Homework 3 Solutions

1. (This problem was worth 30 points: five for each part.)

(a) From the table we have $L[\sin x] = \frac{1}{p^2+1}$ so by linearity $L[6\sin x] = \frac{6}{p^2+1}$ and therefore $L^{-1}[\frac{6}{p^2+1}] = 6\sin x$.

(b) We have that $\frac{p^2+4}{p^3} = \frac{1}{p} + \frac{4}{p^3}$ so $L^{-1}[\frac{p^2+4}{p^3}] = L^{-1}[\frac{1}{p}] + 2L^{-1}[\frac{2}{p^3}]$ (by linearity) $= 1 + 2x^2$ (from the table).

(c) Note that $L^{-1}[\frac{5p-6}{p^2+4} + \frac{2}{p}] = 5L^{-1}[\frac{p}{p^2+4}] - 3L^{-1}[\frac{2}{p^2+4}] + 2L^{-1}[\frac{1}{p}]$ (by linearity) $= 5\cos 2x - 3\sin 2x + 2$ (from the table).

(d) We note that $L[x^2] = \frac{2}{p^3}$ by the table; the first shift formula therefore implies that $L[e^{-x}x^2] = \frac{2}{(p+1)^3}$ and therefore $L^{-1}[\frac{4p+3}{p^2+9} + \frac{6}{(p+1)^3}] = 4L^{-1}[\frac{p}{p^2+9}] + L^{-1}[\frac{3}{p^2+9}] + 3L^{-1}[\frac{2}{(p+1)^3}]$ (by linearity) $= 4\cos 3x + \sin 3x + 3e^{-x}x^2$.

(e) We note that

$$L^{-1}[\frac{7p-5}{p^2-6p+13}] = L^{-1}[\frac{7p-5}{(p-3)^2+2^2}]$$

$$= L^{-1}[\frac{7(p-3)+16}{(p-3)^2+2^2}]$$

$$= 7L^{-1}[\frac{p-3}{(p-3)^2+2^2}] + 8L^{-1}[\frac{2}{(p-3)^2+2^2}]$$

(by linearity) $= 7e^{3x}\cos 2x + 8e^{3x}\sin 2x$ (by the table and the first shift formula).

(f) We note that

$$L^{-1}[\frac{6p-1}{p^2+6p+25}] = L^{-1}[\frac{6p-1}{(p+3)^2+4^2}]$$

$$= L^{-1}[\frac{6(p+3)-19}{(p+3)^2+4^2}]$$

$$= 6L^{-1}[\frac{p+3}{(p+3)^2+4^2}] - \frac{19}{4}L^{-1}[\frac{4}{(p+3)^2+4^2}]$$

(by linearity) $= 6e^{-3x}\cos 4x - \frac{19}{4}e^{-3x}\sin 4x$ (by the table and the first shift formula).

2. (I graded this problem based on completion; a reasonable attempt at each part was worth five points.)

(a) If $0 \le x < a$ then $u(x) = 1$ and $u(x-a) = 0$ (as $x - a < 0$) so $f_0(x) = u(x)f(x) - u(x-a)f(x) = f(x) - 0 = f(x)$ so $f_0(x)$ coincides with $f(x)$ for such x. However, if $x \ge a$ then $u(x) = 1 = u(x-a)$ so $f_0(x) = u(x)f(x) - u(x-a)f(x) = f(x) - f(x) = 0$ so $f_0(x)$ vanishes for such x.

(b) If $f(x) = -f(x-a)$ for all $x \ge a$ then

$$L[f_0(x)] = L[u(x)f(x) + u(x-a)f(x-a)] = L[u(x)f(x)] + L[u(x-a)f(x-a)]$$

(by linearity) $= F(p) + e^{-ap}F(p)$ (by the second shift formula; remember F is notation for $L[f]$) $= F(p)[1 + e^{-ap}]$.

