Problems graded: p97 3, p120 2, p123 4.

p88 2. Case 1: $c \neq 0$.

$S$ is parabolic if and only if the equation $z = \frac{az+b}{cz+d}$ has a unique root, i.e. iff the quadratic equation $cz^2 + dz = az + b$ has a unique root, or iff $cz^2 + (d-a)z - b$ has a unique zero. However, its discriminant is

$$(d-a)^2 + 4bc = a^2 - 2ad + d^2 + 4bc = a^2 + 2ad + d^2 - 4(ad - bc) = (a+d)^2 - 4$$

(by normalization); as quadratics with unique roots are precisely those with discriminants equal to zero, $S$ is parabolic iff $a + d = \pm 2$.

In general, the fixed points of $S$ are therefore equal to $\frac{a-d \pm \sqrt{(a+d)^2-4}}{2c}$.

If these two are distinct (i.e. the discriminant was nonzero) we can set $\alpha = \frac{a-d+\sqrt{(a+d)^2-4}}{2c}$ and $\beta = \frac{a-d-\sqrt{(a+d)^2-4}}{2c}$. Then the transformation is of the form $\frac{w-\alpha}{w-\beta} = k\frac{z-\alpha}{z-\beta}$ where $w = Sz$. Setting $z = \infty$ ($\infty$ is not a fixed point because we presumed $c \neq 0$), $w = \frac{a}{c} = \frac{2a}{2c}$.

Then

$$k = \frac{w-\alpha}{w-\beta} = \frac{a+d-\sqrt{(a+d)^2-4}}{a+d+\sqrt{(a+d)^2-4}}.$$

We have that $S$ is elliptic iff $\sqrt{(a+d)^2-4}$ is pure imaginary (and nonzero), i.e. iff $(a+d)^2 - 4$ is negative, which happens iff $-2 < a + d < 2$.

Further, $S$ is hyperbolic iff $\sqrt{(a+d)^2-4}$ is real (and nonzero), i.e. iff $(a+d)^2 - 4$ is strictly positive, which happens iff $a + d < -2$ or $a + d > 2$.

Case 2: $c = 0$. In this case, the transformation is of the form $\frac{az+b}{d}$; this always has a fixed point at infinity.

This is unique iff $\frac{a}{d} = 1$; as $bc = 0$, we know that $ad = ad - bc = 1$. Therefore, $d = a^{-1}$ so $\frac{a}{d} = 1$ iff $a = d$, which happens iff $a + d = 2$ or $-2$ (if $(a+d)^2 = 4$ and $ad = 1$ then $a = d$ by subtracting $4ad$ from $(a+d)^2$ to get $(a-d)^2$).

If the fixed point is not unique, the standard form is $w - \alpha = k(z - \alpha)$ where $\alpha$ is the finite fixed point, and ellipticity and hyperbolicity now rest on $k$ from this form.

However, it is clear that $k = \frac{a}{d}$ (because $w = \frac{a}{d}z + \frac{c}{d}$) so, for the non-parabolic case, $S$ is elliptic iff $|a| = |d|$, i.e. iff $a$ and $d$ are complex conjugates (alternatively, iff $a$ is nonreal complex with $|a| = 1$), which happens iff $a + d \in (-2, 2)$ . Further, $S$ is hyperbolic iff $\frac{a}{d}$ is real and not equal to 1, i.e. iff $a$ and $d$ are distinct reals, which happens iff $-2 < a + d$ or $a + d < 2$, exactly as desired.

3. Because the standard form for parabolic transformations involves translations (which are not periodic), $S$ cannot be parabolic. Using the standard form from equation 12, we conclude $k^n = 1$ from which $|k| = 1$ and therefore $S$ is elliptic.

