}[x_2]$

③ Have fibrations

$$\begin{tikzcd} K(\mathbb{Q},n-1) = \Omega K(\mathbb{Q},n) \arrow[r] & * \arrow[d] \\ & K(\mathbb{Q},n) \end{tikzcd}$$

so by induction, can compute $H^*(K(\mathbb{Q},n))$

Prop 2 As an algebra $H^*(K(\mathbb{Q},2n+1)) = E(x_{2n+1})$

$$H^*(K(\mathbb{Q},2n)) = \mathbb{Q}[x_{2n}]$$

When we use the Yoneda Lemma, this says the only natural transformations of $H^*(-;\mathbb{Q})$ are the cup-powers.

Want integral information, so we'll look at each prime. ⇒ Steenrod ops.

Our method, though, will be essentially the same as in the $\mathbb{Q}$ case.

At $p=2$, the Steenrod algebra is fundamental here.

Def The Steenrod algebra is the graded $\mathbb{F}_2$ algebra generated by classes Sq^i & subject to the Adem relations. Denote this $\mathcal{A}$.

Pause here to describe another form of the Adem relations:

Prop 3 $Sq^{2n-1}Sq^n = 0$

The others all follow from this by using Pascal's triangle:

$$\begin{tikzcd}[column sep=tiny] & & Sq^{2n-1}Sq^n = 0 \arrow[dl] \arrow[dr] & & \\ & Sq^{2n-2}Sq^n \arrow[dl] \arrow[dr] & + & Sq^{2n-1}Sq^{n-1} = 0 \arrow[dl] \arrow[dr] & \\ Sq^{2n-3}Sq^n & + & \overset{0}{\cancel{Sq^{2n-2}Sq^{n-1} + Sq^{2n-2}Sq^{n-1}}} & + & Sq^{2n-1}Sq^{n-2} = 0 \end{tikzcd}$$

This will follow easily from our analysis of the dual algebra and the cap product.

<u>Ex</u>: $Sq^3 Sq^2 = 0$ $\qquad$ $Sq^1 Sq^1 = 0$.

$$Sq^2 Sq^2 + Sq^3 Sq^1 = 0$$

$$Sq^1 Sq^2 + Sq^3 = 0$$

<u>Prop 4</u> $Sq^1 Sq^{2n} = Sq^{2n+1}$

$$Sq^1 Sq^{2n+1} = 0$$

<u>Def</u> The total squaring operation $Sq : H^* \to H^*$ is defined by

$$Sq(x) = \sum_{i \geq 0} Sq^i(x).$$

The Cartan formula ensures this is a ring hom.

Though this is <u>not</u> homogeneous, the homogeneous parts record the various steenrod actions.

<u>Ex</u>: $\mathbb{RP}^\infty$: $H^*(\mathbb{RP}^\infty; \mathbb{F}_2) = \mathbb{F}_2[x_1]$

By axioms (1-3), $Sq(x_1) = x_1 + x_1^2$

So $Sq(x_1^2) = (Sq(x_1))^2 = x_1^2 + x_1^4 = \underset{\overset{\uparrow}{Sq^0}}{x_1^2} + \underset{\overset{\uparrow}{Sq^1}}{0} + \underset{\overset{\uparrow}{Sq^2}}{x_1^4}$

$$Sq(x_1^3) = (Sq(x_1))^3 = (x_1 + x_1^2)(x_1^2 + x_1^4) = \underset{\overset{\uparrow}{Sq^0}}{x_1^3} + \underset{\overset{\uparrow}{Sq^1}}{x_1^4} + \underset{\overset{\uparrow}{Sq^2}}{x_1^5} + \underset{\overset{\uparrow}{Sq^3}}{x_1^6}$$

More generally,

<u>Prop 5</u> $Sq^i(x^k) = \binom{k}{i} x^{k+i}$

We can prove this & compute binom coeffs by writing out the dyadic (=binary) expansions of k & i. If i is 1 in any place $k = 0$, the binom coeff is zero.

<u>Ex</u>: $\mathbb{CP}^\infty$ $\quad H^*(\mathbb{CP}^\infty) = \mathbb{F}_2[x_2] \Rightarrow Sq(x_2) = x_2 + \overset{H^3=0}{0} + x_2^2$

So same analysis for $\mathbb{RP}^\infty$ works here!

Some notation

Def If $I = (i_1, i_2, \ldots)$ is a finite sequence of pos. ints, let $Sq^I = Sq^{i_1} Sq^{i_2} \cdots$

Say I is admissible if $i_j > 2 i_{j-1}$

If I is admissible, let $|I| = \sum_{j \geq 1} i_j$ be the degree &

$e(I) = 2i_1 - |I| = (i_1 - 2i_2) + (i_2 - 2i_3) + \cdots$ denote the excess.

Prop 6 ① Sq^I raises degree by $|I|$

② If $|x| < e(I)$, then $Sq^I(x) = 0$

③ If $|x| = e(I)$, then $Sq^I(x) = (Sq^{I'}(x))^2$ I' a subsequence.

① is obvious, ② & ③ will follow from our analysis of the SS for $K(\mathbb{Z}/2, n)$.

Thm 1 $\{Sq^I \text{ for } I \text{ admissible}\}$ forms a basis for $\mathcal{A}$.

This will also follow. We check this on test spaces like $K(\mathbb{Z}/2, n)$ or $(\mathbb{R}P^\infty)^n$.

We'll finish with the dual. Since $\mathcal{A}$ is locally finite (each degree is finite dim), $\mathcal{A}_*$ is well behaved.

Prop 7 The Cartan formula $Sq^i \longrightarrow \sum_{j+k=i} Sq^j \otimes Sq^k$ induces a coproduct on $\mathcal{A}$ that is an alg hom.

This makes $\mathcal{A}$ into a cocommutative Hopf algebra $\Rightarrow$ $\mathcal{A}_*$ is a commutative Hopf alg.

Thm 2 (Milnor) $\mathcal{A}_* = \mathbb{F}_2[\xi_1, \xi_2, \ldots]$, $|\xi_i| = 2^i - 1$

$$\Delta(\xi_i) = \sum_{j+k=i} \xi_j^{2^k} \otimes \xi_k$$

The class ξ_k is dual to Sq^{I_k}, $I_k = (2^{k-1}, 2^{k-2}, \ldots, 2, 1)$

Capping with ξ_1 gives the easier Adem relations.