In the case where $g_0(x) = \sin x$ for $0 \le x < \pi$ and $g_0(x) = 0$ for $x > \pi$ we let $f(x)$ be defined as $\sin x$ for all positive x; with f_0 defined as in part (a) (where $a = \pi$) we have by (a) that $f_0 = g_0$ so $G_0(p) = F_0(p) = F(p)[1 + e^{-ap}]$ (by Equation (2)) $= \frac{1+e^{-\pi p}}{1+p^2}$ (using that $L[\sin x] = \frac{1}{1+p^2}$.)

(c) If $g(x)$ is periodic with period a then

$$G_0(p) = L[u(x)g(x) - u(x-a)g(x)] = L[u(x)g(x) - u(x-a)g(x-a)]$$

(by periodicity) $= L[u(x)g(x)] - L[u(x-a)g(x-a)]$ (by linearity) $= G(p) - G(p)e^{-ap}$ (by the second shift formula) $= G(p)[1-e^{-ap}]$ so $G(p) = \frac{G_0(p)}{[1-e^{-ap}]}$.

(d) If $g(x) = |\sin x|$ and $f(x) = \sin x$ then as $g(x) = f(x)$ for $0 \le x < \pi$ then setting $a = \pi$ gives $g_0(x) = f_0(x)$ for all nonnegative x so $G_0(p) = F_0(p)$. Therefore, we have that $G(p) = \frac{G_0(p)}{1-e^{-\pi p}}$ (by (3))$= \frac{F_0(p)}{1-e^{-\pi p}} = \frac{1+e^{-\pi p}}{1+p^2} * \frac{1}{1-e^{-\pi p}}$ (by (2)) $= \frac{1}{1+p^2} * \frac{1+e^{-\pi p}}{1-e^{-\pi p}}$.

3. (I also graded this problem based on completeness: a reasonable attempt was worth ten points.)

Solution 1 (This solution follows the method of Professor Roberts in class.): We have that

$$L[f'(x)] = \int_0^{c-} f'(x)e^{-px}dx + \int_{c+}^{\infty} f'(x)e^{-px}dx$$

$$= (f(c-)e^{-pc} - f(0)e^{-p*0}) + \int_0^{c-} pf(x)e^{-px}dx$$

$$+(0 - f(c+)e^{-pc}) + \int_{c+}^{\infty} pf(x)e^{-px}dx$$

(this is an integration by parts) $= e^{-pc}(f(c-) - f(c+)) - f(0) + pL[f(x)] = pL[f(x)] - f(0) - J(c)e^{-cp}$ as desired.

Solution 2 (This solution uses the result from class but not Professor Roberts' method.): We define the function g as follows: $g(x) = f(x) - J(c)u(x-c)$. We note that g satisfies (b) (as $J(c)u(x-c)$ has the same jump discontinuity as f at c) so by (4), $L[g'(x)] = pL[g(x)] - g(0)$. However, as $g' = f'$ except at a single point and $g(0) = f(0)$ this implies that $L[f'(x)] = pL[g(x)] - f(0)$. However, as $g(x) = f(x) - J(c)u(x-c), L[g(x)] = L[f(x)] - \frac{J(c)}{p}e^{-cp}$ by linearity and the second shift formula. Therefore, $L[f'(x)] = p(L[f(x)] - \frac{J(c)}{p}e^{-cp}) - f(0) = pL[f(x)] - f(0) - J(c)e^{-cp}$ as desired.

4. (This problem was worth 10 points: five for each part.)

Before we begin, we collect the following formulae for Laplace transforms: $L[y'] = pL[y] - y(0)$ and $L[y''] = p^2L[y] - py(0) - y'(0)$ we already know. In addition, we have $L[xy] = -L'[y]$, $L[xy'] = -(pL[y] - y(0))' = -L[y] - pL'[y]$, and $L[xy''] = -(p^2L[y] - py(0) - y'(0))' = -2pL[y] - p^2L'[y] + y(0)$.

