Math 246A
Homework 7 Solutions
Due Thursday, May 24

Note the solutions are often just an outline, but I have tried to address all of the key points and you should be able to fill in the details. If you have questions about them feel free to ask me; also if there are any errors in my solutions please let me know. Of course there are usually many other correct ways to solve each problem as well.

5. Prove that $f(z) = -\frac{1}{2}(z + 1/z)$ is a conformal map from the half-disc $\{z = x + iy : |z| < 1, y > 0\}$ to the upper half-plane.

   [Hint: The equation $f(z) = w$ reduces to the quadratic equation $z^2 + 2wz + 1 = 0$, which has two distinct roots in $\mathbb{C}$ whenever $w \neq \pm 1$. This is certainly the case if $w \in \mathbb{H}$.]

   First $f(z)$ maps the half-disc into $\mathbb{H}$ since if $z = re^{i\theta}$ where $r < 1$ and $0 < \theta < \pi$, then Im $z = -\frac{1}{2}\sin\theta(r - 1/r)$, and $\sin\theta > 0$ while $r - 1/r < 0$. Following the hint $z^2 + 2wz + 1 = 0$ has two distinct roots if $w \neq \pm 1$, and as their product is 1 one must lie in $\mathbb{D}$ and one in $\mathbb{C} - \bar{\mathbb{D}}$. Also if z is a root then $z + 1/z = -2w$, where $w \in \mathbb{H}$. Since Im $(z + 1/z) =$ (Im $z)(1 - 1/|z|)$ and $|z| < 1$ we must have Im $z > 0$ as well and z is in the upper half-disc. Let $g(w) = -w + \sqrt{w^2 - 1}$ where we choose a branch of square root so that $\sqrt{i^2 - 1} = i\sqrt{2}$. Then $g(i) \in \mathbb{D}$, and by continuity $g(w) \in \mathbb{D}$ (and also Im $g(w) > 0$) for all $w \in \mathbb{H}$. The functions f and g are inverses by calculation.

10 Let $F : \mathbb{H} \to \mathbb{C}$ be a holomorphic function that satisfies $|F(z)| \leq 1$ and $F(i) = 0$. Prove that

$$|F(z)| \leq |\frac{z - i}{z + i}| \text{ for all } z \in \mathbb{H}.$$

   Define $G : \mathbb{D} \to \mathbb{H}$ by $G(z) = i\frac{1-w}{1+w}$ This is a conformal map from $\mathbb{D}$ into $\mathbb{H}$ and $f_r(w) = r * (F \circ G)$ for $r < 1$ is a holomorphic map from $\mathbb{D}$ into $\mathbb{C}$ such that $f_r(0) = r * (F(i)) = 0$ and $|f_r(w)| \leq r < 1$ for all $w \in \mathbb{D}$. Then by Schwarz's lemma $|r * (F \circ G(w))| = |f_r(w)| \leq |w|$ for all $w \in \mathbb{D}$. Letting r go to 1 we get $|F \circ G(w)| \leq |w|$ for all $w \in \mathbb{D}$. letting $w = G^{-1}(z) = \frac{i-z}{i+z}$ gives $|F(z)| \leq |\frac{z-i}{z+i}|$.

13 The **pseudo-hyperbolic distance** between two points $z, w \in \mathbb{D}$ is defined by

$$\rho(z, w) = \left|\frac{z - w}{1 - \bar{w}z}\right|.$$

   (a) Prove that if $f : \mathbb{D} \to \mathbb{D}$ is holomorphic, then $\rho(f(z), f(w)) \leq \rho(z, w)$ for all $z, w \in \mathbb{D}$. Moreover, prove that if f is an automorphism of $\mathbb{D}$ then f preserves the pseudo-hyperbolic distance $\rho(f(z), f(w)) = \rho(z, w)$ for all $z, w \in \mathbb{D}$.

   [Hint: Consider the automorphism $\psi_\alpha(z) = (z - \alpha)/(1 - \bar{\alpha}z)$ and apply the Schwarz lemma to $\psi_{f(w)} \circ f \circ \psi_w^{-1}$.]

   Following the hint, if $g = \psi_{f(w)} \circ f \circ \psi_w^{-1}$ then $g : \mathbb{D} \to \mathbb{D}$ is holomorphic and $g(0) = 0$, and so by the Schwarz lemma it follows $|g(z')| \leq |z'|$ for all $z' \in D$. Let $z' = \psi_w(z)$ for $z \in \mathbb{D}$, and $\rho(f(z), f(w)) \leq \rho(z, w)$ follows. For f an automorphism of $\mathbb{D}$, then $\rho(f(z), f(w)) \leq \rho(z, w)$ and $\rho(z, w) = \rho(f^{-1}(f(z))), f^{-1}(f(w))) \leq \rho(f(z), f(w))$; therefore $\rho(f(z), f(w)) = \rho(z, w)$ for all $z, w \in \mathbb{D}$.

