The following problems were graded in detail and worth ten points: p32 2, p78 3, p82 8.

p28 5. Defining $g(z) = \overline{f(\bar{z})}$, we write $f(z) = u(z) + iv(z)$ and $z = x + iy$ with u, v, x, y real. At (x, y), we note that for real h, the following four sets of equations hold:

$$\frac{\mathrm{Re}(f(x+h+iy)) - \mathrm{Re}(f(x+iy))}{h} = \frac{u(x+h,y) - u(x,y)}{h}$$
$$= \frac{\mathrm{Re}(g(x+h-iy)) - \mathrm{Re}(g(x-iy))}{h}$$

$$\frac{\mathrm{Im}(f(x+h+iy)) - \mathrm{Im}(f(x+iy))}{h} = \frac{v(x+h,y) - v(x,y)}{h}$$
$$= -\frac{\mathrm{Im}(g(x+h-iy)) - \mathrm{Im}(g(x-iy))}{h}$$

$$\frac{\mathrm{Re}(f(x+i(y+h))) - \mathrm{Re}(f(x+iy))}{h} = \frac{u(x,y+h) - u(x,y+h)}{h}$$
$$= -\frac{\mathrm{Re}(g(x+i(-y+h))) - \mathrm{Re}(g(x+i(-y)))}{h}$$

$$\frac{\mathrm{Im}(f(x+i(y+h))) - \mathrm{Im}(f(x+iy))}{h} = \frac{u(x,y+h) - u(x,y+h)}{h}$$
$$= \frac{\mathrm{Im}(g(x+i(-y+h))) - \mathrm{Im}(g(x+i(-y)))}{h}$$

(where $-h$ is substituted for h in the last equality of the third and fourth set of equations); therefore, letting u^* and v^* denote the real and imaginary parts of g and letting h approach zero, we conclude

$$u_x(z) = u^*_x(\bar{z}), u_y(z) = -u^*_y(\bar{z}), v_x(z) = -v^*_x(\bar{z}), v_y(z) = v^*_y(\bar{z})$$

so f satisfies the Cauchy-Riemann equations at z iff g does at $\bar{z}$, which implies that f and g are simultaneously analytic.

7. We compute:

$$\frac{\delta^2 u}{\delta z \delta \bar{z}} = \frac{1}{2}(\frac{\delta}{\delta x} - i\frac{\delta}{\delta y})\frac{1}{2}(\frac{\delta}{\delta x} + i\frac{\delta}{\delta y})u$$
$$= \frac{1}{4}(\frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2}))u$$

(because the partial differentiation operators with respect to x and y commute)

$$= \frac{1}{4}\Delta u = 0.$$

p32 2. We note that as z approaches α_k,

$$\frac{P(z)}{Q(z)}(z-\alpha_k) = \frac{P(z)}{\frac{Q(z)-Q(\alpha_k)}{z-\alpha}}$$

(as $Q(\alpha_k) = 0$) approaches $\frac{P(\alpha_k)}{Q'(\alpha_k)}$ by the definition of the derivative. Therefore, $\frac{P(\alpha_k)}{Q'(\alpha_k)(z-\alpha_k)}$ has only one pole, at α_k, with singular part exactly the same as that of $\frac{P(z)}{Q(z)}$. This implies that

$$\frac{P(z)}{Q(z)} - \Sigma_{k=1}^{n} \frac{P(\alpha_k)}{Q'(\alpha_k)(z-\alpha_k)}$$

is a rational function which has no finite poles (and approaches zero as z approaches infinity so this can't be a pole either) so it must be the constant function zero. We conclude

$$\frac{P(z)}{Q(z)} = \Sigma_{k=1}^{n} \frac{P(\alpha_k)}{Q'(\alpha_k)(z-\alpha_k)}.$$

3. Setting $P(z) = \Sigma_{k=1}^{n} c_k \frac{Q(z)}{Q'(\alpha_k)(z-\alpha_k)}$ (each summand is a polynomial of degree $n-1$ because $z-\alpha_k$ divides Q), the formula from the preceding part tells us that

$$\Sigma_{k=1}^{n} \frac{c_k}{Q'(\alpha_k)(z-\alpha_k)} = \frac{P(z)}{Q(z)} = \Sigma_{k=1}^{n} \frac{P(\alpha_k)}{Q'(\alpha_k)(z-\alpha_k)};$$

by uniqueness of partial fraction decompositions we conclude $P(\alpha_k) = c_k$ for each k. If P' were another polynomial of degree $< n$ with values c_k at the α_k, $P - P'$ has degree $< n$ with at least n zeroes (at the α_k) so it is the zero polynomial and P is unique.