(a) We apply the Laplace transform to both sides of $xy'' + (3x-1)y' - (4x+9)y = 0$, using Y to denote $L[y]$ for ease of notation. By the formulae above and linearity, this gives us $-2pY - p^2Y' + y(0) - 3Y - 3pY' - pY + y(0) +$

$4Y' - 9Y = 0$, or (multiplying by -1 and using the initial condition of $y(0) = 0$) $Y'(p^2 + 3p - 4) + Y(3p + 12) = 0$. This is a first order ODE; we place it in standard form by dividing both sides by $p^2 + 3p - 4 = (p + 4)(p - 1)$ and get $Y' + \frac{3Y}{p-1} = 0$. Multiplying both sides by the integrating factor $(p - 1)^3$ gives us $0 = Y'(p - 1)^3 + 3Y(p - 1)^2 = (Y(p - 1)^3)'$ so $Y(p - 1)^3 = C$ for some constant C. This implies that $y = L^{-1}[C(p - 1)^{-3}] = \frac{C}{2}x^2e^x$.

NOTE: The problem only asks you to find "one of the two independent solutions" so you can take any nonzero value for C; for example $\frac{1}{2}x^2e^x$ and x^2e^x are both acceptable solutions.

(b) Once again we apply the Laplace transform to both sides of the equation, $xy'' + (2x + 3)y' + (x + 3)y = 3e^{-x}$ and get $-2pY - p^2Y' + y(0) - 2Y - 2pY' + 3pY - 3y(0) - Y' + 3Y = \frac{3}{p+1}$, or, realizing $y(0) = 0$ from the initial condition and multiplying both sides by -1, $Y'(p^2 + 2p + 1) - Y(p + 1) = \frac{-3}{p+1}$. This is also a first order linear ODE; we place it in standard form by dividing both sides by $p^2 + 2p + 1 = (p + 1)^2$ to get $Y' - \frac{Y}{p+1} = \frac{-3}{(p+1)^3}$. After multiplying both sides by the integrating factor $\frac{1}{p+1}$, we get $\frac{Y'}{p+1} - \frac{Y}{(p+1)^2} = \frac{-3}{(p+1)^4}$; as the LHS is equal to $(\frac{Y}{p+1})'$, integrating gives us $\frac{Y}{p+1} = \frac{1}{(p+1)^3} + C$ or $Y = \frac{1}{(p+1)^2} + C(p + 1)$. As Y approaches 0 as p approaches infinity, $C = 0$ and therefore $Y = L^{-1}(\frac{1}{(p+1)^2}) = xe^{-x}$.

NOTE: Since this is not a homogeneous equation, scalar multiples of the solution xe^{-x} are not necessarily solutions (in fact, no nontrivial scalar multiple of xe^{-x} solves the differential equation.)

5. (I graded this problem based on completion: of the three parts you were supposed to do, (a), (b), and one of (c) - (f), a reasonably complete attempt was worth five points.)

(a) We begin by decomposing $\frac{120}{(p+1)(p+4)(p+9)}$ into partial fractions, i.e. we want to find real A, B, C with $\frac{A}{p+1} + \frac{B}{p+4} + \frac{C}{p+9} = \frac{120}{(p+1)(p+4)(p+9)}$. Multiplying both sides by $(p+1)(p+4)(p+9)$ gives $A(p+4)(p+9)+B(p+1)(p+9)+C(p+1)(p + 4) = 120$. Evaluating both sides at $p = -1$ gives $24A = 120$ so $A = 5$; evaluation at $p = -4$ yields $-15B = 120$ so $B = -8$ and evaluation at $p = -9$ yields $40C = 120$ so $C = 3$ and we have that $\frac{120}{(p+1)(p+4)(p+9)} = \frac{5}{p+1} - \frac{8}{p+4} + \frac{3}{p+9}$. Therefore, by linearity, the inverse Laplace transform of our desired expression is $5L^{-1}[\frac{1}{p+1}] - 8L^{-1}[\frac{1}{p+4}] + 3L^{-1}[\frac{1}{p+9}] = 5e^{-x} - 8e^{-4x} + 3e^{-9x}$.