4. By conjugation (this preserves the hyperbolic/loxodromic property by Problem 2; conjugation of matrices preserves their trace and determinant) we may suppose one of the fixed points is at infinity, i.e. the transformation is of the form $az+b$ (with $a \neq 1$; otherwise the transformation would be parabolic). With a fixed point of $b/(1-a)$, conjugating by a translation of this amount switches us to a transformation of the form $az$ with $|a| \neq 0$. From this formation, the statement is clear; if $|a| > 1$ then $S^n z$ approaches infinity unless $z = 0$ and if $|a| < 1$ then $S^n z$ approaches zero unless $z = \infty$. For $n \to -\infty$ the roles of the attractive and repellent fixed point are reversed because $S^{-1}$ sends $z$ to $a^{-1}z$. In the parabolic case (where there is only one fixed point), from the standard form in (14) one can see that as $n$ approaches either positive or negative infinity, $S^n z$ will always approach the fixed point (repeatedly adding a nonzero number to some other number will cause an infinite limit); the same holds even for a fixed point at infinity (when the map is just translation).

p97 3. (NOTE: I shall describe the conformal mappings via a series of steps).

Step 1: Use the conformal map $\frac{z-1}{z+1}$, which maps our region to the complement of the nonnegative imaginary axis and the point at infinity to 1.

Step 2: Multiply by i and then apply the square root map (in particular, the branch thereof with positive real part); our region is mapped to the set with Re $z > 0$ and what was the point at infinity now goes to $e^{\frac{\pi i}{4}}$.

Step 3: Apply the conformal map $\frac{z+1}{z-1}$; this maps our region to the outside of the unit circle (the boundary goes to the unit circle and 1, which was inside it, goes to $\infty$) and what was the point at infinity goes to $\frac{1+i+\sqrt{2}}{1+i-\sqrt{2}}$; call this number $\gamma$.

Step 4: Apply the conformal map $\frac{z-\overline{\gamma^{-1}}}{1-\gamma^{-1}z}$ (a conformal self-map of the unit disc, and therefore a conformal self-map of the outside of the unit disc, sending $\gamma$ to $\infty$); this allows the points at infinity to correspond to each other.

4. We note that $(r+i)^2 = (r^2-1)+(2ri)$; this is on the parabola $y^2 = 4x+4$ and all points on this parabola can be thusly described. Consequently, our procedure is as follows:

Step 1: Divide by $\frac{p}{2}$; then the region is the outside of $y^2 = 4x$ with $z = 0$ sent to 0 and $z = -\frac{p}{2}$ sent to $-1$.

Step 2: Subtract 1; then the region is the outside of $y^2 = 4x+4$ with $z = 0$ sent to $-1$ and $z = -\frac{p}{2}$ sent to $-2$.

Step 3: Take square roots (in particular, the branch of the square root where $\sqrt{-1} = i$); then (as the parabola is sent to the line $y = 1$) the region is now the set of all points with $y > 1$, with $z = 0$ sent to $i$ and $z = -\frac{p}{2}$ sent to $i\sqrt{2}$.

Step 4: Subtract $i$: the region is now the upper half plane with $z = 0$ sent to 0 and $z = -\frac{p}{2}$ sent to $i(\sqrt{2}-1)$.

Step 5: Apply the conformal map $\frac{z-i}{z+i}$; the region is now the inside of the unit disc (the boundary gets sent to the unit disc and $i$ gets sent to 0) with $z = 0$ sent to $-1$ and $z = -\frac{p}{2}$ sent to $\frac{\sqrt{2}-2}{\sqrt{2}} = 1-\sqrt{2}$.

Step 6: Take additive inverses; the image is still the unit disc but $z = 0$ is sent to 1 and $z = -\frac{p}{2}$ goes to $\sqrt{2}-1$.

Step 7: Apply the conformal map $\frac{z-(\sqrt{2}-1)}{1-(\sqrt{2}-1)z}$; although this is a conformal map of the unit disc and $z = 0$ still corresponds to 1, we finally have $z = -\frac{p}{2}$ corresponding to 0 as desired.