   (b) Prove that

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2} \quad \text{for all } z \in \mathbb{D}$$

   This result is called the Schwarz-Pick lemma.

   By part (a), for all $z, w \in \mathbb{D}$ we have $\left|\frac{f(z)-f(w)}{1-\overline{f(w)}f(z)}\right| \leq \left|\frac{z-w}{1-\bar{w}z}\right|$, which implies $\left|\frac{f(z)-f(w)}{z-w}\frac{1}{1-\overline{f(w)}f(z)}\right| \leq \left|\frac{1}{1-\bar{w}z}\right|$ if $z \neq w$. Take the limit as $w \to z$ to get the result.

   One can also use that $|g'(0)| \leq 1$ and then the result follows from the chain rule.

14 Prove that all conformal mappings from the upper half plane $\mathbb{H}$ to the unit disc $\mathbb{D}$ take the form

$$e^{i\theta}\frac{z-\beta}{z-\bar{\beta}}, \quad \theta \in \mathbb{R} \text{ and } \beta \in \mathbb{H}.$$

First we show that $T_\beta(z) = \frac{z-\beta}{z-\bar{\beta}}$, $\beta \in \mathbb{H}$, is a conformal map from $\mathbb{H}$ to $\mathbb{D}$. Clearly T_β maps $\mathbb{H}$ into $\mathbb{D}$ since $|z-\beta| < |z-\bar{\beta}|$ for $z \in \mathbb{H}$. The inverse function is $T_\beta^{-1}(z) = \frac{z\bar{\beta}-\beta}{z-1}$ which maps $\mathbb{D}$ into $\mathbb{H}$ as a calculation shows: If $\beta = a+bi$ and $z = x+iy$ where $b>0$ and $x^2+y^2<1$, then the imaginary part of $\frac{z\bar{\beta}-\beta}{z-1}$ is $\frac{b(1-x^2)}{(x-1)^2+y^2} > 0$. It follows immediately that $T_{\theta,\beta} = e^{i\theta}\frac{z-\beta}{z-\bar{\beta}}$ is also a conformal map from $\mathbb{H}$ to $\mathbb{D}$. Let $T = T_{0,i} = \frac{z-i}{z+i}$ and let S be some other conformal map from $\mathbb{H}$ to $\mathbb{D}$. Then $S \circ T^{-1} = e^{i\varphi}\frac{\alpha-z}{1-\bar{\alpha}z}$ for some $\varphi \in \mathbb{R}$ and $\alpha \in \mathbb{D}$. It follows
$S = e^{i\varphi}\frac{z(\alpha-1)+i(1+\alpha)}{z(1-\bar{\alpha})+i(1+\bar{\alpha})} = -e^{i\varphi}\frac{z+i\left(\frac{1+\alpha}{\alpha-1}\right)}{z\left(\frac{\bar{\alpha}-1}{\alpha-1}\right)-i\left(\frac{1+\bar{\alpha}}{\alpha-1}\right)} = -e^{i\varphi}\left(\frac{\alpha-1}{\bar{\alpha}-1}\right)\frac{z-i\left(\frac{1+\alpha}{1-\alpha}\right)}{z+i\left(\frac{1+\bar{\alpha}}{1-\bar{\alpha}}\right)} = e^{i\theta}\frac{z-\beta}{z-\bar{\beta}}$, where
$e^{i\theta} = -e^{i\varphi}\left(\frac{\alpha-1}{\bar{\alpha}-1}\right)$ and $\beta = i\left(\frac{1+\alpha}{1-\alpha}\right) \in \mathbb{H}$.

15 Here are two properties enjoyed by automorphisms of the upper half-plane.

(a) Suppose Φ is an automorphism of $\mathbb{H}$ that fixes three distinct points on the real axis. Then Φ is the identity.
Let the three points be a, b, c. Then the map $T = \frac{z-a}{z-c}\frac{b-c}{b-a}$ is an LFT (possibly in $PSL_2(\mathbb{C})$, but that's okay) mapping a, b, c to $0, 1, \infty$ respectively. It follows $T^{-1} \circ \Phi \circ T$ fixes $0, 1, \infty$. The only LFT that does this is the identity map z. It follows $\Phi(z) = z$ as well.