(b) We use the decomposition of $\frac{120}{(p+1)(p+4)(p+9)}$ in (a) to find the decomposition of $\frac{120p}{(p^2+1)(p^2+4)(p^2+9)}$; replacing p by p^2 in $\frac{120}{(p+1)(p+4)(p+9)} = \frac{5}{p+1} - \frac{8}{p+4} + \frac{3}{p+9}$ yields $\frac{120}{(p^2+1)(p^2+4)(p^2+9)} = \frac{5}{p^2+1} - \frac{8}{p^2+4} + \frac{3}{p^2+9}$ and then multiplying both sides by p gives us $\frac{120p}{(p^2+1)(p^2+4)(p^2+9)} = \frac{5p}{p^2+1} - \frac{8p}{p^2+4} + \frac{3p}{p^2+9}$ so by linearity, the inverse Laplace transform of our desired expression is expression is $5L^{-1}[\frac{p}{p^2+1}] - 8L^{-1}[\frac{p}{p^2+4}] + 3L^{-1}[\frac{p}{p^2+9}] = 5\cos(x) - 8\cos(2x) + 3\cos(3x)$.

(c) We seek to express $\frac{p^3+3p^2+10p-2}{(p-1)^2(p+2)(p+3)}$ in partial fractions, which we do as follows.

We seek real numbers A, B, C, D such that $\frac{p^3+3p^2+10p-2}{(p-1)^2(p+2)(p+3)} = \frac{A}{p-1} + \frac{B}{(p-1)^2} +$

$\frac{C}{p+2} + \frac{D}{p+3}$. Multiplying both sides by $(p-1)^2(p+2)(p+3)$ yields

$p^3+3p^2+10p-2 = A(p-1)(p+2)(p+3)+B(p+2)(p+3)+C(p-1)^2(p+3)+D(p-1)^2(p+2)$. Evaluating both sides at $p=1$ yields $12 = 12B$ so $B=1$, evaluating both sides at $p=-2$ yields $-18 = 9C$ so $C=-2$, and evaluating both sides at $p=-3$ yields $-32 = -16D$ so $D=2$. Comparing the leading coefficients of $p^3+3p^2+10p-2 = A(p-1)(p+2)(p+3)+(p+2)(p+3)-2(p-1)^2(p+3)+2(p-1)^2(p+2)$ yields $A=1$ and our decomposition is therefore $\frac{p^3+3p^2+10p-2}{(p-1)^2(p+2)(p+3)} = \frac{1}{p-1} + \frac{1}{(p-1)^2} + \frac{-2}{p+2} + \frac{2}{p+3}$. By linearity, we therefore have that our desired inverse Laplace transform is $L^{-1}[\frac{1}{p-1}] + L^{-1}[\frac{1}{(p-1)^2}] - 2L^{-1}[\frac{1}{p+2}] + 2L^{-1}[\frac{1}{p+3}] = e^x + xe^x - 2e^{-2x} + 2e^{-3x}$.