5. We begin by noting that the vertex of this branch is $a$ and the focus is the positive square root of $a^2 + a^2$, or $a\sqrt{2}$.

Step 1: Apply the squaring operation from $z$ to $z^2$ (which is conformal and injective on the inside of the hyperbola, a subset of Re $z > 0$) to transform the region into the set $x > a^2$ (as $(x + yi)^2$ has real part $x^2 - y^2$), with the vertex sent to $a^2$ and the focus to $2a^2$.

Step 2: Subtract (and then divide by) $a^2$ to send our region to the right half-plane; the vertex goes to 0 and the focus to 1.

Step 3: Use the conformal map $\frac{z-1}{z+1}$ to send the region to the unit disc (the boundary goes to the unit circle and 1, which was in the region, goes to 0); the vertex goes to 1 and the focus goes to 0.

Step 4: Take additive inverses; the focus now corresponds to $w = 0$ and the vertex to $w = -1$ as desired.

6. For this map, we first seek to find a map from the upper half of our lemniscate (i.e. where the imaginary part is positive) to the upper half of the unit disc which extends continuously to the real axis. This proceeds as follows.

Step 1: Use the squaring map (which is conformal on its restricted domain); our lemniscate goes to the circle $|z - a^2| = p^2$ with the line $[0, a^2 + p^2]$ in the positive real direction removed while points just above the real axis corresponding to points just above (if $x$ were negative) or just below (if $x$ were positive) the radius in the positive real direction.

Step 2: Subtract $a^2$ and divide by $p^2$; the circle is now the unit circle with the line $[-\frac{a^2}{p^2}, 1]$ removed.

Step 3: Apply the fractional linear transformation $\frac{z+\frac{a^2}{p^2}}{1+\frac{a^2}{p^2}z}$; the image is now the unit circle with the radius $[0, 1]$ removed.

Step 4: Apply the square root map; the image is now the upper semicircle with points just above the real axis corresponding to points just above the real axis, and the sign of the real coordinate in the image of such points is the same of that in the preimage. (This gives a continous extension to the real axis).

Now that we have a map $f$ defined on the upper half of the region, we extend $f$ to the entire lemniscate by setting $f(z) = \overline{f(\bar{z})}$ on the lower half; this is conformal everywhere by the Schwarz Reflection Principle (the one point where conformality might fail, $z = 0$, was fixed by the square root from Step 4).

p108 3. We parametrize the circle via $z = 2e^{i\theta}$ for $0 \le \theta \le 2\pi$. Then, the integrand becomes

$\frac{ie^{i\theta}d\theta}{4e^{2i\theta}-1}$. This is a function whose value at $\theta$ is the additive inverse of its value at $\theta + \pi$ (the numerator gets multiplied by $-1$ and the denominator is unchanged) so the integral from 0 to $\pi$ is the additive inverse of the integral from $\pi$ to $2\pi$, canceling and yielding an integral of zero.

4. We use the parametrization $z = e^{i\theta} = \cos\theta + i\sin\theta$ for $0 \le \theta \le 2\pi$. Then, the integrand becomes $|(\cos\theta - 1) + i\sin\theta|d\theta = \sqrt{2 - 2\cos\theta}d\theta$.

We now try to simplify the integrand to make it integrable; to do this we note that $1 - \cos\theta = \frac{1-\cos^2\theta}{1+\cos\theta} = \frac{\sin^2\theta}{1+\cos\theta}$.

Therefore, for $0 \le \theta \le \pi$, the integrand becomes $\sqrt{2}\frac{\sin\theta d\theta}{\sqrt{1+\cos\theta}}$.

We next substitute $u = 1 + \cos\theta$, yielding $du = -\sin\theta$ and an integrand going from 2 to 0; reversing sign appropriately yields

$\int_0^2 \sqrt{2}\frac{du}{\sqrt{u}}$. This integral can now be evaluated by calculus to be equal to 4; to complete the problem we note that the integrand has the same value at $2\pi - \theta$ as $\theta$ so the total integral is twice 4, or 8.