(b) Suppose (x_1, x_2, x_3) and (y_1, y_2, y_3) are two pairs of three distinct points on the real axis with $x_1 < x_2 < x_3$ and $y_1 < y_2 < y_3$. Prove that there exists (a unique) automorphism Φ of $\mathbb{H}$ so that $\Phi(x_j) = y_j$, $j = 1, 2, 3$. The same conclusion holds if $y_3 < y_1 < y_2$ or $y_2 < y_3 < y_1$.
Map x_1, x_2, x_3 to $0, 1, \infty$ using $S = \frac{z-x_1}{z-x_3}\frac{x_2-x_3}{x_2-x_1}$ and map y_1, y_2, y_3 to $0, 1, \infty$ using $T = \frac{z-y_1}{z-y_3}\frac{y_2-y_3}{y_2-y_1}$. Note that both S and T have positive determinant due to the ordering of the points. Divide top and bottom by the square root of the determinant and call the new maps S and T; now they both have determinant 1 and so are in $PSL_2(\mathbb{R})$. Now let $\Phi = T^{-1} \circ S$. For the other orderings use different maps S and T mapping the points to $0, 1, \infty$ but in a different order and still with positive determinant. Uniqueness follows since if there were two maps Φ, Ψ then $\Psi^{-1} \circ \Phi$ fixes three points and thus must be the identity by part (a), and so $\Phi = \Psi$.

16 Let

$$f(z) = \frac{i-z}{i+z} \text{ and } f^{-1}(z) = i\frac{1-w}{1+w}.$$

(a) Given $\theta \in \mathbb{R}$, find real numbers a, b, c, d such that $ad - bc = 1$, and so that for any $z \in \mathbb{H}$

$$\frac{az+b}{cz+d} = f^{-1}(e^{i\theta}f(z)).$$

Note all of the maps can be thought of as fractional linear transformations so we just need to multiply some matrices in $\mathbb{GL}_2(\mathbb{C})$ project down to $\mathbb{PSL}_2(\mathbb{C})$ and show that the entries are real. The matrices are

$$\begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}\begin{pmatrix} -1 & i \\ 1 & i \end{pmatrix} = \begin{pmatrix} ie^{i\theta/2}+ie^{-i\theta/2} & e^{i\theta/2}-e^{-i\theta/2} \\ -e^{i\theta/2}+e^{-i\theta/2} & ie^{i\theta/2}+ie^{-i\theta/2} \end{pmatrix}$$

This matrix has determinant -4 so normalizing (by dividing each entry by $2i$) we get

$$\begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

(b) Given $\alpha \in \mathbb{D}$, find real numbers a, b, c, d such that $ad - bc = 1$, and so that for any $z \in \mathbb{H}$

$$\frac{az+b}{cz+d} = f^{-1}(\Psi_\alpha(f(z))).$$

Again we follow the same strategy as in part (a). The matrices are

$$\begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & \alpha \\ -\overline{\alpha} & 1 \end{pmatrix}\begin{pmatrix} -1 & i \\ 1 & i \end{pmatrix} = \begin{pmatrix} 2\text{Im } \alpha & 2(\text{Re } \alpha - 1) \\ 2(\text{Re } \alpha + 1) & -2\text{Im } \alpha \end{pmatrix}$$

This has determinant $4(1-|\alpha|^2)$ so normalizing (by dividing each element by $2\sqrt{1-|\alpha|^2}$) we get

$$\begin{pmatrix} \frac{\text{Im } \alpha}{\sqrt{1-|\alpha|^2}} & \frac{\text{Re } \alpha - 1}{\sqrt{1-|\alpha|^2}} \\ \frac{\text{Re } \alpha + 1}{\sqrt{1-|\alpha|^2}}) & \frac{-\text{Im } \alpha}{\sqrt{1-|\alpha|^2}} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

(c) Prove that if g is an automorphism of the unit disc, then there exists real numbers a, b, c, d such that $ad - bc = 1$, and so that for any $z \in \mathbb{H}$

$$\frac{az+b}{cz+d} = f^{-1} \circ g \circ f(z).$$

If g is an automorphism of the unit disc then there exists $\theta \in \mathbb{R}$ and $\alpha \in \mathbb{D}$ such that $g(z) = e^{i\theta}\Psi_\alpha(z)$ Thus $f^{-1} \circ g \circ f(z) = f^{-1} \circ e^{i\theta} * f \circ f^{-1} \circ \Psi_\alpha \circ f(z) = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} \circ z$ where the matrices of a's,b's,c's,d's are in $\mathbb{PSL}_2(\mathbb{R})$ and thus their product is as well.

