Homework 4 solutions Due Feb 14

Problem 5.4.10: Suppose there are 6 different types of fruit available in infinite supply. How many different fruit baskets of 10 pieces of fruit are there?

We assume the fruit within each type is indistinguishable, so the generating functions is $G(x) = (1 + x + x^2 + x^3 + \ldots)^6 = (1 - x)^{-6}$ (Note use an ordinary generating function since the order the fruit gets put in the basket doesn't matter) and we want the $x^{10}$ coefficient. This is just $\binom{6+10-1}{10} = \binom{15}{10} = 3003$.

Problem 5.5.12: Find a simple closed form expression for the exponential generating function if we have p types of objects, each in infinite supply, and we wish to choose k objects with atleast one of each kind.

For each type the generatinf function is $(x/1! + x^2/2! + x^3/3! + x^4/4! + \ldots) = e^x - 1$. So the generating function is $G(x) = (e^x - 1)^p$ and we want to find the $x^k/k!$ coefficient.

Problem 5.5.14: Find the number of codewords of length k from the alphabet $\{a, b, c, d, e\}$ if $b$ occurs an odd number of times.

We use an exponential generating function since order matters for codewords. So we get

$$G(x) = (1 + x + x^2/2! + x^3/3! + x^4/4! + \ldots)^4 * (x + x^3/3! + x^5/5! + \ldots)$$

$$= (e^x)^4 * \frac{e^x - e^{-x}}{2} = \frac{e^{5x} - e^{3x}}{2}$$

and we want the $x^k/x!$ coefficient. The coefficient of $x^k$ is $\frac{5^k}{2*k!} - \frac{3^k}{2*k!}$ so the coefficient of $x^k/x!$ is $\frac{5^k - 3^k}{2}$.

Problem 5.5.20: Suppose $a_{n+1} = (n + 1)b_n$ with $a_0 = b_0 = 1$. If $A(x)$ is the exponential generating function for $\{a_n\}$ and $B(x)$ is the exponential generating function for $\{b_n\}$, derive a relation for $A(x)$ and $B(x)$.

As seen in the sketch in section we think $A(x) = 1 + xB(x)$, lets prove it now.

$$1 + xB(x) = 1 + x\Sigma_{n=0}^{\infty} \frac{b_n x^n}{n!} = 1 + x\Sigma_{n=0}^{\infty} \frac{(n + 1)b_n x^n}{(n + 1)n!}$$

$$= a_0 + x\Sigma_{n=0}^{\infty} \frac{a_{n+1} x^n}{(n + 1)!} = a_0 + \Sigma_{n=0}^{\infty} \frac{a_{n+1} x^{n+1}}{(n + 1)!} = \Sigma_{n=0}^{\infty} \frac{a_n x^n}{n!} = A(x).$$

Problem 6.1.14: In a computer system, seven PIN numbers are deleted from 7 accounts and are reassigned at random.

a: How many ways are there to reassign them so that at least one is reassigned correctly?

Recall $D_n$ is the number of way to permute $n$ element so that none is reassigned to its proper place, so here the total number of ways that at least one is reassigned correctly is the number of ways of assigning the PIN numbers minus $D_7$, that is

$$7! - D_7 = 7! - 7!(1/2! - 1/3! + 1/4! - 1/5! + 1/6! - 1/7!) = 3186.$$

b: How many ways are there to assign them if all seven must be properly assigned?

Clearly there is just one way.

Problem 6.1.15: A lab director can run two different types of experiments, the expensive one costs 8000 dollars and the inexpensive one costs 4000 dollars. If $F(n)$ is equal to the number of sequences of experiments she can run with exactly n dollars.
a: Find a recurrence relation for $F(n)$.

Clearly at each step she can either run the expensive experiment and have $n - 8000$ dollars left to do experiments or the inexpensive experiment and have $n - 4000$ dollars left to do experiments. Thus the relation is $F(n) = F(n-8000)+F(n-4000)$. We also need initial conditions. Clearly $F(4000) = 1$, $F(8000) = 2$ and to take care of all other cases it suffices to let $F(n) = 0$ for $0 \leq n \leq 8000$, $n \neq 4000, 8000$.

b: Suppose there are $p$ kinds of experiments, $E_1, \ldots, E_p$, with $E_i$ costing $d_i$ dollars to run. find a recurrence relation for $F(n)$ in this case.

Similarly we have to choose an experiment $E_i$ to do first, then we have $n-d_i$ dollars left to do experiments. So

$$F(n) = \Sigma_{i=1}^{p} F(n - d_i).$$