1i. (This part was worth ten points.)

Setting $s_0 = 0$, for each positive integer n we (noting that the harmonic series diverges) set $s_n = s_{n-1} + \frac{1}{n}$ modulo 2π (if $s_{n-1} + \frac{1}{n} > \pi$ then $s_n = s_{n-1} + \frac{1}{n} - 2\pi$).

Consequently we can define each y_n to be the (piecewise linear and continuous) function to be defined as follows: $y_n = 1$ on $[s_{n-1}, s_n]$, y_n is linear with slope n on $[s_{n-1} - 1/n, s_{n-1}]$ (joining $(s_{n-1} - 1/n, 0)$ to $(s_{n-1}, 1)$), linear with slope $-n$ on $[s_n, s_n + 1/n]$ (joining $(s_n, 1)$ to $(s_n + 1/n, 0)$) and zero everywhere else (note that all computations here are modulo 2π).

As the y_n are bounded by 1 and nonzero except for a set of length $3/n$, $||y_n||_2 \leq \sqrt{\frac{3}{n}}$, which clearly goes to zero; however, the divergence of the harmonic series tells us that each point will be covered in infinitely many intervals of the form $[s_{n-1}, s_n]$, where $f_n = 1$, so no pointwise limit exists. The y_n are continuous by construction so it is possible for continuous y_n.

ii. (This part was worth ten points).

For each n we let y_n be the piecewise linear function joining $(0, 0)$ to $(1/n, n)$ (with slope n^2 on this interval) and $(1/n, n)$ to $(2/n, 0)$ (with slope $-n^2$ here) and equal to zero outside of $[0, 2/n]$. As every nonzero point is outside of $[0, 2/n]$ for sufficiently large n and $y_n(0) = 0$ for all n, the y_n do converge pointwise to the zero function. However, we note that $|y_n| \geq n/2$ on an interval of length $1/n$ ($[.5/n, 1.5/n]$) which implies that the integral of $|y_n|^2$ is at least $(n^2/4) * (1/n) = n/4$, so $||y_n||_2 \geq \frac{\sqrt{n}}{2}$ which goes to infinity, not zero, as n goes to infinity. As the y_n were continuous, this is again possible for continuous n.

iii. (This part was worth five points.)

Fix $\epsilon > 0$. There exists N such that for each $n > N$, $|y_n(x)| < \frac{\epsilon}{\sqrt{2\pi}}$ (by the definition of uniform convergence); for such n, $(||y_n - 0||_2)^2 < \frac{\epsilon^2}{2\pi} * 2\pi = \epsilon^2$ so $||y_n||_2 < \epsilon$, i.e. y_n converges to 0 in L^2.

2i. (This part was worth five points.)

For the steady state temperature, $f_t = 0 = f_{\theta\theta}$ which implies that f_θ is constant and f is linear in θ. The only linear function joining $(0, 10)$ to $(\pi, 0)$ is $f = 10 - \frac{10}{\pi}\theta$ which is indeed our steady state solution.

ii. (This part was worth ten points.)

Set $g_0(\theta) = f_0(\theta) - (m_1 + \frac{m_2 - m_1}{\pi}\theta)$ and $g(t, \theta) = f(t, \theta) - (m_1 + \frac{m_2 - m_1}{\pi}\theta)$; then (as the function we are subtracting is constant with respect to t and linear with respect to θ) we note that $g_t = g_{\theta\theta}$ with $g(0, \theta) = g_0$, $g(t, 0) = 0 = g(t, \pi)$.

Using separation of variables as in Simmons, we search for solutions of the heat equation subject to the boundary conditions, writing $g = X(t)Y(\theta)$ and note $X'(t)Y(\theta) = X(t)Y''(\theta)$; upon dividing both sides by g, $\frac{X'(t)}{X(t)} = \frac{Y''(\theta)}{Y(\theta)}$; suppose they both equal some constant λ. If λ is positive then Y is of the form $C_1 e^{r\theta} + C_2 e^{-r\theta}$ for some positive r, which cannot vanish at both endpoints unless $C_1 = C_2 = 0$. If $\lambda = 0$ then Y is linear and therefore cannot vanish at both points unless $Y = 0$. Therefore, λ is negative; writing n with $\lambda = -n^2$ gives us that $Y = C_1 \cos n\theta + C_2 \sin n\theta$; $Y(0) = 0$ tells us $C_1 = 0$ and $Y(\pi) = 0$ tells us n is an integer. Therefore, our separable solutions are of the form $\sin n\theta e^{-n^2 t}$.

This tells us $g = \Sigma_n C_n \sin(n\theta) e^{-n^2 t}$ for some constants C_n; we can find C_n

by computing the Fourier sine series of g: $C_n = \frac{2}{\pi} \int_0^\pi g_0(\theta) d\theta$.

With these values of C_n, we therefore have

$$f = m_1 + \frac{m_2 - m_1}{\pi}\theta + \Sigma_n (C_n \sin(n\theta) e^{-n^2 t}).$$