4. Define $z^* = \overline{z^{-1}}$; then $z^* = z$ on the unit circle. If f is rational and has norm 1 on the unit circle, then because $f(\bar{z}^*)$ is a rational function (formed by replacing all z's with z^{-1}'s and all constants with their complex conjugate) satisfying $f(z)\overline{f(z^*)} = 1$ on the unit circle (an infinite set), $f(z)\overline{f(z^*)} = 1$ on the entire complex plane (where $\infty * 0$ is understood in terms of appropriate singular parts).

Therefore, suppose f is not a constant function. We note that f has a zero at z iff f has a pole at z^* (in other words, the zeroes are reflections of the poles in the unit circle and conversely); therefore, let α be a zero of f. One observes that $\frac{z-\alpha}{1-\bar{\alpha}z}$ has a zero at α and a pole at α^* (just like f); further (as can be seen by multiplying the numerator by $\bar{z}$), this new function has norm one on the unit circle.

Therefore we can replace f by $f_1 = f * \frac{1-\bar{\alpha}z}{z-\alpha}$ which has norm one on the unit circle and one fewer zero and pole than f. Repeating this process inductively, we will end up with a constant function of the form β with $|\beta| = 1$. Therefore, the general form of f (which clearly satisfies the required conditions) is

$$\beta\Pi_{k=1}^{n}\frac{z-\alpha_k}{1-\overline{\alpha_k}z}$$

with $|\beta| = 1$.

5. Consider z^* as defined in the previous part. Now, because f is real on the unit circle, $f(z) = \overline{f(z^*)}$ on this set, or, in other words, $f(z) - \overline{f(z^*)}$, which is a rational function, is zero on the unit circle and therefore zero everywhere. This implies that z is a zero iff z^* is a zero, and z is a pole iff z^* is a pole (so the zeroes are symmetric with respect to the unit circle, as are the poles).

p41 8. This series is a geometric series, which converges iff $|\frac{z}{1+z}| < 1$, which happens iff $|z| < |1+z|$, i.e. iff z is closer to 0 than -1, which happens iff Re $z > -.5$.

9. If $|z| > 1$ then the nth term is bounded above in norm by $2|z|^{-n}$ for n sufficiently large (i.e. when $|z|^{-n} < .5$); being bounded by a convergent series, the series is convergent.

If $|z| < 1$ then the nth term is bounded above in norm by $2|z|^n$ for n sufficiently large (i.e. when $|z|^n < .5$); being bounded by a convergent series, the series is convergent.

However, if $|z| = 1$ then each term has norm at least .5; as the terms in the series do not converge to zero, the series does not converge.

In other words, the series converges everywhere except on the unit circle.

p47 1. We show the result by induction on the remainder after the nth degree term is considered. (For n even we deal with the cosine series, which only has even-degree terms; for n odd we deal with the sine series, which only has odd-degree terms).

$n = 0$: Because $\sin^2 x + \cos^2 x = 1$, $\sin x < 1$ so the remainder is negative; as the next term is $-x^2/2$, it has the same sign as the leading term.

For induction, we consider the following cases (for $k \geq 0$; this partitions all strictly positive integers): $n = 4k+1$ (sine series; remainder has sign of $-x$), $n = 4k+2$ (cosine series; positive remainder), $n = 4k+3$ (sine series; remainder has sign of x), and $n = 4k+4$ (cosine series; positive remainder) assuming the result for a given $n-1$.

$n = 4k+1$: The derivative of the remainder is the remainder for $n = 4k$, which is negative by the induction hypothesis; as the remainder is zero at zero, the remainder has the same sign as $-x$.

$n = 4k+2$: The derivative of the remainder is the additive inverse of the remainder for $n = 4k+1$, which has the sign of $-x$ by the induction hypothesis; as the remainder is zero at zero, the remainder is positive.

$n = 4k+3$: The derivative of the remainder is the remainder for $n = 4k+2$, which is positive by the induction hypothesis; as the remainder is zero at zero, the remainder has the same sign as x.

$n = 4k+4$: The derivative of the remainder is the additive inverse of the remainder for $n = 4k+3$, which has the sign of x by the induction hypothesis; as the remainder is zero at zero, the remainder is negative.