(d) Noting that p^2+p+1 is irreducible, we seek to provide a partial fractions decomposition by producing real A, B, C, D with $\frac{4p^2+5p-12}{(p-1)(p-2)(p^2+p+1)} = \frac{A}{p-1} + \frac{B}{p-2} + \frac{Cp+D}{p^2+p+1}$. Multiplying both sides by $(p-1)(p-2)(p^2+p+1)$ yields $A(p-2)(p^2+p+1)+B(p-1)(p^2+p+1)+(Cp+D)(p-1)(p-2) = 4p^2+5p-12$. Evaluating both sides of this expression at $p=1$ gives $-3A=-3$ or $A=1$ and evaluating both sides at $p=2$ yields $7B=14$ or $B=2$. Comparing leading coefficients of $(p-2)(p^2+p+1)+2(p-1)(p^2+p+1)+(Cp+D)(p-1)(p-2) = 4p^2+5p-12$ gives $1+2+C=0$ or $C=-3$ and comparing constant coefficents gives $-2-2+2D=-12$ or $D=-4$ so the desired decomposition is $\frac{4p^2+5p-12}{(p-1)(p-2)(p^2+p+1)} = \frac{1}{p-1} + \frac{2}{p-2} + \frac{-3p-4}{p^2+p+1}$. We note that $p^2+p+1 = (p+.5)^2+.75$ so, writing $-3p-4 = -3(p+.5)-2.5$ gives that the desired inverse transform is (by linearity, as usual) $L^{-1}[\frac{1}{p-1}] + 2L^{-1}[\frac{1}{p-2}] - 3L^{-1}[\frac{p+.5}{(p+.5)^2+.75}] - 2.5L^{-1}[\frac{1}{(p+.5)^2+.75}] = e^x + 2e^{2x} - 3e^{-.5x}\cos(\frac{\sqrt{3}x}{2}) - \frac{5\sqrt{3}}{3}e^{-.5x}\sin(\frac{\sqrt{3}x}{2})$.

(e) As in (d) we proceed by seeking A, B, C, D real with $\frac{2p^3+6p^2+21p+52}{p(p+2)(p^2+4p+13)} = \frac{A}{p} + \frac{B}{p+2} + \frac{Cp+D}{p^2+4p+13}$. Multiplying both sides by $p(p+2)(p^2+4p+13)$ gives us $2p^3+6p^2+21p+52 = A(p+2)(p^2+4p+13)+Bp(p^2+4p+13)+(Cp+D)p(p+2)$. Evaluating both sides at $p=0$ gives $52 = 26A$ or $A=2$ and evaluating both sides at $p=-2$ gives $18=-18B$ or $B=-1$. Comparing leading coefficients of $2p^3+6p^2+21p+52 = 2(p+2)(p^2+4p+13) - p(p^2+4p+13) + (Cp+D)p(p+2)$ gives $2 = 2-1+C$ or $C=1$; comparing LINEAR coefficients gives $21 = 26+16-13+2D$ so $D=-4$ and our partial fractions decomposition becomes $\frac{2p^3+6p^2+21p+52}{p(p+2)(p^2+4p+13)} = \frac{2}{p} + \frac{-1}{p+2} + \frac{p-4}{p^2+4p+13}$. Writing $p^2+4p+13 = (p+2)^2+3^2$ means we can rewrite our expression as $\frac{2}{p} + \frac{-1}{p+2} + \frac{(p+2)-6}{(p+2)^2+3^2}$, whose inverse Lapace transform (by linearity) is $2L^{-1}[\frac{1}{p}] - L^{-1}[\frac{1}{p+2}] + L^{-1}[\frac{p+2}{(p+2)^2+3^2}] - 2L^{-1}[\frac{3}{(p+2)^2+3^2}] = 2 - e^{-2x} + e^{-2x}\cos 3x - 2e^{-2x}\sin 3x$.

