HOMEWORK #6: ADAMS SPECTRAL SEQUENCE II

(1) Using the method described in class, show that $\mathrm{Ext}_{\mathcal{A}(1)}(\mathbb{F}_2, \mathbb{F}_2)$ is periodic with period $(4, 12)$.

(2) We discussed Massey products in a differential graded algebra. If (M, d_M) is a differential graded module over a differential graded algebra (A, d_A), then we can define Massey products $\langle a, b, m\rangle$ whenever $a \cdot b = 0$ in A and $b \cdot m = 0$ in M. The definition is the same as in the DGA case. Since $\mathrm{Ext}_{\mathcal{A}(1)}(\mathbb{F}_2, \mathbb{F}_2)$ is the cohomology of a DGA, and since $\mathrm{Ext}_{\mathcal{A}(1)}(M, \mathbb{F}_2)$ is the cohomology of a DGM over this DGA, we can use these techniques.

Let $M = C(\eta)$ be the module from the previous homework set. Consider the LES in Ext induced by $\Sigma^2\mathbb{F}_2 \to M \to \mathbb{F}_2$. Let a be the class in degree $(2, 0)$ from $\mathrm{Ext}^{0,2}(\Sigma^2\mathbb{F}_2, \mathbb{F}_2)$, and let b denote the class in degree $(7, 3)$ from $\mathrm{Ext}^{3,7}(\mathbb{F}_2, \mathbb{F}_2)$ (we'll denote the class in $(0, 0)$ here by 1). The boundary map δ establishes a null-homotopy of $h_1^2 \cdot 1$: $h_1 \cdot a \mapsto h_1^2 \cdot 1$. Using the definition of $\langle h_1, h_1^2, 1\rangle$, show that $h_0 \cdot (h_1^2 a) = b$. You may use that $b = \langle h_0, h_1, h_1^2\rangle$.

(3) Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequences of R-modules. Show that under the connecting homomorphisms $\delta\colon \mathrm{Hom}_R(M', M') \to \mathrm{Ext}^1_R(M'', M')$, the class of the identity maps to the class of the extension.

(4) This problem and the following concern computing Ext over the full Steenrod algebra. Consider the filtration of the dual Steenrod algebra given by $|\xi_i^{2^j}| = 2i-1$. The associated graded Hopf algebra is the primitively generated exterior algebra on classes $[\xi_i^{2^j}]$ for all i, j. We therefore conclude that $\mathrm{Ext}_{Gr}(\mathbb{F}_2, \mathbb{F}_2) = \mathbb{F}_2[h_{i,j}]$, where $h_{i,j}$ is represented by $\xi_i^{2^j}$. Run the May spectral sequence in low dimensions (through at least $t - s = 10$ and for all s in this range). Use the following facts:
   (a) The differentials arise from taking the coproduct on classes and then looking in successively lower filtrations.
   (b) The differentials "commute" with the algebraic Steenrod operations, so $d_?(Sq^j(x)) = Sq^j d_*(x)$. The relation between $*$ and ? in this formula depends on the filtration of $Sq^j d_*(x)$.

(5) Continuing the previous problem, you will see some basic relations:
   (a) $h_0 \cdot h_1 = 0$,
   (b) $h_0 \cdot h_2^2 = 0$,
   (c) $h_1^3 + h_0^2 h_2 = 0$.

   Using the fact that Sq^0 is a ring homomorphism, deduce the generalizations of these relations to h_i, h_{i+1} and h_{i+2}.