1. (This problem was worth ten points).

Introducing a new variable u equal to $\frac{dx}{dt}$ turns equation (1) with the first two initial conditions from (3) into the pair of equations $\frac{dx}{dt} = u$, $\frac{du}{dt} = \frac{f(t,x,y)}{m}$ with the initial conditions $x(t_0) = x_0, u(t_0) = x'_0$.

Introducing a new variable v equal to $\frac{dy}{dt}$ turns equation (1) with the last two initial conditions from (3) into the pair of equations $\frac{dy}{dt} = v$, $\frac{dy}{dt} = \frac{g(t,x,y)}{m}$ with the initial conditions $y(t_0) = y_0, v(t_0) = w'_0$.

To make this system autonomous, we need to encapsulate t as part of the system; for this we introduce a new dependent variable w that satisfies the differential equation $\frac{dw}{dt} = 1$ subject to the initial condition $w(t_0) = t_0$ which lets us replace the t's in $f(t,x,y)$ and $g(t,x,y)$ by w's, turning our system into

$$\frac{dx}{dt} = u$$
$$\frac{du}{dt} = \frac{f(w,x,y)}{m}$$
$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = \frac{g(w,x,y)}{m}$$
$$\frac{dw}{dt} = 1$$
$$x(t_0) = x_0, x'(t_0) = x'_0, y(t_0) = y_0, y'(t_0) = y'_0, w(t_0) = t_0.$$

2. (This problem was worth 30 points: ten for each model).

For the first model, the auxillary equation is $m^2 + bc = 0$ so $m = \pm i\sqrt{b}c$. For solutions of the form $(Ae^{i\sqrt{-bc}t}, Be^{i\sqrt{-bc}t})$, the conditions to be satisfied are $Ai\sqrt{-bc} = bB$ and $Bi\sqrt{-bc} = cA$ both of which are equivalent to $A/B = -i\sqrt{-b/c}$. This gives us a solution

$$(R,J) = (-iB\sqrt{-b/c}e^{i\sqrt{-bc}t}, Be^{i\sqrt{-bc}t})$$
$$= B(\sqrt{-b/c}\sin\sqrt{-bc}t - i\sqrt{-b/c}\cos\sqrt{-bc}t, \cos\sqrt{-bc}t + i\sin\sqrt{-bc}t$$

from which, upon taking real and imaginary parts, we get the general solution

$$(C_1\sqrt{-b/c}\sin\sqrt{-bc}t - C_2\sqrt{-b/c}\cos\sqrt{-bc}t,$$
$$C_1\sqrt{-b/c}\cos\sqrt{-bc}t + C_2\sqrt{-b/c}\sin\sqrt{-bc}t.$$

No matter what the initial conditions are, after a certain period of time (R,J) will become the additive inverse of their initial values and then they will return to the starting point to repeat the cycle. In other words, we have a harmonic oscillator in action.

For our second model, the auxillary equation becomes $(1-m)(-1-m)-1 = 0$, or $m^2 - 2 = 0$; in other words, $m = \pm\sqrt{2}$. To find the solution $(Ae^{\sqrt{2}t}, Be^{\sqrt{2}t})$

we want to solve $A\sqrt{2} = A + B$ and $B\sqrt{2} = A - B$, both of which tell us $A = B(\sqrt{2}+1)$ so $((\sqrt{2}+1)e^{\sqrt{2}t}, e^{\sqrt{2}t}$ is a solution. To find the solution $(Ae^{-\sqrt{2}t}, Be^{-\sqrt{2}t})$ we want to solve $-A\sqrt{2} = A + B$ and $-B\sqrt{2} = A - B$, both of which tell us $A = B(-\sqrt{2}+1)$ so $((-\sqrt{2}+1)e^{-\sqrt{2}t}, e^{-\sqrt{2}t}$ is a solution. Therefore, the general solution is

$$(C_1(\sqrt{2}+1)e^{\sqrt{2}t} + C_2(-\sqrt{2}+1)e^{-\sqrt{2}t}, C_1e^{\sqrt{2}t} + C_2e^{-\sqrt{2}t}.$$

For $R(0) = J(0) = 1$ we know that $C_1 + C_2 = 1$ from the J condition and therefore $1 = (C_1 - C_2)\sqrt{2} + (C_1 + C_2) = (C_1 - C_2)\sqrt{2} + 1$ so $C_1 = C_2 = .5$, i.e.

$$(R, J) = (.5(\sqrt{2}+1)e^{\sqrt{2}t} + .5(-\sqrt{2}+1)e^{-\sqrt{2}t}, .5e^{\sqrt{2}t} + .5e^{-\sqrt{2}t}).$$

We can see that the $e^{\sqrt{2}t}$ terms dominate for large t so both R and J go off to infinity and their love grows ever stronger.

For $R(0) = -1, J(0) = 1$ we know that $C_1 + C_2 = 1$ from the J condition and therefore $-1 = (C_1 - C_2)\sqrt{2} + (C_1 + C_2) = (C_1 - C_2)\sqrt{2} + 1$ so $C_1 - C_2 = -\sqrt{2}$ and therefore $(C_1, C_2) = (.5 - .5\sqrt{2}, .5 + .5\sqrt{2})$, i.e.

$$(R, J) = (-.5e^{\sqrt{2}t} + -.5e^{-\sqrt{2}t}, (.5 - .5\sqrt{2})e^{\sqrt{2}t} + (.5 + .5\sqrt{2})e^{-\sqrt{2}t}).$$

We can see that the $e^{\sqrt{2}t}$ terms dominate for large t so R and J go off to negative infinity, which means that Romeo's hatred grows ever stronger while Juliet's love turns into hatred and becomes almost as strong as Romeo's.

For our last model, the auxillary equation becomes $(1-m)(-1-m)+2 = 0$, or $m^2 - 2 = 0$; in other words, $m = \pm i$.

For solutions of the form (Ae^{it}, Be^{it}), the conditions to be satisfied are $Ai = A + 2B$ and $Bi = A - B\sqrt{bc} = cA$ both of which are equivalent to $A/B = 1 + i$. This gives us a solution

$$(R, J) = ((1+i)Be^{it}, Be^{it})$$

$$= B(\cos t - \sin t + i(\cos t + \sin t), \cos t + i \sin t$$

from which, upon taking real and imaginary parts, we get the general solution

$$(C_1(\cos t - \sin t) + C_2(\cos t + \sin t), C_1 \cos t + C_2 \sin t).$$

If $R(0) = J(0) = 1$ then $C_1 = 1$ from the second equation so $C_2 = 0$ from the first so we get $(\cos t - \sin t, \cos t)$ and the system oscillates with period 2π. For $R(0) = -1$, $J(0) = 1$ then $C_1 = 1$ from the second equation but $C_2 = -2$ from the first so we get $(-\cos t - 3\sin t, \cos t - 2\sin t)$ and once again the system oscillates with period 2π.

The formula of the general solution says that this type of oscillation with period 2π will always happen, independent of initial conditions so the affair will never degenerate into complete apathy unless that was the starting point.