6. Because $|f(z) - 1| < 1$, we can define a branch log of the logarithm on the image of $f$ (the real part of $f$ is positive; therefore we can let the argument range from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$). Therefore, the integrand can be written as $(\log f(z))'dz$ so the integral only depends on the (coincident) endpoints and therefore is equal to zero.

7. We write $P(z) = \Sigma_0^n c_n(z-a)^n$ and parametrize the circle via $a + Re^{i\theta}$ for $0 \le \theta \le 2\pi$; then $\bar{z} = Re^{-i\theta}$ and $d\bar{z} = -iRe^{-i\theta}d\theta$.

Therefore, the integral of $P(z)$ over the circle is $\int_0^{2\pi} -iR\Sigma_0^n c_n R^n e^{i(n-1)\theta}d\theta$; as $\int_0^{2\pi} e^{ik\theta}d\theta$ is equal to 0 for $k \neq 0$ (by the Fundamental Theorem of Calculus) and $2\pi$ for $k = 0$, the integral is $-iR * 2\pi * R * c_1$ (only the $c_1$ term does not vanish), or $-2\pi i R^2 P'(a)$ (as $P'(a) = c_1$).

p120 2. Note: because of its length, this problem was worth 12 points (four for each part) instead of 10.

(a) Because any two partitions of an interval have a common refinement, it suffices to show that if one segment of a subdivision is divided into two parts (say, $\sigma_k$, with endpoints $a_k$ and $b_k$, is divided by some interior point $c$ into $\sigma_{(k,1)}$, with endpoints $a_k$ and $c$, and $\sigma_{(k,2)}$, with endpoints $c$ and $b_k$), then $n(\sigma, a)$ is unchanged.

However, the only difference that occurs is that the integral of $\frac{1}{z-a}$ over the line segment joining $a_k$ to $b_k$ is replaced by the integral over the line segment joining $a_k$ to $c$ followed by the integral from $c$ to $b_k$. This can be done by adding the integral over the triangle joining $a_k$ to $c$ to $b_k$. As the added integral is zero by Cauchy's theorem in a disk (the disk containing $\sigma_k$ also contains the triangle), $n(\sigma, a)$ is unchanged and therefore independent of the subdivision.

(b) If $\gamma$ is piecewise differentiable, we divide it into $\gamma_1, \ldots, \gamma_n$ as in the new definition. Under the old definition the winding number would be equal to the sum of the integral of $\frac{1}{z-a}$ over the $\gamma_n$. Under the new definition, we replace the integral over $\gamma_n$ with the integral over the line segment joining the endpoints (maintaining the order of the endpoints). Letting $\gamma_k$ join $a_k$ to $b_k$, we let $\gamma_k'$ be the curve consisting of $\gamma_k$ followed by the line segment from $b_k$ to $a_k$; then the integral over $\gamma'$ is equal to zero by Cauchy's theorem in a disk (again, use the disk containing $\gamma_k$) which implies that the integral from the line segment is the same as the integral on $\gamma$ precisely as desired.

(c) For (i) we simply let $\gamma = \gamma_1$ (contained in a disk not including $a$ by assumption) so $\sigma$ is the trivial line segment and $n(\sigma, a) = 0$ so $n(\gamma, a) = 0$.

For (ii) we show that $n(\gamma, a)$ is locally constant: at a given $a$ we use a specific subdivision into subarcs $\gamma_1, \ldots, \gamma_n$, each contained in a (closed) disk $D_n$ that does not include $a$. Letting $\epsilon$ be the minimal distance of $a$ to a $D_n$, we note that if $|b - a| < \frac{\epsilon}{2}$ then the same subdivision can be used to calculate $n(\gamma, b)$; letting $\sigma$ be as in the problem (derived from our division of $\gamma$ into subarcs), $n(\gamma, b) = n(\sigma, b)$ for $|b - a| < \frac{\epsilon}{2}$. As $n(\sigma, b)$ is locally constant by property (ii) for the piecewise differentible case, the same holds in this case as well.

