Problems, 215B

Do 10 problems. Due March 17.

1. Let $a_1, \dots a_r$ be an R-regular sequence. Prove that for any positive integers $k_1, \dots k_r$ the sequence $a_1^{k_1}, \dots, a_r^{k_r}$ is also R-regular.

2. Let $a_1, \dots, a_n$ be a n R-regular sequence of a Noetherian local ring R and $I = \langle a_1, \dots, a_n \rangle$. Prove that for any finitely generated R-module M and any i there are canonical isomorphisms

$$\mathrm{Tor}_i^R\big(R/I, M\big) \simeq H_i(a, M) \simeq \mathrm{Ext}_R^{n-i}\big(R/I, M\big).$$

3. Let $R = F[t_1, t_2, t_3]$ with F a field and $a_1 = t_1(t_2-1)$, $a_2 = t_2$, $a_3 = t_3(t_2-1)$. Prove that a_1, a_2, a_3 is an R-regular sequence while a_1, a_3, a_2 is not.

4. Show that a Noetherian local ring R is regular if and only if the maximal ideal of R can be generated by an R-regular sequence.

5. Let R be a local C.M. ring and G a finite group of ring automorphisms of R. Prove that if the order of G is invertible in R then the subring R^G of G-invariant elements in R is also C.M.

6. Let I be a prime ideal of the ring $R = F[t_1, t_2, t_3]$ with F a field. Prove that the ring R/I is C.M.

7. Prove that the subring $F[t_1^4, t_1^3t_2, t_1t_2^3, t_2^4]$ of $F[t_1, t_2]$ with F a field is not C.M.

8. Let R be the localization of $F[t_1, t_2, t_3, t_4]$ (F a field) at the maximal ideal $\langle t_1, t_2, t_3, t_4 \rangle$ and I the ideal of R generated by t_1t_2, t_3t_4 and $t_1t_3 + t_2t_4$. Determine $\dim(R/I)$ and $\mathrm{depth}(R/I)$.

9. Let $f : L \to R$ and $f' : L' \to R$ be two R-linear maps. Let $g : L \oplus L' \to R$ be the map defined by $g(l + l') = f(l) + f'(l')$. Show that the Koszul complex $K(g)$ is isomorphic to $K(f) \otimes K(f')$.

10. Prove that if I is a prime ideal of a complete intersection ring R then the localization R_I is also a complete intersection ring.

11. Find an example of a Noetherian local ring R and a finitely generated R-module M such that $\operatorname{depth} M > \operatorname{depth} R$.

12. Let M be a finitely generated module over a Noetherian local ring R with maximal ideal P. Prove that M is free if and only if $\operatorname{Tor}_1^R(R/P, M) = 0$.

13. Let R be a Noetherian local ring with maximal ideal P. Show that if the R-module R/P has a finite free resolution of length n then every finitely generated R-module has a finite free resolution of length n.