HOMEWORK 4: APPLICATIONS OF $H^*\big(K(\pi, n)\big)$

(1) Sketch a proof of Cartan's result that $H^*\big(K(\mathbb{Z}/p, n); \mathbb{F}_p\big)$ is the free commutative algebra on classes $\mathcal{P}^I(\iota_n)$, where $e(I) < n$, and classes $\mathcal{P}^I(\iota_n)$, where $e(I) = n$ and $\mathcal{P}^I(\iota_n)$ realizes the p^{th} Massey power of an exterior class.

(2) Using Postnikov sections, compute the first 9 stable homotopy groups of spheres at $p = 3$. Determine the two possibilities for what could happen at the next stage.

(3) (a) Using Postnikov sections, show that the first p-torsion in $\pi_*^s(S^0)$ is a $\mathbb{Z}/p$ and occurs in degree $2p - 3$.
 (b) Show more generally that this is true for the first p-torsion in the unstable homotopy groups: we first see p-torsion in $\pi_*(S^k)$ degree $(2p - 3) + k$.

(4) Using Postnikov sections and results of Serre and Cartan, compute the first 4 homotopy groups of S^3.

(5) Mirror the arguments given for spheres and compute the first few homotopy groups of $\mathbb{C}P^2$. In general, unstable homotopy groups are very difficult to compute. This method is a very useful one for finding them!

(6) Let $M = Sp(5)/SU(5)$. Using the Serre spectral sequence, show that the rational cohomology of M (which is a 31-dimensional manifold) has two non-trivial Massey products. Conclude that
$$M \not\simeq_{\mathbb{Q}} (S^6 \times S^{25}) \# (S^{10} \times S^{21}).$$
You may assume that the map
$$\mathbb{Q}[x_3, x_7, x_{11}, x_{15}, x_{19}] = H^*(Sp(5)) \to H^*(SU(5)) = \mathbb{Q}[x_3, x_5, x_7, x_9]$$
sends x_3 and x_7 to the classes of the same name.