Further, we note that if $|a|$ is greater than twice the maximal value of $|z|$ then $|\frac{1}{z-a}| \leq \frac{2}{|a|}$ on $\gamma$ which implies that the integral of $\frac{1}{z-a}$ over $\gamma$ is bounded above in norm by $|\gamma|\frac{2}{|a|}$ which clearly approaches 0 as $a$ approaches infinity. Therefore, the index, which is locally constant and approaches 0 as $a$ approaches infinity, is zero on the unbounded component.

3. We begin by noting that both $\gamma_1$ and $\gamma_2$ intersect the positive real axis by the Intermediate Value Theorem (applied to imaginary components).

(a) Because $\sigma_1$ does not touch the portion of the x-axis to the right of $x_2$, a branch of $\log z - x_2$, an antiderivative of $\frac{1}{z-x_2}$, can be defined in a neighborhood of $\sigma_1$. Therefore, $\int_{\sigma_1} \frac{1}{z-x_2} dz = 0$. By continuity of index (property (ii)) and pathconnectedness of $\gamma_2$, $n(\sigma_1, z) = 0$ for $z \in \gamma_2$.

(b) Let $\epsilon > 0$ be such that Re $z > \epsilon$ on $\gamma$ (continuous strictly positive functions on compact sets have strictly positive minima). To make notation semi-consistent with Lemma 2, we let $z_1' = z_2$ and $z_2' = z_1$, which are points on the closed curve $\sigma_1$ which does not pass through $\epsilon$ (which is to play the role of the origin). We let $\delta_1 = \gamma_1$ (the subarc from $z_1'$ to $z_2'$, which does not touch the portion of the real axis to the left of $\epsilon$) and $\delta_2$ be the subarc of $\sigma_2$ from $z_2'$ to $z_1'$ (which consists of two straight line segments which only meet the real axis at 0, i.e. never to the right of $\epsilon$) so, by Lemma 2, $n(\sigma_1, \epsilon) = 1$. This argument carries through replacing $\sigma_1$ by $\sigma_2$ and $\gamma_1$ by $\gamma_2$ so $n(\sigma_2, \epsilon) = 1$. In other words, $n(\sigma_1, x) = n(\sigma_2, x) = 1$ for small $x > 0$.

(c) Assume to the contrary that $x_1 \in \gamma_2$. By the argument of (b), $n(\sigma_1, x_1) = 1$ (because $\sigma_1$ does not touch the portion of the real axis to the left of $x_1$). However, $n(\sigma_1, x_1) = 0$ by (a) producing a contradiction so $x_1 \in \gamma_1$.

(d) By the argument of (b), $n(\sigma_2, x_1) = 1$ because $\sigma_1$ does not touch the portion of the real axis to the left of $x_1$. By continuity of index and pathconnectedness of $\gamma_1$ we conclude $n(\sigma_2, z) = 1$ for $z \in \gamma_1$.

(e) Letting $w_1$ be the rightmost point of $\gamma_1$ on the real axis, the portion of the real axis to the right of $w_1$ still hits $\gamma_2$ (for example, at $x_2$); call the leftmost such point $w_2$.

Therefore, $[w_1, w_2]$ is a segment of the positive real axis with one endpoint on $\gamma_1$, the other on $\gamma_2$, and no other point on $\gamma$. For $r$ in $(w_1, w_2)$, $n(\gamma, r) = n(\sigma_2, r) - n(\sigma_1, r)$ (We assume that $\gamma$ is oriented so as to follow $\gamma_1$ from $z_1$ to $z_2$ and then $\gamma_2$ from $z_2$ to $z_1$; if $\gamma$ has the reverse orientation then $n(\gamma, x)$ must be multiplied by 1). However, $n(\sigma_2, w_1) = 1$ by (d); by continuity of $n$ and pathconnectedness of $[w_1, w_2)$, $n(\sigma_2, r) = 1$ on $[w_1, w_2)$. Further, $n(\sigma_1, w_2) = 0$ by (a); by continuity of $n$ and pathconnectedness of $(w_1, w_2]$, $n(\sigma_2, r) = 0$ on $(w_1, w_2]$. Therefore, $n(\gamma, r) = 1 - 0 = 1$ on $(w_1, w_2)$ (or $-1$ if $\gamma$ has the opposite orientation) exactly as desired.