By induction we conclude the remainder has the same sign as the leading term for each series at the appropriate degree, just as desired.

2. Because $1-\frac{(\sqrt{3})^2}{2}+\frac{(\sqrt{3})^4}{4!}=1-3/2+9/24=-1/8<0$ and the remainder of the cosine series after the 4th degree term is negative by the preceding part, we conclude that $\cos(\sqrt{3})<0$ so $\pi<2\sqrt{3}$ (the first zero of cos, $\pi/2$, must be less than $\sqrt{3}$ by the Intermediate Value Theorem).

Similarly, as

$$1-\frac{(3/2)^2}{2}+\frac{(3/2)^4}{4!}-\frac{(3/2)^6}{6!}>1-9/8+27/128-729/10000>.09-.0729>0$$

and the remainder of the cosine teries after the 6th degree term is positive by the preceding part, we conclude that $\cos(3/2)>0$. As $\sin y>y(1-y^2/6)>0$ for $y<2$, the cosine function is decreasing on $[0,2]$ which implies that the first zero of cos, π, is more than $3/2$ so $\pi>3$.

9. Letting A, B, and C be vertices of the triangle (ordered by the counterclockwise direction along the circumcircle), we define the angle at A to be $\arg(\frac{C-A}{B-A})$, the angle at B to be $\arg(\frac{A-B}{C-B})$, and the angle at C to be $\arg(\frac{B-C}{A-C})$. The sum of the arguments of numbers is (modulo 2π) the argument of their product; therefore, the sum of the three angles is $\arg(-1)$ modulo 2π so it's π modulo 2π. However, as each angle is strictly between 0 and π, the sum of the angles is strictly between 0 and 3π while congruent to π modulo 2π and therefore equal to π.

p78 3. Suppose T is a transformation which leaves the origin fixed and preserves distances. Letting θ be such that $T(1)=e^{i\theta}$, we let ρ_θ be the rotation about the origin that sends 1 to $e^{i\theta}$. Then $\rho_{-\theta}T$ fixes 0 and 1 and preserves distances, so it fixes every point on the real line (as each such point is the intersection of two circles which are tangent to each other).

Then, $\rho_{-\theta}T$ must send i to a point which is 1 away from 0, its closest point on the line, i.e. either to i or $-i$. Noting that any transformation fixing the real line and i fixes everything (fixing two points on a line fixes everything on the line), we therefore have that $\rho_{-\theta}T$ is either the identity (if it sends i to itself; then $T=\rho_\theta$) or reflection R in the real axis (if it sends i to $-i$; then $T=\rho_\theta R=R\rho_{-\theta}$) so T is either a rotation or a rotation followed by a reflection in the real axis.

4. Writing $Tz=\frac{az+b}{cz+d}$ we note that if $d=0$ then $Tz=\frac{a}{c}+\frac{b}{cz}$; as this approaches $\frac{a}{c}$ as z approaches infinity, we have $\frac{a}{c}$ is real (and therefore, dividing a, b, and c by c we suppose a, c is real). Further, evaluating at 1 gives that $\frac{b}{c}$ is real so b is real and we use all real coefficients.

Therefore, we assume this is not the case; dividing all coefficients by d we suppose $d=1$. Evaluating at 0, $\frac{b}{d}$ is real so b is real; letting z approach infinity tells us $\frac{a}{c}$ is real. The partial derivative of T with respect to the real component at the origin, $\frac{a}{d}-\frac{bc}{d^2}$, must itself be real (as T is real on the real line) so, as d is real, so is $ad-bc$. However, $\frac{a}{c}-\frac{b}{d}=\frac{ad-bc}{cd}$ which must itself be real, which implies in turn that $cd, c,$and a are real.

p82 1. Suppose C is a circle centered at x of radius r; as reflections preserve distance, if R is a reflection then $|x-y|=r$ iff $|Rx-Ry|=r$ so RC is the circle centered at Rx of radius r.

2. As the imaginary axis and $|z-2|=1$ are both symmetric about the real axis, its reflection is as well so it should have a diameter on the real axis. The reflection of 0 is 1.5 (as the reflection of -2 in the unit circle is .5) and the reflection of the point at infinity is 2 so the image will be a circle with diameter joining 1.5 to 2, i.e. the circle centered at 1.75 of radius .25.

