Problems, 215B

Do 15 problems. Due Dec 5.

1. Let $A = A_0 \oplus A_1 \oplus \dots$ be a graded commutative ring. Prove that A is Noetherian if and only if A_0 is Noetherian and the A_0-algebra A is finitely generated.

2. Let R be a local ring, $f \in R[t_1, \dots, t_n]$ a polynomial. Suppose that one of the coefficients of f is invertible in R. Prove that f is a non-zero-divisor in $R[t_1, \dots, t_n]$.

3. Let R be a complete Noetherian local ring with maximal ideal P. Suppose there is a subfield $K \subset R$ mapping isomorphically onto R/P. Let $x_1, \dots, x_d$ be a system of parameters of R. Prove that R is a finitely generated module over the subring $K[[x_1, \dots, x_d]]$.

4. Let f and g be functions $\{0, 1, 2, \dots\} \to \mathbb{Z}$ such that $f(n+1) - f(n) = g(n)$ for all $n \geq 0$. Prove that if g is a polynomial of degree d then f is a polynomial of degree $d + 1$.

5. Let R be a Noetherian local ring with maximal ideal P and $x \in P$. Prove that

$$\dim R \geq \dim(R/xR) \geq \dim R - 1.$$

6. Let R be a Noetherian local ring with maximal ideal P and $a_1, a_2, \dots, a_k \in P$. Prove that $\{a_1, a_2, \dots, a_k\}$ is a part of a system of parameters if and only if $\dim\bigl(R/\langle a_1, a_2, \dots, a_k\rangle\bigr) = \dim R - k$.

7. Let $R \to S$ be a flat local homomorphism of Noetherian local rings, P the maximal ideal of R. Prove that is R and S/PS are regular then so is S.

8. Let $a_1, \dots a_r$ be an R-regular sequence. Prove that for any positive integers $k_1, \dots k_r$ the sequence $a_1^{k_1}, \dots, a_r^{k_r}$ is also R-regular.

9. Let $a_1, \dots, a_n$ be a n R-regular sequence of a Noetherian local ring R and $I = \langle a_1, \dots, a_n\rangle$. Prove that for any finitely generated R-module M and any i there are canonical isomorphisms

$$\mathrm{Tor}_i^R\bigl(R/I, M\bigr) \simeq H_i(a, M) \simeq \mathrm{Ext}_R^{n-i}\bigl(R/I, M\bigr).$$

10. Let $R = F[t_1, t_2, t_3]$ with F a field and $a_1 = t_1(t_2 - 1)$, $a_2 = t_2$, $a_3 = t_3(t_2 - 1)$. Prove that a_1, a_2, a_3 is an R-regular sequence while a_1, a_3, a_2 is not.

11. Show that a Noetherian local ring R is regular if and only if the maximal ideal of R can be generated by an R-regular sequence.

12. Let R be a local C.M. ring and G a finite group of ring automorphisms of R. Prove that if the order of G is invertible in R then the subring R^G of G-invariant elements in R is also C.M.

13. Let I be a prime ideal of the ring $R = F[t_1, t_2, t_3]$ with F a field. Prove that the ring R/I is C.M.

14. Prove that the subring $F[t_1^4, t_1^3 t_2, t_1 t_2^3, t_2^4]$ of $F[t_1, t_2]$ with F a field is not C.M.

15. Let R be the localization of $F[t_1, t_2, t_3, t_4]$ (F a field) at the maximal ideal $\langle t_1, t_2, t_3, t_4 \rangle$ and I the ideal of R generated by $t_1 t_2$, $t_3 t_4$ and $t_1 t_3 + t_2 t_4$. Determine $\dim(R/I)$ and $\operatorname{depth}(R/I)$.

16. Let $f : L \to R$ and $f' : L' \to R$ be two R-linear maps. Let $g : L \oplus L' \to R$ be the map defined by $g(l + l') = f(l) + f'(l')$. Show that the Koszul complex $K(g)$ is isomorphic to $K(f) \otimes K(f')$.

17. Prove that if I is a prime ideal of a complete intersection ring R then the localization R_I is also a complete intersection ring.

18. Find an example of a Noetherian local ring R and a finitely generated R-module M such that $\operatorname{depth} M > \operatorname{depth} R$.

19. Let M be a finitely generated module over a Noetherian local ring R with maximal ideal P. Prove that M is free if and only if $\operatorname{Tor}_1^R(R/P, M) = 0$.

20. Let R be a Noetherian local ring with maximal ideal P. Show that if the R-module R/P has a finite free resolution of length n then every finitely generated R-module has a finite free resolution of length n.