p120 2. (Note: the easiest way to find this integral is to use symmetry and the fact that the integrand is an odd function; however, the problem asks us to decompose the integrand into partial fractions instead.)

As $z^2+1 = (z+i)(z-i)$, $\frac{1}{z^2+1} = \frac{A}{z+i} + \frac{B}{z-i}$ for some scalars $A, B$. Multiplying by $z^2+1$ and comparing $z$ coefficients, $A + B = 0$ so $B = -A$. (This is all we need for this problem, though one can solve $A = i/2, B = -i/2$.) Therefore,

$$\int_{|z|=2} \frac{dz}{z^2+1} = \int_{|z|=2} \frac{Adz}{z+i} + \int_{|z|=2} \frac{Bdz}{z-i}$$

$= 2\pi i(A + B)$ (by Cauchy's integral formula) $= 0$.

3. By substituting $z = we^{i\theta}$, we compute

$$\int_{|z|=p} \frac{|dz|}{|z-a|^2} = \int_{|w|=p} \frac{|dw|}{|we^{i\theta} - a|^2} = \int_{|z|=p} \frac{|dz|}{|z - e^{-i\theta}a|^2}$$

for any $\theta$ so, in particular, we can our calculations supposing $a$ is positive and real.

Then, one notes that if $z = pe^{i\theta}$ then $dz = ipe^{i\theta}d\theta = izd\theta$ while $|dz| = pd\theta$ so $|dz| = -ip\frac{dz}{z}$.

This turns the integrand into $\frac{-ipdz}{z|z-a|^2}$, or (because $|z-a|^2 = |z|^2 - za - \bar{z}a + a^2$ and $|z| = p$ on the region of integration) $\frac{-ipdz}{zp^2 - z^2a - p^2a + a^2z}$.

To compute our integral, our next step is to find poles of this revised integrand.

The denominator is $-z^2a + z(p^2 + a^2) - p^2a$; by the quadratic formula the roots are $\frac{-(p^2+a^2) \pm (p^2-a^2)}{-2a}$, or in other words: $a$ and $\frac{p^2}{a}$. (In other words, the denominator factors as $-(z-a)(az-p^2)$. Therefore, we isolate two cases.

Case 1: $|a| < p$. In this case, the integrand is equal to the quotient of $\frac{ipdz}{az-p^2}$, analytic on $|z| \leq p$ and equal to $\frac{ip}{a^2-p^2}$ at $a$, by $z - a$ so Cauchy's formula tells us that the integral is $\frac{2\pi p}{p^2-a^2}$.

Case 2: $|a| > p$. In this case, the integrand is equal to the quotient of $\frac{ipdz}{a(z-a)}$, analytic on $|z| \leq p$ and equal to $\frac{ip}{p^2-a^2}$ at $a$, by $z - \frac{p^2}{a}$ so Cauchy's formula tells us that the integral is $\frac{2\pi p}{a^2-p^2}$.