As the line $x=y$ and $|z-2|=1$ are both symmetric about $x+y=2$, its reflection is as well so it should have a diameter on that line. As before, the point at infinity is sent to 2; however, the point $(1,1)$ (which is $\sqrt{2}$ away from the center) is sent to $(1.5, 1.5)$ (which is $.5\sqrt{2})$ away so the image will be a circle with diameter joining $1.5+1.5i$ to 2, i.e. the circle centered at $1.75+.25i$ of radius $.25\sqrt{2}$.

As the unit circle and $|z-2|=1$ are both symmetric about the real axis, its reflection is as well so it should have a diameter on the real axis. The reflection of 1 is 1 (right on the circle) and the reflection of -1 (which is 3 away from the center) is $\frac{5}{3}$ (which is $\frac{1}{3}$ away) so the image will be a circle with diameter joining 1 to $\frac{5}{3}$, i.e. the circle centered at $\frac{4}{3}$ of radius $\frac{1}{3}$.

4. Because the origin gets sent to i, the symmetric point of the origin with respect to the circle $|z|=2$ (the point at infinity) must be sent to the symmetric point of i with respect to $|z+1|=1$, which is $-.5+.5i$ (whose distance from the center is $\sqrt{2}$, as opposed to 1).

Therefore, the linear transformation in question is of the form $\frac{z+2}{cz+d}$ for some scalar c, d by rescaling (as -2 goes to the origin); letting z approach infinity, $\frac{1}{c}=-.5+.5i$ so $c=(-1-i)$. Further, setting $z=0$, $\frac{2}{d}=i$ so $d=-2i$ and therefore our transformation is $\frac{z+2}{(-1-i)z-2i}$.

5. $R=1$: From p33 #4 (note that linear transformations have only one zero and one pole apiece) they are of the form $e^{i\theta}\frac{z-a}{1-\bar{a}z}$ for θ real and $|a|\neq 1$.

General R: If T is such a transformation, right-composing with multiplication by R and left-composing with division by R fixes the unit circle; therefore, the transformation is obtained from the $R=1$ case by doing the exact opposite, yielding

$e^{i\theta}\frac{z-Ra}{R-\bar{a}z}$ for θ real, $|a|\neq 1$.

6. WLOG we suppose (by composing with translation) that the common center is the origin for both the preimage circles and the image circles.

Further (by composing with a dilation) we suppose the outer circle in the preimage is the unit circle; by composing with another dilation and reflection we suppose that it is sent to the unit circle, which is still the outer circle.

Then, the transformation must be of the form

$e^{i\theta}\frac{z-a}{1-\bar{a}z}$ for θ real and $|a|\neq 1$ (in standard form: $\frac{az+b}{cz+d}$, this means $|a|=|d|$ and $|b|=|c|$, so $|ab|=|cd|$).

Because it sends the inner circle (of radius R, say) to another circle (of radius λR), it is also of the form

$\lambda e^{i\theta}\frac{z-Ra}{R-\bar{a}z}$ for θ real, $|a|\neq 1$ (in standard form, this means $|ab|=\lambda^2|cd|$) so $\lambda=1$ and the ratio of $1:R$ is preserved.

7. Consider the transformation $\frac{z-b}{1-bz}$ where $b = 2-\sqrt{3}$. This carries the unit circle into itself; as it preserves the real line, it carries the circle $|z - \frac{1}{4}| = \frac{1}{4}$ into another circle symmetric about the real line. As the origin gets sent to $-(2-\sqrt{3})$ and $\frac{1}{2}$ gets sent to $\frac{-1.5+\sqrt{3}}{.5\sqrt{3}} = 2 - \sqrt{3}$, the image is a circle with diameter joining $2 - \sqrt{3}$ to its additive inverse, i.e. the circle centered at the origin of radius $2 - \sqrt{3}$. These circles are concentric with radii ratio $1 : 2 - \sqrt{3} = 2 + \sqrt{3} : 1$.

8. The transformation $1/z$ sends the unit circle to itself and the line $x = 2$ to $|z - \frac{1}{4}| = \frac{1}{4}$ (infinity goes to zero, 2 goes to $\frac{1}{2}$, and $2 + i$ goes to $.4 - .2i$, all of which are on the circle) so following it with the transformation from the preceding problem, yielding $\frac{1-bz}{z-b}$, will carry the unit circle and $x = 2$ into concentric circles with radii $2 + \sqrt{3} : 1$.