(f) Here we seek to find real numbers A, B, C, D with $\frac{4(p^3-p^2+1)}{(p+1)^2(p-1)^2} = \frac{A}{p+1} + \frac{B}{(p+1)^2} + \frac{C}{p-1} + \frac{D}{(p-1)^2}$; multiplying both sides by $(p+1)^2(p-1)^2$ yields $4(p^3-p^2+1) = A(p+1)(p-1)^2 + B(p-1)^2 + C(p+1)^2(p-1) + D(p+1)^2$. Evaluating both sides at $p=-1$ gives us $-4 = 4B$ or $B=-1$ and evaluating both sides at $p=1$ gives $4=4D$ or $D=1$. Comparing the leading and constant terms of $4(p^3-p^2+1) = A(p+1)(p-1)^2 - (p-1)^2 + C(p+1)^2(p-1) + (p+1)^2$ gives us $A+C=4$ and $A-C=4$ so $A=4$ and $C=0$ making our decomposition $\frac{4(p^3-p^2+1)}{(p+1)^2(p-1)^2} = \frac{4}{p+1} - \frac{1}{(p+1)^2} + \frac{1}{(p-1)^2}$ whose inverse Laplace transform (by linearity) is $4L^{-1}[\frac{1}{p+1}] - L^{-1}[\frac{1}{(p+1)^2}] + L^{-1}[\frac{1}{(p+1)^2}] = 4e^{-x} - xe^{-x} + xe^{x}$.

6. (I also graded this problem based on completeness; a reasonably complete attempt at each transform was worth five points each.)

We begin by recalling the formula $L[\frac{f(x)}{x}] = \int_p^\infty F(p)dp$ where F is the Laplace transform of f. Armed with this formula we can calculate our transforms.

$L[\frac{1-e^{-x}}{x}]$: As $L[1-e^{-x}] = \frac{1}{p} - \frac{1}{p+1}$, our desired Laplace transform is $\int_p^\infty(\frac{1}{p} - \frac{1}{p+1})dp = -\ln p - (-\ln(p+1)) = \ln(\frac{p+1}{p})$.

$L[\frac{1-\cos x}{x}]$: As $L[1-\cos x] = \frac{1}{p} - \frac{p}{p^2+1}$, our desired Laplace transform is $\int_p^\infty(\frac{1}{p} - \frac{p}{p^2+1})dp = -\ln p - -(\frac{1}{2}\ln(p^2+1)) = \ln(\frac{\sqrt{p^2+1}}{p})$.

$L[\frac{\sinh x}{x}]$: As $L[\sinh x] = \frac{1}{p^2-1} = \frac{1}{2}(\frac{1}{p-1} - \frac{1}{p+1})$, our desired Laplace transform is $\frac{1}{2}\int_p^\infty(\frac{1}{p-1} - \frac{1}{p+1})dp = \frac{1}{2}(-\ln(p-1) - (-\ln(p+1)) = \frac{1}{2}\ln(\frac{p+1}{p-1})$.

7. (This problem was worth ten points: five for each part.)

We recall the formula $\int_0^\infty \frac{f(x)}{x}dx = \int_0^\infty F(p)dp$ where F refers to the Laplace transform of f.

(a) As $L[e^{-ax} - e^{-bx}] = \frac{1}{p+a} - \frac{1}{p+b}$,

the above formula gives us $\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}dx$

$= \int_0^\infty \frac{1}{p+a} - \frac{1}{p+b}dp$

which is equal to the limit as r goes to infinity of $\int_0^r \frac{1}{p+a} - \frac{1}{p+b}dp = (\ln r - \ln a) - (\ln r - \ln b) = \ln b - \ln a = \ln\frac{b}{a}$; as this is constant regardless of r our desired integral indeed evaluates to $\ln\frac{b}{a}$.

(b) As $L[e^{ax}\sin bx] = \frac{b}{(p+a)^2+b^2}$, the above formula gives us $\int_0^\infty \frac{e^{ax}\sin bx}{x}dx = \int_0^\infty \frac{b}{(p+a)^2+b^2}dp$.

As the integrand has antiderivative $\arctan(\frac{p+a}{b})$, which approaches $\frac{\pi}{2}$ as p approaches infinity our integral becomes $\frac{\pi}{2} - \arctan(\frac{a}{b})$, which is equal to $\arctan(\frac{b}{a})$ by trigonometry (if θ is an angle in the first quadrant then $\tan(\frac{\pi}{2}-\theta)$ is the reciprocal of $\tan\theta$.)