[Hint: Use parts (a) and (b).]

Problem 2 The angle between two non-zero complex numbers z and w (taken in that order) is simply the oriented angle, in $(-\pi, \pi]$, that is formed between two vectors in $\mathbb{R}^2$ corresponding to the points z and w. The oriented angle, say α, is uniquely determined by the two quantities $\frac{(z,w)}{|z||w|}$ and $\frac{(z,-iw)}{|z||w|}$, which are simply the cosine and sine of α, respectively. Here the notation $(\cdot, \cdot)$ corresponds to the usual Euclidian inner product in $\mathbb{R}^2$, which in terms of complex numbers takes the form $(z, w) = \text{Re } (z\bar{w})$.

In particular, we may now consider two smooth curves $\gamma : [a, b] \to \mathbb{C}$ and $\eta : [a, b] \to \mathbb{C}$, that intersect at z_0, say $\gamma(t_0) = \eta(t_0) = z_0$ for some $t_0 \in (a, b)$. If the quantities $\gamma'(t_0)$ and $\eta'(t_0)$ are non-zero, then they represent the tangents to the curves γ and η at the point z_0, and we say that the two curves intersect at z_0 at the angle formed by the two vectors $\gamma'(t_0)$ and $\eta'(t_0)$.

A holomorphic function f defined near z_0 is said to **preserve angles** at z_0 if for any two smooth curves γ and η intersecting at z_0, the angle formed between the curves γ and η equals the angle formed between the curves $f \circ \gamma$ and $f \circ \eta$ at $f(z_0)$. (See Figure 12 on page 255 of Stein and Shakarchi for an illustration.) In particular, we assume that the tangents to the curves γ, η, $f \circ \gamma$, and $f \circ \eta$ at the point z_0 and $f(z_0)$ are all non-zero.

(a) Prove that if $f : \Omega \to \mathbb{C}$ is holomorphic, and $f'(z_0) \neq 0$, then f preserves angles at z_0. [Hint: Observe that $(f'(z_0)\gamma'(t_0), f'(z_0)\eta'(t_0)) = |f'(z_0)|^2 (\gamma'(t_0), \eta'(t_0))$.]

We want to show $\frac{((f\circ\gamma)'(t_0),(f\circ\eta)'(t_0))}{|(f\circ\gamma)'(t_0)||(f\circ\eta)'(t_0)|} = \frac{(\gamma'(t_0),\eta'(t_0))}{|\gamma'(t_0)||\eta'(t_0)|}$ which follows from the chain rule, the hint, and the fact that $f'(z_0) \neq 0$. The calculation for sine is similar.

(b) Conversely, prove the following: suppose $f : \Omega \to \mathbb{C}$ is a complex-valued function, that is real differentiable at $z_0 \in \Omega$, and $J_f(z_0) \neq 0$. If f preserves angles at z_0, then f is holomorphic at z_0 with $f'(z_0) \neq 0$.

Since Ω is open there exists $\epsilon > 0$ such that $\overline{D_\epsilon(z_0)} \subset \Omega$. Let $\gamma(t) = z_0 + t$ and $\eta(t) = z_0 + e^{i\theta}t$ where $t \in [-\epsilon, \epsilon]$. Suppose $(f \circ \gamma)'(0) = re^{i\alpha}$ and $(f \circ \eta)'(0) = Re^{i\beta(\theta)}$ Since f preserves angles we have $(e^{i\alpha}, e^{i\beta(\theta)}) = (1, e^{i\theta})$ using the cosine which implies $\cos(\beta(\theta) - \alpha) = \cos(\theta)$. Similarly using the sine one can show that $\sin(\beta(\theta) - \alpha) = \sin(\theta)$. Thus $\beta(\theta) - \theta$ must be a constant which implies $\arg \frac{(f\circ\eta)'(0)}{\eta'(0)}$ is independent of θ. One can compute that $(f \circ \eta)'(0) = \frac{\partial f}{\partial z}(z_0)e^{i\theta} + \frac{\partial f}{\partial \bar{z}}(z_0)e^{-i\theta}$, using the chain rule for instance, and this must be non-zero by the condition on the Jacobian. Then $\frac{(f\circ\eta)'(0)}{\eta'(0)} = \frac{\partial f}{\partial z}(z_0) + \frac{\partial f}{\partial \bar{z}}(z_0)e^{-2i\theta}$. In order for the argument to be independent of θ $\frac{\partial f}{\partial \bar{z}}$ must be zero which is precisely the Cauchy Riemann Equations giving holomorphicity and $|f'(z_0)| = \sqrt{|J_f(z_0)|} \neq 0$.