HOMEWORK 2: SERRE SPECTRAL SEQUENCE COMPUTATIONS

(1) Show that $H^*(\Omega S^{2n+1}) = \Gamma(x_{2n})$ and that $H^*(\Omega S^{2n}) = E(x_{2n-1}) \otimes \Gamma(x_{4n-2})$.

(2) Show that $H^*(U(n)) = E(x_1, x_3, \ldots, x_{2n-1})$. Show that $H^*(Sp(n)) = E(x_3, x_7, \ldots, x_{4n-1})$. Verify your computations by checking that the top non-zero degree is in the dimension of the Lie group.

(3) The exceptional Lie group G_2 sits inside 2 fibrations:

$$SU(3) \to G_2 \to S^6 \text{ and } S^3 \to G_2 \to V_2(\mathbb{R}^7)$$

The former is easier to see. G_2 is the automorphism group of the octonions, the division algebra built on $\mathbb{R}^8$. The "imaginary" octonions forms a copy of $\mathbb{R}^7$ inside this, and G_2 preserves this subspace (the other copy of $\mathbb{R}$ is the center). It therefore acts on the unit sphere, S^6, in this $\mathbb{R}^7$. The stabilizer is a copy of $SU(3)$. Using these two fibrations, compute the cohomology of G_2 with mod 2 and rational coefficients. (There is some indeterminacy with the product structure at this stage. When we talk about the Steenrod algebra, this will be resolved).

(4) Compute the homology and cohomology of the fiber of the degree m map $S^n \to S^n$ (here you may assume that we have already replaced the map with a fibration).

(5) Prove the Leray-Hirsch theorem: If $F \xrightarrow{i} E \xrightarrow{\pi} B$ is a fibration and there are classes $a_i \in H^*(E)$ such that $i^*(a_i)$ form a basis for $H^*(F)$, then $H^*(E)$ is a free $H^*(B)$-module on the classes a_i.

(6) Show that if p is a unit in R, then $H^{*>0}(\mathbb{Z}/p; R) = 0$ (hint: universal coefficients).

(7) Complete the computation of $H^*(\Sigma_3)$. You may find it easier to compute $H^*(\Sigma_3; \mathbb{F}_3)$.

(8) Generalizing this, let G be the semi-direct product of $\mathbb{Z}/p$ and $\mathbb{Z}/(p-1)$, where $\mathbb{Z}/(p-1)$ acts on $\mathbb{Z}/p$ via multiplication by $\mathbb{Z}/p^\times = \mathbb{Z}/(p-1)$. Using the short exact sequence $\mathbb{Z}/p \to G \to \mathbb{Z}/(p-1)$, compute $H^*(G; \mathbb{Z}/p)$. Since $\mathbb{Z}/p$ is the p-Sylow subgroup of Σ_p, and since G is the normalizer of that in Σ_p, the cohomology of G is the cohomology of Σ_p, p-locally.