In general (where $a$ is an arbitrary complex number with $|a| \neq p$) we therefore conclude that the integral is $|\frac{2\pi p}{|a|^2-p^2}|$.

p123 2. By Cauchy's integral formula, we note that because $|f(z)| < |z|^n$ for sufficiently large $|z|$ (say $|z| > R_0$), we note that if $a$ is a complex number and $R > |a| + R_0$ then

$$|f^{(n+1)}(a)| \leq \frac{(n+1)!}{2\pi} \int_{C(a,R)} \frac{|f(\zeta)||d\zeta|}{(R-|a|)^{n+2}}$$

(where $C(a, R)$ denotes the circle centered at $a$ of radius $R$)

$$\leq \frac{(n+1)!}{2\pi} \int_{C(a,R)} \frac{(R+|a|)^n}{(R-|a|)^{n+2}} |d\zeta|$$

$$= \frac{(n+1)!}{2\pi} 2\pi R \frac{(R+|a|^n)}{(R-|a|)^{n+2}}$$

which clearly goes to 0 as $R$ goes to infinity.

Therefore, $f^{(n+1)} = 0$ everywhere; performing $n+1$ integrations tells us $f$ is a polynomial of degree $n$.

3. Because $|z| \leq \rho < R$ the closed ball centered at $|z|$ of radius $R - \rho$ is contianed in $\{|z| \leq R\}$. Therefore, from formula (25) we conclude $|f^{(n)}(a)| \leq \frac{Mn!}{(R-\rho)^n}$.

4. Integrating $f$ along the circle $|z| = r$, (25) now tells us $|f^{(n)}(0)| \leq \frac{1}{1-r} n! r^{-n} = \frac{n!}{r^n(1-r)}$.

Being allowed to use $0 < r < 1$, the best bound is attained when the denominator, $r^n(1-r)$, is maximized. Clearly (the function is positive on the interval and zero on the endpoints) the derivative with respect to $r$, $nr^{n-1}(1-r) - r^n = 0$; as $r$ is nonzero we conclude $0 = n(1-r) - r = n - (n+1)r$ so $r = \frac{n}{n+1}$. At this point the maximum is $\frac{n^n}{(n+1)^{n+1}}$ giving us a bound of $\frac{n!(n+1)^{n+1}}{n^n}$.

6. Representing

$$\phi(z,t) = \frac{1}{2\pi i}\int_C \frac{\phi(\zeta,t)}{\zeta - z}d\zeta,$$

we use the definition of $F$ to compute

$$F(z) = \int_\alpha^\beta \phi(z,t)dt = \int_\alpha^\beta (\frac{1}{2\pi i}\int_C \frac{\phi(\zeta,t)}{\zeta - z}d\zeta)dt$$

$$= \int_\alpha^\beta \int_C \frac{1}{2\pi i}\frac{\phi(\zeta,t)}{\zeta - z}d\zeta dt$$

$$= \int_C \int_\alpha^\beta \frac{1}{2\pi i}\frac{\phi(\zeta,t)}{\zeta - z}dt d\zeta$$

(by Fubini's theorem; this applies as the integrand is continuous and bounded in the appropriate region)

$$= \int_C (\frac{1}{2\pi i}\int_\alpha^\beta \phi(\zeta,t)dt)\frac{d\zeta}{\zeta - z}.$$

From this form, we conclude by Lemma 3 that $F'(z)$ is equal to

$$= \int_C (\frac{1}{2\pi i}\int_\alpha^\beta \phi(\zeta,t)dt)\frac{d\zeta}{(\zeta - z)^2}$$

$$= \int_C \int_\alpha^\beta \frac{1}{2\pi i}\frac{\phi(\zeta,t)}{(\zeta - z)^2}dt d\zeta$$

$$= \int_\alpha^\beta \int_C \frac{1}{2\pi i}\frac{\phi(\zeta,t)}{(\zeta - z)^2}d\zeta dt$$

(by Fubini's theorem)

$$= \int_\alpha^\beta (\frac{1}{2\pi i}\int_C \frac{\phi(\zeta,t)}{(\zeta - z)^2}d\zeta)dt$$

$$= \int_\alpha^\beta \frac{\delta\phi(z,t)}{\delta z}dt$$

by another application of Lemma 3. This establishes the analyticity of $F$ and the correct value of $F'$.